O‘ZBEKISTON RESPUBLIKASI
OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

NAMANGAN DAVLAT UNIVERSITETI

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INVOLYUTSIYA QATNASHGAN DIFFERENSIAL TENGLAMALAR

USLUBIY QO‘LLANMA

Namangan – 2023

Ushbu uslubiy qo‘llanma Namangan davlat universiteti Ilmiy Kengashining 2023-yil _ _______dagi majlisida ko‘rib chiqilgan va chop etishga tavsiya etilgan (_________).

Uslubiy qo‘llanma universitetlarning matematika, amaliy matematika, informatika yo‘nalishlari talabalari uchun mo‘ljallangan. Unda differensial tenglamaning argumentida involyutsiya xossasiga ega bo‘lgan funksiya qatnashgan holada yechimi mavjudligi ko‘rsatilgan. Xar bir paragrafda ko‘plab misollar yechimi bilan keltirilgan.

Uslubiy qo‘llanma Namangan davlat universiteti Ilmiy Kengashi tomonidan nashrga tavsiya etilgan (2023 yil __________ bayonnomasi)

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SO‘Z BOSHI

Oddiy differensial tenglamalar fanining XXI asrdagi rivoji va uning xalq xo‘jaligining turli tarmoqlariga tadbiqlari o‘zining ilmiy va amaliy ahamiyati bilan ajralib turadi.

Ushbu uslubiy qo‘llanmada invoyutsiya tushunchasi batafsil keltirib o‘tilgan bo‘lib, involyutsiya qatnashgan oddiy va xususiy hosilalai differensial tenglamalarga oid ko‘plab ma’lumotlar va misollar keltirilgan. Uslubiy qo‘llanma uch bobdan iborat.

Qo‘llanmaning birinchi bobi tayanch tushunchalardan tashkil topgan bo‘lib, involyutsiya tushunchasini o‘rganish uchun boshlang‘ich ma’lumotlar: o‘zgarmas koffisientlnvi chiziqli differensial tenglamalar, o‘zgarmasni variatsiyalash usuli, involyutsiya va uning xossalari, Eyler va Lagranj tenglamalari bayon qilingan.

Ikinchi bobda invoyutsiya xossasini o‘rganish uchun, oddiy differensial tenglamalar nazariyasidan ma’lumotlar, oddiy differensial tenglamalar involyutsiyasi chiziqli differensial tenglamalar invoyutsiyasi, chiziqli differensial tenglamalar sistemasining involyutsiyasi keltirilgan va ularga doir bir qancha misollarni ishlab umumiy yechimlari keltirib chiqarilgan.

Uchinchi bobda involyutsiya xossasiga va maxsus potensialga ega bo‘lgan xususiy hosilali differensial tenglama uchun aralash masalaning qo‘yilishi va uning klassik yechimini toppish ko‘rsatilgan.

1-BOB. ASOSIY TUSHUNCHALAR.

Involyutsiya xossasiga ega bo‘lgan oddiy differensial tenglamalarni yechish jarayonida almashtirishlardan so‘ng Eyler yoki Lagranj tenglamalari hosil bo‘ladi. O‘z navbatida bu tenglamalar almashtirishlar yordamida o‘zgarmas koeffisiyentli chiziqli tenglamalarga olib o‘tilishi va yechilishi mumkin. Shuning uchun bu bobda asosiy tushunchalar qatorida o‘zgarmas koeffisiyentli chiziqli differensial tenglamalar uchun Eyler va Lagranj tenglamalarini ko‘rib chiqamiz.

1.1. O‘zgarmas koeffisiyentli chiziqli differensial tenglamalar.

Biz $n-$tartibli o‘zgarmas koeffisiyentli chiziqli

$$L[y] \equiv y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(t) \qquad (1.1.1)$$

differensial tenglamani qaraymiz, bu yerda $a_1, a_2, ..., a_n -$o‘zgarmas kompleks sonlar, $f(t)$ qandaydir oraliqda berilgan t o‘zgaruvchining kompleks funksiyasi. (1.1.1) tenglamaning chap qismi $n-$tartibli chiziqli differensial operator deyiladi va $L[y]$ kabi belgilanadi. (1.1.1) tenglamaning o‘zi esa

$$L[y] = f(t) \qquad (1.1.2)$$

ko‘rinishda yoziladi.

Dastlab $n-$ tartibli bir jinsli o‘zgarmas koeffisiyentli tenglamani qaraymiz.

1. **Bir jinsli tenglamalar.** Har bir

$$L[y] \equiv y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$

operatorga yoki bir jinsli

$$L[y] = 0 \qquad (1.1.3)$$

tenglamaga (1.1.3) tenglamaning yoki $L[y]$ operatorning xarakteristik ko‘phadi deb nomlangan

$$D(p) = p^n + a_1 p^{n-1} + \cdots + a_0$$

(1.1.4) ko‘phadni mos qo‘yamiz.

Lemma 1.1.1. Ixtiyoriy n marta uzluksiz differensiallanuvchi $f(t)$ funksiya uchun quyidagi:

$$L\left[e^{\lambda t} f(t)\right]=e^{\lambda t}\left(D(\lambda) f+\frac{D'(\lambda)}{1!} f'+\cdots+\frac{D^{(n)}(\lambda)}{n!} f^{(n)}\right) \qquad (1.1.5)$$

formula o‘rinli.

Isboti. $L[y]=y^{(k)}$, $(0 \le k \le n)$ bo‘lsin, u holda $D(p)=p^k$ bo‘lib, agar $l>k$ bo‘lganda $D^{(l)}(\lambda)=0$ bo‘lgani uchun

$$L\left[e^{\lambda t} f(t)\right]=\frac{d^k}{dt^k}\left(e^{\lambda t} f(t)\right)=\sum_{l=0}^{k} C_k^l \frac{d^{k-l}}{dt^{k-l}}(e^{\lambda t}) \cdot f^{(l)}(t)=\sum_{l=0}^{k} \frac{k(k-1)...(k-l+1)}{l!} \lambda^{k-l} e^{\lambda t} f^{(l)}(t)=$$

$$=e^{\lambda t} \sum_{l=0}^{k} \frac{1}{l!} \frac{d^l}{d\lambda^l}(\lambda^k) \cdot f^{(l)}(t)=e^{\lambda t} \sum_{l=0}^{k} \frac{D^{(l)}(\lambda)}{l!} f^{(l)}(t)=e^{\lambda t} \sum_{l=0}^{n} \frac{D^{(l)}(\lambda)}{l!} f^{(l)}(t)$$

Biz bu yerda ikki funksiya ko‘payitmasining $k-$ tartibli hosilasini hisoblashda Leybnits formulasidan foydalandik.

Demak, (1.1.5) formula $L[y]=y^{(k)}$ xususiy hol uchun isbotlandi. (1.5.1) formulaning umumiy holda to‘g‘riligi $L[y]$ operatorning $L[y]=y^{(k)}$, $(0 \le k \le n)$ ko‘rinishdagi operatorlarning chiziqli kombinatsiyasidan iborat ekanligidan kelib chiqadi.

Lemma isbotlandi.

Lemma 1.1.2. $\lambda-$ soni $D(p)$ xarakteristik ko‘phadning k karrali ildizi bo‘lsa, u holda $y_1=e^{\lambda t}$, $y_2=te^{\lambda t}, ..., y_k=t^{k-1}e^{\lambda t}$ funksiyalar bir jinsli (1.1.3) tenglamaning yechimlari bo‘ladi.

Isboti. $\lambda-$ soni $D(p)$ xarakteristik ko‘phadning k karrali ildizi bo‘lgani uchun $D(\lambda)=D'(\lambda)=\cdots=D^{(k-1)}(\lambda)=0, D^{(k)}(\lambda) \ne 0$

(1.1.5) formulani $y_j=t^j e^{\lambda t}$ $(0 \le j \le k-1)$ funksiyalarga tadbiq qilib,

$$L\left[y_{j+1}\right]=L\left[e^{\lambda t} t^j\right]=e^{\lambda t}\left(\frac{D^{(k)}(\lambda)}{k!} \frac{d^k}{dt^k}\left(t^j\right)+\cdots+\frac{D^{(n)}(\lambda)}{n!} \frac{d^n}{dt^n}\left(t^j\right)\right)=0$$

chunki $l>j$ bo‘lganda $\frac{d^l}{dt^l}\left(t^j\right)=0$. Lemma isbotlandi.

Lemma 1.1.3. $r=0,1,2,...,k-1$ uchun $L\left[t^r e^{\lambda t}\right]\Big|_{t=0}=0$ bo‘lsin. U holda λ soni xarakteristik ko‘phadning karraligi k dan kichik bo‘lmagan ildizi bo‘ladi.

Isboti. 1.1.1-lemmaga ko‘ra

$$L\left[t^r e^{\lambda t}\right]\Big|_{t=0}=e^{\lambda t}\left(D(\lambda)t^r+\frac{D'(\lambda)}{1!}\frac{d\left(t^r\right)}{dt}+\cdots+\frac{D^{(r)}(\lambda)}{r!}\frac{d^r\left(t^r\right)}{dt^r}\right)\Bigg|_{t=0}$$

chunki $k>r$ bo‘lganda $\frac{d^k}{dt^k}\left(t^r\right)=0$. Bundan tashqari $k<r$ bo‘lganda $\frac{d^k}{dt^k}\left(t^r\right)\Big|_{t=0}=0$. Shuning uchun

$$L\left[t^r e^{\lambda t}\right]\Big|_{t=0}=e^{\lambda t}\frac{D^{(r)}(\lambda)}{r!}r\left(r-1\right)...1\Big|_{t=0}=D^{(r)}=0 \qquad \left(r=0,1,...,k-1\right)$$

Demak, λ xarakteristik ko‘phadning k dan kichik bo‘lmagan ildizi bo‘ladi. Lemma isbotlandi.

Endi bir jinsli (1.1.3) tenglamaga qaytamiz. Aytaylik $D(p)$ xarakteristik ko‘phad m $(m\le n)$ ta turli $\lambda_1,\lambda_2,\ ...,\lambda_m$ ildizlarga ega bo‘lsin. Bu ildizlarning karraliklarini
$k_1,k_2,...,k_m$ bilan belgilaymiz. U holda 1.1.2-lemmaga ko‘ra

$$\begin{aligned}
&y_1=e^{\lambda_1 t},\ y_2=te^{\lambda_1 t},\ ...,y_{k_1}=t^{k_1-1}e^{\lambda_1 t},\\
&y_{k_1+1}=e^{\lambda_2 t},\ y_{k_1+2}=te^{\lambda_2 t},\ ...,y_{k_1+k_2}=t^{k_2-1}e^{\lambda_2 t},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&y_{k_1+...+k_{m-1}+1}=e^{\lambda_m t},\ y_2=te^{\lambda_1 t},\ ...,y_{k_1+...+k_m}=t^{k_m-1}e^{\lambda_m t},
\end{aligned} \tag{1.1.6}$$

funksiyalar ham bir jinsli (1.1.3) tenglamaning yechimlari bo‘ladi. $k_1+\cdots+k_m=n$ bo‘lgani uchun (1.1.6) formula (1.1.3) tenglamaning n ta $y_j(t)$ yechimlarini aniqlaydi.

Lemma 1.1.4. Agar $y_1, y_2, ...y_n$ funksiyalar (1.1.6) tengliklar bilan aniqlansa, u holda

$$\begin{vmatrix} y_1(0) & y_2(0) & \dots & y_n(0) \\ y_1'(0) & y_2'(0) & \dots & y_n'(0) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(0) & y_2^{(n-1)}(0) & \dots & y_n^{(n-1)}(0) \end{vmatrix} \neq 0 \qquad (1.1.7)$$

Isboti. Faraz qilaylik (1.1.7) determinant nolga teng bo‘lsin. U holda bu determinantning satrlari orasida

$$b_0 y_j^{(n-1)}(0) + b_1 y_j^{(n-2)}(0) + \cdots + b_{n-1} y_j(0) = 0 \qquad (1.1.8)$$

$(j = 1, 2, ..., n)$

chiziqli bog‘liqlik mavjud, bu yerda $b_0, b_1, ..., b_{n-1}$ koeffisiyentlarning barchasi bir vaqtda nolga teng emas.

$$L_1[y] = b_0 y^{(n-1)} + b_1 y^{(n-2)} + \cdots + b_{n-1} y$$

differensial operatorni qaraymiz. Bu operatorga darajasi $n-1$ dan ortmagan

$$D_1(p) = b_0 p^{n-1} + b_1 p^{n-2} + \cdots + b_{n-1}$$

xarakteristik ko‘phad mos keladi.

$j = 1, 2, \ldots, k_1$ bo‘lsin, u holda (1.1.8) munosabatni

$$L\left[t^r e^{\lambda_1 t}\right]\Big|_{t=0} = 0 \qquad \left(j = 1, \ldots, k_1\right)$$

yoki

$$L\left[t^r e^{\lambda_1 t}\right]\Big|_{t=0} = 0 \qquad \left(r = 0, 1, ..., k_1 - 1;\ r = j - 1\right)$$

ko‘rinishda yozishimiz mumkin.

1.1.3-lemmaga ko‘ra λ_1 soni $D_1(p)$ xarakteristik ko‘phadning karraligi k_1 dan kichik bo‘lmagan ildizi bo‘ladi. Xuddi shu kabi $j = k_1 + 1, \ldots, k_1 + k_2$ ko‘rinishda tanlab λ_2 soni $D_1(p)$ xarakteristik ko‘phadning karraligi k_2 dan kichik bo‘lmagan ildizi bo‘lishini va hakozo λ_m soni $D_1(p)$ xarakteristik ko‘phadning karraligi k_m dan kichik bo‘lmagan ildizi bo‘lishini isbotlashimiz mumkin. Shuning uchun har bir ildizni karraligi bilan hisoblab $D_1(p)$ xarakteristik ko‘phadning ildizlari soni $k_1 + \cdots + k_m = n$ ga teng bo‘ladi degan xulosaga kelamiz. Ammo bu mumkin emas, chunki $D_1(p)$ xarakteristik ko‘phadning darajasi $n-1$ dan ortmaydi. Hosil bo‘lgan ziddiyat lemmani isbotlaydi.

1.2. O‘zgarmasni variantsiyalash usuli.

Bir jinsli bo‘lmagan chiziqli tenglamalarning xususiy yechimlarini topish ko‘p hollarda o‘zgarmasni variantsiyalash usuli bilan amalga oshiriladi. Biz ikkinchi tartibli tenglama va matritsa ko‘rinishida berilgan tenglamalar sistemasi uchun ko‘rib o‘tamiz.

Dastlab ikkinchi tartibli

$$y'' + a(x)y' + b(x)y = f(x) \tag{1.2.1}$$

tenglamani qaraymiz. Aytaylik $y_1(x), y_2(x)$ funksiyalar (1) tenglamaga mos bir jinsli qismi bo‘lgan

$$y'' + a(x)y' + b(x)y = 0 \tag{1.2.2}$$

tenglamaning yechimlari bo‘lsin. (1.2.1) tenglamaning xususiy yechimini

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x) \tag{1.2.3}$$

ko‘rinishda izlaymiz, bu yerda $C_1(x)$, $C_2(x)$ noma’lum funksiyalar.

$$y'(x) = [C_1'(x)y_1(x) + C_2'(x)y_2(x)] + C_1(x)y_1'(x) + C_2(x)y_2'(x)$$

$C_1(x)$, $C_2(x)$ noma’lum funksiyalarni shunday tanlaymizki kvadrat qavs ichida joylashgan funksiya nolga teng bo‘lsin, ya’ni

$$C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \tag{1.2.4}$$

bo‘lsin. U holda

$$y'(x) = C_1(x)y_1'(x) + C_2(x)y_2'(x) \tag{1.2.5}$$

(1.2.3) va (1.2.5) ni (1.2.1) tenglamaga qo‘yilsa va guruhlansa

$$C_1(x)[y_1''(x) + a(x)y_1'(x) + b(x)y_1(x)] + C_2(x)[y_2''(x) + a(x)y_2'(x) + b(x)y_2(x)] +$$
$$+ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x)$$

tenglikni va bundan $y_1(x), y_2(x)$ funksiyalar (1.2.2) tenglamaning yechimi ekanligidan

$$C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x) \tag{1.2.6}$$

tenglikni hosil qilamiz. Natijada $C_1(x)$, $C_2(x)$ noma'lum funksiyalarni aniqlash uchun (1.2.4) va (1.2.6) dan iborat bo'lgan

$$\begin{cases} C_1'(x)y_1(x)+C_2'(x)y_2(x)=0, \\ C_1'(x)y_1'(x)+C_2'(x)y_2'(x)=f(x) \end{cases} \tag{1.2.7}$$

tenglamalar sistemasini hosil qilamiz. (1.2.7) sistemani Kramer qoidasi bo'yicha

$$C_1'(x)=-\frac{f(x)y_2(x)}{\omega(x)}, \qquad C_2'(x)=\frac{f(x)y_1(x)}{\omega(x)} \tag{1.2.8}$$

munosabatlarni hosil qilamiz, bu yerda

$$\omega(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \tag{1.2.9}$$

(1.2.8) tengliklarni $[x_0, x]$ kesmada integrallash va hosil bo'lgan $C_1(x)$, $C_2(x)$ noma'lum funksiyalarning ifodalarini (1.2.3) tenglikka qo'yib (1.2.1) tenglamaning

$$y(x)=\int_{x_0}^{x}\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_{21}(x) \end{vmatrix}\cdot\frac{f(t)}{\omega(t)}dt \tag{1.2.10}$$

xususiy yechimini hosil qilamiz.

Misol 1.2.1. $y''+\mu^2 y=g(x)$ tenglamaning $y(0,\mu)=0,\ y'(0,\mu)=\mu$ shartlarni qanoatlantiruvchi yechimini topamiz.

Yechilishi. Berilgan tenglamaning mos bir jinsli qismi: $y''+\mu^2 y=0$ bo'lib, uning umumiy yechimi $y_1(x)=\cos\mu x$, $y_2(x)=\sin\mu x$ ildizlarga va bizda $f(x)=g(x)$ bo'lgani uchun (1.2.10) formulaga ko'ra berilgan tenglamaning xususiy yechimi

$$y(x)=\int_0^x\begin{vmatrix} \cos\mu t & \sin\mu t \\ \cos\mu x & \sin\mu x \end{vmatrix}\cdot\frac{g(t)}{\omega(t)}dt=\frac{1}{\mu}\int_0^x \sin\mu(x-t)g(t)dt$$

ko'rinishga ega bo'ladi, chunki

$$\omega = \begin{vmatrix} \cos\mu x & \sin\mu x \\ -\mu\sin\mu x & \mu\cos\mu x \end{vmatrix} = \mu$$

Berilgan tenglamaning umumiy yechimi mos bir jinsli qismining umumiy yechimi bilan bu tenglama xususiy yechimlari yig‘indisiga teng bo‘lgani uchun, u

$$y(x) = C_1\cos\mu x + C_2\sin\mu x + \frac{1}{\mu}\int_0^x \sin\mu(x-t)g(t)dt$$

ko‘rinishga ega. $y(0,\mu)=0,\ y'(0,\mu)=\mu$ boshlang‘ich shartlarga ko‘ra $C_1 = 0$, $C_2 = 1$ ekanligini aniqlaymiz.

Javob. $y(x) = \sin\mu x + \frac{1}{\mu}\int_0^x \sin\mu(x-t)g(t)dt$

Endi sistema uchun umumiyroq bo‘lgan holni qaraymiz:

$$X' = A(t)X + f(t)\,,\ X(0) = x_0 \qquad (1.2.11)$$

sistemaning umumiy yechimini topamiz. (11) sistema yechimini $X(t) = \Phi(t)c(t)$ ko‘rinishda izlaymiz, bu yerda $\Phi(t)$ bilan $X' = A(t)X$ sistemaning $\Phi(0) = E$ shartni qanoatlantiruvchi fundamental matritsasi belgilangan. U holda

$$X'(t) = \Phi'(t)c(t) + \Phi(t)c'(t)$$

bo‘lib, (1.2.1) ga ko‘ra

$$\Phi'(t)c(t) + \Phi(t)c'(t) = A(t)X + f(t)\,,$$

yoki

$$c'(t) = \Phi^{-1}(t)f(t)$$

Bundan

$$c(t) = \int_0^t \Phi^{-1}(s)f(s)ds$$

bo‘lgani uchun berilgan sistemaning xususiy yechimi

$$x(t) = \Phi(t)c(t) = \Phi(t)\int_0^t \Phi^{-1}(s)f(s)ds$$

va umumiy yechimi esa

$$X(t)=\Phi(t)c+\Phi(t)\int_0^t \Phi^{-1}(s)f(s)ds$$

ko‘rinishga ega bo‘ladi. $X(0)=x_0$, $\Phi(0)=E$ boshlang‘ich shartga ko‘ra izlanayotgan yechim

$$X(t)=\Phi(t)x_0+\Phi(t)\int_0^t \Phi^{-1}(s)f(s)ds$$

dan iborat.

1.3. Eyler va Lagranj tenglamalari.

Differensial tenglamalar orasida oddiy almashtirishlar vositasida o‘zgarmas koeffisiyentli tenglamalarga o‘tuvchi o‘zgaruvchi koeffisiyentli tenglamalar ham uchraydi.

$$a_0t^n\frac{d^ny}{dt^n}+a_1t^{n-1}\frac{dy^{n-1}}{dt^{n-1}}+\cdots+a_{n-1}t\frac{dy}{dt}+a_ny=0 \qquad (1.3.1)$$

ko‘rinishdagi tenglamaga Eyler tenglamasi deyiladi, bu yerda a_0, $a_1,....,a_n$ o‘zgarmas sonlar. Agar (1.3.1) tenglamada t ni e^x bilan almashtirsak tenglamaning ko‘rinishi o‘zgarmaydi. Demak, (1.3.1) tenglamada x erkli o‘zgaruvchini

$$x=\ln t\ ,\ t=e^x \qquad (1.3.2)$$

almashtirish bilan kiritsak, u holda x ni $x+C$ bilan almashtirishda tenglama o‘zgarmaydi, ya’ni hosil bo‘lgan yangi tenglama x ni oshkor ko‘rinishda saqlamaydi. Erkli o‘zgaruvchini almashtirishda tenglama chiziqli tenglamaga o‘tmaganligi uchun, biz o‘zgarmas koeffisiyentli chiziqli tenglamaga ega bo‘lamiz. Bu tasdiqni hisoblashlar vositasida bevosita tekshirishimiz mumkin. Biz y funksiyaning t bo‘yicha hosilalarini (1.3.2) formula bo‘yicha x bo‘yicha hosilalari orqali ketma-ket ifodalaymiz:

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=e^{-x}\frac{dy}{dx};$$

$$\frac{d^2y}{dt^2}=e^{-x}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right)=e^{-2x}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}\right);$$

$$\frac{d^3y}{dt^3}=e^{-x}\frac{d}{dx}\left[e^{-2x}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)\right]=e^{-3x}\left(\frac{d^3y}{dx^3}-3x\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right);$$

Biz ko‘ramizki, t bo‘yicha olingan birinchi, ikkinchi va uchinchi tartibli hosilalarni qatnashgan ifodalar mos ravishda e^{-x}, e^{-2x} va e^{-3x} ko‘paytuvchilarga ega. Faraz qilaylik t bo‘yicha olingan $k-$ tartibli hosila

$$\frac{d^ky}{dt^k}=e^{-kx}\left(\frac{d^ky}{dx^k}+\alpha_1\frac{d^{k-1}y}{dx^{k-1}}+\cdots+\alpha_{k-1}\frac{dy}{dx}\right)$$

ko‘rinishga ega bo‘lsin, bu yerda $\alpha_1,\alpha_2,....,\alpha_{k-1}-$ o‘zgarmas sonlar. U holda t bo‘yicha olingan $(k+1)-$ tartibli hosila

$$\frac{d^{k+1}y}{dt^{k+1}}=e^{-x}\frac{d}{dx}\left(\frac{d^ky}{dt^k}\right)=e^{-(k+1)x}\left(\frac{d^{k+1}y}{dx^{k+1}}+(\alpha_1-k)\frac{d^ky}{dx^k}+\cdots-k\alpha_{k-1}\frac{dy}{dx}\right)$$

ko‘rinishga ega bo‘ladi va yana qavs oldida $e^{-(k+1)x}$ ko‘paytuvchi , qavslar ichida esa x bo‘yicha birinchi tartibli hosiladan boshlab $(k+1)-$tartibli hosilagacha ifodalarning chiziqli kombinatsiyalari joylashgan. Demak ko‘rsatilgan xossa ixtiyoriy k natural soni uchun isbotlandi. Biz hisoblangan hosilalarni (1.3.1) tenglamaga qo‘ysak, har bir k uchun $\frac{d^ky}{dt^k}$ ifodani $a_kt^k=a_ke^{kx}$ ko‘paytirishlozim bo‘ladi va shu bilan birga x ni o‘zida saqlovchi ko‘rsatkichli ko‘paytuvchilar qisqaradi hamda o‘zgarmas koeffisiyentli chiziqli tenglama hosil bo‘ladi.

Misol 1.3.1. Ushbu

$$t^2\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+y=0$$

tenglamani qaraymiz. $t=e^x$ almashtirish bizga

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$$

tenglamani beradi. Bu tenglamaning xarakteristik tenglamasi: $k^2+2k+1=0$ bir xil

$k_1=k_2=-1$ ildizga ega bo‘lgani uchun x o‘zgaruvchi bo‘yicha umumiy yechim

$$y=e^{-x}\left(C_1+C_2x\right)$$

ko‘rinishga ega. Kiritilgan almashtirishga ko‘ra t o‘zgaruvchi bo‘yicha umumiy yechim

$$y=\frac{1}{t}\left(C_1+C_2\ln t\right)$$

ko‘rinishda bo‘ladi.

Biz almashtirilgan tenglamaning xarakteristik tenglamasi karrali ildizlarga ega bo‘lmagan holda $e^{kx}=\left(e^x\right)^k$ xususiy yechimga ega bo‘ladi, demak dastlabki tenglamada bu yechim t^k ko‘rinishda bo‘ladi. Shuning uchun bevosita xususiy yechimni bu ko‘rinishda izlash va uni (1.3.1) tenglamaga qo‘yish mumkin. Agar

$$t^m\frac{d^m\left(t^k\right)}{dt^m}=k(k-1)....(k-m+1)t^k\ ,\quad m\le k$$

ekanligini e’tiborga olib bu ko‘rinishdagi ifodalar (1.3.1) tenglamaga qo‘yilsa va hosil bo‘lgan tenglik x^k ga qisqartirilsa k ni aniqlash uchun $n-$ darajali

$$\begin{aligned}&k(k-1).....(k-n+1)+a_1k(k-1).....(k-n+2)+ :::+\\&\qquad+a_{n-2}k(k-1)+a_{n-1}k+a_n=0\end{aligned} \tag{1.3.3}$$

algebraik tenglamani hosil qilamiz. Avvalgi mulohazalardan (1.3.3) tenglama x o‘zgaruvchi bo‘yicha topilgan xarakteristik tenglama bilan ustma ust tushadi. (1.3.3) tenglamaning har bir k oddiy ildiziga (1.3.1) tenglamaning t^k xususiy yechimi, (1.3.3) tenglamaning ikki karrali k ildiziga (1.3.1) tenglamaning t^k va $t^k\ln t$ xususiy yechimlari mos keladi va hakozo. $k=\alpha\pm i\beta$ qo‘shma kompleks ildiziga $t^{i\beta}=e^{i\beta\ln t}$ tenglikka binoan (1.3.1) tenglamaning ikkita $y=t^\alpha\cos(\beta\ln t)$ va $y=t^\alpha\sin(\beta\ln t)$ xususiy yechimilari mos keladi.

Misol 1.3.2. Ushbu

$$t^2\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+5y=0$$

tenglamani qaraymiz. Bu tenglamaning xususiy yechimini $y=t^k$ ko‘rinishda izlaymiz va berilgan tenglamadan

$$k(k-1)+3k+5=0$$

yoki

$$k^2+2k+5=0$$

xarakteristik tenglamani hosil qilamiz. Bu tenglama $k=-1\pm 2i$ qo‘shma kompleks ildizlarga ega bo‘lgani uchun berilgan tenglamaning umumiy yechimi

$$y=\frac{1}{t}\left[C_1\cos(2\ln t)+C_2\sin(2\ln t)\right]$$

ko‘rinishga ega bo‘ladi.

Differensial tenglamalar orasida oddiy almashtirishlar vositasida o‘zgarmas koeffisiyentli tenglamalarga o‘tuvchi o‘zgaruvchi koeffisiyentli tenglamalar orasida Lagranj tenglamasi deb nomlangan

$$(at+b)^n\frac{d^ny}{dt^n}+a_1(at+b)^{n-1}\frac{dy^{n-1}}{dt^{n-1}}+\cdots+a_{n-1}(at+b)\frac{dy}{dt}+a_ny=0 \qquad (1.3.4)$$

ko‘rinishdagi tenglamalar ham uchraydi bu yerda a_0, $a_1,....,a_n$ o‘zgarmas sonlar. (1.3.4) Lagranj tenglamasida x erkli o‘zgaruvchini

$$x=\ln(at+b)\ ,\ \ at+b=e^x \qquad (1.3.4)$$

tengliklar yordamida almashtirilsa o‘zgarmas koeffisiyentli chiziqli tenglama hosil bo‘ladi.

Bir jinsli bo‘lmagan Eyler tenglamasining o‘ng tomoni $P(t)$ ko‘phadning chekli sondagi arifmetik amallardan tashkil topgan $\sum e^{\alpha t}P(t)$ ifodasidan iborat bo‘lsa, u holda almashtirish natijasida hosil bo‘lgan o‘zgarmas koeffisiyentli bir jinsli tenglamaning o‘ng tomoni $\sum t^{\alpha}P(\ln t)$ ko‘rinishga o‘tsa bunda ham xususiy yechimlarni topish bilan integrallashni amalga oshirilishi mumkinligini eslatamiz. Endi Eyler va Lagrang tenglamalarini yechishga oid misollardan namunalar keltiramiz.

Misol 1.3.3. Quyidagi tenglamani yeching.

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{n(n+1)}{r^2}R=0$$

Yechilishi. Berilgan tenglamaning xususiy yechimini $R=r^k$ ko‘rinishda izlaymiz. Natijada

$$k^2+k-n(n+1)=0,$$

xarakteristik tenglama tenglamani hosil qilamiz. Uning ildizlari

$$k_1=-\frac{1}{2}+\frac{\sqrt{4n(n+1)+1}}{2}\ ,\ k_2=-\frac{1}{2}-\frac{\sqrt{4n(n+1)+1}}{2}$$

bo‘lgani uchun tenglamaning umumiy yechimi

$$R=\frac{1}{\sqrt{r}}\left[C_1r^{\alpha}+C_2r^{-\alpha}\right]\ ,\ \alpha=\frac{1}{2}\sqrt{4n(n+1)+1}$$

ko‘rinishga ega bo‘ladi.

Misol 1.3.4. Quyidagi tenglamani yeching:

$$t^2y''-4ty'+6y=t$$

Yechilishi. Berilgan tenglamada $x=\ln t$, $t=e^x$ almashtirishi qo‘llash bilan bu tenglama bir jinsli bo‘lmagan

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^x$$

tenglamaga o‘tadi. Bu tenglama mos bir jinsli qismining umumiy yechimi

$$y=C_1e^{2x}+C_2e^{3x}$$

Xususiy yechimini esa $y=Ae^x$ ko‘rinishda izlaymiz va bu xususiy yechim $y=\frac{1}{2}e^x$ bo‘lgani uchun almashtirish nati-jasida hosil bo‘lgan tenglamaning yechimi

$$y=C_1e^{2x}+C_2e^{3x}+\frac{1}{2}e^x$$

bo‘lib, (1.3.2) almashtirishga ko‘ra berilgan tenglamaning yechimi

$$y=C_1t^2+C_2t^3+\frac{1}{2}t$$

funksiyadan iborat bo‘ladi.

Misol 1.3.5. Quyidagi tenglamani yeching.

$$t^2y'' - ty' + 2y = t\ln t$$

Yechilishi. (1.3.2) almashtirish bu tenglamani

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = xe^x$$

tenglamaga o‘tadi. Bu tenglama mos bir jinsli qismining xarakteristik tenglamasi

$$k^2 - 2k + 2 = 0$$

o‘zaro qo‘shma $k = -1 \pm i$ kompleks ildizlarga ega bo‘lgani uchun uning umumiy yechimi

$$y = e^{-x}\left[C_1 cosx + C_2 \sin x\right]$$

ko‘rinishga ega. Xususiy yechimini esa $y = (Ax + B)e^x$ ko‘rinishda izlaymiz va bu xususiy yechim $y = xe^x$ bo‘lgani uchun almashtirish natijasida hosil bo‘lgan tenglamaning yechimi

$$y = e^{-x}\left[C_1 cosx + C_2 \sin x\right] + xe^x$$

bo‘lib, (1.3.2) almashtirishga ko‘ra dastlabki tenglamaning yechimi

$$y = \frac{1}{t}\left[C_1 cos\ln t + C_2 \sin\ln t\right] + t\ln t$$

funksiyadan iborat.

1.4. Involyutsiya tushunchasi va uning asosiy xossalari.

Qandaydir f akslantirish berilgan bo‘lib, bu akslantirishda P nuqtaning tasviri $P' = f(P)$ nuqta bo‘lsin. O‘z navbatida $P'' = f(P'), P'' = P$ ya’ni $f \circ f(P) = P$ bo‘lsin. Demak, f akslantirish involyutiv akslantirish bo‘lishi uchun quyidagi shartlarning biri o‘rinli bo‘lishi kerak:

1) ixtiyoriy P nuqta uchun

$$f\left(f(P)\right) = P \tag{1.4.1}$$

tenglikning bajarilishi yoki

2) ixtiyoriy P nuqta uchun $P' = f'(x)$ munosabat bilan birgalikda

$$P' = f^{-1}(P) \qquad (1.4.2)$$

munosabat bajarilishi, ya'ni har qanday akslantirish o'ziga teskari akslantirish bilan ustma ust tushishi lozim.

Shu sababli ko'plab geometrik adabiyotlarda, masalan [1] da o'ziga teskari akslantirishlar bilan bir xil bo'lgan akslantirishlarga involyutiv akslantirishlar deyiladi. Shuningdek geometriyada butun son o'qida aniqlangan haqiqiy argumentli (1.4.1) tenglikni qanoatlantiruvchi $f(x)$ funksiyaga kuchli involyutsiya deyiladi.

Kuchli invoyutsiyalar to'plamini Υ bilan belgilasak, u holda har bir $f(x) \in \Upsilon$ funksiyaning grafigi $y = x$ to'g'ri chiziqga nisbatan simmetrik joylasgan bo'ladi. Agar G bilan Oxy tekislikning $y = x$ to'g'ri chiziqga nisbatan simmetrik joylashgan funksiyalar to'plami bo'lib, bunda har bir x element uchun bu to'plamning x absissaga ega bo'lgan yagona nuqtasi mos kelsa, u holda G to'plam Υ to'plamdagi birorta $f(x)$ involyutiv akslantirishning grafigi bo'ladi.

Kuchli invoyutiv akslantirish bo'ladigan $f(x)$ akslantirishni quyidagi tartibda hosil qilishimiz mumkin. Faraz qilaylik haqiqiy o'zgaruvchili $g(x, y)$ funksiya barcha tartiblangan (x, y) haqiqiy nuqtalar to'plamida aniqlangan bo'lib, $g(x, y) = 0$ tenglikdan $g(y, x) = 0$ tenglik kelib chiqsin. Ma'lumki, xususiy holda $g(x, y) = g(y, x)$ tenglik bajarilsa, odatda $g(x, y)$ funksiyaga simmetrik funksiya deyiladi. Agar har bir x uchun $g(x, y) = 0$ tenglamani qanoatlantiruvchi $y = f(x)$ funksiya mos kelsa, u holda $y = f(x) \in \Upsilon$ bo'ladi. misollar keltiramiz:

1. $g(x, y) = x + y - C$ bo'lsin. U holda $g(x, y) = 0$ tenglikdan $g(y, x) = 0$ tenglik kelib chiqganligi uchun $y = f(x) = x - C$ bo'ladi;

2. $g(x, y) = x^3 + y^3 - C$ bo'lsin. U holda $g(x, y) = 0$ tenglikdan $g(y, x) = 0$ tenglik kelib chiqganligi uchun $y = f(x) = \sqrt[3]{x - C}$ bo'ladi. Yuqorida bayon etilganlardan tashqari $f(x) \in \Upsilon$, $f(x) \equiv x$ munisabatlarni qanoatlantiruvchi

involyutiv funksiyalarning to‘plamining elementlari monoton lamayuvchi funksiyalardir, ya’ni

$$\lim_{x\to+\infty} f(x) = -\infty,\ \lim_{x\to-\infty} f(x) = +\infty \qquad (1.4.3)$$

munosabatlar o‘rinli. Endi quyidagi tasdiqni keltiramiz.

Teorema 1.4.1. $f(x)\equiv x$ munosabatni qanoatlantiruvchi har bir uzliksiz $f(x)-$ kuchli involyutsiya yagona qo‘zg‘almas nuqtaga ega.

Isboti. $\varphi(x) = f(x) - x$ kuchli involyutsiya xossasiga ega bo‘lgan uzluksiz monoton funksiya bo‘lgani uchun uning grafigi $y = x$ to‘g‘ri chiziqga nisbatan simmetrik joylashgandir, ya’ni bu funksiya (1.4.3) munosabatni qanoatlantiradi. Ma’lumki, (1.4.3) tenglik yagona $x = x_0$ nuqta uchun o‘rinli bo‘lgani uchun $f(x_0) - x_0 = 0$, ya’ni $f(x_0) = x_0$. Teorema isbotlandi.

Demak, $f(x)-$ kuchli involyutsiya bo‘lsa, u holda u yagona qo‘zg‘almas nuqtaga ega bo‘lishini ko‘rib o‘tdik. Endi involyutsiyaning turlarini ko‘rib chiqamiz. Buning uchun $f(x) = \dfrac{\alpha x + \beta}{\gamma x + \delta}$ kasr chiziqli almashtirishni qaraymiz.

Kasr chiziqli almashtirish proyektiv tekislikning har bir M nuqtasini uning M' nuqtasiga o‘tkazsin, ya’ni $M(x_0)$ va $M'(x_0')$ nuqtalari uchun

$$f(M) = M',\ f(M') = M$$

tengliklar bajarilsin va shu bilan birgalikda $x_0 \neq x_0'$ bo‘lsin. Aytilganlarga ko‘ra koordinatalar bo‘yicha

$$x_0' = \frac{\alpha x_0 + \beta}{\gamma x_0 + \delta}, \qquad x_0 = \frac{\alpha x_0' + \beta}{\gamma x_0' + \delta}$$

tengliklarni yozamiz. O‘z navbatida bu tengliklardan

$$\begin{cases} \gamma x_0 x_0' + \delta x_0' - \alpha x_0 - \beta = 0, \\ \gamma x_0 x_0' + \delta x_0 - \alpha x_0' - \beta = 0 \end{cases}$$

tenglamalar sistemasini hosil qilamiz. Sistemaning birinchi tenglamasidan ikkinchi tenglamasini hadlab ayirib

$$\delta(x_0' - x_0) + \alpha(x_0' - x_0) = 0, \quad \text{yoki} \quad (\delta + \alpha)(x_0' - x_0) = 0$$

tenglikni hosil qilamiz. Ammo $x_0 \neq x_0'$ bo‘lgani uchun $\delta = -\alpha$ bo‘ladi.

Demak, $f(x)$ kuchli involyutiv akslantirish bo‘lganligi uchun , u

$$f(x) = \frac{\alpha x + \beta}{\gamma x - \alpha}$$

ko‘rinishda bo‘lishi lozim. Endi bu invoyutsiyaning qo‘zg‘almas nuqtalarini topamiz.

$f(x) = x$ tenglik bajarilishi uchun

$$\frac{\alpha x + \beta}{\gamma x - \alpha} = x,$$

ya’ni

$$\gamma x^2 - 2\alpha x - \beta = 0$$

tenglik o‘rinli bo‘lishi kerak. $\Delta = \alpha^2 + \beta\gamma$ belgilash kiritamiz. U holda quyidagi hollar bo‘lishi mumkin:

a) agar $\Delta > 0$ bo‘lsa, u holda qaralayotgan involyutsiya ikkita haqiqiy qo‘zg‘almas nuqtalarga ega bo‘ladi va bu involyutsiya giperbolik involyutsiya deyiladi;

b) agar $\Delta = 0$ bo‘lsa, u holda qaralayotgan involyutsiya yagona haqiqiy qo‘zg‘almas nuqtaga ega bo‘ladi va bu involyutsiya parabolik involyutsiya deyiladi;

c) agar $\Delta < 0$ bo‘lsa, u holda qaralayotgan involyutsiya haqiqiy qo‘zg‘almas nuqtaga ega bo‘lmaydi va bu involyutsiya elliptik involyutsiya deyiladi;

Bu mulohazalardan $f-$kuchli involyutsiya bo‘lishi uchun u parabolik involyutsiya bo‘lishi lozim. Demak, quyidagi tasdiq o‘rinli.

Teorema 1.4.2. Agar

$$f(x) = \frac{\alpha x + \beta}{\gamma x - \alpha}$$

akslantirishda $\alpha^2 + \beta\gamma = 0$ bo‘lsa, u holda bu akslantirish kuchli involyutsiya bo‘ladi.

2-BOB. INVOLYUTSIYA XOSSASIGA EGA BO‘LGAN ODDIY DIFFERENSIAL TENGLAMALAR

2.1. Differensial tenglamalar involyutsiyasi

Dastlab biz differensial tenglamalar involyutsiyasining ta’rifini keltiramiz.

Ta’rif 2.1.1. Agar $f_1(x), f_2(x),..., f_m(x)$ akslantirishlar involyutsiyalar bo‘lsa, u holda

$$F\Big(x, y\big(f_1(x)\big), y\big(f_2(x)\big),..., y\big(f_m(x)\big),..., y^{(n)}\big(f_1(x)\big), y^{(n)}\big(f_2\big),..., y^{(n)}\big(f_m(x)\big)\Big) = 0 \qquad (2.1.1)$$

ko‘rinishdagi tenglamalarga involyutsiya xossasiga ega bo‘lgan tenglamalar deyiladi.

Endi involyutsiya xossasiga ega bo‘lgan differensial tenglamalar uchun ba’zi mulohazalarni keltiramiz.

Teorema 2.1.1. Agar

$$y' = F\big(x, y(x), y(f(x))\big) \qquad (2.1.2)$$

tenglama uchun quyidagi shartlar bajarilsin:

1) $f(x)-$ yagona qo‘zg‘almas nuqtaga ega bo‘lgan uzluksiz differensiallanuvchi funksiya;

2) $F\big(x, y(x), y(f(x))\big)-$ barcha argumentlari bo‘yicha aniqlangan va bu argumentlar bo‘yicha uzluksiz differensiallanuvchi funksiya;

3) (2.1.2) tenglama $y\big(f(x)\big)$ argumentga nisbatan bir qiymatli yechimga ega, ya’ni

$$y\big(f(x)\big) = G\big(x, y(x), y'(x)\big) \qquad (2.1.3)$$

U holda

$$y''(x) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y(x)} \cdot y'(x) + \frac{\partial F}{\partial y\big(f(x)\big)} \cdot f'(x) \cdot F\big(x, y(x), y'(f(x))\big) \qquad (2.1.4)$$

munosabat o‘rinli bo‘lib, bu yerda $y\big(f(x)\big)$ (2.1.3) tenglik bilan beriladi hamda (2.1.2) tenglamaning

$$y(x_0) = y_0, y'(x_0) = F(x_0, y_0, y_0') \qquad (2.1.5)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimi bo‘ladi.

Isboti. Dastlab (2.1.4) tenglama (2.1.2) tenglamani differensiallash yo‘li bilan hosil qilinishini ko‘rsatamiz. Buning uchun (2.1.2) tenglamani x bo‘yicha differensiallaymiz:

$$y''(x)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y(x)}\cdot y'(x)+\frac{\partial F}{\partial y(f(x))}\cdot y'(f(x))\cdot f'(x)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y(x)}\cdot y'(x)+$$

$$+\frac{\partial F}{\partial y(f(x))}\cdot F(x,y(f(x)),y(f(f(x))))\cdot f'(x)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y(x)}\cdot y'(x)+$$

$$+\frac{\partial F}{\partial y(f(x))}\cdot F(x,y(f(x)),y(x))\cdot f'(x)$$

ya’ni (2.1.4) tenglama o‘rinli ekanligini ko‘ramiz. Teoremani isbotlashda (2.1.2) ning o‘ng tomonidagi ifodadan hamda involyutsiyaning $f(f(x))=x$ xossasidan foydalandik. Boshlang‘ich shartlardan (2.1.5) ni hosil qilish uchun esa

$$y'(f(x))=F(x,y(f(x)),y(x)) \tag{2.1.6}$$

tenglamada x ning o‘rniga $x=x_0$ qiymarni qo‘yamiz va $f(x_0)=x_0$ ekanligini e’tiborga olamiz. Teorema isbotlandi.

Endi $y(x)$ funksiyani oshkormas holda saqlovchi

$$y'=F(y(f(x))) \tag{2.1.7}$$

ko‘rinishdagi differensial tenglamalarni qaraymiz.

Teorema 2.1.2. Agar (2.1.7) tenglamada quyidagi shartlar bajarilsa:

1) $f(x)-$yagona qo‘zg‘almas x_0 nuqtaga ega bo‘lgan uzluksiz differensiallanuvchi kuchli involyutsiya;

2) $F(y(f(x)))-$butun son o‘qida aniqlangan uzliksiz differensiallanuvchi qat’iy monoton funksiya bo‘lsin. U holda

$$y''(f(x))=F'(y(f(x)))\cdot F(y(x))\cdot f'(x) \tag{2.1.8}$$

oddiy differensial tenglamaning

$$y(x_0)=y_0,\ y'(x_0)=F(y_0)$$

boshlzng‘ich shartlarni qanoatlantiruvchi yechimi (2.1.7) tenglamaning $y(x_0)=y_0$ boshlangich shartni qanoatlantiruvchi yechimidan iborat bo‘ladi.

Isboti. Berilgan tenglamani x bo‘yicha differensiallaymiz. Natijada

$$y''(x) = F'(y(f(x))) \cdot y'(f(x)) \cdot f'(x)$$

tenglikni hosil qilamiz. Ammo

$$f(f(x)) = x; \quad y'(x_0) = y'(f(x_0)) = F(y(f(x_0))) = F(y(x_0))$$

bo‘lgani uchun

$$y'(f(x)) = F(y(f(f(x)))) = F(y(x))$$

va shu bilan birga

$$y' = F(y(f(x)))$$

tenglikdan

$$y(f(x)) = F^{-1}(y'(x))$$

kelib chiqadi. Teorema isbotlandi.

Yuqorida keltirilgan teoremalardan quyidagi natija kelib chiqadi.

Natija. Agar yuqoridagi 2.1.1-chi va 2.1.2-chi teoremalarda

$$f(x) = \frac{\alpha x + \beta}{\gamma x - \alpha}, \ \alpha^2 + \beta\gamma > 0$$

akslantirishlar giperbolik involyutsiya bo‘lib, (2.1.2) va (2.1.7) tenglamalar $(-\infty, \alpha / \gamma)$ yoki $(\alpha / \gamma, +\infty)$ oraliqlarda aniqlangan bo‘lsa, u holda bu teoremalar o‘z kuchini saqlaydi.

Eslatma.1) Agar x_0 nuqta $f(x)$ involyutsiyaning qo‘zg‘almas nuqtasi bo‘lib, $x > x_0$ bo‘lsa, (2.1.2) va (2.1.7) tenglamalar kechikkan argumentli differensial tenglamalar bo‘lad.

3) Agar x_0 nuqta $f(x)$ involyutsiyaning qo‘zg‘almas nuqtasi bo‘lib, $x < x_0$ bo‘lsa, (2.1.2) va (2.1.7) tenglamalar ortgan argumentli differensial tenglamalar bo‘ladi.

2.2. Argumentida involyutsiya qatnashgan bir jinsli bo‘lmagan birinchi tartibli boshlang‘ich shartli masalaning yechimi.

Quyidagi boshlang‘ich shartli masalani qaraylik

$$y'(x) + ax^n \cdot y\left(\frac{1}{x}\right) = g(x) + \frac{1}{g(x)} \tag{2.2.1}$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{2.2.2}$$

Bunda $a = const, a \neq 0, n\epsilon R, x_0 \neq 0,\ g(x) -$ berilgan funksiya $(g(x) \neq 0)$. $y_1 = g(x_0) + \frac{1}{g(x_0)} - ax_0^n \cdot y_0$

Endi biz yuqoridagi (2.2.1), (2.2.2) boshlang‘ich shartli masalani $y(x) = \varphi(x)$ yechimini topish bilan shug‘ullanamiz. $\varphi(x) -$ ikkinchi tartibli hosilasigacha uzluksiz bo‘lgan funksiya. Quyidagi teoremani keltiramiz.

Teorema 2.2.1. Yuqoridagi (2.2.1) tenglamani involyutsiyadan qutqarish natijasida ikkinchi tartibli Eyler tenglamasini integrallash masalasiga keladi.

Isboti. Yuqoridagi (2.2.1) tenglamadan bir martta x bo‘yicha hosila olamiz. Natijada

$$y''(x) + nax^{n-1}y\left(\frac{1}{x}\right) - ax^{n-2}y'\left(\frac{1}{x}\right) = g'(x) - \frac{g'(x)}{g^2(x)} \tag{2.2.3}$$

va (2.2.1) dan $y\left(\frac{1}{x}\right) = \frac{1}{ax^n}\left[g(x) + \frac{1}{g(x)} - y'(x)\right]$ ga tengligi ma’lum.

(2.2.3) tenglikdan

$$y'\left(\frac{1}{x}\right) = -\frac{1}{ax^{n-2}}\left[g'(x) - \frac{g'(x)}{g^2(x)} - y''(x) - \frac{n}{x}\left(g(x) + \frac{1}{g(x)} - y'(x)\right)\right]$$

(2.2.1) tenglikda involyutsiya hossasidan foydalanib $f: x \to \frac{1}{x}$ akslantirish bajarsak,

$$y'\left(\frac{1}{x}\right) + \frac{a}{x^n}y(x) = g\left(\frac{1}{x}\right) + \frac{1}{g\left(\frac{1}{x}\right)} \tag{2.2.4}$$

tenglik hosil bo‘ladi.

Yuqoridagi $y\left(\frac{1}{x}\right)$ va $y'(\frac{1}{x})$ larni (2.2.4) tenglikka olib kelib qo‘yamiz.

$$-\frac{1}{ax^{n-2}}\left[g'(x) - \frac{g'(x)}{g^2(x)} - y''(x) - \frac{n}{x}\left(g(x) + \frac{1}{g(x)} - y'(x)\right)\right] + \frac{a}{x^n}y(x) =$$

$= g\left(\frac{1}{x}\right) + \frac{1}{g\left(\frac{1}{x}\right)}$ =>

$x^2y''(x) - nxy'(x) + a^2y(x) = ax^n\left(g\left(\frac{1}{x}\right) + \frac{1}{g\left(\frac{1}{x}\right)}\right) + x^2\left(g'(x) - \frac{g'(x)}{g^2(x)}\right) -$

$-nx\left(g(x) + \frac{1}{g(x)}\right)$ bunda

$d(x) = ax^n\left(g\left(\frac{1}{x}\right) + \frac{1}{g\left(\frac{1}{x}\right)}\right) + x^2\left(g'(x) - \frac{g'(x)}{g^2(x)}\right) - nx\left(g(x) + \frac{1}{g(x)}\right)$ desak,

berilgan (2.2.1) tenglamamiz ushbu

$$x^2y''(x) - nxy'(x) + a^2y(x) = d(x) \tag{2.2.5}$$

ko‘rinishga keladi. (2.2.5) tenenglama esa ikkinchi tartibli Eyler tenglamasini integrallash masalasiga keldi. Teorema isbotlandi.

Endilikda (2.2.5) tenglamani umumiy yechimini topish bilan shug‘ullanamiz. (2.2.5) tenglamada $x = e^t$ almashtirish bajarib quyidagi:

$$\frac{d^2y}{dt^2} - (n+1)\frac{dy}{dt} + a^2y = d(e^t) = h(t) \tag{2.2.6}$$

(2.2.6) tenglamani bir jinsli qismini $\frac{d^2y}{dt^2} - (n+1)\frac{dy}{dt} + a^2y = 0$ (2.2.7)

xarakteristik tenglamasini tuzsak:

$$k^2 - (n+1)k + a^2 = 0 \quad => \Delta = (n+1)^2 - 4a^2$$

1-hol. $\Delta > 0$ bo‘lsin u holda $k_1 = \frac{n+1-\sqrt{\Delta}}{2}$ $k_2 = \frac{n+1+\sqrt{\Delta}}{2}$

Bu holda (2.2.7) tenglamaning umumiy yechimi: $y_1(t) = C_1e^{k_1t} + C_2e^{k_2t}$

(2.2.6) tenglamaning umumiy yechimini o‘zgarmasni variatsiyalash usulida topamiz. Quyidagi:

$$\begin{cases} C_1'(t)\,e^{k_1t} + C_2'(t)e^{k_2t} = 0 \\ k_1C_1'(t)\,e^{k_1t} + k_2C_2'(t)e^{k_2t} = h(t) \end{cases}$$

tenglamalar sistemasini qaraylik, algebreyik hisoblashlar natijasida sistemadan

$$C_1(t) = -\int_0^t \frac{h(z)}{k_2e^{k_1z} - k_1}dz + C1_1$$

$$C_2(t) = \int_0^t \frac{h(z)e^{(k_1-k_2)z}}{k_2e^{k_1z} - k_1}dz + C1_2$$

Demak, $(n+1)^2-4a^2>0$ bo‘lganda (2.2.6) tenglamaning umumiy yechimida $t=lnx$ almashtirish bajarganimizda (2.2.5) ning umumiy yechimi:

$$y(x)=[C1_1-\int_0^{lnx}\frac{h(z)}{k_2e^{k_1z}-k_1}dz]x^{k_1}+[\int_0^{lnx}\frac{h(z)e^{(k_1-k_2)z}}{k_2e^{k_1z}-k_1}dz+C_2]x^{k_2} \quad (2.2.8)$$

$$y(x_0)=y_0, \quad y'(x_0)=y_1 \quad (2.2.9)$$

Bunda $k_1=\frac{n+1-\sqrt{\Delta}}{2}$ $\quad k_2=\frac{n+1+\sqrt{\Delta}}{2}$, $\Delta=(n+1)^2-4a^2$ va $h(lnx)=d(x)$.

2-hol. $\Delta<0$ bo‘lsin u holda $k_1=\frac{n+1-i\sqrt{-\Delta}}{2}$ $\quad k_2=\frac{n+1+i\sqrt{-\Delta}}{2}$

Bu holda (2.2.7) tenglamaning umumiy yechimi:

$$y_2(t)=e^{\frac{(n+1)}{2}t}(C_1coc(\frac{\sqrt{-\Delta}}{2})t+C_2sin(\frac{\sqrt{-\Delta}}{2})t)$$

(2.2.6) tenglamaning umumiy yechimini o‘zgarmasni variatsiyalash usulida topamiz. Quyidagi:

$$\begin{cases} C_1'(t)coc(\frac{\sqrt{-\Delta}}{2})t+C_2'(t)sin(\frac{\sqrt{-\Delta}}{2})t=0 \\ \frac{-\sqrt{-\Delta}}{2}C_1'(t)sin(\frac{\sqrt{-\Delta}}{2})t+\frac{\sqrt{-\Delta}}{2}C_2'(t)coc(\frac{\sqrt{-\Delta}}{2})t=h(t) \end{cases}$$

tenglamalar sistemasini qaraylik, algebreyik hisoblashlar natijasida sistemadan

$$C_1(t)=-\frac{2}{\sqrt{-\Delta}}\int_0^t h(z)\sin\left(\frac{\sqrt{-\Delta}}{2}\right)zdz+C2_1$$

$$C_2(t)=\frac{2}{\sqrt{-\Delta}}\int_0^t h(z)\cos\left(\frac{\sqrt{-\Delta}}{2}\right)z\,dz+C2_2$$

Demak, $(n+1)^2-4a^2<0$ bo‘lganda (2.2.6) tenglamaning umumiy yechimida $t=lnx$ almashtirishbajarganimizda (2.2.5) ning umumiy yechimi:

$$y(x)=e^{\frac{(n+1)}{2}t}[(C2_1-\frac{2}{\sqrt{-\Delta}}\int_0^{lnx}h(z)\sin\left(\frac{\sqrt{-\Delta}}{2}\right)zdz)coc(\frac{\sqrt{-\Delta}}{2})lnx+$$

$$+(\frac{2}{\sqrt{-\Delta}}\int_0^{lnx}h(z)\cos\left(\frac{\sqrt{-\Delta}}{2}\right)z\,dz+C2_2)sin(\frac{\sqrt{-\Delta}}{2})lnx] \quad (2.2.10)$$

$$y(x_0)=y_0, \quad y'\left(\frac{1}{x_0}\right)=y_1 \quad (2.2.2)$$

Bunda $\Delta=(n+1)^2-4a^2<0$ va $h(lnx)=d(x)$.

3-hol. $\Delta=0$ bo‘lsin u holda $k_1=k_2=\frac{n+1}{2}$

Bu holda (2.2.7) tenglamaning umumiy yechimi: $y_3(t) = (C_1 + tC_2)e^{\frac{n+1}{2}t}$

(2.2.6) tenglamaning umumiy yechimini o‘zgarmasni variatsiyalash usulida topamiz. Quyidagi

$$\begin{cases} C_1'(t)e^{\frac{n+1}{2}t} + C_2'(t)e^{\frac{n+1}{2}t}t = 0 \\ \frac{n+1}{2}C_1'(t)e^{\frac{n+1}{2}t} + C_2'(t)(e^{\frac{n+1}{2}t} + t\frac{n+1}{2}e^{\frac{n+1}{2}t}) = h(t) \end{cases}$$

tenglamalar sistemasini qaraylik, algebreyik hisoblashlar natijasida sistemadan

$$C_1(t) = -\int_0^t t \cdot h(z)e^{-\frac{n+1}{2}t}dz + C3_1$$

$$C_2(t) = \int_0^t h(z)e^{-\frac{n+1}{2}t}dz + C3_2$$

Demak, $(n+1)^2 - 4a^2 = 0$ bo‘lganda (2.2.6) tenglamaning umumiy yechimida $t = lnx$ almashtirish bajarganimizda (2.2.5) ning umumiy yechimi (2.2.11)

$$y(x) = x^{\frac{n+1}{2}}\,[C3_1 - \int_0^{ln|x|} t \cdot h(z)e^{-\frac{n+1}{2}t}dz + ln|x|(\int_0^{ln|x|} h(z)e^{-\frac{n+1}{2}t}dz + C3_2)]$$

$$y(x_0) = y_0, \quad y'\left(\frac{1}{x_0}\right) = y_1 \qquad (2.2.2)$$

Bunda $\Delta = (n+1)^2 - 4a^2 = 0$ va $h(lnx) = d(x)$.

Biz berilgan masalani yechimini umumiy holda topdik. Yuqoridagi umumiy yechimda qatnashgan o‘zgarmaslarni (2.2.2) boshlang‘ich shartdan topiladi.

Misol 2.2.1 (Zilbershteyin). Quyidagi tenglamani yeching:

$$y'(x) = y\left(\frac{1}{x}\right)$$

Yechilishi. Berilgan tenglamani yechish uchun $x = e^t$, $y(x) = g(t)$ almashtirish kiritamiz. U holda

$$y(x) = y(e^t) = g(t)$$

bo‘lgani uchun

$$y\left(\frac{1}{x}\right) = y(e^{-t}) = g(-t)$$

bo‘ladi va bundan tashqari $t = \ln x$ bo‘lgani uchun

$$y'(x) = g'(t)\cdot\frac{dt}{dx} = \frac{1}{x}g'(t) = e^{-t}g'(t)$$

bo‘lib, Zilbershteyin tenglamasi

$$g'(t) = e^{t}g(-t)$$

ko‘tinishni oladi. Hosil bo‘lgan tenglamani yana bir bor differensiallash bilan

$$g''(t) = e^{t}g(-t) - e^{t}g'(-t) = g'(t) - g(t),$$

ya’ni

$$g''(t) - g'(t) + g(t) = 0$$

o‘zgarmas koeffisiyentli differensial tenglamani hosil qilamiz. Bu tenglamaning xarakteristik tenglamasi

$$k^2 - k + 1 = 0$$

va uning ildizlari

$$k_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

bo‘lgani uchun hosil qilingan oddiy differensial tenglamaning umumiy yechimi

$$g(t) = e^{\frac{t}{2}}\left[C_1\cos\frac{t\sqrt{3}}{2} + C_2\sin\frac{t\sqrt{3}}{2}\right]$$

bo‘ladi. Bu tenglamadan

$$g'(t) = \frac{1}{2}g(t) + \frac{\sqrt{3}}{2}e^{\frac{t}{2}}\left[-C_1\sin\frac{t\sqrt{3}}{2} + C_2\cos\frac{t\sqrt{3}}{2}\right] =$$

$$= e^{\frac{t}{2}}\left[\frac{C_1 + \sqrt{3}C_2}{2}\cos\frac{t\sqrt{3}}{2} + \frac{C_2 - \sqrt{3}C_1}{2}\sin\frac{t\sqrt{3}}{2}\right]$$

bo‘lgani uchun

$$e^{-t}g'(t) = g(-t)$$

tenglikka ko‘ra

$$e^{-\frac{t}{2}}\left[C_1\cos\frac{t\sqrt{3}}{2} - C_2\sin\frac{t\sqrt{3}}{2}\right] = e^{\frac{t}{2}}\left[\frac{C_1 + \sqrt{3}C_2}{2}\cos\frac{t\sqrt{3}}{2} + \frac{C_2 - \sqrt{3}C_1}{2}\sin\frac{t\sqrt{3}}{2}\right]$$

Bu tenglikdan $C_1 = \frac{C_1 + \sqrt{3}C_2}{2}$, $\quad -C_2 = \frac{C_2 - \sqrt{3}C_1}{2}$, ya'ni $C_1 = \sqrt{3}C_2$ ekanligini aniqlaymiz. Demak,

$$g(t) = C_2 e^{\frac{t}{2}}\left[\sqrt{3}\cos\frac{t\sqrt{3}}{2} + \sin\frac{t\sqrt{3}}{2}\right] = 2C_2 e^{\frac{t}{2}}\left[\cos\frac{\pi}{6}\cos\frac{t\sqrt{3}}{2} + \sin\frac{\pi}{6}\sin\frac{t\sqrt{3}}{2}\right] =$$

$$= Ce^{\frac{t}{2}}\cos\left(\frac{\pi}{6} - \frac{t\sqrt{3}}{2}\right)$$

bu yerda $C = 2C_2$.

Ammo $g(t) = y(x)$, $t = \ln x$ bo‘lgani uchun berilgan tenglamaning yechimi

$$y(x) = C\sqrt{x}\cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\ln|x|\right)$$

dan iborat.

Misol 2.2.2. Quyidagi tenglamani yeching:

$$f'(x) + \alpha^2 f\left(\frac{1}{x}\right) = 0$$

Yechilishi. Berilgan tenglamani yechish uchun $x = e^t$, $f(x) = g(t)$ almashtirish kiritamiz. U holda

$$f(x) = f\left(e^t\right) = g(t)$$

bo‘lgani uchun

$$f\left(\frac{1}{x}\right) = f\left(e^{-t}\right) = g(-t)$$

bo‘ladi va bundan tashqari $t = \ln x$ ekanligi e'tiborga olinsa

$$f'(x) = g'(t)\cdot\frac{dt}{dx} = \frac{1}{x}g'(t) = e^{-t}g'(t)$$

bo‘lib,berilgan tenglama

$$e^{-t}g'(t) + \alpha^2 g(-t) = 0,$$

yoki

$$g'(t) = -\alpha^2 e^t g(-t)$$

ko‘tinishni oladi. Hosil bo‘lgan tenglamani yana bir bor differensiallash bilan

$$g''(t) = -\alpha^2 e^t g(-t) + \alpha^2 e^t g'(-t) = g'(t) - \alpha^4 g(t),$$

ya'ni

$$g''(t) - g'(t) + \alpha^4 g(t) = 0$$

o'zgarmas koeffisiyentli differensial tenglamani hosil qilamiz. Bu tenglamaning xarakteristik tenglamasi

$$k^2 - k + \alpha^4 = 0$$

va uning ildizlari

$$k_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{1-\alpha^4}}{2}$$

bo'lgani uchun hosil qilingan oddiy differensial tenglamaning umumiy yechimi

$$g(t) = \begin{cases} e^{\frac{t}{2}}[C_1 + C_2 t], \ agar\ |\alpha| = \frac{1}{\sqrt{2}}\ bo'lsa, \\ e^{\frac{t}{2}}\left[C_1 \cos\frac{t\sqrt{4\alpha^4 - 1}}{2} + C_2 \sin\frac{t\sqrt{4\alpha^4 - 1}}{2}\right], agar\ |\alpha| > \frac{1}{\sqrt{2}}\ bo'lsa, \\ C_1 e^{\frac{1+\sqrt{1-4\alpha^4}}{2}t} + C_1 e^{\frac{1-\sqrt{1-4\alpha^4}}{2}t}, \ agar\ |\alpha| < \frac{1}{\sqrt{2}}\ bo'lsa \end{cases}$$

bo'ladi. Ammo $g(t) = f(x),\ t = \ln x$ bo'lgani uchun berilgan tenglamaning yechimi

$$f(x) = \begin{cases} \sqrt{x}[C_1 + C_2 \ln|x|], \ agar\ |\alpha| = \frac{1}{\sqrt{2}}\ bo'lsa, \\ \sqrt{x}\left[C_1 \cos\frac{\sqrt{4\alpha^4 - 1}\ln|x|}{2} + C_2 \sin\frac{\sqrt{4\alpha^4 - 1}\ln|x|}{2}\right], agar\ |\alpha| > \frac{1}{\sqrt{2}}\ bo'lsa, \\ C_1 x^{\frac{1+\sqrt{1-4\alpha^4}}{2}} + C_1 x^{\frac{1-\sqrt{1-4\alpha^4}}{2}}, \ agar\ |\alpha| < \frac{1}{\sqrt{2}}\ bo'lsa \end{cases}$$

ko'rinishga ega.

Yuqorida keltirilgan misollarni yechilishidan foydalanib quyidagi teoremani isbotlash mumkin.

Teorema 2.2.2. Ushbu

$$y'(x) = \alpha x^{\beta} y\left(\frac{1}{x}\right), \quad 0 < x < +\infty, \quad y(1) = y_0 \qquad (2.2.12)$$

ko‘rinishdagi tenglamani integrallash mumkin.

Isboti. Berilgan tenglamani x bo‘yicha differensiallash natijasida

$$y''(x) = \alpha\beta x^{\beta-1} y\left(\frac{1}{x}\right) - \alpha^2 x^{-2} y(x) \qquad (2.2.13)$$

tenglama hosil bo‘ladi. Berilgan (2.2.12) tenglamadan

$$y\left(\frac{1}{x}\right) = \frac{1}{\alpha} x^{-\beta} y'(x)$$

bo‘lgani uchun bu tenglamani bir qator murakkab bo‘lmagan hisoblaslar yordamida

$$x^2 y''(x) - \beta x y'(x) + \alpha^2 y(x) = 0 \qquad (2.2.14)$$

Eyler tenglamasi ko‘rinishida ifodalashimiz mumkin.

(2.2.14) tenglamada $x = e^t$, $y(x) = g(t)$ almashtirish kiritamiz. U holda

$$y(x) = y(e^t) = g(t)$$

bo‘lgani uchun

$$y\left(\frac{1}{x}\right) = y(e^{-t}) = g(-t)$$

bo‘ladi va bundan tashqari $t = \ln x$ ekanligi e’tiborga olinsa

$$y'(x) = g'(t) \cdot \frac{dt}{dx} = \frac{1}{x} g'(t),$$

$$y''(x) = -\frac{1}{x^2} g'(t) + \frac{1}{x^2} g''(t)$$

tengliklar hosil bo‘ladi. Bu tengliklarga ko‘ra (17) tenglama

$$g''(t) - (\beta - 1) g'(t) + \alpha^2 g(t) = 0 \qquad (2.2.15)$$

ko‘rinishni oladi.

Endi $y(1)=y_0$ boshlang‘ich shart e’tiborga olinsa, u holda $x=e^t, x_0=1$ bo‘lgani uchun $t=0$ bo‘lganda $g(0)=y(1)=y_0$ va

$$y'(x)=\alpha x^{\beta} y\left(\frac{1}{x}\right)$$

berilgan tenglamadan $y'(1)=\alpha y_0$ bo‘lgani uchun $g'(t)=xy'(x)$ tenglikdan $g'(0)=y'(1)=\alpha\, y_0$ ekanligini ko‘ramiz.

Demak, (2.2.15) tenglamani

$$g(0)=y_0,\; g'(0)=\alpha y_0 \tag{2.2.16}$$

boshlang‘ich shartlar bo‘yicha integrallash mumkin. Teorema isbotlandi.

Isbotlangan teoremada hosil bo‘lgan differensial tenglamani integrallaylik.

Misol 2.2.3. Ushbu

$$g''(t)-(\beta-1)g'(t)+\alpha^2 g(t)=0$$

(2.2.15) tenglamani

$$g(0)=y_0,\; g'(0)=\alpha y_0$$

(2.2.16) boshlang‘ich shartlar bo‘yicha yeching.

Yechilishi. Xarakteristik tenglama

$$k^2-(\beta-1)k+\alpha^2=0$$

ko‘rinishdagi kvadrat tenglama bo‘lib, uning diskriminanti

$$\Delta=(\beta-1)^2-4\alpha^2$$

Quyidagi hollar bo‘lishi mumkin:

1-hol. $\Delta>0$ bo‘lsa, u holda xarakteristik tenglama ikkita haqiqiy

$$k_1=\frac{1}{2}\left(\beta+1-\sqrt{\Delta}\right),\quad k_2=\frac{1}{2}\left(\beta+1+\sqrt{\Delta}\right)$$

ildizlarga ega. Bu holda (2.2.15) tenglamaning umumiy yechimi

$$g(t)=C_1e^{k_1 t}+C_2e^{k_2 t}$$

bo‘lgani uchun

$$g'(t)=C_1k_1e^{k_1 t}+C_2k_2e^{k_2 t}$$

bo‘lib, (2.2.16) chegaraviy shartlarga asosan

$$\begin{cases} C_1 + C_2 = y_0, \\ k_1C_1 + k_2C_2 = \alpha y_0 \end{cases}$$

tenglamalar sistemasini yozamiz. Bu sistemani yechib, noma’lum

$$C_1 = \frac{\alpha - k_1}{k_2 - k_1} y_0, \; C_2 = \frac{\alpha - k_2}{k_2 - k_1} y_0$$

koeffisiyentlarni aniqlaymiz. Demak,

$$g(t) = \frac{y_0}{k_2 - k_1}\left[(\alpha - k_1)e^{k_1 t} - (\alpha - k_2)e^{k_2 t}\right]$$

(2.2.15),(2.2.16) masalaning yechimi bo‘ladi. Endi $x = e^t, \; y(x) = g(t)$ belgilashlarni va

$$k_1 = \frac{1}{2}\left(\beta + 1 - \sqrt{\Delta}\right), \; k_2 = \frac{1}{2}\left(\beta + 1 + \sqrt{\Delta}\right)$$

ekanligi e’tiborga olinsa, u holda (2.2.12) masalaning yechimi

$$y(x) = \frac{y_0}{\sqrt{\Delta}}\left[(\alpha - k_1)x^{k_1} - (\alpha - k_2)x^{k_2}\right],$$

yoki to‘laroq holda

$$y(x) = \frac{y_0 x^{\frac{\beta+1}{2}}}{2\sqrt{(\beta+1)^2 - 4\alpha^2}}\left[\left(\beta + 1 - 2\alpha + \sqrt{(\beta+1)^2 - 4\alpha^2}\right)x^{\frac{\sqrt{(\beta+1)^2 - 4\alpha^2}}{2}} + \right.$$

$$\left. + \left(2\alpha + \beta + 1 - \sqrt{(\beta+1)^2 - 4\alpha^2}\right)x^{-\frac{\sqrt{(\beta+1)^2 - 4\alpha^2}}{2}}\right]$$

ko‘rinishda ega bo‘ladi.

2-hol. $\Delta = 0$ bo‘lsa, u holda xarakteristik tenglama ikkita teng haqiqiy

$$k_1 = k_2 = \frac{1}{2}(\beta + 1)$$

ildizlarga ega bo‘lgani uchun (2.2.15) tenglamaning umumiy yechimi

$$g(t) = (C_1 + C_2 t)e^{\frac{\beta+1}{2}t}$$

bo‘lgani uchun va

$$g'(t)=\left[\frac{(C_1+C_2t)(\beta+1)}{2}+C_2\right]e^{\frac{\beta+1}{2}t}$$

bo‘lib, (2.2.16) chegaraviy shartlaega asosan

$$\begin{cases} C_1=y_0, \\ C_2+C_1\cdot\dfrac{\beta+1}{2}=\alpha y_0 \end{cases}$$

tenglamalar sistemasini yozamiz. Bu sistemani yechib, noma’lum

$$C_1=y_0,\ C_2=\frac{\alpha-\beta-1}{2}y_0$$

koeffisiyentlarni aniqlaymiz. Demak,

$$g(t)=\frac{y_0}{2}\left[2+(2\alpha-\beta-1)t\right]e^{\frac{\beta+1}{2}t}$$

(2.2.15),(2.2.16) masalaning yechimi bo‘ladi. Endi $x=e^t,\ y(x)=g(t)$ belgilashlarni va ekanligi e’tiborga olinsa, u holda (2.2.12) masalaning yechimi

$$y(x)=\frac{y_0x^{\frac{\beta+1}{2}}}{2}\left[2+(2\alpha-\beta-1)\ln|x|\right]$$

ko‘rinishda ega bo‘ladi.

3-hol. $\Delta<0$ bo‘lsa, u holda xarakteristik tenglama ikkita lompleks

$$k_{1,2}=\frac{1}{2}\left(\beta+1\pm i\sqrt{4\alpha^2-(\beta+1)}\right)$$

ildizlarga ega. Bu holda (2.2.15) tenglamaning umumiy yechimi

$$g(t)=e^{\frac{\beta+1}{2}t}\left[C_1\cos\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}+C_2\sin\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}\right]$$

bo‘lgani uchun $g(0)=y_0, g'(0)=\alpha y_0$ boshlang‘ich shartlar e’tiborga olinsa, u holda

$$g'(t)=e^{\frac{\beta+1}{2}t}\cdot\frac{\beta+1}{2}\left[C_1\cos\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}+C_2\sin\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}\right]+$$

$$+e^{\frac{\beta+1}{2}t}\left[-C_1\sin\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}+C_2\cos\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}\right]\cdot\frac{\sqrt{4\alpha^2-(\beta+1)}}{2}$$

bo‘lib, (2.2.16) chegaraviy shartlaega asosan

$$\begin{cases} C_1=y_0, \\ \frac{\beta+1}{2}C_1+\frac{\sqrt{4\alpha^2-(\beta+1)}}{2}C_2=\alpha y_0 \end{cases}$$

tenglamalar sistemasini yozamiz. Bu sistemani yechib, noma’lum

$$C_1=y_0,\ C_2=\frac{2\alpha-\beta-1}{\sqrt{4\alpha^2-(\beta+1)}}y_0$$

koeffisiyentlarni aniqlaymiz. Demak, (2.2.15),(2.2.16) masalaning yechimi

$$g(t)=\frac{y_0\sqrt{x^{\beta+1}}}{\sqrt{4\alpha^2-(\beta+1)}}\left[\sqrt{4\alpha^2-(\beta+1)}\cos\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}+\right.$$

$$\left.+\left(2\alpha^2-\beta-1\right)\sin\frac{t\sqrt{4\alpha^2-(\beta+1)}}{2}\right]$$

bo‘ladi. Endi $x=e^t$, $y(x)=g(t)$ belgilashlarni e’tiborga olinsa, u holda (2.2.12) masalaning yechimi

$$y(x)=\frac{y_0\sqrt{x^{\beta+1}}}{\sqrt{4\alpha^2-(\beta+1)}}\left[\sqrt{4\alpha^2-(\beta+1)}\cos\frac{\sqrt{4\alpha^2-(\beta+1)}\ln|x|}{2}+\right.$$

$$\left.+\left(2\alpha^2-\beta-1\right)\sin\frac{\sqrt{4\alpha^2-(\beta+1)}\ln|x|}{2}\right]$$

ko‘rinishda bo‘ladi.

2.3. $n-$ tartibli involyutsiya qatnashgan Eyler tenglamasi

Quyidagi

$$a_nx^ny^{(n)}(x)+a_{n-1}x^{n-1}y^{(n-1)}(x)+\cdots+a_1xy'(x)+y\left(\frac{1}{x}\right)=q(x) \quad (2.3.1)$$

tenglamani qaraylik. Bunda $a_k = const, k = \overline{1,n}\,, n \epsilon N, q(x) \neq 0$ −berilgan funksiya.

Teorema 2.3.1. Yuqoridagi (2.3.1) tenglamani involyutsiyadan qutqarish natijasida $2n$ −tartibli bir jinsli bo‘lmagan Eyler tenglamasini integrallash masalasiga keladi.

Isboti. (2.3.1) tenglamadan

$$y\left(\frac{1}{x}\right) = q(x) - a_1 x y'(x) - \cdots - a_n x^n y^{(n)}(x) \qquad (2.3.2)$$

ni topamiz va (2.3.2) dan ketma-ket n martta x bo‘yicha xosila olib birin ketinlikda yozamiz. Bizda talab qilingan narsa (2.3.1) tenglamani $2n$ −ta’rtibli Eyler tenglamasiga kelishini ko‘rsatish, shuning uchun uning o‘zgarmas koeffitsiyentlarini aniq son bilan keltirib o‘tmaymiz. Olingan natijalarni yozamiz:

1) $y'\left(\frac{1}{x}\right) = q_1(x) + a_{10}x^2y'(x) + a_{11}x^3y''(x) + \cdots +$
$a_{1(n-1)}x^{n+1}y^{(n)}(x) + a_{1n}x^{n+2}y^{(n+1)}(x)$;

2) $y''\left(\frac{1}{x}\right) = q_2(x) + a_{20}x^3y'(x) + a_{21}x^4y''(x) + \cdots +$
$a_{2(n-1)}x^{n+2}y^{(n)}(x) + a_{2n}x^{n+3}y^{(n+1)}(x) + a_{2(n+1)}x^{n+4}y^{(n+2)}(x)$;

3).$y'''\left(\frac{1}{x}\right) = q_3(x) + a_{30}x^4y'(x) + a_{21}x^5y''(x) + \cdots + a_{2(n-1)}x^{n+3}y^{(n)}(x) +$
$a_{3n}x^{n+4}y^{(n+1)}(x) + a_{3(n+1)}x^{n+5}y^{(n+2)}(x) + a_{3(n+2)}x^{n+6}y^{(n+3)}(x)$;

… … … … … … …

n) $y^{(n)}\left(\frac{1}{x}\right) = q_n(x) + a_{n0}x^{(n+1)}y'(x) + a_{n1}x^{(n+2)}y''(x) + \cdots +$
$a_{n(2n-2)}x^{3n-2}y^{(2n-2)} + a_{n(2n-1)}x^{3n-1}y^{(2n-1)}(x) + a_{n(2n)}x^{3n}y^{(2n)}(x)$;

Bunda $a_{ij} = const, i = \overline{1,n}, j = \overline{0,2n}\,, q_i(x) -$ berilgan funksiyalar.

Yuqorida berilgan (2.3.1) tenglamada $f\colon x \to \frac{1}{x}$ akslantirish bajarsak quyidagi

$$a_n \frac{1}{x^n} y^{(n)}\left(\frac{1}{x}\right) + a_{n-1}\frac{1}{x^{n-1}} y^{(n-1)}\left(\frac{1}{x}\right) + \cdots + a_1 \frac{1}{x} y'\left(\frac{1}{x}\right) + y(x) = q\left(\frac{1}{x}\right) \quad (2.3.3)$$

(2.3.3) tenglikdagi $y^{(n)}(\frac{1}{x}), y^{(n-1)}(\frac{1}{x}), \ldots, y'(\frac{1}{x})$ lar o‘rniga yuqorida olingan natijalarni olib kelib qo‘yamiz:

$$a_n \frac{1}{x^n}[q_n(x) + a_{n0}x^{(n+1)}y'(x) + a_{n1}x^{(n+2)}y''(x) + \cdots +$$
$$a_{n(2n-2)}x^{3n-2}y^{(2n-2)} + a_{n(2n-1)}x^{3n-1}y^{(2n-1)}(x) + a_{n(2n)}x^{3n}y^{(2n)}(x)] +$$
$$a_{n-1}\frac{1}{x^{n-1}}[q_n(x) + a_{n0}x^{(n+1)}y'(x) + a_{n1}x^{(n+2)}y''(x) + \cdots +$$
$$a_{n(2n-2)}x^{3n-2}y^{(2n-2)} + a_{n(2n-1)}x^{3n-1}y^{(2n-1)}(x) + a_{n(2n)}x^{3n}y^{(2n)}(x)] + \cdots +$$
$$a_1\frac{1}{x}[q_1(x) + a_{10}x^2y'(x) + a_{11}x^3y''(x) + \cdots + a_{1(n-1)}x^{n+1}y^{(n)}(x) +$$
$$a_{1n}x^{n+2}y^{(n+1)}(x)] + y(x) = q(\tfrac{1}{x}) \qquad (2.3.4)$$

(2.3.4) tenglamani soddalashtirib ixchamlasak,

$$b_{2n}x^{2n}y^{(2n)}(x) + b_{2n-1}x^{2n-1}y^{(2n-1)}(x) + \cdots + b_1xy'(x) + y(x) = g(x) \quad (2.3.5)$$

ko‘rinishga keladi. Bunda $b_i = const, i = \overline{1,2n}, n\epsilon N, g(x)$ −berilgan funksiya. $g(x) -$ funksiya va berilgan o‘zgarmas koeffitsiyentlarni aniq ko‘rinishini hisoblab topish ham mumkin. Lekin, biz berilgan (2.3.1) tenglamani $2n -$ tartibli Eyler tenglamasini integrallash masalasiga kelishini ko‘rsatishimiz kerak edi. Bu teoremadan xulosa qilib shuni aytish mumkin ekanki, agar (2.3.1) tenglama bir jinsli bo‘lsa, u holda (2.3.5) tenglama ham bir jinsli bo‘lar ekan.

Biz quyidagi tenglamani qaraymiz

$$a \cdot y^{(n)}(x) = y(x) + q(x) \qquad (2.3.6)$$

bunda $a = const, n \in N,\ q(x)$ −ozod had.

Endi (2.3.6) tenglamaga $f(x)$ involyutsiyani ta’sir etkazsak. Natijada quyidagi tenglama hosil bo‘ladi:

$$a \cdot y^{(n)}(x) = y(f(x)) + q(x) \qquad (2.3.7)$$

Teorema 2.3.2. Agar (2.3.7) tenglamada $f(x) = \frac{1}{x}$ bo‘lsa, u holda (2.3.7) tenglamani involyutsiyadan qutqarish natijasida $(2n) -$ tartibli Eyler tenglamasiga keladi.

Isboti. (2.3.7) tenglamada $f(x) = \frac{1}{x}$ bo‘lsa, u holda quyidagi ko‘rinishga keladi.

$$a \cdot y^{(n)}(x) = y\left(\frac{1}{x}\right) + q(x) \qquad (2.3.8)$$

Bu tenglamani n ning hususiy hollarida tekshirib chiqamiz.

a) $n = 1$ da (2.3.8) tenglama ushbu ko‘rinishga keladi:

$$a \cdot y'(x) = y\left(\frac{1}{x}\right) + q(x) \qquad (2.3.9)$$

Bundan esa $y\left(\frac{1}{x}\right) = a \cdot y'(x) - q(x)$ (2.3.10) ni topamiz va undan bir martta x bo‘yisha hosila olamiz natijada ushbu tenglik hosil bo‘ladi.

$y'\left(\frac{1}{x}\right) = -x^2(ay''(x) - q'(x))$ (2.3.11). (2.3.9) tenglamada involyutsiya xossasidan foydalansak ushbu tenglikka ega bo‘lamiz:

$$a \cdot y'\left(\frac{1}{x}\right) = y(x) + q(\frac{1}{x}) \qquad (2.3.12)$$

(2.3.11) va (2.3.12) tengliklardan quyidagi natijaga ega bo‘lamiz:

$(ax)^2 y''(x) + y(x) = g(x)$ (2.3.13), bunda $g(x) = -ax^2 q'(x) - q(\frac{1}{x})$.

2) $n = 2$ da (2.3.9) tenglama ushbu ko‘rinishga keladi:

$$a \cdot y''(x) = y\left(\frac{1}{x}\right) + q(x) \qquad (2.3.14)$$

Bundan esa $y\left(\frac{1}{x}\right) = a \cdot y''(x) - q(x)$ (2.3.15) ni topamiz va undan ikki martta x bo‘yisha hosila olamiz natijada ushbu tengliklar hosil bo‘ladi.

$$y'\left(\frac{1}{x}\right) = -ax^2 y'''(x) + x^2 q'(x) \qquad (2.3.16)$$

$$y''\left(\frac{1}{x}\right) = ax^4 y^{(4)}(x) + 2ax^3 y^{(3)}(x) - x^4 q''(x) - 2x^3 q'(x) \qquad (2.3.17)$$

(2.3.14) tenglamada involyutsiya xossasidan foydalansak ushbu tenglikka ega bo‘lamiz:

$$a \cdot y''\left(\frac{1}{x}\right) = y(x) + q(\frac{1}{x}) \qquad (2.3.18)$$

(2.3.17) va (2.3.18) tengliklardan quyidagi natijaga ega bo‘lamiz:

$$a^2 x^4 y^{(4)}(x) + 2a^2 x^3 y^{(3)}(x) - y(x) = g_1(x) \qquad (2.3.19)$$

Bunda $g_1(x) = x^4 q''(x) + 2x^3 q'(x) + q(\frac{1}{x})$.

Demak, yuqoridagi a) va b) lardan kelib chiqqan holda $n-$ tartibli uchun ham quydagini yozishimiz mumkin. Ya’ni (2.3.9) tenglamani involyutsiyadan qutqarsak natijada quyidagi tenglik hosil bo‘ladi.

1) Agar $n-$ juft son bo‘lsa, bunda $a_i = const$, $i = \overline{0, (n-1)}, n\epsilon N$

$$a_0x^{2n}y^{(2n)}(x)+a_1x^{2n-1}y^{(2n-1)}(x)+\cdots+a_{n-1}x^{n+1}y^{(n+1)}(x)-y(x)=f(x)$$

2) Agar n –toq son bo‘lsa, bunda $a_i=const\,,i=\overline{0,(n-1)},n\epsilon N$

$$a_0x^{2n}y^{(2n)}(x)+a_1x^{2n-1}y^{(2n-1)}(x)+\cdots+a_{n-1}x^{n+1}y^{(n+1)}(x)+y(x)=f(x)$$

Yuqoridagi 1) va 2) tengliklarni ham induksiya metodidan foydalanib isbotlash mumkin bu unchalik qiyinchilik tug‘dirmaydi.

Misol 2.3.1. Quyidagi ikkinchi tartibli

$$y''(x)=y\left(\frac{1}{x}\right)$$

tenglamaning umumiy yechimini toping.

Yechilishi. Berilgan tenglamani x bo‘yicha ketma-ket differensiallab

$$y'''(x)=-\frac{1}{x^2}y'\left(\frac{1}{x}\right),\ y^{(IV)}(x)=\frac{2}{x^3}y'\left(\frac{1}{x}\right)+\frac{1}{x^4}y''\left(\frac{1}{x}\right)$$

tengliklarni hosil qilamiz.

Endi berilgan tenglamada $f:x\to\frac{1}{x}$ almashtirish bajarsak, berilgan tenglama

$$y''\left(\frac{1}{x}\right)=y(x)$$

ko‘rinishni oladi. Bundan va yuqorida hosil qilingan ikki tengliklardan

$$x^4y^{(IV)}(x)+2x^3y'''(x)-y(x)=0$$

Eylerning oddiy differensial tenglamasini hosil qilamiz. Tenglamaning xarakteristik ko‘phadi: $k(k-1)(k-2)(k-3)+2k(k-1)(k-2)-1=0$

$$k_{1,2}=1\pm\sqrt{\frac{1+\sqrt{5}}{2}}=1\pm\frac{\sqrt{2+2\sqrt{5}}}{2},\ k_{3,4}=1\pm i\sqrt{\frac{-1+\sqrt{5}}{2}}=1\pm i\frac{\sqrt{-2+2\sqrt{5}}}{2}$$

ildizlarga ega bo‘lgani uchun berilgan differensial tenglamaning umumiy yechimi:

$$y(x)=x\left[C_1ch\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)+C_2sh\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)\right]+$$

$$+x\left[C_3\cos\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)+C_4\sin\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)\right]$$

2.4-§. Yuqori tartibli involyutsiya qatnashgan differensial tenglamalar

Ta'rif 2.4.1. Ushbu

$$Q_n y^{(n)}(x) + Q_{n-1} y^{(n-1)}(x) + \cdots + Q_1 y'(x) + y(f(x)) = g(x) \quad (2.4.1)$$

tenglamaga n- tartibli o'zgarmas koeffitsiyentli differensial tenglamalarni bitta argumentiga involyutsiyani ta'siri deymiz.

Bunda $Q_i = const, i = \overline{1,n}$, $n\epsilon N$, $f(x)$ −involyutsiya, ya'ni $f\big(f(x)\big) = x$. $g(x)$ −ozod had.

Ta'rif 2.4.2. Ushbu

$$P_n y^{(n)}(x) + P_{n-1} y^{(n-1)}(f(x)) + \cdots + P_1 y'(f(x)) + y(f(x)) = q(x) \quad (2.4.2)$$

tenglamaga n- tartibli o'zgarmas koeffitsiyentli differensial tenglamalarni n ta argumentiga involyutsiyani ta'siri deymiz.

Bunda $P_i = const, i = \overline{1,n}$, $n\epsilon N$, $f(x)$ −involyutsiya, ya'ni $f\big(f(x)\big) = x$, $q(x)$ −ozod had.

Teorema 2.4.1. Agar (2.4.1) tenglamada $f(x) = a - x, a\epsilon R$ ko'rinishda bo'lsa, u holda (2.4.1) tenglama chekli qadamlardan so'ng $(2n)-$ tartibli $n\epsilon N$ o'zgarmas koeffitsiyentli oddiy differensial tenglamani yechish masalasiga keladi.

Isboti. (2.4.1) tenglamada $f(x) = a - x, a\epsilon R$ ga teng bo'lsa, unda (2.4.1) tenglama quydagicha bo'ladi.

$$Q_n y^{(n)}(x) + Q_{n-1} y^{(n-1)}(x) + \cdots + Q_1 y'(x) + y(a - x) = g(x) \quad (2.4.3)$$

$$y(a - x) = g(x) - Q_1 y'(x) - \cdots - Q_{n-1} y^{(n-1)}(x) - Q_n y^{(n)}(x) \quad (2.4.4)$$

Hosil bo'lgan (2.4.4) tenglikdan ketma-ket n marta x bo'yicha differensiyallaymiz:

$$-y'(a - x) = g'(x) - Q_1 y''(x) - \cdots - Q_{n-1} y^{(n)}(x) - Q_n y^{(n+1)}(x) \quad (2.4.5)$$

... ...

$$(-1)^n \cdot y^{(n)}(a - x) = g^{(n)}(x) - Q_1 y^{(n+1)}(x) - \cdots - Q_{n-1} y^{(2n-1)}(x) - Q_n y^{(2n)}(x) \quad (2.4.6)$$

Agar (2.4.3) tenglikda $(a - x)$ involyutsiya bo'lganligidan $x \sim (a - x)$ almashtirish bajarsak, natijada:

$Q_n y^{(n)}(a-x) + Q_{n-1} y^{(n-1)}(a-x) + \cdots + Q_1 y'(a-x) + y(x) = g(a-x)$ (2.4.7)

bizda (2.4.7) tenglikdagi ushbu: $y^{(n)}(a-x), y^{(n-1)}(a-x), \dots, y'(a-x)$ ifodalar yuqoridagi tengliklarda topilgan, ularni (2.4.7) tenglikka olib kelib qo‘yamiz.

$$(-1)^n Q_n[g^{(n)}(x) - Q_1 y^{(n+1)}(x) - \cdots - Q_{n-1} y^{(2n-1)}(x) - Q_n y^{(2n)}(x)]$$
$$+(-1)^{n-1} Q_{n-1}[g^{(n-1)}(x) - Q_1 y^{(n)}(x) - \cdots - Q_{n-1} y^{(2n-2)}(x) -$$
$$Q_n y^{(2n-1)}(x)] + \cdots + (-1) Q_1[g'(x) - Q_1 y''(x) - \cdots - Q_{n-1} y^{(n)}(x) -$$
$$Q_n y^{(n+1)}(x)] + y(x) = g(a-x) \qquad (2.4.8)$$

(2.4.8) tenglama esa $(2n)-$ tartibli o‘zgarmas koeffitsiyentli oddiy differensial tenglamaga keldi

Teorema 2.4.2. Agar (2.4.2) tenglamada $f(x) = a - x$, $a\epsilon R$ ko‘rinishda bo‘lsa, u holda (2.4.2) tenglama chekli qadamlardan so‘ng $(2n+1)-$ tartibli $n\epsilon N$ o‘zgarmas koeffitsiyentli oddiy differensial tenglamani yechish masalasiga keladi.

Isboti. (2.4.2) tenglamada $(x) = a - x$, $a\epsilon R$ desak quyidagi tenglama hosil bo‘ladi.

$$P_n y^{(n)}(x) + P_{n-1} y^{(n-1)}(a-x) + \cdots + P_1 y'(a-x) + y(a-x) = q(x) \quad (2.4.9)$$

(2.4.3) tenglamadan x bo‘yicha bir martta xosila olamiz. Bunda $(a-x)' = -1$

$P_n y^{(n+1)}(x) - P_{n-1} y^{(n)}(a-x) - \cdots - P_1 y''(a-x) - y'(a-x) = q'(x)$ (2.4.10)

$(2.4.9) - va\ (2.4.10) -$ tengliklarni qo‘shamiz. Natijada quyidagi tenglik hosil bo‘ladi:

$$y^{(n+1)}(x) + y^{(n)}(x) - P_{n-1} y^{(n)}(a-x) + y(a-x) = q(x) + q'(x) \quad (2.4.11)$$

(2.4.9) tenglamada $(a-x)$ involyutsiya bo‘lganligidan $x \sim (a-x)$ almashtirish bajaramiz:

$$y^{(n)}(a-x) + P_{n-1} y^{(n-1)}(x) + \cdots + P_1 y'(x) + y(x) = q(a-x) \quad (2.4.12)$$

(2.4.12) tenglikni P_{n-1} ga ko‘paytirib songra hosil bo‘lgan tenglikni (2.4.11) ga qoshish natijasida quydagi tenglikni hosil qilamiz:

$$y^{(n+1)}(x) + y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) + y(a-x) =$$
$$q(x) + q'(x) + a_0 q(a-x) \qquad (2.4.13)$$

(2.4.13) tenglikda $a_0 = P_{n-1}, a_1 = P_1 P_{n-1}, \dots, a_{n-1} = P_{n-1} P_{n-1}$.

(2.4.13) tenglamani teorema 2.4.1 ga ko‘ra yechsa ham bo‘ladi. Lekin, bizda ba’zi funksiyalar borligidan (2.4.13) tenglamadan n martta x bo‘yicha differensiyallaymiz:

$$y^{(2n+1)}(x) + y^{(2n)}(x) + a_{n-1} y^{(2n-1)}(x) + \dots + a_1 y^{(n+1)}(x) + a_0 y^{(n)}(x) + (-1)^n y^{(n)}(a-x) = q^{(n)}(x) + q^{(n+1)}(x) + a_0 q^{(n)}(a-x) \quad (2.4.14)$$

(2.4.12) tenglikni har ikki tarafini $(-1)^n$ ga ko‘paytirib quydagini hosil qilamiz:

$$(-1)^n y^{(n)}(a-x) + (-1)^n [P_{n-1} y^{(n-1)}(x) + \dots + P_1 y'(x) + y(x)] = (-1)^n q(a-x) \quad (2.4.15)$$

Endi (2.4.14) tenglikdan (2.4.15) ni ayiramiz, natijada:

$$y^{(2n+1)}(x) + y^{(2n)}(x) + a_{n-1} y^{(2n-1)}(x) + \dots + a_1 y^{(n+1)}(x) + a_0 y^{(n)}(x) - \\ -(-1)^n \left[P_{n-1} y^{(n-1)}(x) + \dots + P_1 y'(x) + y(x)\right] = q^{(n)}(x) + q^{(n+1)}(x) + \\ + a_0 q^{(n)}(a-x) — 1nq(a-x) \quad (2.4.16)$$

Agar $q^{(n)}(x) + q^{(n+1)}(x) + a_0 q^{(n)}(a-x) — 1nq(a-x) = a(x)$ desak, u holda quydagi

$$y^{(2n+1)}(x) + y^{(2n)}(x) + a_{n-1} y^{(2n-1)}(x) + \dots + a_1 y^{(n+1)}(x) + a_0 y^{(n)}(x) - \\ -(-1)^n \left[P_{n-1} y^{(n-1)}(x) + \dots + P_1 y'(x) + y(x)\right] = a(x) \quad (2.4.17)$$

tenglamaga ega bo‘lamiz.

(2.4.17) tenglama esa $(2n+1)-$ tartibli o‘zgarmas koeffitsiyentli oddiy differensial tenglamaga keldi.

Teorema 2.4.3. Agar (2.4.1) tenglamada $f(x)$ ixtiyoriy involyutsiya ya’ni $f\left(f(x)\right) = x$ bo‘lsa, u holda (2.4.1) tenglama chekli qadamlardan so‘ng $m-$ tartibli oddiy differensial tenglamani yechish masalasiga keladi.

Bunda $m \geq n, m, n \epsilon N$.

Bu teoremani hususiy isboti sifatida quydagi:

a) $n = 1$ bo‘lganda

$$Q_1 y'(x) + y(f(x)) = g(x) \quad (2.4.18)$$

b) $n = 2$ bo‘lganda

$$Q_2 y''(x) + Q_1 y'(x) + y(f(x)) = g(x) \quad (2.4.19)$$

c) $n = 3$ bo‘lganda

$$Q_3 y'''(x) + Q_2 y''(x) + Q_1 y'(x) + y(f(x)) = g(x) \qquad (2.4.20)$$

tenglamalar [36], [37] va [38] maqolalarda involyutsiyadan qutqarilib oddiy differensial tenglamani yechish masalasiga olib kelingan. Lekin, umumiy $n-$ tartibli uchun umumiy holda ko‘rilmagan.

Quyidagi

$$ay'(x) + by(x) + y\left(\frac{1}{x}\right) = q(x) \qquad (2.4.21)$$

tenglamani qaraylik. Bunda $a, b = const, a \neq 0,\ q(x) -$ozod had.

Teorema 2.4.4. (2.4.21) tenglama chekli qadamlardan so‘ng ikkinchi tartibli oddiy differensial tenglamani integrallash masalasiga keladi.

Isboti. (2.4.21) tenglamadan

$$y\left(\frac{1}{x}\right) = q(x) - ay'(x) - by(x) \qquad (2.4.22)$$

topamiz va (2.4.22) tenglikni bir martta differensiyallaymiz:

$-\frac{1}{x^2} y'\left(\frac{1}{x}\right) = q'(x) - ay''(x) - by'(x)$ bundan quydagiga ega bo‘lamiz:

$$y'\left(\frac{1}{x}\right) = -x^2(q'(x) - ay''(x) - by'(x)) \qquad (2.4.23)$$

yuqoridagi (2.4.21) tenglamada involyutsiya hossasidan foydalanib x ni $\frac{1}{x}$ bilan almashtiramiz:

$$ay'\left(\frac{1}{x}\right) + by\left(\frac{1}{x}\right) + y(x) = q\left(\frac{1}{x}\right) \qquad (2.4.24)$$

(2.4.24) tenglikka yuqoridagi (2.4.22) va (2.4.23) tengliklarda topilgan $y\left(\frac{1}{x}\right), y'\left(\frac{1}{x}\right)$ ifodalarni olib kelib qo‘yamiz. Natijada quydagi

$$-ax^2(q'(x) - ay''(x) - by'(x)) + b(q(x) - ay'(x) - by(x)) + y(x) = q\left(\frac{1}{x}\right)$$

tenglikka ega bo‘lamiz va uni soddalashtirsak,

$$(ax)^2 y''(x) + ab(x^2 - 1)y'(x) + (1 - b^2)y(x) = q\left(\frac{1}{x}\right) + (ax^2 - b)q(x)$$

Qulaylig uchun $g(x) = q\left(\frac{1}{x}\right) + (ax^2 - b)q(x)$ desak,

$$(ax)^2 y''(x) + ab(x^2 - 1)y'(x) + (1 - b^2)y(x) = g(x) \qquad (2.4.25)$$

tenglik hosil bo‘ladi. (2.4.25) tenglik oddiy differensial tenglamani integrallash masalasiga keldi.

Yuqoridagi (2.4.25) tenglamani yengillashtirish uchun uning yechimlarini $b = -1, b = 0, b = 1$ bo‘lgan hollarda qaraymiz.

1. $b = -1$ bo‘lsa, unda (2.4.25) tenglama quydagicha bo‘ladi

$$(ax)^2 y''(x) - a(x^2 - 1)y'(x) = g(x) \qquad (2.4.26)$$

$y'(x) = P(x)$ deb belgilash kiritsak, $y''(x) = P'(x)$ bo‘ladi.

$$P'(x) - \frac{1}{a}\left(1 - \frac{1}{x^2}\right)P(x) = \frac{g(x)}{(ax)^2}$$

$P'(x) - \frac{1}{a}\left(1 - \frac{1}{x^2}\right)P(x) = 0 \quad => \int \frac{dP}{P} = \frac{1}{a}\int\left(1 - \frac{1}{x^2}\right)dx$

$P(x) = e^{\frac{1}{a}\left(x+\frac{1}{x}\right)} \cdot h(x) \quad => h'(x) = \frac{g(x)}{(ax)^2} e^{-\frac{1}{a}\left(x+\frac{1}{x}\right)}$

$h(x) = \int_0^x \frac{g(z)}{(az)^2} e^{-\frac{1}{a}\left(z+\frac{1}{z}\right)}dz + C_1$, $P(x) = e^{\frac{1}{a}\left(x+\frac{1}{x}\right)}(\int_0^x \frac{g(z)}{(az)^2} e^{-\frac{1}{a}\left(z+\frac{1}{z}\right)}dz + C_1)$

$y(x) = \int_0^x (e^{\frac{1}{a}\left(t+\frac{1}{t}\right)} \int_0^t \frac{g(z)}{(az)^2} e^{-\frac{1}{a}\left(z+\frac{1}{z}\right)}dz)dt + C_1 \int_0^x e^{\frac{1}{a}\left(t+\frac{1}{t}\right)}dt + C_2$

2. $b = 1$ bo‘lsa, unda (2.4.25) tenglama quydagicha bo‘ladi

$$(ax)^2 y''(x) + a(x^2 - 1)y'(x) = g(x) \qquad (2.4.27)$$

$y'(x) = P(x)$ deb belgilash kiritsak, $y''(x) = P'(x)$ bo‘ladi.

$$P'(x) + \frac{1}{a}\left(1 - \frac{1}{x^2}\right)P(x) = \frac{g(x)}{(ax)^2}$$

$P'(x) + \frac{1}{a}\left(1 - \frac{1}{x^2}\right)P(x) = 0 \quad => \int \frac{dP}{P} = -\frac{1}{a}\int\left(1 - \frac{1}{x^2}\right)dx$

$P(x) = e^{-\frac{1}{a}\left(x+\frac{1}{x}\right)} \cdot h(x) \quad => h'(x) = \frac{g(x)}{(ax)^2} e^{\frac{1}{a}\left(x+\frac{1}{x}\right)}$

$h(x) = \int_0^x \frac{g(z)}{(az)^2} e^{\frac{1}{a}\left(z+\frac{1}{z}\right)}dz + C_1$, $P(x) = e^{-\frac{1}{a}\left(x+\frac{1}{x}\right)}(\int_0^x \frac{g(z)}{(az)^2} e^{\frac{1}{a}\left(z+\frac{1}{z}\right)}dz + C_1)$

$y(x) = \int_0^x (e^{-\frac{1}{a}\left(t+\frac{1}{t}\right)} \int_0^t \frac{g(z)}{(az)^2} e^{\frac{1}{a}\left(z+\frac{1}{z}\right)}dz)dt + C_1 \int_0^x e^{-\frac{1}{a}\left(t+\frac{1}{t}\right)}dt + C_2$

3. $b = 0$ bo‘lsa, unda (2.4.25) tenglama quydagicha bo‘ladi

$$(ax)^2 y''(x) + y(x) = g(x) \qquad (2.4.28)$$

Bundan $x = e^t$ almashtirish bajarsak u holda quyidagi ifodalar hosil bo‘ladi.

$$y(x) = y(e^t),\ x^2y''(x) = \frac{d^2y}{dt^2} - \frac{dy}{dt} \ =>$$

$a^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + y = g(e^t)$ (2.4.29), (2.4.29) tenglamani bir jinsli qismini umumiy yechmini topamz:

$$a^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + y = 0 \ => a^2k^2 - a^2k + 1 = 0,\ k_{1,2} = \frac{a^2 \pm \sqrt{a^4 - 4a^2}}{2a^2}$$

$$y_1 = C_1 e^{\frac{a^2+\sqrt{a^4-4a^2}}{2a^2}t} + C_2 e^{\frac{a^2-\sqrt{a^4-4a^2}}{2a^2}t} = C_1 x^{\frac{a^2+\sqrt{a^4-4a^2}}{2a^2}} + C_2 x^{\frac{a^2-\sqrt{a^4-4a^2}}{2a^2}}$$

(agar $a^4 - 4a^2 \geq 0$ bo‘lsa). (2.4.28) tenglamaning hususiy yechmini y_2 desak, u holda (2.4.28) tenglamaning umumiy yechimi quyidagi ko‘rinishda bo‘ladi.

$$y(x) = \begin{cases} y_2 + C_1 x^{\frac{a^2+\sqrt{a^4-4a^2}}{2a^2}} + C_2 x^{\frac{a^2-\sqrt{a^4-4a^2}}{2a^2}}, & a^4 - 4a^2 \geq 0 \\ y_2 + x^{\frac{1}{2}}[C_1 cos lnx + C_2 sin lnx], & a^4 - 4a^2 < 0. \end{cases}$$

2.5-§. Misollar yechish

Biz bu bo‘limda involyutsiya xossasiga ega bo‘lgan birinchi va ikkinchi tartibli differensial tenglamalarni yechishga oid misollar yechishdan namunalar keltiramiz.

Dastavval agar $\alpha(\alpha(x)) = x$ bo‘lsa, u holda $\alpha(x)$ involyutsoya eslagan holda involyutsiyalarga misollar keltiramiz.

Masalan $\alpha(x) = \sqrt[m]{1 - x^m}$ $m \in N$, $\alpha(x) = \dfrac{ax+b}{cx-a}$ funksiya va uning xususiy hollari bo‘lgan $\alpha(x) = 1 - x, \alpha(x) = \dfrac{1}{x}, \alpha(x) = \dfrac{x}{x-1}, \alpha(x) = \dfrac{x+1}{x-1}$ funksiyalar involyutsiyaga misol bo‘la oladi. Shuning uchun birinchi darajali involyutsiyaga ega bo‘lgan birinchi tartibli oddiy diferensial tenglamalarni umumiy holda

$$y'(x) = y(\alpha(x))$$

ko‘rinishda berilishi mumkin. Agar x ni $\alpha(x)$ bilan almashtirsak

$$y'(\alpha(x)) = y(x)$$

ifodani va differensial tenglamani differensiallash bilan

$$y''(x) = \alpha'(x)y'(\alpha(x)) = \alpha'(x)y(x),$$

ya’ni ikkinchi tartibli Eyler tipidagi

$$y''(x)-\alpha'(x)y(x)=0$$

tenglamani hosil qilamiz. Endi involyutsiya qatnashgan oddiy differensial tenglamalarni yechishga misollar keltiramiz:

1-misol. Ushbu

$$y'(x)=y(1-x)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $1-x$ bilan almashtirsak,

$$y'(1-x)=y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan

$$y''(x)=-y'(1-x)=-y(x)$$

yoki

$$y''(x)+y(x)=0$$

tenglamani hosil qilamiz. Bu tenglamaning umumiy yechimi

$$y(x)=C_1 cosx+C_2\sin x$$

ko‘rinishga ega va bu yechimda x ni $1-x$ bilan almashtirosh va differensiallash bizga

$$y(1-x)=C_1\cos(1-x)+C_2\sin(1-x) \quad \text{va} \quad y'(x)=-C_1\sin x+C_2\cos x$$

tengliklarni beradi. Berilgan tenglamaga ko‘ra

$$-C_1\sin x+C_2\cos x=C_1\cos(1-x)+C_2\sin(1-x)$$

tenglikni hosil qilamiz. Bir qator hisoblashlardan so‘ng bu tenglikdan

$$\begin{cases} C_1\cos 1+C_2\sin 1=C_2, \\ C_1\sin 1-C_2\cos 1=-C_1 \end{cases}$$

Bundan $C_2=\dfrac{1+\sin 1}{\cos 1}C_1$ bo‘lgani uchun berilgan tenglamaning

umumiy yechimini

$$y(x)=C_1\left(\frac{1+\sin 1}{\cos 1}\sin x+\cos x\right)=C(\sin x+\cos(1-x))$$

ko‘rinishda ifodalashimiz mumkin, bu yerda $C=\frac{1}{\cos 1}C_1$.

Javob: $y(x)=C(\sin x+\cos(1-x))$

2-misol. Ushbu

$$y'(x)=\frac{1}{y(a-x)}$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $a-x$ bilan almashtirsak,

$$y'(a-x)=\frac{1}{y(x)}$$

tenglikni hosil qilamiz. . $\alpha(x)=a-x$ akslantirishning qo‘zg‘almas nuqtasi $a-x=x$ tenglikdan $x=\frac{a}{2}$ bo‘lgani uchun berilgan tenglama uchun boshlang‘ich shartlarni $y\left(\frac{a}{2}\right)=y_0$, $y'\left(\frac{a}{2}\right)=\frac{1}{y\left(a-\frac{a}{2}\right)}=\frac{1}{y_0}$ ko‘rinishida olishimiz mumkin. Berilgan tenglamani differensiallash bilan

$$\frac{y''(x)y(x)-y'^2(x)}{y(x)}=0$$

tenglamaga kelamiz. Bu tenglamani

$$\left(\frac{y'(x)}{y(x)}\right)'=0$$

ko‘rinishda yozib integrallash bilan

$$y'(x)=\frac{1}{y_0^2}y(x)$$

tenglamani hosil qilamiz. Bu tenglamani yana bir bor integrallash bilan

$$y(x) = C exp\left(\frac{x}{y_0^2}\right)$$

umumiy yechimni va $y\left(\frac{a}{2}\right) = y_0$ boshlang‘ich shartga ko‘ra $C = y_0 exp\left(-\frac{a}{2y_0^2}\right)$ ekanligini aniqlaymiz. Shuning uchun berilgan tenglamaning yechimi

$$y(x) = y_0 exp\left(\frac{x - \frac{a}{2}}{y_0^2}\right)$$

ko‘rinishga ega.

Javob: $y(x) = y_0 exp\left(\frac{x - \frac{a}{2}}{y_0^2}\right)$

3-misol. Ushbu

$$y'(x) = y\left(\frac{x}{x-1}\right)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $\frac{x}{x-1}$ bilan almashtirsak,

$$y'\left(\frac{x}{x-1}\right) = y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan

$$y''(x) = -\frac{1}{(x-1)^2} y'\left(\frac{x}{x-1}\right) = -\frac{1}{(x-1)^2} y(x)$$

yoki bundan

$$(x-1)^2 y''(x) + y(x) = 0$$

tenglamani hosil qilamiz.

Bu tenglamaning yechimini $y = (x-1)^k$ ko‘rinishida izlasak, xarakteristik tenglama $k(k-1)+1=0$ ko‘rinishda bo‘lib, uning oldizlari $k=\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$ bo‘lgani uchun bu tenglamaning umumiy yechimini

$$y(x)=\sqrt{x-1}\left[C_1 cos\sqrt{3}\ln\sqrt{x-1}+C_2\sin\sqrt{3}\ln\sqrt{x-1}\right]$$

ko‘rinishda ifodalashimiz mumkin.

Endi bu funksiyani berilgan tenglamaga qo‘yamiz. Buning uchun x ni $\frac{x}{x-1}$ bilan almashtirsak bu funksiya

$$y\left(\frac{x}{x-1}\right)=\frac{1}{\sqrt{x-1}}\left[C_1 cos\sqrt{3}\ln\sqrt{x-1}-C_2\sin\sqrt{3}\ln\sqrt{x-1}\right]$$

funksiyaga, va bu funksiyaning hosilasi

$$y'(x)=\frac{1}{2\sqrt{x-1}}\left[(C_1+\sqrt{3}C_2)cos\sqrt{3}\ln\sqrt{x-1}+(C_2-\sqrt{3}C_1)\sin\sqrt{3}\ln\sqrt{x-1}\right]$$

bo‘lgani uchun berilgan tenglamaga ko‘ra

$$2C_1=C_1+\sqrt{3}C_2,\ 2C_2=C_2-\sqrt{3}C_1$$

Bundan $C_1^2+C_2^2=0$ bo‘lgani uchun berilgan tenglamaning yechimi $y(x)=0$ dan iborat.

Javob: $y(x)=0$

4-misol. Ushbu

$$y'(x)=y\left(\frac{x+1}{x-1}\right)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $\frac{x+1}{x-1}$ bilan almashtirsak,

$$y'\left(\frac{x+1}{x-1}\right)=y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan

$$y''(x) = -\frac{2}{(x-1)^2}\, y'\left(\frac{x+1}{x-1}\right) = -\frac{2}{(x-1)^2}\, y(x)$$

yoki bundan

$$(x-1)^2 y''(x) + 2y(x) = 0$$

tenglamani hosil qilamiz.

Bu tenglamaning yechimini $y = (x-1)^k$ ko‘rinishida izlasak, xarakteristik tenglama $k(k-1) + 2 = 0$ ko‘rinishda bo‘lib, uning oldizlari $k = \frac{1}{2} \pm \frac{i\sqrt{7}}{2}$ bo‘lgani uchun bu tenglamaning umumiy yechimini

$$y(x) = \sqrt{x-1}\left[C_1 cos\sqrt{7}\ln\sqrt{x-1} + C_2 \sin\sqrt{7}\ln\sqrt{x-1}\right]$$

ko‘rinishda ifodalashimiz mumkin.

Endi bu funksiyani berilgan tenglamaga qo‘yamiz. Buning uchun x ni $\frac{x+1}{x-1}$ bilan almashtirsak bu funksiya

$$y\left(\frac{x+1}{x-1}\right) = \frac{\sqrt{2}}{\sqrt{x-1}}\left[(C_1 cos\sqrt{7}\ln 2 + C_2 \sin\sqrt{7}\ln 2)\cos\sqrt{7}\ln\sqrt{x-1}\right] +$$

$$+\frac{\sqrt{2}}{\sqrt{x-1}}\left[(C_1 \sin\sqrt{7}\ln 2 - C_2 \cos\sqrt{7}\ln 2)\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

uning hosilasi esa

$$y'(x) = \frac{1}{2\sqrt{x-1}}\left[(C_1 + \sqrt{7}C_2) cos\sqrt{7}\ln\sqrt{x-1} + (C_2 - \sqrt{7}C_1)\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

bo‘lgani uchun berilgan tenglamadan

$$\begin{cases} C_1 + \sqrt{7}C_2 = 2\sqrt{2}(C_1 cos\sqrt{7}\ln 2 + C_2 \sin\sqrt{7}\ln 2 \\ C_2 - \sqrt{7}C_1 = 2\sqrt{2}(C_1 \sin\sqrt{7}\ln 2 - C_2 \cos\sqrt{7}\ln 2 \end{cases}$$

tenglamalar sistemasini hosil qilamiz. Bu sistemadan $C_2 = C_1\sqrt{\frac{\alpha + \sqrt{7}}{\alpha - \sqrt{7}}}$ munosabatni hosil qilamiz, bu yerda

$$\alpha = 2\sqrt{2}\sin(\sqrt{7}\ln 2) - 4\sin(2\sqrt{2}\ln 2) - 2\sqrt{14}\cos(\sqrt{7}\ln 2).$$

Shuning uchun berilgan tenglamaning umumiy yechimi

$$y(x) = C\sqrt{x-1}\left[\left(\alpha + \sqrt{7}\right)cos\sqrt{7}\ln\sqrt{x-1} + \left(\alpha - \sqrt{7}\right)\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

ko‘rinishga ega.

Javob: $y(x) = C\sqrt{x-1}\left[\left(\alpha + \sqrt{7}\right)cos\sqrt{7}\ln\sqrt{x-1} + \left(\alpha - \sqrt{7}\right)\sin\sqrt{7}\ln\sqrt{x-1}\right]$,

bu yerda $\alpha = 2\sqrt{2}\sin(\sqrt{7}\ln 2) - 4\sin(2\sqrt{2}\ln 2) - 2\sqrt{14}\cos(\sqrt{7}\ln 2)$

5-misol. Ushbu

$$y''(x) = y\left(\frac{1}{x}\right)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamani x bo‘yicha ketma-ket differensiallab

$$y'''(x) = -\frac{1}{x^2}y'\left(\frac{1}{x}\right),$$

$$y^{(IV)}(x) = \frac{2}{x^3}y'\left(\frac{1}{x}\right) + \frac{1}{x^4}y''\left(\frac{1}{x}\right)$$

tengliklarni hosil qilamiz.

Endi berilgan tenglamada $f : x \to \frac{1}{x}$ almashtirish bajarsak, tenglama

$$y''\left(\frac{1}{x}\right) = y(x)$$

ko‘rinishni oladi. Agar bu tenglikni hisobga olsak, yuqorida hosil qilingan ikki tengliklardan

$$y^{(IV)}(x) = -\frac{2}{x}y'''(x) + \frac{1}{x^4}y(x)$$

ya’ni

$$x^4 y^{(IV)}(x) + 2x^3 y'''(x) - y(x) = 0$$

Eyler tenglamasini hosil qilamiz. Bu tenglama uchun xarakteristik tenglama

$$k(k-1)(k-2)(k-3) + 2k(k-1)(k-2) - 1 = 0,$$

yoki

$$k(k-1)(k-2)(k-1)-1=0$$

ko‘rinishda bo‘ladi. Bu tenglamani

$$\left(k^2-2k\right)^2+\left(k^2-2k\right)-1=0$$

ko‘rinishda yozsak,

$$k^2-2k=\frac{-1+\sqrt{5}}{2},\ k^2-2k=\frac{-1-\sqrt{5}}{2}$$

Bu tengliklarning har ikkala qismiga 1 ni qo‘shish natijasida

$$(k-1)^2=\frac{1+\sqrt{5}}{2},\ (k-1)^2=-\frac{\sqrt{5}-1}{2}$$

tengliklarni hosil qilamiz. Bundan xarakteristik tenglama ikkita haqiqiy va ikkita kompleks:

$$k_{1,2}=1\pm\sqrt{\frac{1+\sqrt{5}}{2}}=1\pm\frac{\sqrt{2+2\sqrt{5}}}{2},\ k_{3,4}=1\pm i\sqrt{\frac{-1+\sqrt{5}}{2}}=1\pm i\frac{\sqrt{-2+2\sqrt{5}}}{2}$$

ildizlarga ega bo‘lgani uchun berilgan differensial tenglamaning umumiy yechimi

$$y(x)=x\left[C_1ch\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)+C_2sh\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)\right]+$$

$$+x\left[C_3\cos\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)+C_4\sin\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)\right]$$

ko‘rinishga ega bo‘ladi.

3-BOB. INVOLYUTSIYA XOSSASIGA EGA BO‘LGAN XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR

Biz ushbu bobda involyutsoya va maxsus ko‘rinishdagi potensialga ega bo‘lgan differensial tenglama uchun aralash masalani tadqiq qilamiz.

3.1-§. $f(x)=1-x$ involyutsiya xossasiga ega bo‘lgan birinchi tartibli xususiy hosilali differensial tenglama uchun aralash masalaning qo‘yilishi

Biz quyidagi:

$$\frac{1}{\beta i}\frac{\partial u(x,t)}{\partial t}=\left.\frac{\partial u(\xi,t)}{\partial \xi}\right|_{\xi=1-x}+q(x)u(x,t),\ x\in[0,1],\ t\in(-\infty;+\infty),\qquad (3.1.1)$$

$$u(x,0)=\varphi(x),\qquad u(0,t)=0\qquad (3.1.2)$$

aralash masalani qaraymiz. Bu yerda quyidagi shartlar:

1) $\beta -$ haqiqiy son va $\beta\neq 0$;

2) $q(x)\in C^1[0,1]$, $q(x)=q(1-x)$, $q(x)-$ Haqiqiy funksiya;

3) $\varphi(x)\in C^1[0,1]$ va $\varphi(0)=0$, $\varphi'(1)=0$

bajariladi deb hisoblaymiz.

(3.1.1) tenglama $\nu(x)=1-x$ involyutsiyani o‘zida saqlovchi eng soda xususoy hosilali differensial tenglamadir. Involyutsoyaga ega bo‘lgan tenglamalar ustida ko‘plab tadqiqotlar olib borilmoqda (masalan [1] va unda keltirilgan adabiyot-larga qarang).

(3.1.1)-(3.1.2) masalaning yechimini topish uchun Furye usulidan foydalanamiz. Biz keltirgan shartlar masalaning klassik yechimini, ya’ni har ikkala argument bo‘yicha uzluksiz differensialanuvchi yechimni toppish imkoniyatini beradi. $\varphi(x)$ funksiyaga nisbatan qo‘yilgan shartlar tabbiy bo‘lib , bu funksiya (3.1.1)-(3.1.2) chegaraviy masaladan kelib chiquvchi xos funksiani qanoatlantiradi. $q(x)$ funksiyaga qo‘yolgan shart esa masalani tadqiq qilishdagi ko‘plab muammolarni osonlashtiradi va yechim ko‘rinishining yaxshi shaklini beradi.

Ishda [2] ning usullaridan foydalaniladi va bu usul funksional qatorni hadlab differensiallashdan chetlanishga imkoniyat yaratadi.

Fur'e usuliga ko'ra $u(x,t)=y(x)\cdot T(t)$ belgilash kiritamiz. Natijada (3.1.1) tenglama

$$\frac{1}{\beta i}y(x)\cdot T'(t)=y'(1-x)\cdot T(t)+q(x)y(x)\cdot T(t)$$

yoki

$$\frac{T'(t)}{\beta iT(t)}=\frac{y'(1-x)+q(x)y(x)}{y(x)}$$

ko'rinishda yozilishi mumkin. Oxirgi tenglikning chap qismi faqat $t$ga , o'ng qismi esa faqat x ga bog'liq funksiyalar bo'lgani uchun , bu tenglik o'zgarmas sondan iborat, ya'ni

$$\frac{T'(t)}{\beta iT(t)}=\frac{y'(1-x)+q(x)y(x)}{y(x)}=\lambda$$

Bundan $y(x)$ funksiya uchun

$$y'(1-x)+q(x)y(x)=\lambda y(x), \qquad (3.1.3)$$

$$y(0)=0 \qquad (3.1.4)$$

xos qiymatlar masalasini, $T(t)$ funksiya uchun esa $T(t)=ce^{\lambda i\beta t}$ ifodani hosil qilamiz.

2. (3.1.3)- (3.1.4) masalaning yechimini topamiz. (3.1.3) tenglamada x ni $1-x$ ga almashtirib, $y(x)=z_1(x)$, $y(1-x)=z_2(x)$ belgilashlar kiritsak,

$$z_1'(x)+q(1-x)z_2(x)=\lambda z_2(x),$$

$$-z_2'(x)+q(x)z_1(x)=\lambda z_1(x)$$

tenglamalarni hosil qilamiz. Agar $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$ belgilash kiritsak, bu tenglamalarni vector matritsa usuli bilan

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix}+\begin{pmatrix} q(x) & 0 \\ 0 & q(1-x) \end{pmatrix}\cdot\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}=\lambda\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix},$$

yoki $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$ ga nisbatan

$$Bz'(x)+P(x)z(x)=\lambda z(x) \qquad (3.1.5)$$

Drak sistemasi ko‘rinishida yozishimiz mumkin, bu yerda

$$B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ P(x)=\begin{pmatrix}q(x) & 0\\ 0 & q(1-x)\end{pmatrix} \text{ va } z_2(x)=z_1(1-x).$$

Teskarisi ham o‘rinli: agar $z(x)=\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix}$ funksiya (3.1.5) sistemaning yechimi bo‘lib, $z_1(x)=z_2(1-x)$ tenglik bajarilsa, u holda $y(x)=z_1(x)$ funksiya (3.1.3) tenglamaning yechimi bo‘ladi.

3.1.1-lemma. (3.1.5) tenglamaning umumiy yechimi

$$z(x)=z(x,\lambda)=TV(x,\lambda)c \qquad (3.1.6)$$

ko‘rinishga ega, bu yerda

$$T=\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix},\ V(x,\lambda)=diag\left(u_1(x)e^{-\lambda ix},\ u_2(x)e^{\lambda ix}\right),$$

$$u_1(x)=\exp\left(i\int_0^x q(t)dt\right), \qquad u_2(x)=\exp\left(-i\int_0^x q(t)dt\right),$$

$c=\begin{pmatrix}c_1\\ c_2\end{pmatrix}$, c_k, c_2 – ixtiyoriy o‘zgarmas sonlar.

Eslatma 1). B matritsani diognal ko‘rinishga keltiruvchi, ya’ni $T^{-1}BT=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ tenglikni qanoatlantiruvchi

$$T_1=\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}, T_2=\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}, T_3=\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}, T_4=\begin{pmatrix}i & 1\\ 1 & i\end{pmatrix}$$

matritsalar mavjud bo‘lib, bu matritsalarga teskari matritsalar mos ravishda

$$T_1^{-1}=\frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}, T_2^{-1}=\frac{1}{2}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}, T_3^{-1}=\frac{1}{2}\begin{pmatrix}-i & 1\\ i & 1\end{pmatrix}, T_4^{-1}=\frac{1}{2}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix}$$

ko‘rinishga ega.

Eslatma 2). (3.1.5) tenglamaning umumiy yechimi komponentalari bo‘yicha quyidagi ko‘rinishga ega:

$$z_1(x)=c_1u_1(x)e^{-\lambda ix}-c_2iu_2(x)e^{\lambda ix},\ \ z_2(x)=-c_1iu_1(x)e^{-\lambda ix}+c_2iu_2(x)e^{\lambda ix}$$

bu yerda

$$u_1(x)=\exp\left(i\int_0^x q(t)dt\right),\ u_1(x)=\exp\left(-i\int_0^x q(t)dt\right)$$

Isboti. Dastlab (3.1.5) sistemadagi $B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ matritsani kanonik ko‘ronoshga keltiramiz. B matritsaning xarakteristik tenglamasi $|B-kE|=0\Leftrightarrow k^2+1=0$ tenglama $\pm i$ ildizlarga ega bo‘lgani uchun uning kanonik shakli $D=diag(i,-i)$ ko‘rinishda bo‘lib bunga $T^{-1}BT=D$ tenglik orqali erishiladi.

Agar $T=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ almashtirish matritsasi bo‘lsa, u holda bu tenglikni

$$BT=TD\Leftrightarrow\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\cdot\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\Leftrightarrow\begin{pmatrix}-a_{21} & -a_{22}\\ a_{11} & a_{12}\end{pmatrix}=\begin{pmatrix}a_{11}i & -a_{12}i\\ a_{21}i & -a_{22}i\end{pmatrix}$$

ko‘rinishda ifodalashimiz mumkin. Oxirgi tenglikdan $a_{11}=1, a_{21}=-i, a_{12}=-i, a_{22}=1$ bo‘lgani uchun almashtirish atritsasi $T=\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$ dan iborat/ Bu matritsaga teskari matritsani topamiz.

$$\left(\begin{array}{cc|cc}1 & -i & 1 & 0\\ -i & 1 & 0 & 1\end{array}\right)\to\left(\begin{array}{cc|cc}1 & -i & 1 & 0\\ 0 & 2 & i & 1\end{array}\right)\to\left(\begin{array}{cc|cc}1 & -i & 1 & 0\\ 0 & 1 & i/2 & 1/2\end{array}\right)\to\left(\begin{array}{cc|cc}1 & 0 & 1/2 & i/2\\ 0 & 1 & i/2 & 1/2\end{array}\right)$$

bo‘lgani uchun bu tenglikdan $T^{-1}=\begin{pmatrix}1/2 & i/2\\ i/2 & 1/2\end{pmatrix}$ ekanligini aniqlaymiz. Endi (3.1.5) tenglamada $z=Tv$ almashtirish kiritamiz. U holda

$$BTv'(x)+P(x)Tv(x)=\lambda Tv(x)$$

tenglik hosil bo‘ladi. Hosil bo‘lgan tenglikni chapdan T^{-1} matritsaga ko‘paytirilsa

$$T^{-1}BTv'(x)+T^{-1}P(x)Tv(x)=\lambda T^{-1}Tv(x),$$

yoki

$$Dv'(x)+T^{-1}P(x)Tv(x)=\lambda v(x),$$

yoki nihoyat

$$v'(x)+P_1(x)v(x)=\lambda D^{-1}v(x) \tag{3.1.7}$$

tenglamani hosil qilamiz, bu yerda

$$P_1(x)=D^{-1}T^{-1}P(x)T=D^{-1}q(x),$$

chunki $q(x)=q(1-x)$ simmetriklikdan $P_1(x)$ ham diognal matritsa bo‘ladi.

Endi (3.1.7) sistemani komponentalari bo‘yicha yozsak bu sistema ikkita

$$v_1'(x)-iq(x)v_1(x)=-\lambda v_1(x), \qquad v_2'(x)+iq(x)v_2(x)=\lambda v_2(x)$$

tenglamalarga ajraladi. Bu tenglamalarning umumiy yechimlari mos ravishda

$$v_1(x)=v_1(x,\lambda)=c_1u_1(x)e^{-\lambda ix} \quad , \; v_2(x)=v_2(x,\lambda)=c_2u_2(x)e^{\lambda ix}$$

Bu yerda $u_1(x), u_2(x)$ lemma shartida aniqlangan funksiyalar, c_1, c_2-ixtiyoriy o‘zgarmas sonlar.

Nihoyat (3.1.7) tenglamaning yechimini matritsa ko‘rinishida yozib (3.1.6) formulani hosil qilamiz. Lemma isbotlandi.

3.1.2-lemma. (3.1.3) sistemaning umumiy yechimi

$$y(x)=y(x,\lambda)=c\varphi(x,\lambda) \qquad (3.1.8)$$

ko‘rinishga ega, bu yerda

$$\varphi(x,\lambda)=u_1(x)e^{-i\int_0^1 q(t)dt}e^{\lambda i(1-x)}-iu_2(x)e^{\lambda ix}, \; c-ixtiyoriy\ o'zgarmas\ son$$

Isboti. Agar $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$ funksiya (3.1.5) sistemaning yechimi bo‘lsa va $z_2(x)=z_1(1-x)$ bo‘lsa, u holda $y(x)=z_1(x)$ funksiya (3.1.3) sistemaning yechimi bo‘lishi yuqorida ko‘rsatolgan edi. Xususiy holda bundan

$$z_2(1)=z_1(0) \qquad (3.1.9)$$

tenglikni hosil qilamiz. (3.1.6) va (3.1.9) dan

$$c_1u_1(0)-ic_2(0)=-c_1iu_1(1)e^{-\lambda i}+c_2u_2(1)e^{\lambda i}$$

yoki

$$c_1\left[u_1(0)+iu_1(1)e^{-\lambda i}\right]=c_2\left[u_2(1)e^{\lambda i}+iu_2(0)\right] \qquad (3.1.10)$$

$u_1(0)=1,\ u_2(0)=1,\ u_2(1)e^{-i\int_0^1 q(t)dt}=u_1^{-1}(1)$ bo‘lgani uchun (3.1.10) dan

$$c_1\left[1+iu_1(1)e^{-\lambda i}\right]=c_2u_2(1)e^{\lambda i}\left[1+iu_1(1)e^{-\lambda i}\right]$$

Bundan $c_1=c_2u_2(1)e^{\lambda i}$ bo‘lgani uchun

$$y(x)=z_1(x)=c_1u_1(x)e^{-\lambda ix}-c_2iu_2(x)e^{\lambda ix}=c_2\varphi(x,\lambda)$$

bo‘lib, bu (3.1.8) ni isbotlaydi.

Endi masalaning xos qiymati va xos funksiyalarini topamiz.

3.1.3-lemma. (3.1.3)-(3.1.4) chegaraviy masalaning xos qiymati

$$\lambda_n = 2\pi n + a, n \in N \text{, bu yerda} \qquad a = \frac{\pi}{2} + \int_0^1 q(t)dt \ , \qquad (3.1.11)$$

va unga mos xos funksiya

$$y_n(x) = p(1-x)e^{2\pi ni(1-x)} - ip(x)e^{2\pi nix}\ , \qquad (3.1.12)$$

bu yerda $p(x) = u_2(x)e^{aox}$.

Isboti. (3.1.4) va (3.1.8) ga ko‘ra xos qiymatlarni topish uchun $\varphi(0,\lambda) = 0$, yoki

$$u_1(0)e^{-i\int_0^1 q(t)dt} e^{\lambda i(1-0)} - iu_2(0)e^{\lambda i 0} = 0 \text{, yoki } e^{i(\frac{\pi}{2}+\int_0^1 q(t)dt)} = e^{i\lambda}$$

Bu tenglikdan

$$\lambda_n = 2\pi n + a,\ n \in Z$$

xos qiymatlarni topamiz, bu yerda $a = \frac{\pi}{2} + \int_0^1 q(t)dt$.

Endi masalaning $y_n(x) = \varphi(x,\lambda_n)$ xos funksiyalarini topamiz.

$$u_1(x) = \exp\left(i\int_0^x q(t)dt\right) = \exp\left(i\int_0^1 q(t)dt - i\int_x^1 q(t)dt\right) = \exp\left(a - i\int_0^{1-x} q(t)dt\right) = e^{ix}u_2(1-x)$$

Biz bu yerda yana $q(x)$ potensialni simmetrikligidan foydalandik.

Demak, yuqoridagi belgilashlarga asosan

$$y_n(x,\lambda_n) = u_1(x)e^{-i\int_0^1 q(t)dt} e^{\lambda_n i(1-x)} - iu_2(x)e^{\lambda_n ix} = u_1(x)e^{-ia}\cdot e^{2n\pi i(1-x)}\cdot e^{ia(1-x)} -$$
$$-iu_2(x)\cdot e^{2\pi nix}\cdot e^{aix} = u_2(1-x)e^{ia(1-x)}e^{2\pi ni(1-x)} - iu_2(x)e^{aix}e^{2\pi nix} =$$
$$= p(1-x)e^{2\pi ni(1-x)} - ip(x)e^{2\pi nix}$$

Lemma isbotlandi.

3.2-§.Masala xos qiymati va xos funksiyalarining xossalari

Dastlab $\{y_n(x)\}$ funksiyalar sistemasining xossalarini tekshiramiz.

3.2.1-lemma. $\{y_n(x)\}$ funksiyalar sistemasi $L_2[0,1]$ fazoda to‘liq ortonormal sistemani tashkil qiladi.

Isboti. Xos funksiyalari $\{y_n(x)\}$ bo‘lgan L operatorni qaraymiz:

$$Ly \equiv y'(1-x)+q(x)y(x),\quad y(0)=0$$

Bu operatorga qo‘shma bo‘lgan L^* operatorni topamiz. Aytaylik $z(x)\in W_2^1[0,1]$ bo‘lsin. U holda

$$(Ly,z)=\int_0^1[y'(1-x)+q(x)y(x)]\overline{z(x)}dx=\int_0^1 y'(1-x)\overline{z(x)}dx+\int_0^1 q(x)y(x)\overline{z(x)}dx=$$

$$=\int_0^1 y'(x)\overline{z(1-x)}dx+\int_0^1 q(x)y(x)\overline{z(x)}dx=y(1)\overline{z(0)}+\int_0^1 y(x)\overline{z'(1-x)+z(x)q(x)}dx$$

Bundan $L^*z(x)=z'(1-x)+\overline{q(x)}z(x),\quad z(0)=0$, ammo $q(x)$ haqiqiy funksiya bo‘lgani uchun, u holda $L=L^*$ bo‘lib bundan xos funksiyalarning ortogonalligi kelib chiqadi.

Endi $\{y_n(x)\}$ funksiyalar sistemasining to‘laligini ko‘rsatamiz.

Aytaylik $f\in L[0,1]$ bo‘lib, f funksiya $y_n, n\in Z$ funklsiyaga orthogonal bo‘lsin. U holda

$$(y_n,f)=\int_0^1 y_n(x)\overline{f(x)}dx=\int_0^1\overline{f(x)}\left[p(1-x)e^{2\pi ni(1-x)}-ip(x)e^{2\pi nix}\right]dx=\int_0^1\overline{f(1-x)}p(x)e^{2\pi nix}-$$

$$-i\int_0^1\overline{f(x)}p(x)e^{2\pi nix}dx=\int_0^1\left[\overline{f(1-x)}-i\overline{f(x)}\right]p(x)e^{2\pi nix}=0$$

$\{e^{2\pi nix}\}$ trigonometrik sistema tola bo‘lgani uchun oxirgi tenglikdan

$$f(1-x)+if(x)\equiv 0\ ,x\in[0,1] \qquad (3.2.1)$$

ayniyatga ega bo‘lamiz. (3.2.1) tenglikda x ni $1-x$ ga almashtirib hosil bo‘lgan tenglikni i ga ko‘paytirib

$$if(x)+i^2 f(1-x)\equiv 0 \qquad (3.2.2)$$

ayniyatga ega bo‘lamiz. (3.2.1) va (3.2.1) tengliklarni hadlab qo‘shib $f(x)\equiv 0$ ekanligini ko‘ramiz. Lemma isbotlandi.

Eslatma. 3.2.1-lemmadan (3.1.11) xos qiymatlar bir karralliligi kelib chiqadi.

3.2.2-lemma. Aytaylik $y_n^0(x)=\dfrac{y_n(x)}{\|y_n\|}$ bo‘lsin, bu yerda $\|y_n\|-L_2[0;1]$ fazodagi norma. U holda $y_n^0(x)=\gamma\, y_n(x)$, bu yerda

$$\gamma_n = \frac{1}{\sqrt{2}}[1], \quad [1] = 1 + O\left(\frac{1}{n}\right)$$

Isboti.

$$\|y_n\|^2 = \int_0^1 y_n(x)\overline{y_n(x)}dx = \int_0^1 p(1-x)\overline{p(1-x)}dx + \int_0^1 p(x)\overline{p(x)}dx +$$

$$+ i\int_0^1 p(1-x)\overline{p(x)}e^{-4\pi inx}dx - \int_0^1 p(x)\overline{p(1-x)}e^{4\pi nix}dx$$

Uchinchi va to‘rtinchi integrallarni bo‘laklab integrallash, ularga kiruvchi eksponentalarni chegaralanganligi hamda $\|p(x)\| = 1$ ekanligini e’tiborga olsak $\|y_n\|^2 = 2 + O(1/n)$ ekanligini ko‘ramiz, bundan esa lemmaning tasdig‘I kelib chiqadi.

L operatorning $L_2[0;1]$ fazodagi aniqlanish sohasini D_L bilan belgilaymiz.

3.2.3-lemma. Agar $f(x) \in D_L$ bo‘lsa, u holda bu funksiyaning $\{y_n(x)\}$ xos funksiyalar bo‘yicha Fur’ye qatori [0;1] kesmada absolyut va tekis yaqinlashadi.

Isboti. $f(x)$ funksiyaning $\{y_n(x)\}$ xos funksiyalari bo‘yicha Fur’ye qatori

$$\sum_{-\infty}^{\infty}(f, y_n)\gamma_n^2 y_n(x) = \sum_{-\infty}^{\infty}(f, y_n^0)\gamma_n^2 y_n^0(x)$$

ko‘rinishga ega. Aytaylik μ_0 haqiqiy soni L operatorning xos qiymati bo‘lmasin. E birlik operator uchun $(L - \mu_0 E)f = g$ bo‘lsin. U holda $f = R_{\mu_0} g$ bu yerda R_λ – L operatorning rezolventasi. Shu bilan birga

$$(L - \mu_0 E)y_n^0 = (\lambda_n - \mu_0)y_n^0,$$

bndan $y_n^0 = (\lambda_0 - \mu_0)R_{\mu_0} y_n^0$ va

$$(f, y_n^0) = (R_{\mu_0} g, y_n^0) = (g, R_{\mu_0} y_n^0) = \frac{1}{\lambda_0 - \mu_0}(g, y_n^0)$$

Shuning uchun

$$\sum_{-\infty}^{\infty}(f, y_n^0)y_n^0(x) = \sum_{-\infty}^{\infty}\frac{1}{\lambda_0 - \mu_0}(g, y_n^0)y_n^0(x)$$

$\frac{1}{\lambda_0 - \mu_0} = O\left(\frac{1}{n}\right)$ va $\sum_{-\infty}^{\infty}\left|(g, y_n^0)\right|^2 < \infty$ bo‘lgani uchun lemmaning tasdig‘i

Koshi Bunyakovskiy tengsizligi va $y_n^0(x)$ xos funksiyaning chegaralanganligidan kelib chiqadi.

3.2.3-lemmadan $\sum|c_n|$ qatorning yaqinlashishi kelib chiqadi. Shuning uchun $f_0(x)=\sum c_n e^{2\pi nix}$ funksiya $(-\infty,+\infty)$ oraliqda uzluksiz va davri 1 ga teng bo‘lgan davriy funsiyadir, bu yerda $c_n=(f(x),y_n(x))\gamma_n^2$.

3.2.4-lemma. $x\in[0;1]$ bo‘lganda

$$f_0(x)=\frac{1}{2p(x)}[i\varphi(x)+\varphi(1-x)] \qquad (3.2.3)$$

formula o‘rinli.

Isboti. 3.2.3-lemmaga ko‘ra $x\in[0;1]$ bo‘lganda

$$\varphi(x)=\sum_{-\infty}^{\infty}(\varphi,y_n)\gamma_n^2 y_n(x)=\sum_{-\infty}^{\infty}c_n y_n(x)=\sum_{-\infty}^{\infty}c_n\left[p(1-x)e^{2\pi ni(1-x)}-ip(x)e^{2\pi nix}\right],$$

bundan

$$\varphi(x)=p(1-x)f_0(1-x)-ip(x)f_0(x) \qquad (3.2.4)$$

va

$$\varphi(1-x)=p(x)f_0(x)-ip(1-x)f_0(1-x) \qquad (3.2.5)$$

(3.2.4) va (3.2.5) dan

$$i\varphi(x)+\varphi(1-x)=2p(x)f_0(x) \qquad (3.2.6)$$

(3.2.6) dan esa (3.2.3) kelib chiqadi.

Eslatma. $f_0(x)$ funksiya davriy bo‘lgani uchun [0;1] kesmadagina berilgan qi1ymati bilan butun son o‘qida bir qiymatli aniqlanadi. Shuning uchun $f_0(x)$ funksiya qator bilan emas balki (15) formula bilan beriladi.

3.2.5-lemma. Agar $\varphi(x)\in C^1[0;1],\ \varphi(0)=\varphi'(1)=0$ bo‘lsa, u holda $f_0(x)$ funksiya butun son o‘qida uzluksiz differensiallanuvchi bo‘ladi.

Isboti. (3.2.3) formuladan $f_0(x)$ funksiyaning [0;1] kesmada uzluksiz differensiallanuvchanligi (kesma chetlarida bir tonlamali hosilalar tushuniladi) kelib chiqadi. $f_0(x)$ davriy funksiya bo‘lgani uchun, bu funksiya $x=n,n\in N$ nuqtalardan boshqa butun $(-\infty,+\infty)$ son o‘qida uzluksiz differensiallanuvchi bo‘ladi. $f_0'(n-0)=f_0'(n+0)$ ekanligini ko‘rsatamiz. $f_0(x)$ funksiyaning davriyligiga ko‘ra

$$f_0'(0+0)=f_0'(0-0) \tag{3.2.7}$$

ekanligini ko‘rsatish yetarli. (3.2.6) ifodani differensiallab

$$i\varphi'(x)-\varphi'(1-x)=2p'(x)f_0(x)+2p(x)f_0'(x) \tag{3.2.8}$$

tenglikni hosil qilamiz. (3.2.8), lemma shartlaridan va

$$f_0(0)=f_0(1),\ f_0'(1-0)=f_0'(0-0)$$

shartlardan

$$2p'(0)f_0(0)+2p(0)f_0'(0+0)=i\varphi'(0),\quad 2p'(1)f_0(0)+2p(1)f_0'(0-0)=-i\varphi'(0)$$

tengliklarga va bulardan

$$2[p'(0)+ip'(1)]f_0(0)+2[p(0)f_0'(0+0)+ip'(1)f_0'(0-0)]=0 \tag{3.2.9}$$

tenglikni hosil qilamiz. Shu bilan birga

$$p(0)=1,\ p(1)=\exp\left(-i\int q(t)dt\right)e^{ia}=e^{\pi i/2}=i,\ u_2'(x)=-iq(x)u_2(x),$$

$$p'(0)=-iq(0)+ia,\ \ p'(1)=q(1)=a$$

hamda $q(0)=q(1)$. U holda $p'(0)+ip'(1)=0$ bo‘lib (3.2.9) dan (3.2.7) kelib chiqadi. Lemma isbotlandi.

Eslatma. $\varphi'(1)=0$ shart tabbiy, chunki barcha xos funksiyalar bu shartni qanoatlantiradi.

3.3-§.Masalaning klassik yechimi

Furye usuliga ko‘ra (3.1.1)-(3.1.2) masalaning $u(x,t)$ yechimi formal ko‘rinishda

$$\sum_{-\infty}^{\infty}(\varphi,y_n^0)y_n^0(x)e^{\lambda_n\beta it}=\sum_{-\infty}^{\infty}c_ny_n(x)e^{\lambda_n\beta it}, \tag{3.3.1}$$

qator ko‘rinishida ifodalanadi, bu yerda $c_n=(f(x),y_n(x))\gamma_n^2$.

3.3.1-lemma. Barcha $x\in[0;1]$ va $t\in(-\infty,+\infty)$ lar uchun (3.3.1) qator absolyut va tekis yaqinlashadi va uning uchun quyidagi

$$\sum_{-\infty}^{\infty} c_n y_n(x) e^{\lambda_n \beta it} = e^{a\beta it}\left[p(1-x) f_0(1-x+\beta t) - ip(x) f_0(x+\beta t)\right] \qquad (3.3.2)$$

formula o‘rinli, bu yerda $p(x) = u_2(x) e^{iax}$.

Isboti. (3.3.1) qatorning yaqinlashishi 3.2.3-lemmadan kelib chiqadi. Keyin

$$\sum_{-\infty}^{\infty} c_n y_n(x) e^{\lambda_n \beta ie} = \sum_{-\infty}^{\infty} c_n \left[p(1-x) e^{2\pi ni(1-x)} - ip(x) e^{2\pi nix}\right] e^{\lambda_n \beta it} =$$

$$= e^{a\beta it}\left[p(1-x)\sum_{-\infty}^{\infty} c_n e^{2\pi ni(1-x+\beta t)} - ip(x)\sum_{-\infty}^{\infty} c_n e^{2\pi ni(x+\beta t)}\right]$$

Bundan (23) kelib chiqadi.

3.3.1-teorema. Agar $\varphi(x) \in C^1[0;1],\ \varphi(0) = \varphi'(1) = 0.\ q(x) \in C[0,1], q(x) = q(1-x)$ bo‘lsa, u holda (3.1.1)-(3.1.2) masalaning klassik yechimi mavjud va u

$$u(x,t) = e^{a\beta it}\left[p(1-x) f_0(1-x+\beta t) - ip(x) f_0(x+\beta t)\right], \qquad (3.3.3)$$

ko‘rinishga ega, bu yerda $p(x) = \exp\left(aix - i\int_0^x q(t)dt\right)$, $f_0(x)$ davri 1 ga teng bo‘lgan davriy funksiya bo‘lib [0,1] kesmada

$$f_0(x) = \frac{1}{2p(x)}\left[i\varphi(x) + \varphi(1-x)\right] \qquad (3.3.4)$$

Isboti. Yuqorida agar (3.3.4) tenglik bilan berilgan $f_0(x)$ funksiyani butun son o‘qida 1 ga teng davr bo‘yicha davom ettirsak, uholda son o‘qining barcha nuqtasida uzluksiz differensiallanuvchi funksiya bo‘lishi ko‘rsatilgan edi. Endi (3.3.3) formula bilan berilgan $u(x,t)$ funksiya (3.1.1)-(3.1.2) aralash masalaning yechimi bo‘lishini ko‘rsatamiz.

Dastlab $u(x,t)$ funksiya (3.1.1) tenglamani qanoatlantirishini ko‘rsatamiz.

$$\frac{1}{\beta i} u(x,t) = a e^{a\beta it}\left[p(1-x) f_0(1-x+\beta t) - ip(x) f_0(x+\beta t)\right] +$$

$$+ \frac{1}{i} e^{a\beta it}\left[p(1-x) f_0'(1-x+\beta t) - ip(x) f_0'(x+\beta t)\right],$$

$$u_\xi(\xi,t)\Big|_{\xi=1-x}=$$

$$=ae^{a\beta it}\left[-p'(1-\xi)f_0(1-\xi+\beta t)-p(1-\xi)f_0'(1-\xi+\beta t)-p'(1-x)f_0(\xi+\beta t)-ip(\xi)f_0'(\xi+\beta t)\right]\Big|_{\xi=1-x}=$$

$$=e^{a\beta it}\left[-p'(x)f_0(x+\beta t)-p(x)f_0'(x+\beta t)-p'(1-x)f_0(1-x+\beta t)-ip(1-x)f_0'(1-x+\beta t)\right]$$

Hisoblanganlarni (3.1.1) formulaga qo‘yib

$$e^{a\beta it}\{f_0(1-x+\beta t)[ap(1-x)+ip'(1-x)-p(1-x)q(x)]+f_0(x+\beta t)[-aip(x)+p'(x)+ip(x)q(x)]+\}$$

$$+f_0'(1-x+\beta t)\left[\frac{1}{i}p(1-x)f_0+ip(1-x)\right]+f_0'(x+\beta t)[-p(x)+p(x)]\ \}$$

tenglikni hosil qilamiz.Oxirgi ikkita kvadrat qavslardagi ifodalar nolga teng. $p(x)$ va $p'(x)$ ifodalarinig oshkor ko‘rinishlarini qo‘ysak birinchi va ikkinchi kvadratdagi ifodalarining ham nolga tengligini ko‘ramiz. Demak $u(x,t)$ funksiya (3.1.1) tenglamani qanoatlantiradi.

Shu bilan birga $x\in[0,1]$ bo‘lganda

$$u(x,0)=p(1-x)f_0(1-x)-ip(x)f_0(x)=\varphi(x)$$

ya’ni boshlang‘ich shart bajariladi.

Nihoyat,

$$u(0,t)=e^{a\beta it}[p(1)f_0(\beta t)-ip(0)f_0(\beta t)]=0,$$

ya’ni chegaraviy shart ham bajariladi. Teorema isbotlandi.

3.4. $f(x)=1/x$ involyutsiya xossasiga ega bo‘lgan birinchi tartibli xususiy hosilali differensial tenglama uchun aralash masalaning yechimini topish

Quyidagi

$$\frac{\partial u(x,t)}{\partial t}=\frac{i}{\beta}\frac{\partial u(\varepsilon,t)}{\partial \varepsilon}\ ,\qquad \varepsilon=\frac{1}{x},1\le x\le c,\qquad t>0 \tag{3.4.1}$$

$$u(x,0)=\varphi_0(x),u_t(x,0)=\varphi_1(x) \tag{3.4.2}$$

$$u(1,t)=u(c,t)=0 \tag{3.4.3}$$

aralash masalani qaraylik. Bunda $i^2=-1, c>1, \beta=const, \beta\neq 0$, $\varphi_0(x), \varphi_1(x)\epsilon C^2[1,c]$, $\varphi_1(x)=\frac{i}{\beta}\frac{\partial u(\varepsilon,0)}{\partial\varepsilon}$ va $u(x,t)$ berilgan soxada x va t o‘zgaruvchilar bo‘yicha ikki marta uzluksiz differensiallanuvchi funksiya bo‘lsin.

3.4.1-teorema. (3.4.1) tenglamani involyutsiyadan qutqarish natijasida ikki o‘zgaruvchili ikkinchi tartibli bir jinsli giperbolik tipdagi xususiy hosilali differensial tenglamaga keladi.

Isboti. Bizga berilgan (3.3.1) tenglamadan t boyicha bir martta hosila olamiz:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{i}{\beta}\frac{\partial}{\partial t}\left(\frac{\partial u(\varepsilon,t)}{\partial\varepsilon}\right)=\frac{i}{\beta}\frac{\partial}{\partial\varepsilon}\left(\frac{\partial u(\varepsilon,t)}{\partial t}\right) \qquad (3.4.4)$$

(3.4.1) tenglamada involyutsiya hossasidan foydalanib $f: x\to\frac{1}{x}$ akslantirish bajarsak ushbu $\frac{\partial u(\varepsilon,t)}{\partial t}=\frac{i}{\beta}\frac{\partial u(x,t)}{\partial x}$ tenglikka ega bo‘lamiz. Buni (3.4.4) teklikka olib borib qo‘yamiz.

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{i^2}{\beta^2}\frac{\partial}{\partial\varepsilon}\left(\frac{\partial u(x,t)}{\partial x}\right) \qquad (3.4.5)$$

Eslatma: $\frac{\partial f(x,t)}{\partial\varepsilon}=-x^2\frac{\partial f(x,t)}{\partial x}$, $\varepsilon=\frac{1}{x}$ ga tengligini yodga olamiz, bunday tenglik bajarilishiga sabab biz doimo x ni $\frac{1}{x}$ orqali ifodalay olamiz.

Yuqoridagi eslatmadan foydalanib (3.4.5) tenglamamiz quyidagicha

$$\frac{\partial^2 u(x,t)}{\partial t^2}=(\frac{x}{\beta})^2\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (3.4.6)$$

ikki o‘zgaruvchili ikkinchi tartibli giperbolik tipdagi xususiy hosilali differensial tenglamaga keladi. Teorema isbotlandi.

Endi berilgan aralash masalamizni yechish bilan shug‘ullanamiz, yuqoridagi (3.4.1) tenglama involyutsiyadan qutqarilish natijasida (3.4.6) ko‘rinishga kelganini hisobga olib quyidagi masalaga duch kelamiz:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=(\frac{x}{\beta})^2\frac{\partial^2 u(x,t)}{\partial x^2}\ ,\ x\epsilon[1,\pi],\ t>0 \qquad (3.4.7)$$

tenglamani (3.4.2) va (3.4.3) shartlarni qanoatlantiruvchi yechimini topishdan iborat bo‘lsin.

Bu aralash maslamizni chegaraviy shartlari 0 bo‘lganligidan Furye usulidan foydalanib yechamiz. Unga ko‘ra yechim

$$u(x,t) = y(x) \cdot T(t) \qquad (3.4.8)$$

ko‘rinishda qidiriladi. (3.4.8) ni (3.4.7) ga olib borib qo‘yamiz, va

$$\frac{T''(t)}{T(t)} = (\frac{x}{\beta})^2 \frac{y''(x)}{y(x)} = -a = const \qquad (3.4.9)$$

ga ega bo‘lamiz. (3.4.9) dan:

$$\begin{cases} x^2 y''(x) + \beta^2 a y(x) = 0 & (3.4.10) \\ T''(t) + \beta^2 a T(t) = 0 & (3.4.11) \end{cases}$$

(3.4.3) chegaraviy shartdan: $u(1,t) = y(1)T(t) = 0,\ u(c,t) = y(c)T(t) = 0$

Biz $T(t) \neq 0$ yechim qidiramiz. Bundan ushbu

$$x^2 y''(x) + \beta^2 a y(x) = 0 \ , \qquad (3.4.12)$$

tenglama (3.4.2) va (3.4.3) shartlarni qanoatlantiruvchi masalaga ega bo‘lamiz. $\exists\ a$ topish kerakki, shu a ga mos yechim $y(x) \neq 0$ chiqishi kerak. (3.4.12) da $y(x) = x^k$ almashtirsak

$$x^2 k(k-1)x^{k-2} + \beta^2 a x^k = 0 \qquad (3.4.13)$$

tenglikka kelamiz, bundan xaraktiristik tenglamani tuzsak $k^2 - k + \beta^2 a = 0$ tenglama hosil bo‘ladi. (3.4.10), (3.4.3) chegaraviy masala faqat $a > \frac{1}{4\beta^2}$ dagina 0 dan farqli yechimga ega bo‘ladi. Shuning uchun $a > \frac{1}{4\beta^2}$ da

$$y(x) = \sqrt{x}[C_1 cos\,(\sqrt{4\beta^2 a - 1} lnx) + C_2 sin\,(\sqrt{4\beta^2 a - 1} lnx)]$$

(3.4.3) chegaraviy shartdan

$$a_n = \frac{1}{4\beta^2}\left(\left(\frac{\pi n}{lnc}\right)^2 + 1\right), y_n(x) = C_n\sqrt{\frac{x}{c}} sin\,\frac{\pi n}{lnc} lnx, n = 1,2, ... \qquad (3.4.14)$$

yechimga ega bo‘lamiz. (3.4.11) tenglamadan esa quyidagini

$$T_n(t) = A_n cos\,\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2 + 1}}{2\beta} t + B_n sin\,\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2 + 1}}{2\beta} t \qquad (3.4.15)$$

hosil qilamiz. (3.4.15) va (3.4.14) dan:

$$u_n(x,t) = [A_n cos\,\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2 + 1}}{2\beta} t + B_n sin\,\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2 + 1}}{2\beta} t]\ C_n\sqrt{\frac{x}{c}} sin\,(\frac{\pi n}{lnc} lnx) \qquad (3.4.16)$$

Yechimlarga ega bo‘lamiz. (3.4.1) tenglama chiziqli va bir jinsli bo‘lganligi sababli, (3.4.16) yechimlarning cheksiz yig‘indisi ham yechim bo‘ladi.

Endi (3.4.1), (3.4.2) va (3.4.3) masalaning yechimini

$$u(x,t)=\sum_{n=1}^{\infty}\left[\alpha_n cos\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2+1}}{2\beta}t+\beta_n sin\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2+1}}{2\beta}t\right]\sqrt{\frac{x}{c}}sin\left(\frac{\pi n}{lnc}lnx\right) \qquad (3.4.17)$$

qator ko‘rinishida izlaymiz. Agar bu qator tekis yaqinlashuvchi bo‘lib, uni $x$va t bo‘yicha ikki marta hadlab differensiallash mumkin bo‘lsa, qatorning yig‘indisi ham (3.4.1) tenglamani qanoatlantiradi. (3.4.17) qatorning har bir hadi (3.4.3) chegaraviy shartlarni qanoatlantirgani uchun uning yig‘indisi $u(x,t)$ funksiya ham bu shartni qanoatlantiradi.

(3.4.17) qatorni t bo‘yicha differensiallaymiz:

$$u_t(x,t)=$$

$$\sum_{n=1}^{\infty}\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2+1}}{2\beta}\left[-\alpha_n sin\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2+1}}{2\beta}t+\beta_n cos\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2+1}}{2\beta}t\right]\sqrt{\frac{x}{c}}sin\left(\frac{\pi n}{lnc}lnx\right) \qquad (3.4.18)$$

(3.4.17) va (3.4.18) da $t=0$ deb, (3.4.2) boshlang‘ich shartlarga asosan ushbu

$$\varphi_0(x)=\sum_{n=1}^{\infty}\alpha_n\sqrt{\frac{x}{c}}sin\left(\frac{\pi n}{lnc}lnx\right),\ \varphi_1(x)=\sum_{n=1}^{\infty}\frac{\sqrt{\left(\frac{\pi n}{lnc}\right)^2+1}}{2\beta}\beta_n\sqrt{\frac{x}{c}}sin\left(\frac{\pi n}{lnc}lnx\right)$$

tengliklarni hosil qilamiz. Bu formulalardan α_n,β_n koeffitsientlar topiladi.

Endi (3.4.17) qatorni va uni ikki marta differensiallash natijasida hosil bo‘lgan qatorlarning tekis yaqinlashuvchanligini ko‘rsatsak, (3.4.17) qator bilan aniqlangan $u(x,t)$ funksiya haqiqatdan ham (3.4.1), (3.4.2), (3.4.3) masalaning yechimidan iborat bo‘ladi. Quyidagi teorema o‘rinlidir.

3.4.2-teorema. Agar $\varphi_0(x)$ funksiya $[1,c]$ segmentda ikki marta uzluksiz differensiallanuvchi bo‘lib, uchinchi tartibli bo‘lak-bo‘lak uzluksiz hosilaga ega bo‘lsa, $\varphi_1(x)$ esa uzluksiz differensiallanuvchi bo‘lib, ikkinchi tartibli bo‘lak-bo‘lak uzluksiz hosilaga ega bo‘lsa, hamda

$$\varphi_0(1)=\varphi_0(c)=0,\ \ \varphi_1(1)=\varphi_1(c),\quad \varphi_0''(1)=\varphi_0{}''(c) \qquad (3.4.19)$$

muvofiqlashtirish shartlari bajarilsa, u holda (3.4.17) qator bilan aniqlangan $u(x,t)$ funksiya ikkinchi tartibli uzluksiz hosilaga ega bo‘lib, (3.4.1) tenglamani, (3.4.2) boshlang‘ich va (3.4.3) chegaraviy shartlarni qanoatlantiradi.. Shu bilan birga (3.4.17) qatorni x va t bo‘yicha ikki marta hadlab differensiallash mumkin bo‘lib, xosil bo‘lgan qatorlar ixtiyoriy t da $1 \leq x \leq c$ oraliqda obsalyut va tekis yaqinlashuvchi bo‘ladi.

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