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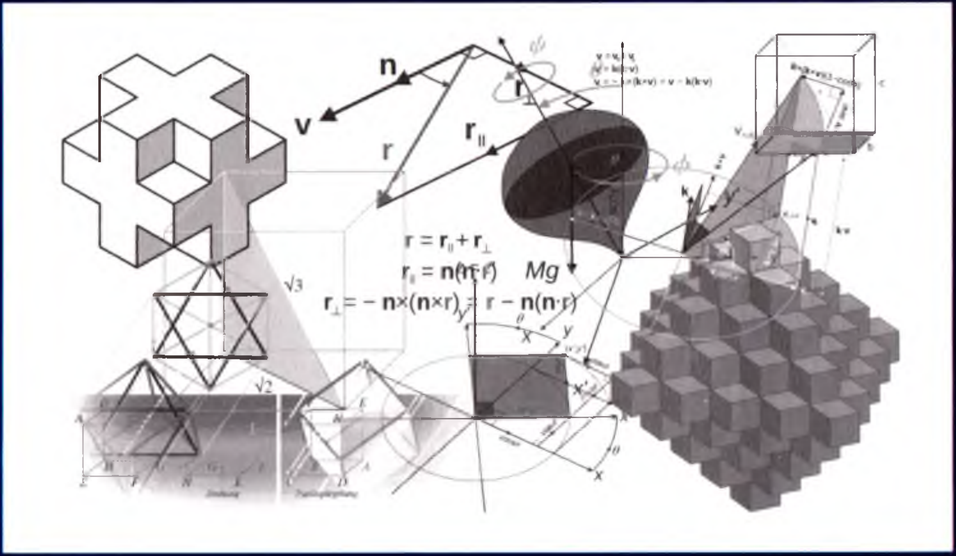
OLIY MATEMATIKA



O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS
TA'LIM VAZIRLIGI
TOSHKENT ARXITEKTURA QURILISH INSTITUTI

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OLIY MATEMATIKA

Uch jildlik

I jild

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi barcha texnika yo'nalishlari uchun darslik sifatida tavsiya etgan

Cho'lpon nomidagi nashriyot-matbaa ijodiy uyi
Toshkent — 2018

UDK 517(075)
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Ushbu darslik oliy texnika o'quv yurtlarining oliy matematika fani dasturi asosida yozilgan va bakalavrlar Davlat ta'lim standartlari talablariga mos keladi.

Darslik uch jilddan iborat. Birinchi jild oliy matematikaning chiziqli algebra elementlari, vektorli algebra elementlari, analitik geometriya, matematik analizga kirish, bir o'zgaruvchi funksiyasining differensial hisobi, oliy algebra elementlari va bir o'zgaruvchi funksiyasining integral hisobi bo'limlariga oid materillarni o'z ichiga oladi. Darslikning har bir mavzusi zamonaviy xorijiy adabiyotlar va o'qitish texnologiyalari tahlili asosida yozilgan.

Kitob oliy ta'lim muassasalarining talabalari va o'qituvchilari uchun mo'ljallangan.

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SO‘Z BOSHI

Matematika barcha tabiiy bilimlar asosidir.
David Gilberd

Hozirgi jadal rivojlanish davrida aql-zakovatli, ijodiy fikrlovchi va mustaqil qaror qabul qiluvchi mutaxassislarni tayyorlashda matematik ta’lim asosiy o‘rin egallaydi. Talabalarni matematik tayyorlash ularning kasbiy faoliyatida zarur bo‘ladigan boshqa tabiiy-ilmiy, umumkasbiy va ixtisoslik fanlarini o‘rganishlari uchun nazariy asoslarni ta’minlashi kerak. Bu esa o‘z navbatida samarali o‘qitishning muhim omillaridan biri bo‘lgan zamon talablariga javob beruvchi darsliklar va o‘quv qo‘llanmalarni yaratishni taqozo etmoqda.

Ushbu darslik oliy texnika o‘quv yurtlarining oliy matematika fani dasturi asosida yozilgan va bakalavrlar Davlat ta’lim standartlari talablariga mos keladi.

Darslik uch jilddan iborat. Darslikning birinchi jildi yettita bobdan iborat bo‘lib, u oliy matematikaning chiziqli algebra elementlari, vektorli algebra elementlari, analitik geometriya, matematik analizga kirish, bir o‘zgaruvchi funksiyasining differensial hisobi, oliy algebra elementlari va bir o‘zgaruvchi funksiyasining integral hisobi bo‘limlariga bag‘ishlangan. Bu bo‘limlar zamonaviy xorijiy adabiyotlar va o‘qitish texnologiyalari tahlili asosida yaratilgangan bo‘lib, har bir mavzuni yozishda bir qancha xorijiy adabiyotlardan foydalanilgan, mavzular to‘liq yoritilgan, tegishli bilimlar talabalar tomonidan mustaqil o‘zlashtirilishiga, ularda ko‘nikma va malakalarning shakllantirilishiga hamda ijodiy qobiliyatlarni rivojlantirishga yo‘naltirigan.

Darslikda talaba asosiy tushunchalarni, ta’riflarni, teoremalarni va tipik masalalarni yechish usullarini ko‘rsatuvchi misollarni topadi. Bunda biror tasdiqning isboti keltirilmagan bo‘lsa, natijalarning ifodasi uning ma’nosini tushuntiradigan misollar bilan to‘ldirilgan.

Darslikning har bir mavzusi ko‘p sondagi misol va masalalar yechimlarida tushuntirilgan, ularni o‘zlashtirishni mustahkamlashga yo‘naltirilgan mashqlar bilan to‘ldirilgan. Ayrim misol va masalalarni matematik paketlar yordamida yechish usullari keltirilgan.

Muallif darslik qo'lyozmasini o'qib, uning sifatini oshirish borasida bildirgan fikr va mulohazalari uchun Qarshi Davlat universitetining professori B.A.Shoimqulovga, O'zbekiston Milliy universitetining o'qituvchilari – professor A.Narmanovga, dotsentlar N.Jabborov va J.Tishaboyevlarga o'z minnatdorchiligini izhor qiladi.

Ayniqsa, darslikning ushbu jildini tuzishda va mazmunini yaxshilashda fizika-matematika fanlari doktori A. Quchqarovning beminnat yordamini muallif e'tirof etishni o'zining burchi deb biladi.

Darslik haqidagi tanqidiy fikr va mulohazalarini bildirgan barcha kitobxonlarga muallif oldindan o'z tashakkurini bildiradi.

1 CHIZIQLI ALGEBRA ELEMENTLARI

- Matritsalar
- Determinantlar
- Matritsa ustida almashtirishlar
- Chiziqli tenglamalar sistemasi

Al-Xorazmiy – Muhammad ibn Musa Xorazmiy (783–850) – xorazmlik matematik, astronom va geograf.

Al-Xorazmiy algebra faniga asos soldi, ilmiy ma'lumot va traktatlarni bayon etishning aniq qoidalarini ishlab chiqdi. U o'nlik pozitsion hisoblash tizimini, nol belgisini va qutblar koordinatalarini birinchilardan bo'lib asoslab berdi va amaliyotga tatbiq etdi.

I.Karimov

Chiziqli algebraning daslabki masalasi chiziqli tenglamalar haqidagi masala hisoblanadi. Bunday tenglamalarni yechish jarayonida determinant tushunchasi paydo bo'ldi.

Chiziqli tenglamalar sistemasi va ularning determinantlarini o'rganish natijasida matritsa tushunchasi kiritildi. G.Frobennus tomonidan matritsaning rangi tushunchasi kiritilishi chiziqli tenglamalar sistemasining birgalikda va aniq bo'lshi shartlarini olish imkonini berdi. Shu zaylda XIX asrning oxirlariga kelib, chiziqli tenglamalar sistemasi nazariyasini barpo qilish jarayoni tugatildi.

1.1.MATRITSALAR

Matritsa tushunchasi *1850-yilda James Joseph Sylvester* tomonidan kiritilgan. Kelming *1858-yilda* chop etilgan *«Matritsalar nazariyasi haqida memuar»* asarida matritsalar nazariyasi mufassal bayon qilingan.

Daslabki vaqtlarda matritsa geometrik obyektlarni almashtirish va chiziqli tenglamalarni yechish bilan bog'liq holda rivojlantirildi. Hozirgi vaqtda matritsalar matematikaning muhim tatbiqiy vositalaridan biri hisoblanadi.

Matritsalar matematika, texnika va iqtisodiyotning turli sohalarida keng qo'llaniladi. Masalan, ulardan matematikada algebraik va differensial tenglamalar sistemasini yechishda, kvant nazariyasida fizik kattaliklarni oldindan aytishda, aviatsiyada zamonaviy samolyotlarni yaratishda foydalaniladi.

1.1.1. Matritsa va uning turlari

Matritsalar sonlar, algebraik belgilar va matematik funksiyalarning katta massivlarini yagona obyekt sifatida qarash va bunday massivlarni o'z ichiga olgan masalalarni qisqa ko'rinishda yozish va yechish imkonini beradi.

Matritsa – bu elementlar (sonlar, algebraik belgilar, matematik funksiyalar) massivining satr hamda ustunlarda berilgan va kichik qavslarga olingan to'g'ri burchakli jadvalidir.

Matritsaning o'lchami uning satrlari soni va ustunlari soni bilan aniqlanadi. Matritsaning o'lchamini ifodalash uchun $m\times n$ belgi ishlatiladi. Bu belgi matritsaning m ta satr va n ta ustundan tashkil topganini bildiradi.

Matritsa lotin alifbosining bosh harflaridan biri bilan belgilanadi.

Masalan,

3×2 *o'lchamli matritsa*	2×3 *o'lchamli matritsa*	2×2 *o'lchamli matritsa*
$A=\begin{pmatrix}2 & 5\\ 0 & 7\\ 3 & 1\end{pmatrix}$	$B=\begin{pmatrix}-1 & 4 & 7\\ 2 & 5 & 6\end{pmatrix}$	$C=\begin{pmatrix}\sin x & -\cos x\\ \cos x & \sin x\end{pmatrix}$

A matritsaning i-satr va j-ustunda joylashgan elementi a_{ij} bilan belgilanadi.

$A=(a_{ij})$, $(i=\overline{1,m}, j=\overline{1,n})$ yoki $A=\| a_{ij} \|$, $(i=\overline{1,m}, j=\overline{1,n})$ yozuv A matritsa a_{ij} elementlardan tashkil topganini bildiradi:

$$A=(a_{ij})=\begin{pmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & \dots & \dots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{pmatrix};\quad A=\| a_{ij} \|=\begin{Vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & \dots & \dots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{Vmatrix}$$

$1\times n$ o'lchamli $A=(a_{11} \quad a_{12} \quad \dots \quad a_{1n})$ matritsaga *satr matritsa* yoki *satr-vektor* deyiladi.

$m\times 1$ o'lchamli $A=\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix}$ matritsaga *ustun matritsa* yoki *ustun-vektor* deyiladi.

$n\times n$ o'lchamli maritsaga n - *tartibli kvadrat matritsa* deyiladi.

Kvadrat matritsaning chap yuqori burchagidan o'ng quyi burchagiga yo'nalgan $a_{11}, a_{22}, \dots, a_{nn}$ elementlaridan tuzilgan diagonaliga uning *bosh diagonali*, o'ng yuqori burchagidan chap quyi burchagiga yo'nalgan $a_{1n}, a_{2(n-1)}, \dots, a_{n1}$ elementlardan tuzilgan diagonaliga uning *yordamchi diagonali* deyiladi.

Bosh diagonalidan yuqorida (pastda) joylashgan barcha elementlari nolga teng bo'lgan

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \quad \left(A'=\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \right)$$

matritsaga *yuqoridan uchburchak* (*quyidan uchburchak*) *matritsa* deyiladi.

Bosh diagonalda joylashmagan barcha elementlari nolga teng bo'lgan

$$A=\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

matritsaga *diagonal matritsa* deyiladi.

Barcha elementlari birga teng bo'lgan diagonal matritsaga *birlik matritsa* deyiladi va I (yoki E) harfi bilan belgilanadi.

Barcha elementlari nolga teng bo'lgan ixtiyoriy o'lchamdagi matritsaga *nol matritsa* deyiladi va O harfi bilan belgilanadi.

A matritsada barcha satrlarni mos ustunlar bilan almashtirish natijasida hosil qilingan A^T matritsaga A matritsaning *transponirlangan matritsasi* deyiladi: $(a_{ij})^T=(a_{ji})$.

Agar $A = A^T$ bo'lsa, A matritsaga *simmetrik matritsa* deyiladi.

1.1.2. Matritsalar ustida arifmetik amallar

Maritsalarning tengligi

Bir xil o'lchamli $A=(a_{ij})$ va $B=(b_{ij})$ matritsalarning barcha mos elementlari teng, ya'ni $a_{ij}=b_{ij}$ bo'lsa, bu matritsalarga *teng matritsalar* deyiladi va $A=B$ deb yoziladi.

$$\boxed{\begin{array}{c} A=B \quad \Leftrightarrow \quad a_{ij}=b_{ij} \\ \text{barcha } i=\overline{1,m},\ \ j=\overline{1,n} \text{ uchun} \end{array}}$$

Matritsani songa ko'paytirish

1-ta'rif. $A=(a_{ij})$ *matritsaning* λ *songa ko'paytmasi* deb, elementlari $c_{ij}=\lambda a_{ij}$ kabi aniqlanadigan $C=\lambda A$ matritsaga aytiladi.

$$\boxed{C=\lambda A \quad \Leftrightarrow \quad c_{ij}=\lambda a_{ij}.}$$

1-misol. $A=\begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & -1 \end{pmatrix}$ bo'lsin. $3A$ ni toping.

Yechish.

$$3A=3\cdot\begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & -1 \end{pmatrix}=\begin{pmatrix} 3\cdot 2 & 3\cdot(-1) & 3\cdot 0 \\ 3\cdot 3 & 3\cdot 4 & 3\cdot(-1) \end{pmatrix}=\begin{pmatrix} 6 & -3 & 0 \\ 9 & 12 & -3 \end{pmatrix}.$$

Matritsalarni qo'shish va ayirish

Matritsalarni qo'shish va ayirish amallari *bir xil o'lchamli matritsalar* uchun kiritiladi. Bunda yig'indi matrisa qo'shiluvchi matritsalar bilan bir xil o'lchamga ega bo'ladi.

2-ta'rif. $A=(a_{ij})$ va $B=(b_{ij})$ *matritsalarning yig'indisi* deb, elementlari $c_{ij}=a_{ij}+b_{ij}$ kabi aniqlanadigan $C=A+B$ matritsaga aytiladi.

$$\boxed{C=A+B \quad \Leftrightarrow \quad c_{ij}=a_{ij}+b_{ij}.}$$

2-misol. $A=\begin{pmatrix}1 & -1 & 4\\ 3 & 0 & 1\end{pmatrix}$ va $B=\begin{pmatrix}2 & 3 & 2\\ 1 & 0 & -2\end{pmatrix}$ bo'lsin. $A+B$ ni toping.

Yechish.

$$A+B=\begin{pmatrix}1 & -1 & 4\\ 3 & 0 & 1\end{pmatrix}+\begin{pmatrix}2 & 3 & 2\\ 1 & 0 & -2\end{pmatrix}=\begin{pmatrix}1+2 & -1+3 & 4+2\\ 3+1 & 0+0 & 1+(-2)\end{pmatrix}=\begin{pmatrix}3 & 2 & 6\\ 4 & 0 & -1\end{pmatrix}.$$

$-A=(-1)\cdot A$ matritsa A *matritsaga qarama-qarshi matritsa* deb ataladi.

3-ta'rif. $A=(a_{ij})$ va $B=(b_{ij})$ *matritsalarning ayirmasi* deb $C=A-B=A+(-B)$ matritsaga aytiladi. Bunda C matritsaning elementlari $c_{ij}=a_{ij}+(-b_{ij})=a_{ij}-b_{ij}$ kabi topiladi.

$$\boxed{C=A-B \quad \Leftrightarrow \quad c_{ij}=a_{ij}-b_{ij}.}$$

3-misol. $A=\begin{pmatrix}2 & -3 & 2\\ 2 & -1 & 4\end{pmatrix}$ va $B=\begin{pmatrix}1 & 3 & 2\\ 2 & 1 & -1\end{pmatrix}$ bo'lsin. $A-B$ ni toping.

Yechish.

$$A-B=\begin{pmatrix}2 & -3 & 2\\ 2 & -1 & 4\end{pmatrix}-\begin{pmatrix}1 & 3 & 2\\ 2 & 1 & -1\end{pmatrix}=\begin{pmatrix}2-1 & -3-3 & 2-2\\ 2-2 & -1-1 & 4-(-1)\end{pmatrix}=\begin{pmatrix}1 & -6 & 0\\ 0 & -2 & 5\end{pmatrix}.$$

Matritsalar ustida chiziqli amallar quyidagi xossalarga ega.

A,B,C,O matritsalar $m\times n$ o'lchamli va λ,μ- skalyar sonlar bo'lsa, u holda:

1^o. $A+B=B+A$;

2^o. $(A+B)+C=A+(B+C)$;

3^o. $A+O=A$;

4^o. $A+(-A)=O$;

5^o. $\lambda(A+B)=\lambda A+\lambda B$;

6^o. $(\lambda+\mu)A=\lambda A+\mu A$;

7^o. $\mu(\lambda A)=\lambda(\mu A)=(\lambda\mu)A$;

8^o. $1\cdot A=A$;

9^o. $(A+B)^T=A^T+B^T$;

10^o. $(\lambda A)^T=\lambda A^T$;

11^o. $A+C=B$ bo'lsa, $C=B-A$ bo'ladi;

12^o. $\lambda A=O$ bo'lsa, $\lambda=0$ yoki $A=O$ bo'ladi;

13°. $\lambda A = \lambda B$ va $\lambda \neq 0$ bo'lsa, $A = B$ bo'ladi.

Xossalardan ayrimlarini isbotlaymiz.
Birinchi to'rtta xossaning isboti bevosita 2-ta'rifdan kelib chiqadi.
5-xossani qaraymiz.
A va B bir xil o'lchamli matritsalar bo'lsin.
U holda

$$A + B = (a_{ij} + b_{ij})$$

yoki

$$\lambda(A+B) = \lambda(a_{ij} + b_{ij}) = \lambda(a_{ij}) + \lambda(b_{ij})$$

va ikkinchidan

$$\lambda(a_{ij}) + \lambda(b_{ij}) = \lambda A + \lambda B.$$

bo'ladi.

Bu ikkita tenglikdan $\lambda(A+B) = \lambda A + \lambda B$ bo'lishi kelib chiqadi.

Matritsalarni ko'paytirish

Bir xil sondagi elementlarga ega bo'lgan A satr matritsa va B ustun matritsa berilgan bo'lsin deylik. Bunda A satrning B ustunga ko'paytmasi quyidagicha aniqlanadi:

$$AB = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{pmatrix} \cdot \begin{pmatrix} b_{11} \\ b_{12} \\ \ldots \\ b_{1n} \end{pmatrix} = a_{11}b_{11} + a_{12}b_{12} + \ldots + a_{1n}b_{1n},$$

ya'ni ko'paytma berilgan matritsalar mos elementlari ko'paytmalarining yig'indisiga teng bo'ladi.

Matritsalarni ko'paytirishning bu qoidasi *satrni ustunga ko'paytirish qoidasi* deb yuritiladi.

Ikki matritsani ko'paytirish amali *moslashtirilgan matritsalar* uchun kiritiladi.

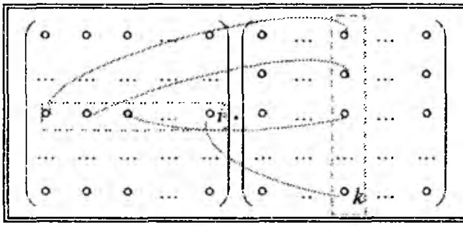
A matritsaning ustunlari soni B matritsaning satrlari soniga teng bo'lsa, A va B *matritsalar moslashtirilgan* deyiladi.

4-ta'rif. $m \times p$ o'lchamli $A = (a_{ij})$ *matritsaning* $p \times n$ o'lchamli $B = (b_{jk})$ *matritsaga ko'paytmasi* AB deb, c_{ik} elementi A matritsaning

i-satrini B matritsaning j-ustuniga satrni ustunga ko'paytirish qoidasi bilan, ya'ni

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ip}b_{pk} = \sum_{r=1}^{p} a_{ir}b_{rk},\ i = 1,\ldots,m,\ k = 1,\ldots,n$$

(qo'shiluvchlari quyidagi sxemada keltirilgan) kabi aniqlanadigan $m \times n$ o'lchamli $C = (c_{ik})$ matritsaga aytiladi.



4-misol. Berilgan matritsalarni ko'paytiring.

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot5+2\cdot7 & 1\cdot6+2\cdot8 \\ 3\cdot5+4\cdot7 & 3\cdot6+4\cdot8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix};$

2. $\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & 4 \\ -1 & 0 & 3 \end{pmatrix} =$

$$= \begin{pmatrix} 2\cdot3+1\cdot(-1) & 2\cdot2+1\cdot0 & 2\cdot4+1\cdot3 \\ -3\cdot3+4\cdot(-1) & -3\cdot2+4\cdot0 & -3\cdot4+4\cdot3 \\ 0\cdot3+2\cdot(-1) & 0\cdot2+2\cdot0 & 0\cdot4+2\cdot3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 11 \\ -13 & -6 & 0 \\ -2 & 0 & 6 \end{pmatrix}.$$

Agar A matritsaning satrlari $A_1, A_2, \ldots, A_m$ bilan va B matritsaning ustulari $B_1, B_2, \ldots, B_n$ bilan belgilansa, u holda matritsalarni ko'paytirish qoidasini quyidagi ko'rinishda yozish mumkin:

$$C = AB = \begin{pmatrix} A_1 \\ A_2 \\ \ldots \\ A_m \end{pmatrix} \cdot \begin{pmatrix} B_1 & B_2 & \ldots & B_n \end{pmatrix} = \begin{pmatrix} A_1B_1 & A_1B_2 & \ldots & A_1B_n \\ A_2B_1 & A_2B_2 & \ldots & A_2B_n \\ \ldots & \ldots & \ldots & \ldots \\ A_mB_1 & A_mB_2 & \ldots & A_mB_n \end{pmatrix}.$$

Matritsalarni ko'paytirishda A^2 yozuv ikkita bir xil matritsaning ko'paytmasini bildiradi:

$$A^2 = A \cdot A.$$

Shu kabi

$$A^3 = A \cdot A \cdot A, \ \dots, \ A^n = \underbrace{A \cdot A \cdot \dots \cdot A}_{n \text{ marta}}.$$

5-misol. $f(x) = 2x - x^2 + 5$ va $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ bo'lsin. $f(A)$ ni toping.

Yechish. Matritsa ko'rinishdagi $f(A)$ funksiyaga o'tishda λ sonli qo'shiluvchi λI ko'paytma bilan almashtiriladi, bu yerda I-birlik matritsa.

$$f(A) = 2A - A^2 + 5I = 2 \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 0 & 5 \end{pmatrix}.$$

Matritsalarni ko'paytirish amali ushbu xossalarga bo'ysunadi

1°. A matritsa $m \times n$ o'lchamli va B, C matritsalar $n \times p$ o'lchamli bo'lsa, $A(B+C) = AB + AC$ bo'ladi;

2°. A, B, C matritsalar mos ravishda $m \times n$, $n \times p$, $p \times q$ o'lchamli bo'lsa, $A(BC) = (AB)C$ bo'ladi;

3°. A, B, I, O moslashtirilgan matritsalar va λ, μ skalyar sonlar bo'lsa, u holda:

1) $(\lambda A)(\mu B) = (\lambda\mu)(AB)$; 2) $A(\lambda B) = (\lambda A)B = \lambda(AB)$;

3) $AI = IA = A$; 4) $AO = OA = O$;

5) $(AB)^T = B^T A^T$.

4°. $A, I, O - n$- tartibli kvadrat matritsalar va p, q manfiy bo'lmagan butun sonlar bo'lsa, u holda:

1) $A^p A^q = A^{p+q}$; 2) $(A^p)^q = (A)^{pq}$;

3) $A^1 = A$; 4) $A^0 = I$.

Xossalardan ayrimlarini ta'riflar yordamida isbotlaymiz va ayrimlarining to'g'riligiga misollarni yechish orqali ishonch hosil qilamiz.

1-xossani qaraylik.

$A=(a_{ij})$ matritsa $m\times n$ o'lchamli va $B=(b_{ij})$, $C=(c_{ij})$ matritsalar $n\times p$ o'lchamli bo'lsin. U holda 2- va 3-ta'riflarga ko'ra istalgan i,j da birinchidan

$$B+C=(b_{ij}+c_{ij})$$

yoki

$$A(B+C)=\sum_{k=1}^{n}a_{ik}(b_{kj}+c_{kj})=\sum_{k=1}^{n}(a_{ik}b_{kj}+a_{ik}c_{kj})=\sum_{k=1}^{n}a_{ik}b_{kj}+\sum_{k=1}^{n}a_{ik}c_{kj}$$

va ikkinchidan

$$\sum_{k=1}^{n}a_{ik}b_{kj}+\sum_{k=1}^{n}a_{ik}c_{kj}=AC+BC$$

bo'ladi.

Oxirgi ikkita tenglikdan $A(B+C)=AB+AC$ bo'lishi kelib chiqadi.

3- xossaning 5-bandini qaraylik.

$A=(a_{ij})$ va $B=(b_{ij})$ bo'lsin. Bundan $A^T=(a'_{ij})$ va $B^T=(b'_{ij})$ bo'ladi, bu yerda $a'_{ij}=a_{ji}$, $b'_{ij}=b_{ji}$.

U holda 3-ta'rifga ko'ra istalgan i,j da birinchidan

$$AB=\sum_{k=1}^{n}a_{ik}b_{kj} \text{ yoki } (AB)^T=\sum_{k=1}^{n}a_{jk}b_{ki}$$

va ikkinchidan,

$$\sum_{k=1}^{n}a_{jk}b_{ki}=\sum_{k=1}^{n}b_{ki}a_{jk}=\sum_{k=1}^{n}b'_{ik}a'_{ik}=B^TA^T$$

bo'ladi. Bundan $(AB)^T=B^TA^T$ bo'lishi kelib chiqadi.

2-xossani to'g'riligiga misol yechish orqali ishonch hosil qilamiz.

$$A=(1\quad 2),\quad B=\begin{pmatrix}3&-1\\0&4\end{pmatrix},\quad C=\begin{pmatrix}-2&4&1\\5&0&2\end{pmatrix} \text{ bo'lsin.}$$

U holda

$$BC=\begin{pmatrix}3&-1\\0&4\end{pmatrix}\cdot\begin{pmatrix}-2&4&1\\5&0&2\end{pmatrix}=\begin{pmatrix}-11&12&1\\20&0&8\end{pmatrix},$$

$$A(BC)=(1\quad 2)\cdot\begin{pmatrix}-11&12&1\\20&0&8\end{pmatrix}=(29\quad 12\quad 17),$$

$$AB=(1\quad 2)\cdot\begin{pmatrix}3&-1\\0&4\end{pmatrix}=(3\quad 7),$$

$$(AB)C = \begin{pmatrix} 3 & 7 \end{pmatrix} \cdot \begin{pmatrix} -2 & 4 & 1 \\ 5 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 29 & 12 & 17 \end{pmatrix}.$$

Demak, $A(BC) = (AB)C$.

Umuman olganda matritsalarni ko'paytirish nokommutativ, ya'ni $AB \neq BA$. Masalan, $1 \times n$ o'lchamli A matritsaning $n \times 1$ o'lchamli B matritsaga AB ko'paytmasi sondan, ya'ni 1×1 o'lchamli matritsadan iborat bo'lsa, BA ko'paytmasi n - tartibli kvadrat matritsa bo'ladi.

Bir xil tartibli A va B kvadrat matritsalar uchun $AB = BA$ bo'lsa, A va B matritsalarga *kommutativ matritsalar*, $AB - BA$ ayirmaga *kommutator* deyiladi.

1.1.3. Mashqlar

1. A kvadrat matritsa bo'lsin. $A + A^T$ simmetrik matritsa bo'lishini ko'rsating.

2. $A = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ matritsani $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ va $Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ matritsalarning chiziqli kombinatsiyasi ko'rinishida ifodalang.

3. $a\begin{pmatrix} 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$ bo'lsa, a va b ni toping.

4. Matritsa 30 ta elementga ega bo'lsa, u qanday tartiblarda berilishi mumkin?

5 - 8. A, B matritsalar va λ, μ sonlar berilgan. $\lambda A + \mu B$ matritsani toping:

5. $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & -1 \\ -1 & 0 & 2 \end{pmatrix}$, $\lambda = -1$, $\mu = 2$.

6. $A = \begin{pmatrix} 0 & -3 \\ -2 & 1 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 3 & -1 \\ 2 & -5 \end{pmatrix}$, $\lambda = 2$, $\mu = -3$.

7. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -1 & 0 \\ -4 & -3 & 2 \end{pmatrix}$, $\lambda = -3$, $\mu = -2$.

8. $A=\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$, $B=I$, $\lambda=1$, $\mu=-v$.

9. A va B moslashtirilgan matritsalar bo'lsin. Quyidagilarni ko'rsating:
(a) agar A matritsa satr matritsa bo'lsa, u holda AB satr matritsa bo'ladi;
(b) agar B matritsa ustun matritsa bo'lsa, u holda AB ustun matritsa bo'ladi.

10. $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 9 & 0 \end{pmatrix}$ bo'lsa, x va y ni toping.

11. Agar A matritsa 3×3 o'lchamli va C esa 5×5 o'lchamli bo'lsa, ABC ko'paytma ma'noga ega bo'lishi uchun B matritsa qanday o'lchamda bo'lishi kerak?

12. $A=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ matritsa berilgan. AB ko'paytmani nol matritsaga aylantiruvchi B matritsani toping.

13-16. A va B matritsalar berilgan. AB matritsani toping:

13. $A=\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & -1 & -2 \\ 3 & 2 & 0 \end{pmatrix}$.

14. $A=\begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 4 & -2 \\ 2 & 3 \end{pmatrix}$.

15. $A=\begin{pmatrix} 1 & 1 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, $B=\begin{pmatrix} -1 & 3 \\ 0 & -1 \\ 2 & 1 \end{pmatrix}$.

16. $A=\begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 3 \\ 1 & -1 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 4 & 0 & -2 \\ 2 & -1 & 0 \\ 0 & -1 & 3 \end{pmatrix}$.

17. $A=\begin{pmatrix} 2 & -2 \\ 2 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$, $C=B-3I$ bo'lsa, $(AB)C$ matritsani toping.

18. $A=\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, $B=\begin{pmatrix} 4 & 5 \\ 2 & 6 \end{pmatrix}$, $C=\begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix}$ bo'lsa, $A(BC)$ matritsani toping.

19. $A=\begin{pmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 0 \end{pmatrix}$ matritsalar berilgan. AB, $B^T B$, A^2 matritsalarni toping.

20. $A=\begin{pmatrix}1 & -2\\ 0 & 3\end{pmatrix}$ va $f(x)=3x^2+5x-4$ bo‘lsin. $f(A)$ ni toping.

21. $A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ bo‘lsa, A^{20} ni toping.

22. Agar $A^2=I$ va A matritsa 2×2 o‘lchamli bo‘lsa, A ni toping.

1.2. DETERMINANTLAR

Determinant tushunchasidan dastlab chiziqli tenglamalar sistemasini yechishda foydalanilgan bo‘lib, keyinchalik determinantlar matematikaning bir qancha masalalarini yechishga, jumladan xos sonlarni topishga, differensial tenglamalarni yechishga, vektor hisobiga, keng tatbiq etildi.

Biz avval ikkinchi va uchinchi tartibli determinantlar tushunchalari bilan tanishamiz. Bu tushunchalar keyinchalik yuqori tartibli determinantlarni hisoblash uchun asos bo‘ladi.

1.2.1. Ikkinchi va uchunchi tartibli determinantlar

Ikkinchi tartibli determinant

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22}-a_{12}a_{21}$$

kabi belgilanadi va aniqlanadi.

Bunda $a_{11}, a_{12}, a_{21}, a_{22}$ lar determinantning elementlari deb ataladi. a_{ij} determinantning i-satr va j-ustunda joylashgan elementini ifodalaydi.

a_{11}, a_{22} elementlar joylashgan diagonalga determinantning bosh diagonali, a_{21}, a_{12} elementlar joylashgan diagonalga determinantning yordamchi diagonali deyiladi.

Shunday qilib, ikkinchi tartibli determinant bosh diagonal elementlari ko‘paytmasidan yordamchi diagonal elementlari ko‘paytmasining ayrilganiga teng:

$$\boxed{\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} - \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}}$$

1-misol. Berilgan determinantlarni hisoblang.

1. $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \cdot 5 - 4 \cdot (-2) = 15 + 8 = 23;$

2. $\begin{vmatrix} tg\alpha & 2\sin\alpha \\ \sin\alpha & ctg\alpha \end{vmatrix} = tg\alpha \cdot ctg\alpha - \sin\alpha \cdot 2\sin\alpha = 1 - 2\sin^2\alpha = \cos 2\alpha.$

Matritsaning muhim tavsiflaridan biri determinant hisoblanadi. Determinant faqat kvadrat matritsalar uchun kiritiladi.

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ kvadrat matritsaning determinanti $\det A$ bilan belgilanadi va $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ kabi aniqlanadi.

Bunda matritsani uning determinanti bilan adashtirmaslik kerak: A – bu sonlar massivi; $\det A$ – bu bitta son (yoki ifoda).

Uchinchi tartibli determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

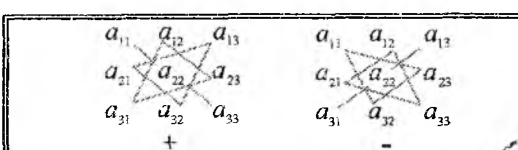
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

kabi belgilanadi va aniqlanadi.

Uchinchi tartibli determinant uchun satr, ustun, bosh diagonal, yordamchi diagonal tushunchalari ikkinchi tartibli determinantdagi kabi kiritiladi.

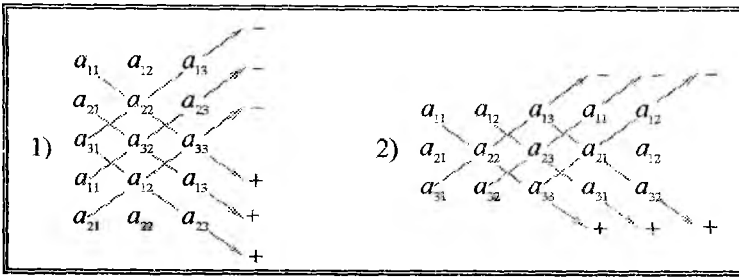
Uchinchi tartibli determinantlarni hisoblashda o‘ng tomonidagi birhadlarni topishning yodda saqlash uchun oson bo‘lgan qoidalaridan foydalaniladi.

«*Uchburchak qoidasi*» ushbu sxema bilan tasvirlanadi:



Bunda diagonallardagi yoki asoslari diagonallarga parallel bo‘lgan uchburchaklar uchlaridagi elementlar uchta elementning ko‘paytmasini hosil qiladi. Agar uchburchaklarning asoslari bosh diagonalga parallel bo‘lsa, u holda elementlarning ko‘paytmasi ishorasini saqlaydi. Agar uchburchaklarning asoslari yordamchi diagonalga parallel bo‘lsa, u holda elementlarning ko‘paytmasi teskari ishora bilan olinadi.

«*Sarryus qoidalari*» quyidagi sxemalar bilan ifodalanadi:



1-qoidada avval determinant tagiga uning birinchi ikkita satri yoziladi, 2-qoidada esa determinantning o‘ng tomoniga uning birinchi ikkita ustuni yoziladi. Bunda diagonallardagi yoki diagonallarga parallel bo‘lgan to‘g‘ri chiziqlardagi elementlar uchta ko‘paytuvchini hosil qiladi. Agar to‘g‘ri chiziqlar bosh diagonalga parallel bo‘lsa, u holda elementlarning ko‘paytmasi ishorasini saqlaydi. Agar to‘g‘ri chiziqlar yordamchi diagonalga parallel bo‘lsa, u holda elementlarning ko‘paytmasi teskari ishora bilan olinadi.

2-*misol*. 1. $\det A = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix}$ determinantni uchburchak qoidasi bilan hisoblang.

2. $\det B = \begin{vmatrix} 1 & 5 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 1 \end{vmatrix}$ determinantni Sarryusning 1-qoidasi bilan hisoblang.

3. $\det C=\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{vmatrix}$ determinantni Sarryusning 2-qoidasi bilan hisoblang.

Yechish.

1. $\begin{array}{ccc} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{array} \Rightarrow -8+1+27=20,\ \begin{array}{ccc} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{array} \Rightarrow 6-6+6=6,$

$\det A=20-6=14.$

2. $\begin{array}{ccccc} 3 & 4 & -1^{-} & 3^{-} & 4^{-} \\ 2 & 0 & 3 & 2 & 0 \\ 3 & -1 & 2 & 3 & -1 \\ & & + & + & + \end{array} \Rightarrow \det B=0+36+2-0+9-16=31.$

3. $\begin{array}{cccc} 1 & 5 & 3 & - \\ 3 & 1 & -2 & - \\ 2 & -4 & 1 & \\ 1 & 5 & 3 & + \\ 3 & 1 & -2 & + \\ & & & + \end{array} \Rightarrow \det C=1-36-20-6-8-15=-84.$

1.2.2. n - tartibli determinant tushunchasi

n-tartibli determinant

$$\det A=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

kabi belgilanadi va maʼlum qoida asosida hisoblanadi.

n-tartibli determinant har bir satr va har bir ustundan faqat bittadan olingan n ta elementning koʻpaytmasidan tuzilgan $n!$ ta qoʻshiluvchilar yigʻindisidan iborat boʻladi, bunda koʻpaytmalar bir-biridan elementlarining tarkibi bilan farq qiladi va har bir koʻpaytma

oldiga inversiya tushunchasi asosida plus yoki minus ishora qo'yiladi.

n-tartibli determinantni bu qoida asosida ifodalash ancha noqulay. Shu sababli yuqori tartibli determinantlarni hisoblashda bir nechta ekvivalent qoidalardan foydalaniladi. Bunday qoidalardan biri yuqori tartibli determinantlarni quyi tartibli determinantlar asosida hisoblash usuli hisoblanadi. Bu usulda determinant biror satr (yoki ustun) bo'yicha yoyiladi. Bunda quyi (ikkinchi va uchunchi) tartibli determinantlar yuqorida keltirilgan ta'riflar asosida topiladi.

n-tartibli determinantlarni yoyishda minor va algebraik to'ldiruvchi tushunchalaridan foydalaniladi.

n-tartibli determinant a_{ij} *elementining minori* deb, shu element joylashgan satr va ustunni o'chirishdan hosil bo'lgan $(n-1)$-tartibli determinantga aytiladi va M_{ij} bilan belgilanadi.

Determinant a_{ij} *elementining* A_{ij} *algebraik to'ldiruvchisi* deb,

$$A_{ij} = (-1)^{i+j} M_{ij}$$

songa aytiladi.

Masalan, $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix}$ determinantning $a_{21} = 2$ elementining minori va algebraik to'ldiruvchisi quyidagicha topiladi:

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix} \Rightarrow M_{21} = \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} = -10, \quad A_{21} = (-1)^{2+1} M_{21} = 10.$$

1.2.3. Determinantning xossalari

Determinantning xossalarini uchinchi tartibli determinant uchun keltiramiz. Bu xossalar ixtiyoriy n-tartibli determinant uchun ham o'rinli bo'ladi.

1-xossa. Transponirlash (barcha satrlarni mos ustunlar bilan almashtirish) natijasida determinantning qiymati o'zgarmaydi, ya'ni

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \det A^T.$$

Isboti. Xossani isbotlash uchun tenglikning chap va o'ng tomonidagi determinantlarning qiymatlarini uchburchak qoidasi orqali yozib olish va olingan ifodalarning tengligiga ishonch hosil qilish kifoya.

1-xossa satr va ustunlarning teng huquqligini belgilab beradi. Boshqacha aytganda, satrlar uchun isbotlangan xossalar ustunlar uchun ham o'rinli bo'ladi va aksincha. Shu sababli keyingi xossalarni ham satrlar va ham ustunlar uchun ifodalab, ularning isbotini faqat satrlar yoki faqat ustunlar uchun ko'rsatamiz.

2-xossa. Determinant ikkita satrining (ustunining) o'rinlari almashtirilsa, uning qiymati qarama-qarshi ishoraga o'zgaradi. Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Bu xossa ham 1-xossa kabi isbotlanadi.

3-xossa. Agar determinant ikkita bir xil satrga (ustunga) ega bo'lsa, u nolga teng bo'ladi.

Isboti. Ikkita bir xil satrning o'rinlari almashtirilsa, determinant (shuningdek, uning qiymati) o'zgarmaydi. Ikkinchi tomondan 2-xossaga ko'ra determinant qiymatining ishorasi o'zgaradi. Demak $\det A = -\det A$, yoki $2\det A = 0$. Bundan $\det A = 0$.

4-xossa. Determinantning biror satri (ustuni) elementlari λ songa ko'paytirilsa, determinant shu songa ko'payadi va aksincha determinant biror satr (ustun) elementlarining umumiy ko'paytuvchisini determinant belgisidan tashqariga chiqarish mumkin.
Masalan,

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Isboti. Tenglikning chap tomondagi determinant hisoblanganida oltita qo'shiluvchining hammasida λ ko'paytuvchi qatnashadi. Bu ko'paytuvchini qavsdan tashqariga chiqarib, qavslar ichidagi qo'shiluvchilardan determinant tuzilsa, tenglikning o'ng tomondagi ifoda hosil bo'ladi.

5-xossa. Agar determinant biror satrining (ustunining) barcha elementlari nolga teng bo'lsa, u nolga teng bo'ladi.

Xossaning *isboti* 4-xossadan $\lambda = 0$ da kelib chiqadi.

6-xossa. Agar determinantning ikki satri (ustuni) proporsional bo'lsa, u nolga teng bo'ladi.

Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

Isboti. 4-xossaga ko'ra determinant ikkinchi satrining λ ko'paytuvchisini determinant belgisidan chiqarish mumkin. Natijada ikkita bir xil satrli determinant qoladi va u 3-xossaga ko'ra nolga teng bo'ladi.

7-xossa. Agar determinant biror satrining (ustunining) har bir elementi ikki qo'shiluvchining yig'indisidan iborat bo'lsa, bu determinant ikki determinant yig'indisiga teng bo'lib, ulardan birinchisining tegishli satri (ustuni) elementlari birinchi qo'shiluvchilardan, ikkinchisining tegishli satri (ustuni) elementlari ikkinchi qo'shiluvchilardan tashkil topadi.

Isboti. Determinant birinchi satrining har bir elementi ikkita qo'shiluvchi yig'indisidan iborat bo'lsin.

U holda

$$\begin{vmatrix} a'_{11} + a''_{11} & a'_{12} + a''_{12} & a'_{13} + a''_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a'_{11} + a''_{11})a_{22}a_{33} + (a'_{12} + a''_{12})a_{23}a_{31} +$$

$$+ (a'_{13} + a''_{13})a_{21}a_{32} - (a'_{13} + a''_{13})a_{22}a_{31} - (a'_{12} + a''_{12})a_{21}a_{33} - (a'_{11} + a''_{11})a_{23}a_{32} =$$

$$= a'_{11}a_{22}a_{33} + a'_{12}a_{23}a_{31} + a'_{13}a_{21}a_{32} - a'_{13}a_{22}a_{31} - a'_{12}a_{21}a_{33} - a'_{11}a_{23}a_{32} + (a''_{11}a_{22}a_{33} + a''_{12}a_{23}a_{31} +$$

$$+ a''_{13}a_{21}a_{32} - a''_{13}a_{22}a_{31} - a''_{12}a_{21}a_{33} - a''_{11}a_{23}a_{32}) = \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a''_{11} & a''_{12} & a''_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

8-xossa. Agar determinantning biror satri (ustuni) elementlariga boshqa satrining (ustunining) mos elementlarini biror songa ko'paytirib qo'shilsa, determinantning qiymati o'zgarmaydi.

Isboti. $\det A$ determinantning ikkinchi satri elementlariga λ ga ko'paytirilgan birinchi satrning mos elementlari qo'shilgan bo'lsin:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+\lambda a_{11} & a_{22}+\lambda a_{12} & a_{23}+\lambda a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Qo'shiluvchilardan birinchisi $\det A$ ga va ikkinchisi esa 3-xossaga ko'ra nolga teng. Demak, yig'indi $\det A$ ga teng.

9-xossa. Determinantning qiymati uning biror satri (ustuni) elementlari bilan shu elementlarga mos algebraik to'ldiruvchilar ko'paytmalarining yig'indisiga teng bo'ladi.

Masalan, $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

yoki

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Isboti. Tenglikning o'ng tomonida almashtirishlar bajaramiz:

$$a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

10-xossa. Determinant biror satri (ustuni) elementlari bilan boshqa satri (ustuni) mos elementlari algebraik to'ldiruvchilari ko'paytmalarining yig'indisi nolga teng bo'ladi.

Masalan, $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = 0$.

Isboti. Determinantni 9-xossani qo'llab, topamiz:

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Bunda a_{11}, a_{12}, a_{13} mos ravishda a_{21}, a_{22}, a_{23} bilan bilan almashtirilsa, 3-xossaga ko'ra, determinant nolga teng bo'ladi.

1-izoh. Determinantning xossalari asosida quyidagi teorema isbotlangan.

1-teorema. Bir xil tartibli A va B kvadrat matritsalar ko'paytmasining determinanti bu matritsalar determinantlarining ko'paytmasiga teng, ya'ni

$$\det(A \cdot B) = \det A \cdot \det B.$$

1.2.4. n-tartibli determinantlarni hisoblash

n- tartibli determinantni xossalar yordamida soddalashtirib, keyin tartibini pasaytirish yoki uchburchak ko'rinishga keltirish usullaridan biri bilan hisoblash mumkin.

Tartibini pasaytirish usuli

n-tartibli determinant, 9-xossaga asosan, biror satr yoki ustun bo'yicha yoyilsa, yoyilmada $(n-1)$-tartibli algebraik to'ldiruvchilar hosil bo'ladi, ya'ni n-tartibli determinantni hisoblash tartibi bittaga past bo'lgan determinantlarni hisoblashga keltiriladi.

Umuman olganda, quyidagi teoremalar o'rinli bo'ladi.

2-teorema. i satrining nomeri qanday bo'lishidan qat'iy nazar, n-tartibli determinant uchun bu determinantni i-satr bo'yicha yoyilmasi deb ataluvchi

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}, \quad i = \overline{1,n}$$

formula o'rinli.

3-teorema. j ustunining nomeri qanday bo'lishidan qat'iy nazar, n-tartibli determinant uchun bu determinantni j-ustun bo'yicha yoyilmasi deb ataluvchi

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}, \quad j = \overline{1,n}$$

formula o'rinli.

Determinantni biror satr yoki ustun bo'yicha yoyishga *Laplas yoyilmalari usuli* deyiladi.

Laplas yoyilmalari usulida determinantning qaysi bir satrida (ustunida) nollar ko'p bo'lsa, u holda yoyishni shu satr (ustun) bo'yicha bajarish qulay bo'ladi.

Bundan tashqari, 8-xossani qo'llab, determinantning biror satrida (ustunida) bitta elementdan boshqa elementlarni nollarga keltirish mumkin. Bunda determinantning qiymati shu satrdagi (ustundagi) noldan farqli element bilan uning algebraik to'ldiruvchisining ko'paytmasidan iborat bo'ladi. Shunday qilib, n-tartibli determinant bitta $(n-1)$-tartibli determinantga keltirib, hisoblanadi.

3-misol.

$$\det A = \begin{vmatrix} 2 & -1 & 0 & 4 \\ 4 & 2 & -1 & 3 \\ -2 & 0 & 3 & -4 \\ 1 & 1 & 0 & -2 \end{vmatrix}$$

determinantni tartibini pasaytirish usuli bilan hisoblang.

Yechish. Bunda: 1) Ikkita elementi nolga teng bo'lgan uchinchi ustunni tanlaymiz va uning ikkinchi satrida joylashgan elementidan boshqa barcha elementlarini nolga aylantiramiz. Buning uchun ikkinchi satr elementlarini 3 ga ko'paytirib, uchunchi satrning mos elementlariga qo'shamiz va hosil bo'lgan determinantni uchinchi ustun elementlari bo'yicha yoyamiz:

$$\det A = \begin{vmatrix} 2 & -1 & 0 & 4 \\ 4 & 2 & -1 & 3 \\ -2 & 0 & 3 & -4 \\ 1 & 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 4 \\ 4 & 2 & -1 & 3 \\ 10 & 6 & 0 & 5 \\ 1 & 1 & 0 & -2 \end{vmatrix} = (-1)\cdot(-1)^{2+3}\begin{vmatrix} 2 & -1 & 4 \\ 10 & 6 & 5 \\ 1 & 1 & -2 \end{vmatrix};$$

2) Hosil qilingan uchinchi tartibli determinantda birinchi ustunning uchinchi satri elementidan yuqorida joylashgan elementlarini nolga aylantiramiz. Buning uchun avval uchinchi satrni (-2) ga ko'paytirib, birinchi satrga qo'shamiz, keyin uchinchi satrni (-10) ga ko'paytirib, ikkinchi satrga qo'shamiz, hosil bo'lgan determinantni birinchi ustun elementlari bo'yicha yoyamiz va hosil bo'lgan ikkinchi tartibli determinantni hisoblaymiz:

$$\det A = \begin{vmatrix} 0 & -3 & 8 \\ 0 & -4 & 25 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 8 \\ -4 & 25 \end{vmatrix} = -75 + 32 = -43.$$

Uchburchak ko'rinishga keltirish usuli

Bu usulda determinant xossalar yordamida soddalashtiriladi va uchburchak ko'rinishga keltiriladi, ya'ni diagonalidan pastda (yuqorida) joylashgan barcha elementlari nolga aylantiriladi.

Bunda

$$\det A = (-1)^k \det U$$

bo'ladi, bu yerda k – satrlarda va ustunlarda bajarilgan barcha o'rin almashtirishlar soni; $\det U$ – berilgan determinantning uchburchak ko'rininshi va uning qiymati quyidagi xossa asosida hisoblanadi.

Xossa. Uchburchak ko'rinishidagi determinant bosh diagonalda joylashgan elementlarining ko'paytmasiga teng.

4-misol.

$$\det A = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix}$$

determinantni uchburchak ko'rinishga keltirib, hisoblang.

Yechish. Determinant ustida quyidagi soddalashtirishlarni bajaramiz:

– birinchi ustunni o'zidan o'ngda joylashgan ustunlar bilan ketma-ket $k=3$ ta o'rin almashtirib, to'rtinchi ustunga o'tkazamiz;

– birinchi ustunning birinchi satridan pastda joylashgan elementlarini nolga aylantiramiz;

– ikkinchi ustunning ikkinchi satridan pastda joylashgan elementlarini nolga aylantiramiz;

– uchinchi ustunning to'rtinchi satrida joylashgan elementini nolga aylantiramiz;

– $(-1)^k = (-1)^3 = -1$ ko'paytuvchi bilan hosil bo'lgan uchburchak ko'rinishdagi determinantning bosh diagonalda joylashgan elementlarini ko'paytiramiz.

$$\det A = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix} = (-1)^3 \cdot \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{vmatrix} =$$

$$=(-1)\cdot\begin{vmatrix}1&0&0&2\\0&1&0&3\\0&0&1&-5\\0&0&2&-2\end{vmatrix}=(-1)\cdot\begin{vmatrix}1&0&0&2\\0&1&0&3\\0&0&1&-5\\0&0&0&8\end{vmatrix}=(-1)\cdot1\cdot1\cdot1\cdot8=-8.$$

1.2.5. Mashqlar

1. A matritsa $n\times n$ o'lchamli bo'lsin. $\det(\lambda A)$ da λ ni determinant belgisidan tashqariga chiqarish uchun formula keltirib chiqaring.

2. A kvadrat matritsa va $A^T A=I$ bo'lsin. $\det A=\pm1$ bo'lishini ko'rsating.

3. $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ va $B=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ bo'lsin. $\det(A+B)=\det A+\det B$ faqat $a+d=0$ bo'lganida bajarilishini ko'rsating.

4. $A=\begin{pmatrix}5&0\\1&2\end{pmatrix}$ va $B=\begin{pmatrix}7&1\\3&2\end{pmatrix}$ bo'lsin. $\det(A\cdot B)=\det A\cdot\det B$ bo'lishiga ishonch hosil qiling.

5. $A=\begin{pmatrix}3&1\\2&1\end{pmatrix}$ bo'lsin. $\det A^{1000}$ ni toping.

6. A va B matritsalar 3×3 o'lchamli, $\det A=-1$ va $\det B=2$ bo'lsin. Toping:
1) $\det AB$; 2) $\det 5A$; 3) $\det A^T A$; 4) $\det B^3$.

7. A va B matritsalar 4×4 o'lchamli, $\det A=4$ va $\det B=-3$ bo'lsin. Toping:
1) $\det AB$; 2) $\det B^5$; 3) $\det 2A$; 4) $\det IA$.

Ikkinchi tartibli determinantlarni hisoblang:

8. $\begin{vmatrix}y&x-y\\x&-x\end{vmatrix}$.

9. $\begin{vmatrix}1&a+b\\b+1&a+b\end{vmatrix}$.

10. $\begin{vmatrix}\sin^2\alpha&\cos^2\alpha\\\sin^2\beta&\cos^2\beta\end{vmatrix}$.

11. $\begin{vmatrix}tg\alpha+1&ctg\alpha-1\\\sin\alpha&\cos\alpha\end{vmatrix}$.

Uchinchi tartibli determinantlarni uchburchak va Sarryus qoidalari bilan hisoblang:

12. $\begin{vmatrix}5&-1&1\\4&0&-3\\2&-3&1\end{vmatrix}$.

13. $\begin{vmatrix}-2&0&-4\\3&1&1\\-1&2&-3\end{vmatrix}$.

Uchinchi tartibli determinantlarni biror satr yoki ustun elementlari bo‘yicha yoyib hisoblang:

14. $\begin{vmatrix} 1 & b & 1 \\ b & b & 0 \\ b & 0 & -b \end{vmatrix}$ **15.** $\begin{vmatrix} x & -1 & x \\ 1 & x & -1 \\ x & 1 & x \end{vmatrix}$.

16. $\begin{vmatrix} \sin\alpha & \sin\beta & 0 \\ \sin\alpha & 0 & \sin\gamma \\ 0 & \sin\beta & \sin\gamma \end{vmatrix}$. **17.** $\begin{vmatrix} tg\alpha & ctg\beta & 0 \\ tg\alpha & 0 & tg\beta \\ 0 & ctg\alpha & tg\beta \end{vmatrix}$.

Uchinchi tartibli determinantlarni xossalaridan foydalanib hisoblang:

18. $\begin{vmatrix} 1 & c & ab \\ 1 & b & ca \\ 1 & a & bc \end{vmatrix}$. **19.** $\begin{vmatrix} 1 & 1 & 1 \\ ax & ay & az \\ a^2+x^2 & a^2+y^2 & a^2+z^2 \end{vmatrix}$.

20. $\begin{vmatrix} a+b & b & b \\ b & a+b & b \\ b & b & a+b \end{vmatrix}$. **21.** $\begin{vmatrix} x & x+y & x-y \\ x & x+z & x-2z \\ x & x & x \end{vmatrix}$.

22. $\begin{vmatrix} a & a^2+1 & (1+a)^2 \\ b & b^2+1 & (1+b)^2 \\ c & c^2+1 & (1+c)^2 \end{vmatrix}$. **23.** $\begin{vmatrix} 1+\cos\alpha & 1 & 1+\sin\alpha \\ 1-\sin\alpha & 1 & 1-\cos\alpha \\ 1 & 1 & 1 \end{vmatrix}$.

To‘rtinchi tartibli determinantlarni hisoblang:

24. $\begin{vmatrix} 1 & -1 & 2 & 2 \\ 3 & -1 & 5 & -2 \\ -2 & -3 & 0 & 2 \\ 0 & -2 & 4 & 1 \end{vmatrix}$. **25.** $\begin{vmatrix} 1 & 1 & 3 & 2 \\ 2 & 0 & 0 & 8 \\ 3 & 0 & 0 & 2 \\ 4 & 4 & 7 & 5 \end{vmatrix}$.

26. $\begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix}$. **27.** $\begin{vmatrix} 3 & 2 & 2 & 2 \\ 9 & -8 & 5 & 10 \\ 5 & -8 & 5 & 8 \\ 6 & -5 & 4 & 7 \end{vmatrix}$.

1.3. MATRITSA USTIDA ALMASHTIRISHLAR

Matritsa ustida almashtirishlar chiziqli algebrada muhim ro‘l o‘ynaydi. Jumladan, chiziqli algebraik tenglamalar sistemasining umumiy yechimini topishda, teskari matritsani aniqlashda, matritsaning rangini hisoblashda matritsa ustidagi almashtirishlardan keng foydalaniladi.

Matritsa satri (ustuni) ustida elementar almashtirishlar uch tipda bo'ladi:

I. ikkita satrning (ustunning) o'rnini almashtirish;

II. satrni (ustunni) noldan farqli songa ko'paytirish;

III. satrga (ustunga) noldan farqli songa ko'paytirilgan boshqa satrni (ustunni) qo'shish.

Biri ikkinchisidan elementar almashtirishlar natijasida hosil qilingan A va B matritsalarga *ekvivalent matritsalar* deyiladi va $A \sim B$ ko'rinishda yoziladi.

1.3.1. Teskari matritsa

Asosiy ushunchalar

Matritsalarni qo'shish, ayirish va ko'paytirish sonlar ustida bajariladigan mos amallarga monand (hamohang) amallar hisoblanadi. Ushbu bandda matritsalar uchun sonlarni bo'lish amaliga monand amal bilan tanishamiz.

Ma'lumki, agar k soni nolga teng bo'lmasa, u holda har qanday m soni uchun $kx = m$ tenglama yagona $x = \frac{m}{k} = k^{-1}m$ yechimga ega bo'ladi, bu yerda k^{-1} soni k soniga teskari son deb ataladi.

Sonlar uchun keltirilgan bu tasdiq matritsali tenglamalarni sonli tenglamalarga monand yechishda muhim ro'l o'ynaydi. Xususan, sonli tenglamalar uchun $kk^{-1} = 1$ va $k^{-1}k = 1$ shartlarining bajarilishi hal qiluvchi hisoblansa, matritsali tenglamalar uchun $AA^{-1} = I$ va $A^{-1}A = I$ shartlarning bajarilishi muhim hisoblanadi, bu yerda $A, I -$ bir xil o'lchamli kvadrat matritsalar.

Agar A va A^{-1} kvadrat matritsalar uchun $AA^{-1} = A^{-1}A = I$ tenglik bajarilsa, A^{-1} matritsa A matritsaga *teskari matritsa* deyiladi.

Sonlarda, k^{-1} mavjud bo'lishi uchun $k \neq 0$ bo'lishi talab etilgani kabi, matritsalarda, A^{-1} mavjud bo'lishi uchun $\det A \neq 0$ bo'lishi talab qilinadi.

Agar $\det A = 0$ bo'lsa, A matritsaga *singular matritsa* deyiladi. Bunda singular so'ziga sinonim sifatida «*xos*» yoki «*maxsus*» terminlaridan ham foydalaniladi. Agar $\det A \neq 0$ bo'lsa, A matritsa *nosingular* (yoki *xosmas* yoki *maxsusmas*) *matritsa* deb ataladi.

Agar A matritsada avval elementlar mos algebraik to'ldiruvchilar bilan almashtirilsa va keyin transponirlansa, hosil bo'lgan matritsa A matritsaga *biriktirilgan matritsa* deyiladi va $\mathrm{adj}A$ bilan belgilanadi:

$$\mathrm{adj}A=\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}.$$

Teskari matritsa haqida teoremalar

1- teorema. Xos matritsa teskari matritsaga ega bo'lmaydi.

Isboti. A matritsa uchun A^{-1} mavjud bo'lsin deb faraz qilaylik. U holda $AA^{-1}=I$ bo'ladi. Bundan $\det(AA^{-1})=\det I$ yoki $\det A\cdot\det A^{-1}=\det I$ kelib chiqadi. Bunda $\det A=0$ va $\det I=1$ ekanini hisobga olsak, $0=1$ ziddiyat hosil bo'ladi. Bu ziddiyat qilingan faraz noto'g'ri ekanini ko'rsatadi, ya'ni teoremani isbotlaydi.

2- teorema. Har qanday xosmas A matritsa uchun teskari matritsa mavjud va yagona bo'ladi.

Isboti. A matritsa xosmas, ya'ni $\det A\neq 0$ bo'lsin. Avval A^{-1} mavjud bo'lishini ko'rsatamiz. Buning uchun A matritsani $\dfrac{1}{\det A}\mathrm{adj}A$ matritsaga ko'paytiramiz va ko'paytmaga determinantning 9- va 10- xossalarini qo'llaymiz:

$$A\cdot\left(\frac{1}{\det A}adjA\right)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\cdot\begin{pmatrix} \frac{A_{11}}{\det A} & \frac{A_{21}}{\det A} & \dots & \frac{A_{n1}}{\det A} \\ \frac{A_{12}}{\det A} & \frac{A_{22}}{\det A} & \dots & \frac{A_{n2}}{\det A} \\ \dots & \dots & \dots & \dots \\ \frac{A_{1n}}{\det A} & \frac{A_{2n}}{\det A} & \dots & \frac{A_{nn}}{\det A} \end{pmatrix}=$$

$$=\begin{pmatrix} \frac{a_{11}A_{11}+a_{12}A_{12}+\dots+a_{1n}A_{1n}}{\det A} & \frac{a_{11}A_{21}+a_{12}A_{22}+\dots+a_{1n}A_{2n}}{\det A} & \dots & \frac{a_{11}A_{n1}+a_{12}A_{n2}+\dots+a_{1n}A_{nn}}{\det A} \\ \frac{a_{21}A_{11}+a_{22}A_{12}+\dots+a_{2n}A_{1n}}{\det A} & \frac{a_{21}A_{21}+a_{22}A_{22}+\dots+a_{2n}A_{2n}}{\det A} & \dots & \frac{a_{21}A_{n1}+a_{22}A_{n2}+\dots+a_{2n}A_{nn}}{\det A} \\ \dots & \dots & \dots & \dots \\ \frac{a_{n1}A_{11}+a_{n2}A_{12}+\dots+a_{nn}A_{1n}}{\det A} & \frac{a_{n1}A_{21}+a_{n2}A_{22}+\dots+a_{nn}A_{2n}}{\det A} & \dots & \frac{a_{n1}A_{n1}+a_{n2}A_{n2}+\dots+a_{nn}A_{nn}}{\det A} \end{pmatrix}=$$

$$=\begin{pmatrix} \frac{\det A}{\det A} & 0 & \dots & 0 \\ 0 & \frac{\det A}{\det A} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\det A}{\det A} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I = AA^{-1}.$$

Demak, A *matritsaga teskari matritsa mavjud va bu matritsa*

$$A^{-1} = \frac{1}{\det A} adjA$$

formula bilan topiladi. Bunda $AA^{-1} = I$ tenglik bajariladi.

$A^{-1}A = I$ tenglikning bajarilishi shu kabi ko'rsatiladi.

Endi A^{-1} yagona ekanini ko'rsatamiz. Buning uchun A^{-1} dan boshqa A matritsaga teskari C matritsa mavjud bo'lsin deb faraz qilamiz. U holda ta'rifga ko'ra, $AC = I$ bo'ladi. Bu tenglikning har ikkala tamonini A^{-1} ga chapdan ko'paytiramiz:

$$A^{-1}AC = A^{-1}I.$$

$A^{-1}A = I$ bo'lgani uchun $IC = A^{-1}I$ bo'ladi.

Endi $IC = C$ va $A^{-1}I = A^{-1}$ ekanini hisobga olsak, $C = A^{-1}$ kelib chiqadi. Teorema to'liq isbot qilindi.

3-teorema. Teskari matritsa uchun ushbu *xossalar* o'rinli bo'ladi:

1°. A matritsa A^{-1} teskari matritsaga ega bo'lsa, $\det A^{-1} = \frac{1}{\det A}$ bo'ladi;

2°. A matritsa A^{-1} teskari matritsaga ega bo'lsa, $(A^{-1})^{-1} = A$ bo'ladi;

3°. $n \times n$ o'lchamli A va B matritsalar A^{-1} va B^{-1} teskari matritsalarga ega bo'lsa, $(AB)^{-1} = B^{-1}A^{-1}$ bo'ladi;

4°. A matritsa A^{-1} teskari matritsaga ega bo'lsa, $(A^T)^{-1} = (A^{-1})^T$ bo'ladi.

Isboti. 1) A matritsa uchun A^{-1} mavjud bo'lsin. U holda $AA^{-1} = I$ yoki $\det(AA^{-1}) = \det I$ bo'ladi. Bundan $\det A \cdot \det A^{-1} = 1$ yoki $\det A^{-1} = \frac{1}{\det A}$ kelib chiqadi.

2) A matritsa uchun A^{-1} mavjud bo'lsin. U holda $AA^{-1} = I = A^{-1}A$ tengliklarga ko'ra A^{-1} matritsa uchun teskari matritsa mavjud va u A dan iborat, ya'ni $(A^{-1})^{-1} = A$ bo'ladi.

3) $n \times n$ o'lchamli A va B matritsalar A^{-1} va B^{-1} teskari matritsalarga ega bo'lsin.

U holda AB va $B^{-1}A^{-1}$ matritsalar uchun

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

bo'ladi. Demak, AB uchun teskari matritsa mavjud va $(AB)^{-1} = B^{-1}A^{-1}$ bo'ladi.

4) A matritsa uchun A^{-1} mavjud bo'lsin.

U holda A^T va $(A^{-1})^T$ matritsalar uchun

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I,$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

bo'ladi. Demak, A^T uchun teskari matritsa mavjud va $(A^T)^{-1} = (A^{-1})^T$ bo'ladi.

1-izoh. 3-xossani k ta $n \times n$ o'lchamli va teskari matritsalarga ega bo'lgan matritsalar uchun quyidagicha umumlashtirish mumkin:

$$\left(A_1 A_2 \cdot \ldots \cdot A_{k-1} A_k\right)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdot \ldots \cdot A_2^{-1} A_1^{-1}.$$

Bu formula matematik induksiya metodi bilan isbotlanadi.

1-misol. $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ matritsaga teskari matritsani toping va natijani tekshiring.

Yechish. Berilgan matritsaning determinantini hisoblaymiz:

$$\det A = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2.$$

$\det A \neq 0$ va A matritsa uchun teskari matritsa mavjud.

Matritsa elementlarining algebraik to'ldiruvchilarini topamiz:

$$A_{11} = (-1)^{1+1} 2 = 2, \qquad A_{12} = (-1)^{1+2} 1 = -1,$$
$$A_{21} = (-1)^{2+1} 4 = -4, \qquad A_{22} = (-1)^{2+2} 3 = 3.$$

A matritsaga biriktirilgan matritsani tuzamiz:

$$adjA = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}.$$

Shunday qilib,

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix}2&-4\\-1&3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&-4\\-1&3\end{pmatrix}=\begin{pmatrix}1&-2\\-\frac{1}{2}&\frac{3}{2}\end{pmatrix}.$$

2-misol. $A=\begin{pmatrix}1&-2&1\\2&0&-1\\-2&1&1\end{pmatrix}$ matritsaga teskari matritsani toping .

Yechish. Bu matritsa uchun:

$$\det A=\begin{vmatrix}1&-2&1\\2&0&-1\\-2&1&1\end{vmatrix}=0-4+2-0+4+1=3\neq 0.$$

Matritsa elementlarining algebraik to'ldiruvchilarini topamiz:

$$A_{11}=\begin{vmatrix}0&-1\\1&1\end{vmatrix}=1,\qquad A_{21}=-\begin{vmatrix}-2&1\\1&1\end{vmatrix}=3,\qquad A_{31}=-\begin{vmatrix}-2&1\\0&-1\end{vmatrix}=2,$$

$$A_{12}=-\begin{vmatrix}2&-1\\-2&1\end{vmatrix}=0,\qquad A_{22}=\begin{vmatrix}1&1\\-2&1\end{vmatrix}=3,\qquad A_{32}=-\begin{vmatrix}1&1\\2&-1\end{vmatrix}=3,$$

$$A_{13}=\begin{vmatrix}2&0\\-2&1\end{vmatrix}=2,\qquad A_{23}=-\begin{vmatrix}1&-2\\-2&1\end{vmatrix}=3,\qquad A_{33}=\begin{vmatrix}1&-2\\2&0\end{vmatrix}=4.$$

A matritsaga biriktirilgan matritsani topamiz:

$$\operatorname{adj}A=\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}=\begin{pmatrix}1&3&2\\0&3&3\\2&3&4\end{pmatrix}.$$

Demak,

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix}1&3&2\\0&3&3\\2&3&4\end{pmatrix}=\begin{pmatrix}\frac{1}{3}&1&\frac{2}{3}\\0&1&1\\\frac{2}{3}&1&\frac{4}{3}\end{pmatrix}.$$

Teskari matritsani topishning Gauss-Jordan usuli

A xosmas matritsaning A^{-1} teskari matritsasini topishning qulay usullaridan biri matritsa satrlari ustida elementar almashtirishlarga asoslangan *Gauss-Jordan usuli* hisoblanadi.

A^{-1} matritsani topishning Gauss-Jordan usuli ushbu tartibda amalga oshiriladi.

Gauss-Jordan usulining algoritmi

1°. A va I matritsalarni yonma-yon yozib, $(A|I)$ kengaytirilgan matritsa tuziladi;

2°. Elementar almashtirishlar yordamida $(A|I)$ matritsa $(I|B)$ koʻrinishga keltiriladi. Bunda B matritsa A matritsa uchun teskari matritsa boʻladi.

3-misol. $A=\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ matritsaga teskari matritsani Gauss-Jordan usuli bilan toping va natijani tekshiring.

Yechish.

$$(A|I)=\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) r_1 \to r_1+(-2)r_2 \sim$$

$$\sim\left(\begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 1 & 2 & 0 & 1 \end{array}\right) r_2 \to r_2+(-1)r_1 \sim\left(\begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 0 & 5 & -1 & 3 \end{array}\right) r_2 \to r_2 : 5 \sim$$

$$\sim\left(\begin{array}{cc|cc} 1 & -3 & 1 & -2 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array}\right) r_1 \to r_1+3r_2 \sim\left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array}\right)=(I|A^{-1}).$$

Yuqorida keltirilgan $r_i \to r_i+\lambda r_k$ belgilash i-satr bu satrga λ songa koʻpaytirilgan k- satrni qoʻshish natijasida hosil qilinganini, $r_i \to r_i : \lambda$ belgi esa i- satr bu satrni λ songa boʻlish natijasida hosil qilinganini bildiradi.

Demak, $A^{-1}=\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}=\frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$.

4-misol. $A=\begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix}$ matritsaga teskari matritsani Gauss-Jordan usuli bilan toping.

Yechish.

$$(A|I)=\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} r_2 \to r_2 + r_1 \\ r_3 \to r_3 + (-2)r_1 \end{array} \sim$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 3 & 0 & -2 & 0 & 1 \end{array}\right) r_2 \to r_2 : 2 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 0 & -2 & 0 & 1 \end{array}\right)\begin{array}{l} r_1 \to r_1 + r_2 \\ \\ r_3 \to r_3 + (-3)r_2 \end{array} \sim$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -\frac{7}{2} & -\frac{3}{2} & 1 \end{array}\right) r_3 \to r_3 : (-3) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{array}\right)\begin{array}{l} r_1 = r_1 + (-3)r_3 \\ r_2 = r_2 + (-1)r_3 \end{array}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{array}\right)=(I\,|\,A^{-1}).$$

Demak,

$$A^{-1}=\begin{pmatrix} -2 & -1 & 1 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}.$$

1.3.2. Matritsani LU yoyish

Chiziqli algebrada matritsalarning turli yoyilmalari keng qo'llaniladi.

Matrisani yoyish deb, uni biror xossaga (masalan, ortogonallik, simmetriklik, diagonallik xossasiga) ega bo'lgan ikki va ikkidan ortiq martitsalar ko'paytmasi shaklida ifodalashga aytiladi. Bunday yoyishlardan biri *matritsani LU yoyish* hisoblanadi.

Matritsani LU yoyishda $m \times n$ o'lchamli A matritsa $A = LU$ shaklda ifodalanadi, bu yerda L – diagonal elementlari birlardan iborat bo'lgan $m \times m$ o'lchamli quyi uchburchak (Lower-triangular) matritsa; U – $m \times n$ o'lchamli yuqori uchburchak ($m \neq n$ da trapetsiya) (Upper-triangular) matritsa.

Masalan,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \cdot \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matritsaning LU yoyilmasi yana *matritsaning LU faktorizasiyasi* deb ataladi. Matritsaning LU yoyilmasidan chiziqli algebraik tenglamalar sistemasini yechishda va teskari matritsani topishda foydalaniladi.

$m \times n$ o'lchamli A matritsa $A = LU$ shaklga keltirish (LU yoyish), umuman olganda, A matritsaning satrlariga noldan farqli songa ko'paytirilgan boshqa satrni qo'shish orqali quyidagi tartibda amalga oshiriladi.

A matritsani LU yoyish algoritmi

1°. A matritsa satrlarida ketma-ket elementar almashtirishlar bajariladi va U shaklga keltiriladi;

2°. Satrlarda bajarilgan elementar almashtirishlar ketma-ketligi asosida L yozuv hosil qilinadi va bu yozuvda barcha diagonal elementlar ustunlarni bo'lish orqali birlarga aylantiriladi.

5-misol. $A=\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{pmatrix}$ matritsani LU yoying.

Yechish. Matritsa satrlarida ketma-ket elementar almashtirishlar bajaramiz:

$$A=\begin{pmatrix} \boxed{2} & 4 & -1 & 5 & -2 \\ \boxed{-4} & -5 & 3 & -8 & 1 \\ \boxed{2} & -5 & -4 & 1 & 8 \\ \boxed{-6} & 0 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & \boxed{3} & 1 & 2 & -3 \\ 0 & \boxed{-9} & -3 & -4 & 10 \\ 0 & \boxed{12} & 4 & 12 & -5 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & \boxed{4} & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{5} \end{pmatrix} = U.$$

A matritsa 4 ta satrdan tashkil topgani sababli L matritsa 4×4 o'lchamli bo'ladi. Birinchi qadamda belgilangan yozuvlar L matritsa yozuvining ustunlarini tashkil qiladi. Bu yozuvda barcha diagonal elementlarni birlarga aylantiramiz:

$$\begin{pmatrix} 2 & & & \\ -4 & 3 & & \\ 2 & -9 & 2 & \\ -6 & 12 & 4 & 5 \end{pmatrix}$$

$$\begin{matrix} :2 & :3 & :2 & :5 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{matrix}$$

$$\begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{pmatrix}. \quad \text{Bundan } L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}.$$

Demak,

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

n- tartibli kvadrat matritsa berilgan bo'lsin. Bunda A matritsani LU yoyish turli algoritmlar bilan amalga oshirilishi mumkin. Shunday algoritmlardan biri bilan tanishamiz.

A matritsa xosmas bo'lsin. U holda ta'rifga ko'ra.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}}_{A}.$$

Bundan

$$a_{ij} = \sum_{k=1}^{n} l_{ik} u_{kj} = \sum_{k=1}^{\min(i,j)} l_{ik} u_{kj}.$$

Bu yig'indidagi oxirgi qo'shiluvchilarni ajratib, topamiz:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \text{ agar } i \le j \text{ bo'lsa;} \tag{3.1}$$

$$l_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right), \text{ agar } i > j \text{ bo'lsa.} \tag{3.2}$$

Shunday qilib, L va U matritsalarning noma'lum elementlari a_{ij} va topilgan l_{ik}, u_{kj} lar orqali ketma-ket ifodalanadi.

2-izoh. (3.1) va (3.2) formulalar shunday tartiblanganki, bunda avval barcha u_{ij} larni va keyin barcha l_{ij} larni hisoblab bo'lmaydi, va aksincha. Bu formulalar orqali hisoblashlar quyidagi tartibda bajariladi:

$$u_{1j} = a_{1j}, \qquad j = 1,2,\dots,n;$$

$$l_{i1} = \frac{a_{i1}}{u_{11}}, \qquad i = 2,3,\dots,n;$$

$$u_{2j} = a_{2j} - l_{21} u_{1j}, \qquad j = 2,3,\dots,n;$$

$$l_{i2} = \frac{a_{i2} - l_{i1} u_{12}}{u_{22}}, \qquad i = 3,4,\dots,n;$$

va hokazo, ya'ni U ning satrlari va L ning ustunlari almashlab hisoblanadi.

5-misol. $A=\begin{pmatrix}8&2&9\\4&9&4\\6&7&9\end{pmatrix}$ matritsani LU yoying.

Yechish. Berilgan matritsa xosmas, chunki $\det A=166\neq 0$.
$LU=A$ yoyilmani tuzamiz:

$$\begin{pmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{pmatrix}\cdot\begin{pmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{pmatrix}=\begin{pmatrix}8&2&9\\4&9&4\\6&7&9\end{pmatrix}.$$

L va U matritsalarning noma'lum elementlarini (3.1) va (3.2) formulalar bilan aniqlaymiz:

$$u_{11}=a_{11}=8,\qquad u_{12}=a_{12}=2,\qquad u_{13}=a_{13}=9,$$

$$l_{21}=\frac{1}{u_{11}}a_{21}=\frac{4}{8}=\frac{1}{2},\qquad u_{22}=a_{22}-l_{21}u_{12}=9-\frac{1}{2}\cdot 2=8,$$

$$u_{23}=a_{23}-l_{21}u_{13}=4-\frac{1}{2}\cdot 9=-\frac{1}{2},\qquad l_{31}=\frac{1}{u_{11}}a_{31}=\frac{6}{8}=\frac{3}{4},$$

$$l_{32}=\frac{1}{u_{22}}(a_{32}-l_{31}u_{12})=\frac{1}{8}\left(7-\frac{3}{4}\cdot 2\right)=\frac{11}{16},$$

$$u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}=9-\frac{3}{4}\cdot 9-\frac{11}{16}\cdot\left(-\frac{1}{2}\right)=\frac{83}{32}.$$

Demak,

$$\begin{pmatrix}8&2&9\\4&9&4\\6&7&9\end{pmatrix}=\begin{pmatrix}1&0&0\\\frac{1}{2}&1&0\\\frac{3}{4}&\frac{11}{16}&1\end{pmatrix}\cdot\begin{pmatrix}8&2&9\\0&8&-\frac{1}{2}\\0&0&\frac{83}{32}\end{pmatrix}.$$

1.3.3. Matritsaning rangi

$m\times n$ o‘lchamli A matritsa berilgan bo‘lsin. Bu matritsadan biror k $(k\leq\min(m;n))$ ta satr va k ta ustunni ajratamiz. Ajratilgan satr va ustunlarning kesishishida joylashgan elementlardan k-tartibli kvadrat matritsani tuzamiz. Bu matritsaning determinantiga *A matritsaning k-tartibli minori* deyiladi.

A matritsa noldan farqli minorlari tartibining eng kattasiga A *matritsaning rangi* deyiladi va $r(A)$ (yoki $rangA$) kabi belgilanadi.

Tartibi $r(A)$ga teng bo'lgan minorga *A matritsaning bazis minori* deyiladi. Matritsa bir nechta bazis minorga ega bo'lishi mumkin.

Matritsa rangining ta'rifidan quyidagi tasdiqlar kelib chiqadi.

1. Matritsaning rangi 0 bilan m,n sonlarining kichigi orasidagi butun son orqali ifodalanadi, ya'ni $0 \le r(A) \le \min(m;n)$.

2. Faqat $A = O$ matritsa uchun $r(A) = 0$ bo'ladi.

3. n-tartibli kvadrat matritsa xosmas bo'lganidagina $r(A) = n$ bo'ladi.

Matritsanng rangi ushbu xossalarga bo'ysunadi.
1°. Transponirlash natijasida matritsaning rangi o'zgarmaydi;
2°. Elementar almashtirishlar natijasida matritsaning rangi o'zgarmaydi.

Isboti. Bilamizki:

a) transponirlash natijasida determinantning qiymati o'zgarmaydi;

b) ikkita satrning (ustunning) o'rni almashtirilsa, determinantning ishorasi o'zgaradi;

c) satrni (ustunni) noldan farqli songa ko'paytirilsa, determinant shu songa ko'payadi;

d) satrga (ustunga) noldan farqli songa ko'paytirilgan boshqa satrni (ustunni) qo'shilsa determinant o'zgarmaydi.

Demak, transponirlash va elementar almashtirishlar natijasida xos matritsa xosligicha va xosmas matritsa xosmasligicha qoladi, ya'ni uning rangi o'zgarmaydi.

$r(A)$ ni ta'rif asosida topish usuli *minorlar ajratish usuli* deb ataladi. Bu usulda matritsaning rangi quyidagicha topiladi: agar barcha birinchi tartibli minorlar (matritsa elementlari) nolga teng bo'lsa, $r(A) = 0$ bo'ladi; agar birinchi tartibli minorlardan hech bo'lmaganda bittasi noldan farqli va barcha ikkinchi tartibli minorlar nolga teng bo'lsa, $r(A) = 1$ bo'ladi; agar ikkinchi tartibli noldan farqli minor mavjud bo'lsa, uchinchi tartibli minorlar tekshiriladi; bu jarayon yoki barcha k-tartibli minorlar nolga teng bo'lishi aniq bo'lguncha yoki k-tartibli minorlar mavjud bo'lmaguncha davom ettiriladi, bunda $r(A) = k - 1$ bo'ladi.

6-misol. $A=\begin{pmatrix} 2 & -1 & 3 & -2 \\ 4 & -2 & 5 & 1 \\ 2 & -1 & 1 & 8 \end{pmatrix}$ matritsaning rangini minorlar ajratish usuli bilan toping.

Yechish. Ravshanki, $1 \le r(A) \le \min(3;5)=3$.

Ikkinchi tartibli minorlardan biri

$$\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = -5+6=1\ne 0.$$

Uchinchi tartibli minorlarni hisoblaymiz:

$$M_1^{(3)}=\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}=0; \qquad M_2^{(3)}=\begin{vmatrix} 2 & -1 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 8 \end{vmatrix}=0;$$

$$M_3^{(3)}=\begin{vmatrix} 2 & 3 & -2 \\ 4 & 5 & 1 \\ 2 & 1 & 8 \end{vmatrix}=0; \qquad M_4^{(3)}=\begin{vmatrix} -1 & 3 & -2 \\ -2 & 5 & 1 \\ -1 & 1 & 8 \end{vmatrix}=0.$$

Barcha uchinchi tartibli minorlar nolga teng. Demak, $r(A)=2$.

$r(A)$ni topishning minorlar ajratish usuli hamma vaqt ham qulay bo‘lavermaydi, chunki ayrim hollarda bir qancha hisoblashlar bajarishga to‘g‘ri keladi.

Elementar almashtirishlar orqali har qanday matritsani bosh diagonalning birinchi bir nechta elementlari birlardan va qolgan elementlari nollardan iborat bo‘lgan matritsa ko‘rinishiga keltirish mumkin. Masalan, ushbu matritsaga

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Bunday matritsaga *kanonik matritsa* deyiladi. Kanonik matritsaning rangi uning bosh diagonalida joylashgan birlar soniga teng bo‘ladi.

$r(A)$ ni kanonik matritsaga keltirib topish usuli matritsani *kanonik ko‘rinishga keltirish* usuli deb ataladi.

7-misol. $A=\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & 0 & 1 & -1 & 2 \\ -1 & 3 & -5 & -10 & 5 \end{pmatrix}$ matritsaning rangini uni kanonik ko‘rinishga keltirish usuli bilan toping.

Yechish.

$$A=\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & 0 & 1 & -1 & 2 \\ -1 & 3 & -5 & -10 & 5 \end{pmatrix}\begin{matrix} \\ r_2 \to r_2+(-2)r_1 \\ r_3 \to r_3+r_1 \end{matrix} \sim$$

$$\sim\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 2 & -3 & -7 & 4 \\ 0 & 2 & -3 & -7 & 4 \end{pmatrix}\begin{matrix} \\ \\ r_3 \to r_3+(-1)r_2 \end{matrix} \sim\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 2 & -3 & -7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{matrix} \\ \\ A \to A^T \end{matrix} \sim$$

$$\sim\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -3 & 0 \\ 3 & -7 & 0 \\ -1 & 4 & 0 \end{pmatrix}\begin{matrix} \\ r_2 \to r_2+r_1 \\ r_3 \to r_3+(-2)r_1 \\ r_4 \to r_4+(-3)r_1 \\ r_5 \to r_5+r_1 \end{matrix} \sim\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & -7 & 0 \\ 0 & 4 & 0 \end{pmatrix}\begin{matrix} \\ r_2 \to r_2:2 \\ r_3 \to r_3:3 \\ r_4 \to r_4:7 \\ r_5 \to r_5:(-4) \end{matrix} \sim$$

$$\sim\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{matrix} \\ \\ r_3 \to r_3+r_2 \\ r_4 \to r_4+r_2 \\ r_5 \to r_5+r_2 \end{matrix} \sim\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Demak, $r(A)=2$.

1.3.4. Mashqlar

1. Agar A matritsa xosmas va simmetrik bo‘lsa, A^{-1} matritsa ham xosmas va simmetrik bo‘lishini ko‘rsating.

2. Agar A kvadrat matritsa va $(I-A)$ xosmas matritsa bo‘lsa, $A(I-A)^{-1}=(I-A)^{-1}A$ tenglik bajarilishini ko‘rsating.

3. $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ matritsaning teskari matritsaga ega bo‘lishi shartini toping.

4. $A=\begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$ va $C=\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$ bo'lsin. $C=A^{-1}$ ekanini ko'rsating.

5. $B=\begin{pmatrix} 4 & 0 & -5 \\ -18 & 1 & 24 \\ -3 & 0 & 4 \end{pmatrix}$ matritsa $A=\begin{pmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{pmatrix}$ matritsaning teskari matritsasi bo'lishini ko'rsating.

6. $B=\frac{1}{36}\begin{pmatrix} 11 & -3 & 5 \\ -17 & 21 & -11 \\ -10 & 6 & 2 \end{pmatrix}$ matritsa $A=\begin{pmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 5 \end{pmatrix}$ matritsaning teskari matritsasi bo'lishini ko'rsating.

7. Berilgan matritsalardan qaysi birlari uchun teskari matritsa mavjud bo'ladi?

1) $A=\begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$; 2) $B=\begin{pmatrix} 0 & 5 \\ 7 & 2 \end{pmatrix}$; 3) $C=\begin{pmatrix} 1 & -2 & 0 \\ 2 & 2 & 6 \\ 3 & 5 & 11 \end{pmatrix}$; $D=\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 0 & 3 & 10 \end{pmatrix}$.

8. $A=\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ bo'lsin. $A^2=A^{-1}$ va $A^3=I$ bo'lishini ko'rsating.

9. Berilgan matritsalardan qaysi birlari o'zaro teskari matritsalar bo'ladi?

1) $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ va $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$; 2) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ va $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$;

3) $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ va $\frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$; 4) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$ va $\begin{pmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$

10. $A=\begin{pmatrix} -3 & 6 \\ 2 & -5 \end{pmatrix}$ matritsa berilgan. A^{-1} matritsani toping.

11. $A=\begin{pmatrix} 5 & 2 \\ -1 & 4 \end{pmatrix}$ matritsa berilgan. A^{-1} matritsani toping.

12. Berilgan shartlarni qanoatlantiruvchi A matritsani toping:

1) $(3A)^{-1}=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$; 2) $(2A)^T=\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}^{-1}$;

3) $(A^T-2I)^{-1}=\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$; 4) $A^{-1}=\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$.

13. $ABC=\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ bo'lsin. $C^{-1}B^{-1}A^{-1}$ ni toping.

14. $A=\begin{pmatrix} -3 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 5 & -1 \end{pmatrix}$ matritsa berilgan. $C = A\cdot \mathrm{adj}A$ ko'paytmaning barcha nodiagonal elementlarini toping.

15. $A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 0 \end{pmatrix}$ matritsa berilgan. $C = A\cdot \mathrm{adj}A$ ko'paytmaning barcha diagonal elementlarini toping.

A matritsa berilgan. A^{-1} matritsani toping:

16. $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 2 & 4 \end{pmatrix}$.

17. $A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 10 & 8 \end{pmatrix}$.

A matritsa berilgan. A^{-1} matritsani Jordan-Gauss usuli bilan toping:

18. $A=\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix}$.

19. $A=\begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$.

20. A matritsa berilgan. Matritsaning LU yoyilmasini toping:

1) $A=\begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$;

2) $A=\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$;

3) $A=\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$;

4) $A=\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$;

5) $A=\begin{pmatrix} 2 & 0 & 5 & 2 \\ -6 & 3 & -13 & -3 \\ 4 & 6 & 16 & 17 \end{pmatrix}$;

6) $A=\begin{pmatrix} 2 & -3 & 4 \\ -4 & 8 & -7 \\ 6 & -5 & 14 \\ -6 & 9 & -12 \\ 8 & -6 & 10 \end{pmatrix}$.

A matritsa berilgan. $r(A)$ ni minorlar ajratish usuli bilan toping:

21. $A=\begin{pmatrix} 1 & -1 & 2 & 3 \\ -1 & 3 & 0 & 1 \\ 3 & 4 & 1 & 1 \end{pmatrix}$.

22. $A=\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & -2 \\ 2 & -2 & 7 \end{pmatrix}$.

A matritsa berilgan. $r(A)$ ni elementar almashtirishlar usuli bilan toping:

23. $A=\begin{pmatrix} 1 & -3 & 2 & -1 \\ 2 & -1 & 4 & -6 \\ -3 & -1 & -6 & 11 \end{pmatrix}$ **24.** $A=\begin{pmatrix} 1 & -1 & 3 & 4 \\ 2 & -1 & 3 & -2 \\ 1 & -4 & 3 & 1 \\ 1 & -3 & 0 & -9 \end{pmatrix}$

1.4. CHIZIQLI TENGLAMALAR SISTEMASI

Chiziqli tenglamalar sistemasini yechish masalasi chiziqli algebraning asosiy masalalaridan biri hisoblanadi. Matematika, texnika va iqtisodiyotning ko'pchilik masalalari chiziqli tenglamalar sistemasi orqali ifodalanadi, masalan, geodeziyaviy o'lchash, elektr zanjirlarini loyihalash, chiziqli programmalashtirish va iqtisodni rejalashtirish masalalari chiziqli tenglamalar sistemasini yechishga keltiriladi.

1.4.1. Asosiy tushunchalar

Ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \tag{4.1}$$

sistemaga *n noma'lumli m ta chiqziqli tenglamalar sistemasi* deyiladi.

Bu yerda $a_{11}, a_{12}, \ldots, a_{mn}$ haqiqiy sonlarga *sistemaning koeffitsiyentlari*, $x_1, x_2, \ldots, x_n$ *noma'lumlar*, $b_1, b_2, \ldots, b_m$ haqiqiy sonlarga *ozod hadlar* deyiladi.

(4.1) sistema koeffitsiyentlaridan tuzilgan

$$A=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \tag{4.2}$$

matritsaga (4.1) sistemaning *matritsasi* (*asosiy matritsasi*) deyiladi.

Bu matritsaga ozod hadlardan tuzilgan ustunni qo'shish orqali hosil qilingan

$$C=\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) \tag{4.3}$$

matritsaga (4.1) *sistemaning kengaytirilgan matritsasi* deyiladi.

(4.1) sistemani

$$AX = B \tag{4.4}$$

matritsa ko'rinishida yozish mumkin, bu yerda

$$X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \qquad B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}. \tag{4.5}$$

Haqiqatdan ham,

$$AX=\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}=B.$$

(4.1) sistema tenglamalarini ayniyatga aylantiradigan noma'lumlarning tartiblangan $x_1^o, x_2^o, \dots, x_n^o$ qiymatlariga (4.1) *sistemaning yechimi* deyiladi.

Kamida bitta yechimga ega sistemaga *birgalikda bo'lgan sistema*, bitta ham yechimga ega bo'lmagan sistemaga *birgalikda bo'lmagan sistema* deyiladi.

Birgalikda bo'lgan va yagona yechimga ega sistemaga *aniq sistema*, cheksiz ko'p yechimga ega sistemaga *aniqmas sistema* deyiladi. Aniqmas sistemaning har bir yechimi *sistemaning xususiy yechimi* deb ataladi. Barcha xususiy yechimlar to'plami *sistemaning umumiy yechimi* deyiladi.

Sistemani *tekshirish* deganda sistemaning birgalikda yoki birgalikda emasligini aniqlash va agar sistema birgalikda bo'lsa, u holda uning aniq yoki aniqmasligini tekshirish tushuniladi. Birgalikda bo'lgan sistemaning umumiy yechimini topishga *sistemani yechish* deyiladi.

Yechimlari to'plami bir xil bo'lgan, ya'ni birinchisining har bir yechimi ikkinchisining yechimi bo'ladigan, va aksincha, ikkinchisining har bir yechimi birinchisining yechimi bo'ladigan ikkita sistemaga *ekvivalent* (*teng kuchli*) *sistemalar* deyiladi.

Ushbu almashtirishlar *sistemada elementar almashtirishlar* deb yuritiladi:

– sistema istalgan ikkita tenglamasining o‘rinlarini almashtirish;

– sistemaning istalgan tenglamasini noldan farqli songa ko‘paytirish (bo‘lish);

– sistemaning istalgan tenglamasiga noldan farqli songa ko‘paytirilgan boshqa tenglamasini qo‘shish.

Elementar almashtirishlar natijasida ekvivalent sistemalar hosil bo‘ladi.

Ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases} \tag{4.6}$$

n noma’lumli n ta chiziqli tenglamalar sistemasining

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \tag{4.7}$$

matritsasi kvadrat matritsa bo‘ladi.

A matritsaning

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{4.8}$$

determinantiga (4.6) *sistemaning determinanti* deyiladi.

Agar $\det A \neq 0$ bo‘lsa, (4.6) sistemaga *xosmas sistema* deyiladi.

Agar $\det A = 0$ bo‘lsa, (4.6) sistemaga *xos sistema* deyiladi.

1.4.2. Chiziqli tenglamalar sistemasini yechishning Gauss usuli

n noma’lumli m ta chiqziqli tenglamalar sistemasini yechishning qulay usullaridan biri – *noma’lumlarni ketma-ket yo‘qotish*ga (chiqarishga) asoslangan *Gauss usuli*ni ko‘rib chiqamiz.

n noma'lumli m ta chiqziqli tenglamalar sistemasi berilgan bo'lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \tag{4.1}$$

(4.1) sistemani Gauss usuli bilan yechish ikki bosqichda amalga oshiriladi.

Birinchi bosqichda sistema pog'onasimon ko'rinishga keltiriladi.

Pog'onasimon sistema deyilganida

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1k}x_k + \ldots + a_{1n}x_n = b_1, \\ \qquad a_{22}x_2 + \ldots + a_{2k}x_k + \ldots + a_{2n}x_n = b_2, \\ \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \qquad\qquad a_{kk}x_k + \ldots + a_{kn}x_n = b_m \end{cases}$$

ko'rinishdagi sistema tushuniladi, bu yerda $k \le n$, $a_{ii} \ne 0$, $i = \overline{1,k}$.

Ikkinchi bosqichda noma'lumlar pog'onasimon sistemadan ketma-ket topiladi.

1-bosqich. Sistemada quyidagi almashtirishlarni bajaramiz: birinchi tenglamaning chap va o'ng tomonini $a_{11} \ne 0$ ga (agar $a_{11} = 0$ bo'lsa, u holda bu tenglama sistemaning x_1 noma'lum oldidagi koeffitsiyenti nolga teng bo'lmagan tenglamasi bilan almashtiriladi) bo'lamiz. Keyin hosil qilingan tenglamani $(-a_{i1})$ ga ko'paytirib, i-tenglamaga qo'shamiz. Bunda sistema tenglamalarining ikkinchisidan boshlab x_1 qatnashgan hadlar yo'qatiladi va (4.1) sistema quyidagi ko'rinishga keladi:

$$\begin{cases} x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \ldots + a_{1n}^{(1)}x_n = b_1^{(1)}, \\ \qquad a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \ldots + a_{2n}^{(1)}x_n = b_2^{(1)}, \\ \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \qquad a_{m2}^{(1)}x_2 + a_{m3}^{(1)}x_3 + \ldots + a_{mn}^{(1)}x_n = b_m^{(1)}, \end{cases}$$

bu yerda $a_{ij}^{(1)}$, $b_i^{(1)}$ $(i = \overline{1,m}, j = \overline{1,n})$ – sistemaning birinchi almashtirishlardan keyin hosil qilingan koeffitsiyentlari va ozod hadlari.

Sistemada x_1 noma'lum oldidagi koeffitsiyenti birga teng bo'lgan tenglama bor bo'lsa, bu tenglamani birinchi o'rinda yozish orqali hisoblashlar osonlashtirilishi mumkin.

Shu kabi $a_{22}^{(1)} \neq 0$ deb, sistemaning uchinchi tenglamasidan boshlab x_2 noma'lumni yo'qotamiz va bu jarayonni mumkin bo'lguniga qadar davom ettiramiz.

Bu bosqichda, agar:

– $0=0$ ko'rinishdagi tengliklar paydo bo'lsa, u holda bu tengliklar tashlab yuboriladi.

– $0=b_i^{(k)}$ $(b_i^k \neq 0)$ ko'rinishdagi tengliklar paydo bo'lsa, u holda jarayon to'xtatiladi, chunki berilgan sistema birgalikda bo'lmaydi.

2-bosqich. Pog'onasimon sistemani yechamiz. Pog'onasimon sistemada k tenglamalar soni n noma'lumlar soniga teng yoki no'malumlar sonidan kichik bo'lishi mumkin. Shu sababli bu sistema yagona yoki cheksiz ko'p yechimga ega bo'lishi mumkin. Agar sistema uchburchak ko'rinishga kelsa, ya'ni $k=n$ bo'lsa, sistema yagona yechimga ega bo'ladi. Agar sistema trapetsiya ko'rinishga kelsa, ya'ni $k<n$ bo'lsa, sistema cheksiz ko'p yechimga ega bo'ladi.

1-misol. $\begin{cases} 2x_1-4x_2-x_3=-2, \\ 3x_1+\ x_2-2x_3=-11, \\ x_1-2x_2+4x_3=\ 8 \end{cases}$ tenglamalar sistemasini Gauss usuli bilan yeching.

Yechish.

1-bosqich. Sistemada quyidagi almashtirishlarni bajaramiz:

– birinchi va uchinchi tenglamalarning o'rinlarini almashtiramiz;

– (–3) ga ko'paytirilgan birinchi tenglamani ikkinchi tenglamaga va (–2) ga ko'paytirilgan birinchi tenglamani uchinchi tenglamaga hadma-had qo'shamiz;

– ikkinchi va uchinchi tenglama hadlarini mos ravishda 7 ga va (–9) ga bo'lamiz.

2-bosqich. x_3 ning uchinchi tenglamadagi qiymatini birinchi va ikkinchi tenglamalarga qo'yamiz, ikkinchi tenglamadan x_2 ni topamiz va uning qiymatini birinchi tenglamaga qo'yib, x_1 ni topamiz.

Sistemaning yechimlarini x_1, x_2, x_3 ketma-ketlikda yozamiz.

$$\begin{cases} 2x_1-4x_2-x_3=-2, \\ 3x_1+\ x_2-2x_3=-11, \\ x_1-2x_2+4x_3=\ 8 \end{cases} \Rightarrow \begin{cases} x_1-2x_2+4x_3=\ 8, \\ 3x_1+\ x_2-2x_3=-11, \\ 2x_1-4x_2-x_3=-2 \end{cases} \Rightarrow$$

$$\Rightarrow\begin{cases} x_1-2x_2+4x_3=\ 8, \\ \quad 7x_2-14x_3=-35, \\ \quad -9x_3=-18 \end{cases}\Rightarrow\begin{cases} x_1-2x_2+4x_3=\ 8, \\ \quad x_2-2x_3=-5, \\ \quad x_3=\ 2 \end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases} x_3=\ 2, \\ x_2-2\cdot 2=-5, \\ x_1-2x_2+4\cdot 2=\ 8 \end{cases}\Rightarrow\begin{cases} x_3=\ 2, \\ x_2=-1, \\ x_1-2\cdot(-1)=\ 0 \end{cases}\Rightarrow\begin{cases} x_1=-2, \\ x_2=-1, \\ x_3=\ 2. \end{cases}$$

Gauss usulining 1-bosqichini sistemaning o'zida emas, balki uning kengaytirilgan matritsasida bajarish qulaylikka ega.

Masalan, yuqoridagi sistemaning 1-bosqichi quyidagicha bajariladi:

$$\left(\begin{array}{ccc|c} 2 & -4 & -1 & -2 \\ 3 & 1 & -2 & -11 \\ 1 & -2 & 4 & 8 \end{array}\right)\begin{array}{l} r_1\to r_3 \\ \\ r_3\to r_1 \end{array}\sim\left(\begin{array}{ccc|c} 1 & -2 & 4 & 8 \\ 3 & 1 & -2 & -11 \\ 2 & -4 & -1 & -2 \end{array}\right)\begin{array}{l} \\ r_2\to r_2+(-3)r_1\sim \\ r_3\to r_3+(-2)r_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 8 \\ 0 & 7 & -14 & -35 \\ 0 & 0 & -9 & -18 \end{array}\right)\begin{array}{l} \\ r_2\to r_2:7 \\ r_3\to r_3:(-9) \end{array}\sim\left(\begin{array}{ccc|c} 1 & -2 & 4 & 8 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Xosmas tenglamalar sistemasini Gauss usuli bilan yechishda bu usulning boshqa bir turi *Jordan-Gauss usuli* qo'llaniladi. Bu usulda kengaytirilgan $(A\,|\,B)$matritsa ustida elementar almashtirishlar bajariladi va A matritsa o'rnida I matritsa hosil qilinadi, ya'ni u $(I\,|\,X)$ ko'rinishga keltiriladi. Bunda oxirgi kengaytirilgan matritsadagi X matritsa tenglamalar sistemasining yechimi bo'ladi.

2-misol. $\begin{cases} x_1-\ x_2-\ x_3=0, \\ 5x_1-\ x_2+4x_3=3, \\ x_1+2x_2+3x_3=5 \end{cases}$ sistemani Gordan-Gauss usuli bilan yeching.

Yechish.

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 5 & -1 & 4 & 3 \\ 1 & 2 & 3 & 5 \end{array}\right)\begin{array}{l} \\ r_2\to r_2+(-5)r_1\sim \\ r_3\to r_3+(-1)r_1 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & 9 & 3 \\ 0 & 3 & 4 & 5 \end{array}\right)r_2\to r_2:4\sim$$

$$\sim\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{3}{4} \\ 0 & 3 & 4 & 5 \end{array}\right) r_3 \to r_3 + (-3)r_2 \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{3}{4} \\ 0 & 0 & -\frac{11}{4} & \frac{11}{4} \end{array}\right) r_3 \to r_3 : \left(-\frac{11}{4}\right) \sim$$

$$\sim\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{3}{4} \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} r_1 \to r_1 + r_3 \\ r_2 \to r_2 + \left(-\frac{9}{4}\right) r_3 \end{array} \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) r_1 \to r_1 + r_2 \sim$$

$$\sim\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Demak, $x_1 = 2, \quad x_2 = 3, x_3 = -1$.

Shunday qilib, Gaussning noma'lumlarni ketma-ket yo'qotish usulida (4.1) sistema tenglamalarda almashtirishlar bajarish orqali yuqori uchburchak (ayrim hollarda trapetsiya) shaklga keltiriladi va hosil qilingan uchburchak sistema teskari o'rniga qo'yish orqali yechiladi. Bu usul matematik nuqtayi nazardan, sistemani uning matritsasini LU yoyish orqali yechish algoritmiga ekvivalent.

Gauss usulining LU yoyishga asoslangan algoritmi.

1°. *To'g'ri o'rniga qo'yish.* A matritsaning $A = LU$ yoyilmasi topiladi;

2°. *Teskari o'rniga qo'yish.* $LUX = B \quad \Leftrightarrow \quad \begin{cases} LY = B, \\ UX = Y \end{cases}$

sistema yechiladi.

3-misol. $\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 3, \\ 2x_1 + x_2 - 2x_3 + x_4 = -5, \\ x_1 - x_2 - 3x_3 + 2x_4 = -8 \end{cases}$ tenglamalar sistemasini LU yoyish orqali yeching.

Yechish.

$$A=\begin{pmatrix}1&2&3&1\\2&1&-2&1\\1&-1&-3&2\end{pmatrix}\begin{matrix}\\ r_2\to r_2+(-2)r_1\\ r_3\to r_3+(-1)r_1\end{matrix}\sim\begin{pmatrix}1&2&3&1\\0&-3&-8&-1\\0&-3&-6&1\end{pmatrix}\begin{matrix}\\ \\ r_3\to r_3+(-1)r_2\end{matrix}\sim$$

$$\sim\begin{pmatrix}1&2&3&1\\0&-3&-8&-1\\0&0&\boxed{2}&2\end{pmatrix}=U,$$

$$\begin{array}{c}\begin{pmatrix}1&&\\2&-3&\\1&-3&2\end{pmatrix}\\ \quad :-3\ \ :2\end{array}\to\begin{pmatrix}1&&\\2&1&\\1&1&1\end{pmatrix}.\quad \text{Bundan}\quad L=\begin{pmatrix}1&0&0\\2&1&0\\1&1&1\end{pmatrix}.$$

Avval $LY=B$ tenglamani yechamiz:

$$\begin{pmatrix}1&0&0\\2&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}3\\-5\\-8\end{pmatrix}.$$

Bundan

$$\begin{cases}y_1=3,\\ 2y_1+y_2=-5,\\ y_1+y_2+y_3=-8\end{cases}\Rightarrow\begin{cases}y_1=3,\\ y_2=-5-2y_1=-11,\\ y_3=-8-y_2-y_1=0.\end{cases}$$

Endi $UX=Y$ tenglamani yechamiz:

$$\begin{pmatrix}1&2&3&1\\0&-3&-8&-1\\0&0&2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}3\\-11\\0\end{pmatrix},$$

$$\begin{cases}x_1+2x_2+3x_3+x_4=3,\\ 3x_2+8x_3+x_4=11,\\ 2x_3+2x_4=0\end{cases}\Rightarrow\begin{cases}x_4=k,\\ x_3=-k,\\ x_2=\dfrac{11+7k}{3},\\ x_1=-\dfrac{13+8k}{3}.\end{cases}$$

Sistema trapetsiyasimon ko'rinishda. Demak, u cheksiz ko'p yechimga ega. Bunda $k$ning tayin qiymatida sistema xususiy yechimga ega bo'ladi.

Masalan, $k=1$ da $x_1=-7,\ x_2=6,\ x_3=-1,\ x_4=1$.

Agar sistema n ta noma'lumdan va n ta chiziqli tenglamadan iborat hamda xosmas bo'lsa, u holda bu sistemaning yechimi LU yoyish asosida quyidagi formulalar bilan topiladi:

$$y_i=b_i-\sum_{j=1}^{i-1}l_{ij}y_j,\ i=1,2,\dots,n, \tag{4.9}$$

$$x_i=\frac{1}{u_{ii}}\left(y_i-\sum_{j=i+1}^{n}u_{ij}x_j\right),\ i=n,n-1,\dots,1. \tag{4.10}$$

4-misol. $\begin{cases} x_1+3x_2-\ x_3=0, \\ 3x_1-\ x_2+2x_3=5, \\ 2x_1-4x_2+\ x_3=7 \end{cases}$ tenglamalar sistemasini LU yoyish asosida yeching.

Yechish. Avval L va U matritsalarning elementlarini topamiz. Ularni (3.1) va (3.2) formulalar bilan aniqlaymiz:

$$u_{11}=a_{11}=1,\qquad u_{12}=a_{12}=3,\qquad u_{13}=a_{13}=-1,$$

$$l_{21}=\frac{a_{21}}{u_{11}}=\frac{3}{1}=3,\qquad l_{31}=\frac{a_{31}}{u_{11}}=\frac{2}{1}=2,$$

$$u_{22}=a_{22}-l_{21}u_{12}=-1-3\cdot3=-10,\qquad u_{23}=a_{23}-l_{21}u_{13}=2-3\cdot(-1)=5,$$

$$l_{32}=\frac{a_{32}-l_{31}u_{12}}{u_{22}}=\frac{-4-2\cdot3}{-10}=1,$$

$$u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}=1-2\cdot(-1)-1\cdot5=-2.$$

Endi y va x larni (4.9) va (4.10) formulalar bilan topamiz:

$$y_1=b_1=0,\quad y_2=b_2-l_{21}y_1=5-3\cdot0=5,\quad y_3=b_3-l_{31}y_1-l_{32}y_2=7-3\cdot0-1\cdot5=2;$$

$$x_3=\frac{1}{u_{33}}y_3=\frac{2}{-2}=-1,\qquad x_2=\frac{1}{u_{22}}(y_2-u_{23}x_3)=\frac{5-5\cdot(-1)}{-10}=-1,$$

$$x_1=\frac{1}{u_{11}}(y_1-u_{12}x_2-u_{13}x_3)=\frac{0-3\cdot(-1)-(-1)\cdot(-1)}{1}=2.$$

Demak, $x_1=2,\quad x_2=-1,\quad x_3=-1$.

1.4.3. n noma'lumli m ta chiziqli tenglamalar sistemasini tekshirish va yechish

n ta noma'lumdan va m ta chiziqli tenglamadan iborat (4.1) sistemani qaraymiz. Bu sistemaning matritsasi A, kengaytirilgan matritsasi C bo'lsin. Bu matritsalar mos ravishda (4.2) va (4.3) tengliklar bilan ifodalanadi. Quyida (4.1) sistema birgalikda bo'lishining zarur va yetarli shartlarini aniqlovchi teorema bilan tanishamiz. Bu teorema rus va o'zbek tilida yozilgan adabiyotlarda Kroniker-Kapelli teoremasi deb yuritiladi.

1-teorema. (4.1) tenglamalar sistemasi birgalikda bo'lishi uchun sistema asosiy va kengaytirilgan matritsalarining ranglari teng, ya'ni $r(A)=r(C)$ bo'lishi zarur va yetarli.

Isboti. Zarurligi. (4.1) sistema birgalikda bo'lsin. Bu noma'lumlarning qandaydir tartiblangan $x_1^o, x_2^o, \dots, x_n^o$ qiymatlari sistema tenglamalarini ayniyatga aylantirishini anglatadi. C matritsa ustida quyidagi almashtirishlarni bajaramiz: uning oxirgi ustuniga $(-x_1^o)$ ga ko'paytirilgan birinchi ustunni qo'shamiz, keyin $(-x_2^o)$ ga ko'paytirilgan ikkinchi ustunni qo'shamiz va shu kabi davom ettirib, oxirida $(-x_n^o)$ ga ko'paytirilgan n- ustunni qo'shamiz. Natijada, sistema yechimining ta'rifiga ko'ra, C matritsa oxirgi ustun elementlari nollarga aylangan quyidagi C_1 matritsaga o'tadi:

$$C \sim C_1 = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right)$$

Elementar almashtirishlar matritsaning rangini o'zgartirmaydi, ya'ni $r(C)=r(C_1)$ bo'ladi. C_1 matritsaning oxirgi ustuni nollardan iborat bo'lgani uchun $r(C_1)=r(A)$. Bundan $r(A)=r(C)$ kelib chiqadi.

Yetarliligi. $r(A)=r(C)=r$ bo'lsin. (4.1) sistema birgalikda ekanini ko'rsatamiz. A matritsaning noldan farqli r-tartibli minorini sistema tenglamalari va noma'lumlarining o'rinlarini almashtirish orqali chap yuqori burchakda joylashtirish mumkin. Bu minor C matritsa uchun ham minor bo'ladi. Shu sababli noma'lumlarning biror $x_1^o, x_2^o, \dots, x_n^o$ qiymatlari dastlabki r ta tenglamani va shu bilan birga qolgan k-tenglamani ham qanoatlantiradi, bu yerda $k>r$. U holda (4.1) sistemada oxirgi

$m-r$ ta tenglamani tashlab yuborish va uni

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = b_r \end{cases} \tag{4.11}$$

sistema bilan almashtirish mumkin. Bunda ikki hol bo'lishi mumkin: $r=n$ yoki $r<n$ ($r>n$ bo'lmaydi, chunki A matritsa jami n ta ustunga ega).

$r=n$ bo'lganda (4.11) sistemaning noma'lumlari soni tenglamalari soniga teng bo'ladi. Bundan tashqari, sistema determinanti nolga teng emas. Shu sababli (4.11) sistema yagona yechimga ega bo'ladi. Bundan bu sistemaga ekvivalent (4.1) sistemaning birgalikda va aniq sistema bo'lishi kelib chiqadi.

$r<n$ da (4.12) sistemani quyidagi ko'rinishda yozish mumkin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r = b_1 - a_{1r+1}x_{r+1} - \ldots - a_{1n}x_n, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 - a_{2r+1}x_{r+1} - \ldots - a_{2n}x_n, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rr}x_r = b_r - a_{rr+1}x_{r+1} - \ldots - a_{rn}x_n. \end{cases} \tag{4.12}$$

Bu sistemaning koeffitsiyentlaridan tuzilgan determinant nolga teng emas. U holda $x_{r+1}, x_{r+2}, \ldots, x_n$ noma'lumlarga biror qiymatlar berib, (4.12) sistemaning $x_1, x_2, \ldots, x_r$ yechimini topsa bo'ladi.

$x_{r+1}, x_{r+2}, \ldots, x_n$ noma'lumlarga istalgan qiymatlar berish mumkin. Shu sababli (4.12) sistema cheksiz ko'p yechimga ega bo'ladi va bu sistemaga ekvivalent (4.11) sistema birgalikda va aniqmas sistema bo'ladi. Teorema to'liq isbotlandi.

Isbotlangan teoremadan quyidagi natijalar kelib chiqadi.

1-natija. Agar birgalikda bo'lgan sistema matritsasining rangi noma'lumlar soniga teng bo'lsa, sistema yagona yechimga ega bo'ladi.

2-natija. Agar birgalikda bo'lgan sistema matritsasining rangi noma'lumlar sonidan kichik bo'lsa, sistema cheksiz ko'p yechimga ega bo'ladi.

(4.1) sistema birgalikda bo'lishining zarur va yetarli shartiga va bu sistemani yechishning Gauss usuliga asosan *n noma'lumli m ta chiziqli tenglamalar sistemasini tekshirish va yechish* quyidagi tartibda amalga oshiriladi.

Tekshirish (*Gauss usuli*): sistemaning kengaytirilgan matritsasi (mos ravishda asosiy matritsasi) elementar almashtirishlar yordamida pog'onasimon ko'rinishga keltiriladi (pog'onasimon matritsa – bu pog'onasimon sistemaning matritsasi).

Bunda:

– agar $r(A) \neq r(C)$ bo'lsa, sistema birgalikda bo'lmaydi;

– agar $r(A) = r(C) = n$, ya'ni sistemaning rangi uning noma'lumlari soniga teng bo'lsa, sistema birgalikda va aniq bo'ladi;

– agar $r(A) = r(C) < n$, ya'ni sistemaning rangi uning noma'lumlari sonidan kichik bo'lsa, sistema birgalikda va aniqmas bo'ladi.

Yechish: Matritsaning rangi va noma'lumlari soni taqqoslanadi.

Bunda:

– agar $r(A) = r(C) = n$ bo'lsa, ya'ni sistema bazis minorining tartibi noma'lumlar soni bilan ustma-ust tushsa, n no'malumli n ta chiziqli xosmas tenglamalar sistemasi yechiladi;

– agar $r(A) = r(C) = r < n$ bo'lsa, sistemada $(n-r)$ ta erkin noma'lumlar hosil qilinadi. Bunda $x_1, x_2, ..., x_r$ – *bazis* (*asosiy*) *noma'lumlar*, $x_{r+1}, x_{r+2}, ..., x_n$ – *erkin noma'lumlar* bo'ladi.

Erkin noma'lumlarni sistema tenglamalarining o'ng tomoniga o'tkazamiz:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1r}x_r = b_1 - a_{1r+1}x_{r+1} - ... - a_{1n}x_n, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 - a_{2r+1}x_{r+1} - ... - a_{2n}x_n, \\ ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \\ a_{r1}x_1 + a_{r2}x_2 + ... + a_{rr}x_r = b_r - a_{rr+1}x_{r+1} - ... - a_{rn}x_n \end{cases} \tag{4.13}$$

yoki matritsa ko'rinishida

$$\widetilde{A}\widetilde{X}_r = \widetilde{B}_0 + x_{r+1}\widetilde{B}_1 + ... + x_n\widetilde{B}_{n-r}, \tag{4.14}$$

bu yerda

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{r2} & \cdots & a_{rr} \end{pmatrix}, \ \det \widetilde{A} \neq 0,$$

$$\widetilde{X}_r = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_r \end{pmatrix}, \ \widetilde{B}_0 = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_r \end{pmatrix}, \ \widetilde{B}_1 = -\begin{pmatrix} a_{1,r+1} \\ a_{2,r+1} \\ \cdots \\ a_{r,r+1} \end{pmatrix}, ..., \ \widetilde{B}_{n-r} = -\begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \cdots \\ a_{r,n} \end{pmatrix}.$$

(4.14) sistema berilgan (4.1) sistemaga ekvivalent va uning yechimi yuqorida qaralgan usullardan istalgan biri bilan topiladi.

Erkin yechimlar $x_{r+1}=c_1,\ x_{r+2}=c_2,...,x_n=c_{n-r}$ qiymatlar qabul qilsin.

U holda (4.14) sistema

$$\widetilde{A}\widetilde{X}_r=\widetilde{B}_0+c_1\widetilde{B}_1+...+c_{n-r}\widetilde{B}_{n-r} \qquad (4.15)$$

ko'rinishni oladi va $x_1,x_2,...,x_r$ bazis noma'lumlar $c_1,\ c_2,...,c_{n-r}$ qiymatlar orqali ma'lum ko'rinishda ifodalanadi:

$$x_i=x_i(c_1,c_2,...,c_{n-r}),\ i=1,2,...,r.$$

Bir jinsli bo'lmagan $AX=B$ sistemaning yechimini

$$X=\begin{pmatrix} x_1(c_1,c_2,...,c_{n-r}) \\ .... \\ x_r(c_1,c_2,...,c_{n-r}) \\ c_1 \\ ... \\ c_{n-r} \end{pmatrix} \qquad (4.16)$$

ustun matritsa ko'rinishda yozish mumkin.

Erkin noma'lumlar istalgan son qiymatini qabul qilishi mumkin. Shu sababli (4.1) sistema cheksiz ko'p yechimga ega bo'ladi.

(4.16) ifoda (4.1) sistemaning umumiy yechimini aniqlaydi. Bundan o'zgarmaslarning tayin qiymatlariga sistemaning xususiy yechimlari mos keladi.

5-misol. $\begin{cases} x_1+2x_2-4x_3=0, \\ 5x_1+3x_2-7x_3=8, \\ 5x_1-4x_2+6x_3=-1 \end{cases}$ tenglamalar sistemasini tekshiring va yeching.

Yechish. Tekshirish. Sistemaning kengaytirilgan matritsasi ustida elementar almashtirishlar bajaramiz:

$$C=\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 5 & 3 & -7 & 8 \\ 5 & -4 & 6 & -1 \end{array}\right)\begin{array}{l} \\ r_2\to r_2+(-5)r_1 \\ r_3\to r_3+(-5)r_1 \end{array}\sim\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -7 & 13 & 8 \\ 0 & -14 & 26 & -1 \end{array}\right)\begin{array}{l} \\ \\ r_3\to r_3+(-2)r_2 \end{array}\sim$$

$$\sim\left(\begin{array}{ccc|c}1 & 2 & -4 & 0\\ 0 & -7 & 13 & 8\\ 0 & 0 & 0 & -17\end{array}\right).$$

$r(A)=2\neq 3=r(C)$. Demak, sistema birgalikda emas.

6-misol. $\begin{cases} x_1-4x_2+2x_3 \qquad =-1,\\ 2x_1-3x_2-\ x_3-5x_4=-7,\\ 3x_1-7x_2+\ x_3-5x_4=-8\end{cases}$ tenglamalar sistemasini tekshiring va yeching.

Yechish. Tekshirish. Sistemaning kengaytirilgan matritsasi ustida elementar almashtirishlar bajaramiz:

$$C=\left(\begin{array}{cccc|c}1 & -4 & 2 & 0 & -1\\ 2 & -3 & -1 & -5 & -7\\ 3 & -7 & 1 & -5 & -8\end{array}\right)\begin{array}{l}\\ r_2\to r_2+(-2)r_1\\ r_3\to r_3+(-3)r_1\end{array}\sim$$

$$\sim\left(\begin{array}{cccc|c}1 & -4 & 2 & 0 & -1\\ 0 & 5 & -5 & -5 & -5\\ 0 & 5 & -5 & -5 & -5\end{array}\right)_{r_3\to r_3+(-1)r_2}\sim\left(\begin{array}{cccc|c}1 & -4 & 2 & 0 & -1\\ 0 & 5 & -5 & -5 & -5\\ 0 & 0 & 0 & 0 & 0\end{array}\right)r_2\to r_2:5\sim$$

$$\sim\left(\begin{array}{cccc|c}1 & -4 & 2 & 0 & -1\\ 0 & 1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0 & 0\end{array}\right)\begin{array}{l}c_2\to c_2+4c_1\\ c_3\to c_3+(-2)c_1\\ c_5\to c_5+c_1\end{array}\sim\left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0 & 0\end{array}\right)\begin{array}{l}c_3\to c_3+c_2\\ c_4\to c_4+c_2\\ c_5\to c_5+c_2\end{array}\sim$$

$$\sim\left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right).$$

Yuqorida keltirilgan $c_i\to c_i+\lambda c_k$ belgilash i-ustun bu ustunga λ songa koʻpaytirilgan k-ustunni qoʻshish natijasida hosil qilinganini bildiradi.

$r(A)=2=2=r(C)<4$. Demak, sistema birgalikda va aniqmas.

Yechish. x_1 va x_2 nomaʼlumlarni bazis deb olamiz va ekvivalent sistemani yozamiz:

$$\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} x_3 - \begin{pmatrix} 0 \\ -5 \end{pmatrix} x_4 .$$

$x_3 = c_1$ va $x_4 = c_2$ deb berilgan sistemaning umumiy yechimini topamiz:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5 + 2x_3 + 4x_4 \\ -1 + x_3 + x_4 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

bu yerda c_1, c_2 – ixtiyoriy o‘zgarmaslar.

1.4.4. Xosmas tenglamalar sistemasini yechish

n ta noma’lumli va n ta chiziqli tenglamadan iborat (4.6) xosmas chiziqli tenglamalar sistemasi berilgan bo‘lsin.

Xosmas chiziqli tenglamalar sistemasini yechishning ikkita usulini qaraymiz.

Matritsalar usuli

Bu usul (4.6) sistemaning ushbu

$$AX = B \tag{4.17}$$

matritsa ko‘rinishini yechishga asoslanadi.

A matritsa xosmas bo‘lgani uchun A^{-1} mavjud bo‘ladi.

(4.17) tenglikning har ikkala qismini chapdan A^{-1} ga ko‘paytiramiz:

$$A^{-1}AX = A^{-1}B, \quad IX = A^{-1}B.$$

Bundan

$$X = A^{-1}B \tag{4.18}$$

kelib chiqadi.

(4.18) tenglamaga *chiziqli tenglamalar sistemasini matritsalar usuli bilan yechish formulasi* deyiladi.

7-misol. $\begin{cases} x_1 - x_2 + x_3 = 5, \\ 2x_1 + x_2 + x_3 = 6, \\ x_1 + x_2 + 2x_3 = 4 \end{cases}$ tenglamalar sistemasini matritsalar usuli bilan yeching.

Yechish. Sistemaning determinantini hisoblaymiz:

$$\det A=\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & 1\\ 1 & 1 & 2\end{vmatrix}=2-1+2-1+4-1=5\neq 0.$$

Demak, sistema – xosmas.

Determinant elementlarining algebraik to'ldiruvchilarini aniqlaymiz:

$$A_{11}=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1,\qquad A_{12}=-\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=-3,\qquad A_{13}=\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}=1,$$

$$A_{21}=-\begin{vmatrix}-1 & 1\\ 1 & 2\end{vmatrix}=3,\qquad A_{22}=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1,\qquad A_{23}=-\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=-2,$$

$$A_{31}=\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}=-2,\qquad A_{32}=-\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}=1,\qquad A_{33}=\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}=3.$$

U holda

$$A^{-1}=\frac{1}{5}\begin{pmatrix}1 & 3 & -2\\ -3 & 1 & 1\\ 1 & -2 & 3\end{pmatrix}.$$

Sistemaning yechimini (4.18) formula bilan topamiz:

$$X=A^{-1}B=\frac{1}{5}\begin{pmatrix}1 & 3 & -2\\ -3 & 1 & 1\\ 1 & -2 & 3\end{pmatrix}\cdot\begin{pmatrix}5\\ 6\\ 4\end{pmatrix}=\frac{1}{5}\begin{pmatrix}5+18-8\\ -15+6+4\\ 5-12+12\end{pmatrix}=\frac{1}{5}\begin{pmatrix}15\\ -5\\ 5\end{pmatrix}=\begin{pmatrix}3\\ -1\\ 1\end{pmatrix}.$$

Demak, $x_1=3$, $x_2=-1$, $x_3=1$.

Determinantlar usuli yoki Kramer formulalari

Chiziqli tenglamalar sistemasini yechishning bu usuli determinantlar nazariyasiga asoslanadi.

2-teorema. Agar (4.6) sistema xosmas bo'lsa, u holda sistema yagona yechimga ega bo'ladi va bu yechim quyidagi formulalar bilan topiladi:

$$x_1=\frac{D_1}{D},\ x_2=\frac{D_2}{D},\ldots,\ x_n=\frac{Dx_n}{D}, \tag{4.19}$$

bu yerda $D_1, D_2, \ldots, D_n$ determinantlar $D = \det A$ determinantdan mos noma'lum oldidagi koeffitsiyentlarni ozod hadlar bilan almashtirish orqali hosil qilinadi.

Isboti. (4.6) sistemaning matritsa ko'rinishidagi yechimini, ya'ni (4.18) formulani quyidagi ko'rinishda yozib olamiz:

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \frac{1}{D}\begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}$$

yoki

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} \dfrac{A_{11}b_1 + A_{21}b_2 + \ldots + A_{n1}b_n}{D} \\ \dfrac{A_{12}b_1 + A_{22}b_2 + \ldots + A_{n2}b_2}{D} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{A_{1n}b_1 + A_{2n}b_2 + \ldots + A_{nn}b_n}{D} \end{pmatrix}.$$

Bundan

$$x_1 = \frac{A_{11}b_1 + A_{21}b_2 + \ldots + A_{n1}b_n}{D},$$

$$x_2 = \frac{A_{12}b_1 + A_{22}b_2 + \ldots + A_{n2}b_n}{D},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_n = D\frac{A_{1n}b_1 + A_{2n}b_n + \ldots + A_{nn}b_n}{\Delta}$$

kelib chiqadi.

Ikkinchidan, $A_{11}b_1 + A_{21}b_2 + \ldots + A_{n1}b_n$ ifoda

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ b_2 & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_n & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

determinantning 1- ustun elementlari bo'yicha yoyilmasiga, ya'ni D_1 ga teng.

Demak,

$$x_1 = \frac{D_1}{D}.$$

(4.19) tenglikning qolgan formulalari shu kabi isbotlanadi:

$$x_2 = \frac{D_2}{D}, \dots, \; x_n = \frac{Dx_n}{D}.$$

$x_i = \dfrac{D_i}{D}$ $(i = \overline{1,n})$ formulalarga *Kramer formulalari* deyiladi.

8-misol. $\begin{cases} 2x_1 + 3x_2 + 2x_3 = 9, \\ x_1 + 2x_2 - 3x_3 = 14, \\ 3x_1 + 4x_2 + x_3 = 16 \end{cases}$ tenglamalar sistemasini Kramer formulalari bilan yeching.

Yechish. D va D_i, $i = 1,2,3$ determinantlarni hisoblaymiz:

$$D = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & -3 \\ 3 & 4 & 1 \end{vmatrix} = -6 \neq 0, \qquad D_1 = \begin{vmatrix} 9 & 3 & 2 \\ 14 & 2 & -3 \\ 16 & 4 & 1 \end{vmatrix} = -12;$$

$$D_2 = \begin{vmatrix} 2 & 9 & 2 \\ 1 & 14 & -3 \\ 3 & 16 & 1 \end{vmatrix} = -18; \qquad D_3 = \begin{vmatrix} 2 & 3 & 9 \\ 1 & 2 & 14 \\ 3 & 4 & 16 \end{vmatrix} = 12.$$

Kramer formulalari bilan topamiz:

$$x_1 = \frac{D_1}{D} = \frac{-12}{-6} = 2, \quad x_2 = \frac{D_2}{D} = \frac{-18}{-6} = 3, \quad x_3 = \frac{D_3}{D} = \frac{12}{-6} = -2.$$

Shunday qilib, n noma'lumli n ta chiziqli xosmas tenglamalar sistemasi yagona yechimga ega bo'ladi va bu yechimni yoki (4.18) matritsalar usuli bilan yoki (4.19) Kramer formulalari bilan topish mumkin.

1.4.5. Bir jinsli chiziqli tenglamalar sistemasi

Ozod hadlari nolga teng bo'lgan ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases} \tag{4.20}$$

sistemaga *bir jinsli tenglamalar sistemasi* deyiladi.

(4.20) sistema asosiy va kengaytirilgan matritsalarining ranglari teng bo‘ladi. Shu sababli bu sistema hamma vaqt birgalikda va nolga teng bo‘lgan (trivial) $x_1 = x_2 = \ldots, = x_n = 0$ yechimga ega.

Bir jinsli tenglamalar sistemasi qanday shartlar bajarilganida nolga teng bo‘lmagan yechimga ega bo‘ladi? Bu savolga quyidagi teoremalar javob beradi.

3-teorema. (4.20) tenglamalar sistemasi nolga teng bo‘lmagan yechimga ega bo‘lishi uchun sistema matrisasining rangi sistema tenglamalari sonidan kichik bo‘lishi, ya’ni $r < n$ bo‘lishi zarur va yetarli.

Isboti. Zarurligi. Ma‘lumki, $r \leq n$. $r = n$ bo‘lsin deymiz. U holda Kramer formulasidagi $D \neq 0$ va barcha $D_i = 0$ bo‘ladi. Shu sababli tenglamalar sistemasi yagona yechimga ega bo‘ladi:

$$x_i = \frac{D_i}{D} = 0, \; D_i = 0, \; D \neq 0.$$

Demak, (4.20) sistemaning trival yechimlardan boshqa yechimlari yo‘q. Agar ular bor bo‘lsa, u holda $r < n$ bo‘ladi.

Yetaliligi. $r < n$ bo‘lsin. U holda birgalikda bo‘lgan bir jinsli sistema aniqmas bo‘ladi. Demak, sistema cheksiz ko‘p yechimlarga ega bo‘ladi, bunda ulardan ayrimlari nolga teng bo‘lmaydi.

4-teorema. n noma’lumli n ta chiziqli bir jinsli tenglamalar sistemasi nolga teng bo‘lmagan yechimga ega bo‘lishi uchun sistema matrisasining determinanti nolga teng bo‘lishi, ya’ni $\det A = 0$ bo‘lishi zarur va yetarli.

Isboti. Zarurligi. Agar $\det A \neq 0$ (yoki $r = n$) bo‘lsa, sistema yagona trival yechimga ega bo‘ladi. Demak, sistema nolga teng bo‘lmagan yechimga ega bo‘lsa, sistemaning determinanti $\det A = 0$ bo‘ladi.

Yetaliligi. Agar $\det A = 0$ bo‘lsa, $r < n$ bo‘ladi. U holda 3-teoremaga ko‘ra, sistema cheksiz ko‘p yechimlarga ega bo‘ladi, bunda ulardan ayrimlari nolga teng bo‘lmaydi.

5-teorema. Agar X_1 va X_2 ustun matritsalar (4.20) sistemaning yechimlari bo‘lsa, u holda bu yechimlarning chiziqli kombinatsiyasi

$$c_1 X_1 + c_2 X_2$$

ham (4.20) sistemaning yechimi bo‘ladi, bu yerda c_1, c_2 – istalgan haqiqiy sonlar.

Isboti. Teoremaning shartiga ko‘ra,

$$AX_1 = 0 \quad \text{va} \quad AX_2 = 0.$$

U holda istalgan c_1 va c_2 sonlari uchun

$$c_1AX_1=0 \text{ dan } A(c_1X_1)=0,$$

$$c_2AX_2=0 \text{ dan } A(c_2X_2)=0$$

bo'ladi. Bu tengliklarni qo'shamiz:

$$A(c_1X_1)+A(c_2X_2)=0$$

yoki

$$A(c_1X_1+c_2X_2)=0.$$

Demak, X_1 va X_2 yechimlarning $c_1X_1+c_2X_2$ chiziqli kombinatsiyasi (4.20) sistemaning yechimi bo'ladi. Bunda X_1 va X_2 yechimlar *fundamental sistema tashkil qiladi* deyiladi.

9-misol. $\begin{cases} -x_1+x_2-\ x_3=0, \\ 3x_1-x_2-\ x_3=0, \\ 2x_1+x_2-2x_3=0 \end{cases}$ bir jinsli tenglamalar sistemasini yeching.

Yechish. Sistema determinantini hisoblaymiz:

$$\det A=\begin{vmatrix} -1 & 1 & -1 \\ 3 & -1 & -1 \\ 2 & 1 & -2 \end{vmatrix}=-2-2-3-2-1+6=-3\neq 0.$$

Demak, sistema sistema trivial yechimga ega: $x_1=x_2=x_3=0$.

10-misol. $\begin{cases} x_1-\ x_2-\ x_3+\ x_4=0, \\ 2x_1-2x_2+\ x_3+\ x_4=0, \\ 5x_1-5x_2-2x_3+4x_4=0 \end{cases}$ bir jinsli tenglamalar sistemasini yeching.

Yechish. Sistema matritsasini pog'onasimon ko'rinishga keltiramiz:

$$A=\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & -2 & 1 & 1 \\ 5 & -5 & -2 & 4 \end{pmatrix}\begin{matrix} \\ r_2\to r_2+(-2)r_1 \\ r_3\to r_3+(-5)r_1 \end{matrix}\sim\begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & -1 \end{pmatrix}\begin{matrix} \\ \\ r_3\to r_3+(-1)r_2 \end{matrix}\sim$$

$$\sim\begin{pmatrix}1 & -1 & -1 & 1\\ 0 & 0 & 3 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{matrix}c_2\to c_2+c_1\\ c_3\to c_3+c_1\\ c_4\to c_4+(-1)c_1\end{matrix}\sim\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 3 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{matrix}c_2\to c_4\\ c_4\to c_2\end{matrix}\sim$$

$$\sim\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 3 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}c_3\to c_3+3c_2\sim\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}c_2\to c_2:(-1)\sim\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

$r(A)=2,\ n=4,\ r<n.$

Demak, sistema nolga teng bo'lmagan yechimga ega. Bunda yechimlardan ikkitasi bazis o'zgaruvchilar, qolgan ikkita yechim erkin o'zgaruvchilar bo'ladi.

$x_2=c_1,\ x_3=c_2$ deymiz va

$$\begin{cases}x_1-c_1-c_2+x_4=0,\\ \qquad 3c_2-x_4=0\end{cases}$$

sistemaga ega bo'lamiz.

Bu sistema $x_4=3c_2,\ x_1=c_1-2c_2$ yechimga ega.

Berilgan sistemaning umumiy yechimini topamiz:

$$X=\begin{pmatrix}c_1-2c_2\\ c_1\\ c_2\\ 3c_2\end{pmatrix}$$

Bu yechimni xususiy yechimlarning chiziqli kombinatsiyasi ko'rinishiga keltiramiz:

$$X=\begin{pmatrix}c_1\\ c_1\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}-2c_2\\ 0\\ c_2\\ 3c_2\end{pmatrix}=c_1\begin{pmatrix}1\\ 1\\ 0\\ 0\end{pmatrix}+c_2\begin{pmatrix}-2\\ 0\\ 1\\ 3\end{pmatrix}.$$

Bunda $X_1=\begin{pmatrix}1\\ 1\\ 0\\ 0\end{pmatrix}$ va $X_2=\begin{pmatrix}-2\\ 0\\ 1\\ 3\end{pmatrix}$ xususiy yechimlar yechimlarning fundamental sistemasini tashkil qiladi.

1.4.6. Chiziqli tenglamalar sistemasini matematik paketlarda yechish

Kompyuterli matematika sistemalari (KMS) yoki matematik paketlar matematikaning turli masalalarini yechishda keng qo‘llaniladi. Hozirgi vaqtda matematik paketlarning turli variantlari yaratilgan: *Maple, MathCAD, MatLAB, Mathematica, Direve.*

Chiziqli tenglamalar sistemasini bu paketlardan birinchisida, ya’ni *Maple* matematik paketida yechishni qisqa bayon qilamiz. Batafsil bayon maxsus kurslarda beriladi.

$Ax = b$ tenglamalar sistemasi Maple paketida ikki usuldan biri bilan yechilishi mumkin.

1-usul:* solve *buyrug‘i bilan.

Bu buyruq bilan (4.1) ko‘rinishda berilgan chiziqli tenglamalar sistema yechiladi.

11-misol. $\begin{cases} 2x+6y+5z = 0, \\ 2x+5y+6z = -4, \\ 5x+7y+8z = -7 \end{cases}$ tenglamalar sistemasini yeching.

Yechish.

> eq:={2*x + 6*y + 5*z = 0, 2*x + 5*y + 6*z =-4, 5*x + 7*y +8*z =-7}:

> solve {eq, {x,y,z}};

{x=-1,y=2,z=-2}

12-misol. $\begin{cases} x-7y+9z = -6, \\ 2x-3y+5z = -3, \\ 3x+y+z = 0 \end{cases}$ tenglamalar sistemasining umumiy va bitta xususiy yechimini toping.

Yechish.

> eq:={x -7*y + 9*z =-6, 2*x -3*y + 5*z =-3, 3*x +y+z =0}:

> s:=solve {eq, {x,y,z}};

$$s := \left\{x = -\frac{3}{11} - \frac{8}{11}z, y = \frac{9}{11} + \frac{13}{11}z, z = z\right\}.$$

Sistemaning xususiy yechimini topish uchun z o‘zgaruvchiga **subs** buyrug‘i bilan tayin qiymat berish kerak:

> subs {z=1,s}

{x=-1,y=2,z=1}

2-usul: **linsolve(A,b)** ***buyrug'i bilan.***

Bu buyruq bilan **linalg** paketidan $Ax = b$ tenglamaning yechimi topiladi. Bunda buyruqning argumenti: **A** -matritsa; **b** - vektor.

13-misol. $\begin{cases} 2x+\ 7y+13z=\ 0, \\ 3x+14y+12z=18, \\ 5x+25y+16z=39 \end{cases}$ tenglamalar sistemasini Gauss usuli bilan, Kramer formulalari bilan matritsalar usuli bilan yeching.

Yechish.

1) Sistemani Gauss usuli bilan yechamiz:

```
> with(LinearAlgebra):
A := <<2,3,5>|<7,14,25>|<13,12,16>>;
```

$$A := \begin{bmatrix} 2 & 7 & 13 \\ 3 & 14 & 12 \\ 5 & 25 & 16 \end{bmatrix}$$

```
> b := <0,18,39>;
```

$$b = \begin{bmatrix} 0 \\ 18 \\ 39 \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} 2 & 7 & 13 \\ 0 & \frac{7}{3} & -\frac{15}{3} \\ 0 & 0 & -\frac{3}{7} \end{bmatrix}$$

```
> GaussianElimination(A,'method'='FractionFree');
```

$$\begin{bmatrix} 2 & 7 & 13 \\ 0 & 7 & -15 \\ 0 & 0 & -3 \end{bmatrix}$$

```
>ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Demak,

$$x := -4 \quad y := 3 \quad z := -1$$

2) Sistemani Kramer formulalari bilan yechamiz:

```
> with(Student[LinearAlgebra]):
> d := <<2,3,5>|<7,14,25>|<13,12,16>>;
```

$$d = \begin{vmatrix} 2 & 7 & 13 \\ 3 & 14 & 12 \\ 5 & 25 & 16 \end{vmatrix}$$

```
> d:=Determinant(d);
```

$$d := -3$$

```
> dx1:=<<0,18,39>|<7,14,25>|<13,12,16>>;
```

$$dx1 = \begin{vmatrix} 0 & 7 & 13 \\ 18 & 14 & 12 \\ 39 & 25 & 16 \end{vmatrix}$$

```
> d1:=Determinant(dx1);
```

$$d1 := 12$$

```
> dx2 := <<2,3,5>|<0,18,39>|<13,12,16>>;
```

$$dx2 = \begin{vmatrix} 2 & 0 & 13 \\ 3 & 18 & 12 \\ 5 & 39 & 16 \end{vmatrix}$$

```
> d2:=Determinant(dx2);
```

$$d2 := -9$$

```
> dx3 := <<2,3,5>|<7,14,25>|<0,18,39>>;
```

$$dx3 = \begin{vmatrix} 2 & 7 & 0 \\ 3 & 14 & 18 \\ 5 & 25 & 39 \end{vmatrix}$$

```
> d3:=Determinant(dx3);
```

$$d3 := 3$$

```
> x:=d1/d;y:=d2/d;z:=d3/d;
```

$$x := -4 \quad y := 3 \quad z := -1$$

3) Sistemani matritsalar usuli bilan yechamiz:

```
> with(Student[LinearAlgebra]):
> A := <<2,3,5>|<7,14,25>|<13,12,16>>;
```

$$A = \begin{pmatrix} 2 & 7 & 18 \\ 3 & 14 & 12 \\ 5 & 25 & 16 \end{pmatrix}$$

> **A^(-1);**

$$\begin{pmatrix} \frac{76}{3} & -\frac{213}{3} & -\frac{98}{3} \\ -\frac{12}{3} & -\frac{33}{3} & -\frac{15}{3} \\ -\frac{5}{3} & \frac{15}{3} & -\frac{7}{3} \end{pmatrix}$$

> **B := <<0,18,39>>;**

$$B := \begin{pmatrix} 0 \\ 18 \\ 39 \end{pmatrix}$$

> **X:=A^(-1).B;**

$$X := \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}$$

linsolve(A,b) buyrug'ining argumenti sifatida mos ravishda **A** va **B** matritsalar olinsa, bu buyruq bilan $AX = B$ matritsaviy tenglama yechimini ham topish mumkin.

14-misol. $\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix} X = \begin{pmatrix} 13 & -4 & 6 \\ 2 & 4 & 2 \\ -2 & 5 & 5 \end{pmatrix}$ matritsaviy tenglamani yeching.

Yechish.

> **with(Student[LinearAlgebra]):**

> **A := <<1,0,1>|<3,1,-1>|<-1,-2,0>>;**

$$A := \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

> **A^{-1}:=A^(-1);**

$$A^{-1} := \begin{pmatrix} \frac{2}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{1}{7} & -\frac{2}{7} \\ \frac{1}{7} & -\frac{4}{7} & -\frac{1}{7} \end{pmatrix}$$

> **B := <<13,2,-2>|<-4,-4,5>|<6,2,5>>;**

$$B := \begin{pmatrix} 13 & -4 & 6 \\ 2 & 4 & 2 \\ -2 & 5 & 5 \end{pmatrix}$$

> **X:=A^(-1).B;**

$$X:=\begin{pmatrix}2 & 3 & 5\\ 4 & -2 & 0\\ 1 & 1 & -1\end{pmatrix}.$$

A matritsaning yadrosi deb shunday vektorlar to'plami x ga aytiladiki, A matritsaning bu vektorlar to'plamiga ko'paytmasi nolga teng, ya'ni $Ax=0$ bo'ladi. Bunda A matritsaning yadrosini topish bir jinsli tenglamalar sistemasi yechimlarini topishga ekvivalent bo'ladi.

A matritsaning yadrosini **kernel (A)** buyrug'i bilan topish mumkin.

4.15-misol. $\begin{cases} x+\ y \quad =0,\\ \quad 2y-z=0,\\ x+3y-z=0\end{cases}$ tenglamalar sistemasini yeching.

Yechish.

> **with(linalg):**

> **A :=array ([[1,1,0],[0,2,-1],[1,3,-1]])**

$$A:=\begin{pmatrix}1 & 1 & 0\\ 0 & 2 & -1\\ 1 & 3 & -1\end{pmatrix}$$

> **kernel (A) :**

{[-1,1,2]}.

1.4.7. Mashqlar

Tenglamalar sistemasini tekshiring:

4.1. $\begin{cases} x_1-x_2+x_3=2,\\ x_1+x_2-x_3=1,\\ 5x_1-x_2+x_3=7.\end{cases}$

4.2. $\begin{cases} x_1-x_2-\ x_3=-1,\\ 5x_1-x_2+2x_3=\ 3,\\ 4x_1 \qquad +3x_3=\ 4.\end{cases}$

4.3. $\begin{cases} x_1+\ x_2+\ 5x_3+2x_4=\ 1,\\ 2x_1+\ x_2+\ 3x_3+2x_4=-3,\\ 2x_1+3x_2+11x_3+5x_4=\ 2,\\ x_1+\ x_2+\ 3x_3+4x_4=-3.\end{cases}$

4.3. $\begin{cases} x_1+x_2-\ x_3+2x_4=3,\\ 2x_1-x_2+\ x_3-\ x_4=1,\\ 3x_1+x_2+2x_3-\ x_4=5,\\ x_1-x_2+4x_3-5x_4=2.\end{cases}$

Tenglamalar sistemasini Gauss usuli bilan yeching:

4.5. $\begin{cases} 2x_1+\ x_2+3x_3=-13,\\ x_1+2x_2-\ x_3=\ -2,\\ 3x_1+\ x_2-4x_3=\ 7.\end{cases}$

4.6. $\begin{cases} 3x_1+2x_2-3x_3=-1,\\ 2x_1+\ x_2+2x_3=\ 4,\\ x_1-3x_2+\ x_3=\ 9.\end{cases}$

4.7. $\begin{cases} x_1+2x_2+x_3-2x_4=-4, \\ x_2+x_3+3x_4=1, \\ 2x_1+x_3-x_4=0, \\ 3x_1+x_2+4x_3=-2. \end{cases}$ **4.8.** $\begin{cases} 2x_1+x_2+x_4=4, \\ x_1-x_2+2x_3+2x_4=1, \\ x_2+3x_3+2x_4=-5, \\ 3x_1-x_2+2x_3=3. \end{cases}$

4.9. $\begin{cases} 2x_1+3x_2-x_3-x_4=8, \\ 3x_1+x_2-x_3+x_4=8, \\ x_1-x_2+x_3-x_4=0, \\ 3x_1+7x_2-3x_3-x_4=16. \end{cases}$ **4.10.** $\begin{cases} x_1-2x_2-3x_3+5x_4=-1, \\ 2x_1-3x_2+2x_3+5x_4=-3, \\ 5x_1-7x_2+9x_3+10x_4=-8, \\ x_1-x_2+5x_3=-2. \end{cases}$

Tenglamalar sistemasini Jordan-Gauss usuli bilan yeching:

4.11. $\begin{cases} x_1+6x_2+3x_3=21, \\ 4x_1+8x_2+x_3=18, \\ 3x_1+5x_2+4x_3=33. \end{cases}$ **4.12.** $\begin{cases} x_1+3x_2+x_3=-2, \\ 3x_1-2x_2+3x_3=5, \\ 4x_1+3x_2+5x_3=1. \end{cases}$

Tenglamalar sistemasini LU yoyish asosida yeching:

4.13. $\begin{cases} 3x_1-7x_2-2x_3=-7, \\ -3x_1+5x_2+x_3=5, \\ 6x_1-4x_2=2. \end{cases}$ **4.14.** $\begin{cases} x_1-2x_2+x_3=3, \\ -4x_1+5x_2+2x_3=0, \\ 6x_1-9x_2+x_3=6. \end{cases}$

4.15. $\begin{cases} x_1-x_2+2x_3=0, \\ x_1-3x_2+x_3=-5, \\ 3x_1+7x_2+5x_3=7. \end{cases}$ **4.16.** $\begin{cases} x_1-3x_2+2x_3=1, \\ x_1-2x_2=1, \\ 2x_2-3x_3=-3. \end{cases}$

Tenglamalar sistemasini matritsalar usuli bilan yeching:

4.17. $\begin{cases} x_1+2x_2-x_3=3, \\ 2x_1-x_2+2x_3=-1, \\ x_1+3x_2-x_3=6. \end{cases}$ **1.4.18.** $\begin{cases} 2x_1+x_2-x_3=2, \\ 2x_1+2x_2-3x_3=-3, \\ x_1+2x_2-2x_3=-5. \end{cases}$

4.19. $\begin{cases} x_1+2x_2+x_3=8, \\ x_1+2x_2+3x_3=10, \\ 2x_1-3x_2-4x_3=-4. \end{cases}$ **1.4.20.** $\begin{cases} 2x_1+7x_2-x_3=10, \\ x_1+2x_2+x_3=2, \\ 3x_1-5x_2+3x_3=-5. \end{cases}$

Tenglamalar sistemasini Kramer formulalari bilan yeching:

4.21. $\begin{cases} 3x_1-4x_2=17, \\ 5x_1+2x_2=11. \end{cases}$ **1.4.22.** $\begin{cases} 5x_1+7x_2=1, \\ 6x_1+4x_2=10. \end{cases}$

4.23. $\begin{cases} x_1+2x_2+3x_3=5, \\ 3x_1-2x_2+3x_3=-1, \\ 2x_1+3x_2-2x_3=8. \end{cases}$ **1.4.24.** $\begin{cases} 2x_1-2x_2+x_3=8, \\ x_1+3x_2+x_3=-3, \\ 3x_1+2x_2-2x_3=-5. \end{cases}$

4.25. $\begin{cases} x_1+2x_2+3x_3=6, \\ 4x_1+5x_2+6x_3=9, \\ 7x_1+8x_2=-6. \end{cases}$ **1.4.26.** $\begin{cases} ax_1+ax_2+x_3=1, \\ x_1+a^2x_2+x_3=a, \\ x_1+ax_2+ax_3=1. \end{cases}$

Bir jinsli tenglamalar sistemasini yeching:

4.27. $\begin{cases} 2x_1+3x_2+2x_3=0, \\ 3x_1-x_2+3x_3=0. \end{cases}$

1.4.28. $\begin{cases} 3x_1-x_2+4x_3=0, \\ 5x_1+3x_2+3x_3=0. \end{cases}$

4.29. $\begin{cases} 3x_1+x_2+2x_3=0, \\ x_1+2x_2-3x_3=0, \\ 5x_1+5x_2-4x_3=0. \end{cases}$

1.4.30. $\begin{cases} 2x_1+3x_2+x_3=0, \\ 3x_1-2x_2+3x_3=0, \\ 4x_1+3x_2+5x_3=0. \end{cases}$

4.31. $\begin{cases} x_1+3x_2-6x_3+2x_4=0, \\ 2x_1-x_2+2x_3=0, \\ 3x_1-2x_2+2x_3-2x_4=0, \\ 2x_1+x_2+4x_3+8x_4=0. \end{cases}$

1.4.32. $\begin{cases} x_1-x_2-2x_3+3x_4=0, \\ x_1+2x_2-4x_4=0, \\ x_1-4x_2+x_3+10x_4=0, \\ 2x_1+x_2-2x_3-x_4=0. \end{cases}$

Bir jinsli tenglamalar sistemasining fundamental yechimlari sistemasini toping:

4.33. $\begin{cases} x_1-x_2+x_3-x_4=0, \\ 2x_1+x_2+x_3+x_4=0, \\ 4x_1-x_2+3x_3-x_4=0. \end{cases}$

1.4.34. $\begin{cases} x_1+3x_2-x_3-x_4=0, \\ 2x_1-x_2+x_3=0, \\ x_1+10x_2-4x_3-3x_4=0. \end{cases}$

Tenglamalar sistemasini Gauss usuli bilan, Kramer formulalari bilan matritsalar usuli bilan Maple matematik paketida yeching:

4.35. $\begin{cases} 5x_1+8x_2-x_3=-2, \\ x_1+2x_2+3x_3=-4, \\ 2x_1-3x_2+2x_3=3. \end{cases}$

1.4.36. $\begin{cases} x_1+2x_2+x_3=0, \\ 3x_1-5x_2+3x_3=11, \\ 2x_1+7x_2-x_3=-3. \end{cases}$

2

VEKTORLI ALGEBRA ELEMENTLARI

- Vektorlar
- Vektorlarning ko'paytmalari

Uilyam Rouen Gamilton (1805–1865) – Irlandiyalik matematik, mexanik va fizik.

Gamiltonning fizika va matematika sohasidagi kashfiyotlari o'z davrida shuhratli bo'lgan va fanning bir necha o'n yillar olg'a siljishiga xizmat qilgan.

Gamiltonning ishlari vektorli algebraning rivojlanishiga va fanga vektor maydon tushunchasining kiritilishiga turtki bo'lgan.

Gamilton tomonidan kiritilgan godograf tushunchasi kinematika va dinamikada tatbiq qilinadi.

2.1. VEKTORLAR

Vektor nisbatan yangi matematik tushuncha hisoblanadi. «Vektor» terminining o'zi *1845-yilda Uilyam Rouen Gamilton* tomonidan kiritilgan.

Vektor tushunchasiga son qiymati va yo'nalishi bilan xarakterlanuvchi obyektlar bilan ish ko'rilganida duch kelinadi. Bunday obyektlarga kuch, tezlik, tezlanish kabi fizik kattaliklar misol bo'ladi.

Vektor matematikaning turli bo'limlarida, masalan, elementar, analitik va differensial geometriya bo'limlarida qo'llaniladi. Vektorli algebra fizika va mexanikaning turli bo'limlariga, kristallografiyaga, geodeziyaga tatbiq qilinadi. Vektorlarsiz nafaqat klassik matematikani, balki boshqa ko'plab fanlarni tasavvur qilib bo'lmaydi.

2.1.1. Asosiy tushunchalar

Tayin uzunlikka va yo'nalishga ega bo'lgan kesma *vektor* deb ataladi.

Vektor $\overrightarrow{AB}$ yoki $\vec{a}$ bilan belgilanadi. Bunda A nuqtaga vektorning boshi deyilsa, B nuqtaga uning oxiri deyiladi.

Boshi va oxiri orasidagi masofaga *vektorning uzunligi* yoki *moduli* deyiladi. Vektorning moduli $|\overline{AB}|$ yoki $|\bar{a}|$ ko'rinishda belgilanadi.

Boshi va oxiri ustma-ust tushadigan vektor *nol vektor* deb ataladi va $\bar{0}$ bilan belgilanadi. Bunda $|\bar{0}|=0$ bo'ladi. Nol vektor yo'nalishga ega bo'lmaydi.

Uzunligi birga teng vektorga *birlik vektor* deyiladi va ko'p hollarda $\bar{e}$ orqali belgilanadi.

$\bar{a}$ vektor bilan bir xil yo'nalgan birlik vektorga $\bar{a}$ *vektorning orti* deyiladi va $\bar{a}^{\circ}$ bilan belgilanadi.

1-ta'rif. Bir to'g'ri chiziqda yoki parallel to'g'ri chiziqlarda yotuvchi vektorlar *kollinear vektorlar* deb ataladi.

$\vec{a}$ va $\vec{b}$ vektorlarning kollinearligi $\vec{a}\|\vec{b}$ deb yoziladi. Kollinear vektorlar bir tomonga yo'nalgan (yo'nalishdosh, ular $\vec{a}\uparrow\uparrow\vec{b}$ kabi belgilanadi) yoki qarama-qarshi tomonlarga yo'nalgan (ular $\vec{a}\uparrow\downarrow\vec{b}$ kabi belgilanadi) bo'lishi mumkin. Hol vektor har qanday vektorga kollinear hisoblanadi.

2-ta'rif. Bir tekislikda yoki parallel teksliklarda yotuvchi vektorlar *komplanar vektorlar* deb ataladi.

3-ta'rif. $\bar{a}$ va $\vec{b}$ vektorlar kollinear, yo'nalishdosh va uzunliklari teng bo'lsa, ularga *teng vektorlar* deyiladi va $\vec{a}=\vec{b}$ kabi yoziladi.

Vektorlar tengligining bu ta'rifi *erkli vektorlar* deb ataluvchi vektorlarni xarakterlaydi. Bu ta'rifga asosan erkli vektorni fazoning ixtiyoriy nuqtasiga parallel ko'chirish mumkin bo'ladi.

Erkli vektorlar tushunchasidan foydalanib, vektorlarning kollinearligi va komplanarligi uchun boshqa ekvivalent ta'riflarni berish mumkin: agar ikkita nol bo'lmagan vektorlar bir nuqtaga ko'chirilganida bir to'g'ri chiziqda yotsa, bu vektorlarga kollinear vektorlar deyiladi; agar uchta nol bo'lmagan vektorlar bir nuqtaga ko'chirilganida bir tekislikda yotsa, bu vektorlarga komplanar vektorlar deyiladi.

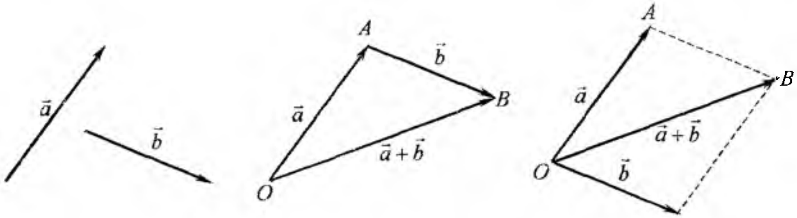
Ayrim hollarda vektorning erkli ko'chirilishi chegaralanishi mumkin. Agar vektorning qo'yilish nuqtasi qat'iy fiksirlangan bo'lsa, bu vektorga *bog'langan vektor* deyiladi. Agar vektorning boshi joylashishi mumkin bo'lgan chiziq berilgan bo'lsa, bu vektorga *sirpanuvchi vektor* deyiladi. Bog'langan va sirpanuvchi vektorlar nazariy mexanikada keng qo'llaniladi. Masalan, M nuqtaning radius vektori bog'langan vektor bo'ladi; aylanma harakatda aylanish o'qida joylashgan burchak tezlik vektori sirpanuvchi vektor bo'ladi.

2.1.2. Vektorlar ustida chiziqli amallar

Vektorlarni qo'shish, ayirish va vektorni songa ko'paytirish amallari *vektorlar ustida chiziqli amallar* hisoblanadi.

Ikkita $\vec{a}$ va $\vec{b}$ vektor berilgan bo'lsin. Istalgan O nuqta olib, bu nuqtaga $\overrightarrow{OA}=\vec{a}$ vektorni parallel ko'chiramiz. A nuqtaga $\overrightarrow{AB}=\vec{b}$ vektorni qo'yamiz. Bunda birinchi vektorning boshini ikkinchi vektorning oxiri bilan tutashtiruvchi $\overrightarrow{OB}$ vektorga *$\vec{a}$ va $\vec{b}$ vektorlarning yig'indisi* deyiladi, ya'ni $\overrightarrow{OB}=\vec{a}+\vec{b}$ (1-shakl). Vektorlarni qo'shishning bu usuli *uchburchak qoidasi* deb ataladi.

Ikkita vektorni *parallelogramm qoidasi* bilan ham qo'shish mumkin. Buning uchun O nuqtaga $\overrightarrow{OA}=\vec{a}$ va $\overrightarrow{OC}=\vec{b}$ vektorlarni qo'yamiz va ulardan parallelogramm yasaymiz. Bunda parallelogrammning O uchidan o'tkazilgan $\overrightarrow{OB}$ diagonal $\vec{a}+\vec{b}$ vektorni beradi (1-shakl).



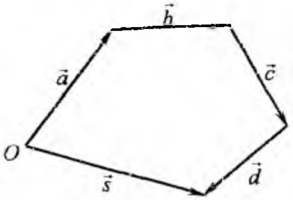
1-shakl.

Bir nechta vektornining yig'indisini topish uchun bu vektorlarga teng vektorlardan ko'pburchak (siniq chiziq) hosil qilinadi. Bunda boshi birinchi
vektorining boshida, oxiri esa oxirgi vektorining oxirida bo'lgan vektor ko'pburchak barcha vektorlarining yig'indisiga teng bo'ladi. Bir necha vektorni bunday qo'shish usuliga *ko'pburchak qoidasi* deyiladi. 2-shaklda to'rtta $\vec{a},\vec{b},\vec{c},\vec{d}$ vektorlarning yig'indisi $\vec{s}$ vektor tasvirlangan.

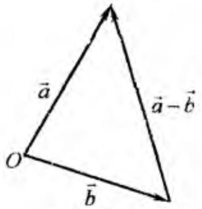
$\vec{a}$ va $\vec{b}$ *vektorlarning ayirmasi* deb, $\vec{a}$ vektor bilan $\vec{b}$ vektorga qarama-qarshi bo'lgan $(-\vec{b})$ vektor yig'indisiga aytiladi, ya'ni $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

$\vec{a}-\vec{b}$ ayirmani topish uchun $\vec{a}$ va $\vec{b}$ vektorni umumiy O nuqtaga qo'yamiz. Bunda $\vec{b}$ vektor oxiridan $\vec{a}$ vektor oxiriga yo'nalgan vektor $\vec{a}-\vec{b}$ vektorni beradi (3-shakl).

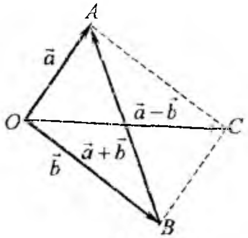
$\overrightarrow{OA}=\vec{a}$ va $\overrightarrow{OB}=\vec{b}$ vektorlarga qurilgan $OACB$ parallelogrammning diagonal vektorlari bu vektorlarning yig'indisidan va ayirmasidan iborat bo'ladi (4-shakl).



2-shakl.



3-shakl.



4-shakl.

$\vec{a}$ *vektorning* $\lambda \neq 0$ *songa ko'paytmasi* deb, $\vec{a}$ vektorga kollinear, uzunligi $|\lambda|\cdot|\vec{a}|$ ga teng va yo'nalishi $\lambda>0$ bo'lganda $\vec{a}$ vektor yo'nalishi bilan bir xil, $\lambda<0$ bo'lganda $\vec{a}$ vektor yo'nalishiga qarama-qarshi bo'lgan $\lambda\vec{a}$ vektorga aytiladi.

Vektorni songa ko'paytirishning bu ta'rifidan quyidagi xossalar kelib chiqadi:

1-xossa. $\vec{a}$ $(\vec{a}\neq 0)$ va $\vec{b}$ vektorlar kollinear bo'lishi uchun $\vec{b}=\lambda\vec{a}$ bo'lishi zarur va yetarli, bu yerda λ – birorta son;

2-xossa. $\vec{a}=|\vec{a}|\cdot\vec{a}^{\circ}\ (\vec{a}\neq 0)$, ya'ni har bir nol bo'lmagan vektor uzunligi bilan ortining ko'paytmasiga teng bo'ladi.

Vektorlar ustida chiziqli amallar ushbu xossalarga ega.

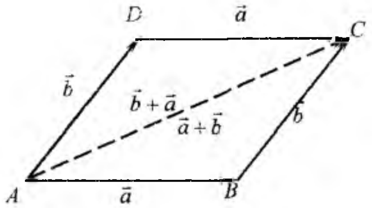
$1^{\circ}.\ \vec{a}+\vec{b}=\vec{b}+\vec{a}$;	$2^{\circ}.\ (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$;
$3^{\circ}.\ \vec{a}+\vec{0}=\vec{a}$;	$4^{\circ}.\ \vec{a}+(-\vec{a})=\vec{0}$;
$5^{\circ}.\ \lambda(\mu\vec{a})=(\lambda\cdot\mu)\vec{a}$;	$6^{\circ}.\ (\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a}$;
$7^{\circ}.\ \lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$;	$8^{\circ}.\ 1\cdot\vec{a}=\vec{a}$.

Xossalarning isboti vektorlarni qo'shish va ayirish qoidalari hamda vektorning songa ko'paytirish ta'rifidan bevosita kelib chiqadi.

Misol tariqasida xossalardan birinchisini isbotlaymiz. $\overrightarrow{AB}$ vektor $\vec{a}$ dan va $\overrightarrow{BC}$ vektor $\vec{b}$ dan iborat bo'lsin (5-shakl). U holda vektorlarni qo'shishning uchburchak xossasiga ko'ra ABC uchburchakda

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

bo'ladi.



5-shakl.

Parallelogrammning xossasiga ko'ra:

$$\overrightarrow{AD} = \overrightarrow{BC}, \quad \overrightarrow{DC} = \overrightarrow{AB}.$$

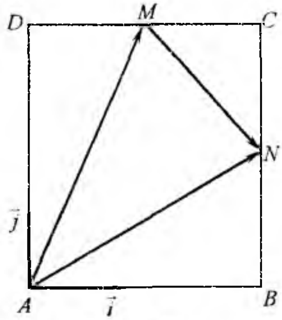
Vektorlarni qo'shishning uchburchak xossasiga ko'ra, ADC uchburchakda

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

bo'ladi.

Bundan, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ kelib chiqadi.

1-misol. $ABCD$ to'g'ri to'rtburchakning tomonlari $AB = 3$, $AD = 4$. $M - DC$ tomonning o'rtasi, $N - CB$ tomonning o'rtasi (6-shakl). $\overrightarrow{AM}, \overrightarrow{AN}, \overrightarrow{MN}$ vektorlarni mos ravishda $\overrightarrow{AB}$ va $\overrightarrow{AD}$ tomonlar bo'ylab yo'nalgan $\vec{i}$ va $\vec{j}$ birlik vektorlar orqali ifodalang.



6-shakl.

Yechish. $\vec{a} = |\vec{a}| \cdot \vec{a}^{\circ}$ bo'lishini hisobga olib, topamiz:

$$\overrightarrow{AB} = |\overrightarrow{AB}| \cdot \vec{i} = 3\vec{i}, \quad \overrightarrow{AD} = |\overrightarrow{AD}| \cdot \vec{j} = 4\vec{j}.$$

6-shaklga ko'ra,

$$\overrightarrow{DM} = \overrightarrow{MC} = \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB} = \frac{3}{2}\vec{i}, \quad \overrightarrow{BN} = \overrightarrow{NC} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AD} = 2\vec{j}.$$

Vektorlarni qo'shish qoidasi bilan topamiz:

$$\overrightarrow{AM} = \overrightarrow{AD} + \overrightarrow{DM} = 4\vec{j} + \frac{3}{2}\vec{i}; \quad \overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN} = 3\vec{i} + 2\vec{j};$$

$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN} = \overrightarrow{MC} - \overrightarrow{NC} = \frac{3}{2}\vec{i} - 2\vec{j}.$$

2.1.3. Vektorning o'qdagi proeksiyasi

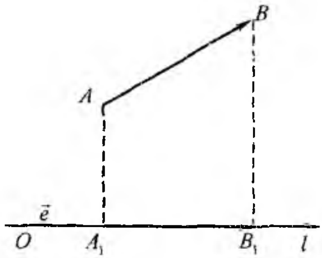
Sanoq boshini aniqlovchi nuqtasi va birlik vektori berilgan to'g'ri chiziqqa *o'q* deyiladi.

$\vec{e}$ birlik vektor va O nuqta l o'qni bir qiymatli aniqlaydi.

M nuqtadan l o'qqa tushirilgan AA_1 perpendikulyarning A_1 asosiga *A nuqtaning l o'qdagi proeksiyasi* deyiladi (7-shakl).

$\overrightarrow{AB}$ $(\overrightarrow{AB} \neq 0)$ ixtiyoriy vektor bo'lsin. A_1 va B_1 bilan mos ravishda $\overrightarrow{AB}$ vektor boshi va oxirining l o'qdagi proeksiyalarini belgilaymiz.

$\overrightarrow{A_1B_1}$ vektorga $\overrightarrow{AB}$ *vektorning l o'qdagi tashkil etuvchisi* deyiladi.



7-shakl.

4-ta'rif. $\overrightarrow{AB}$ *vektorning l o'qdagi proeksiyasi* deb $\pm|\overrightarrow{A_1B_1}|$ songa aytiladi va $\Pr_l \overrightarrow{AB}$ bilan belgilanadi. Bu son $\overrightarrow{A_1B_1}$ tashkil etuvchi va l o'q bir tomonga yo'nalgan bo'lsa musbat ishora bilan, aks holda manfiy ishora olinadi (7-shakl).

Demak,

$$\Pr_l \overrightarrow{AB} = \pm|\overrightarrow{A_1B_1}|.$$

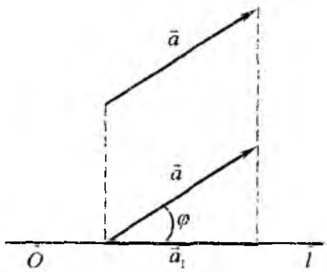
5-ta'rif. $\vec{a}$ vektor bilan uning l o'qdagi tashkil etuvchisi $\vec{a}_1$ vektor orasidagi φ burchakka *$\vec{a}$ vektor bilan l o'q orasidagi burchak* (*ikki $\vec{a}$* va *$\vec{a}_1$ vektor orasidagi burchak*) deyiladi. Ravshanki, $0 \le \varphi \le \pi$ (8-shakl).

Vektorning o'qdagi proeksiyasining asosiy xossalari bilan tanishamiz.

1-xossa. $\vec{a}$ vektorning l o'qdagi proeksiyasi $\vec{a}$ vektor modulining bu vektor bilan o'q orasidagi φ burchak kosinusiga ko'paytmasiga teng, ya'ni

$$\Pr_l \vec{a} = |\vec{a}|\cos\varphi\ .$$

Isboti. Agar $\vec{a}$ vektor bilan l o'q orasidagi burchak o'tkir $\left(0 \le \varphi < \dfrac{\pi}{2}\right)$ bo'lsa, $\vec{a}_1$ tashkil etuvchi va l o'q bir tomonga yo'nalgan bo'ladi.



8-shakl.

U holda

$$\Pr_l \vec{a} = +|\vec{a}_1| = |\vec{a}|\cos\varphi.$$

Agar $\vec{a}$ vektor bilan l o‘q orasidagi burchak o‘tmas $\left(\frac{\pi}{2} < \varphi \le \pi\right)$ bo‘lsa, $\vec{a}_1$ tashkil etuvchi va l o‘q qarama-qarshi tomonga yo‘nalgan bo‘ladi.

U holda

$$\Pr_l \vec{a} = -|\vec{a}_1| = -|\vec{a}|\cos(\pi - \varphi) = |\vec{a}|\cos\varphi.$$

Agar $\varphi = \frac{\pi}{2}$ bo‘lsa, u holda

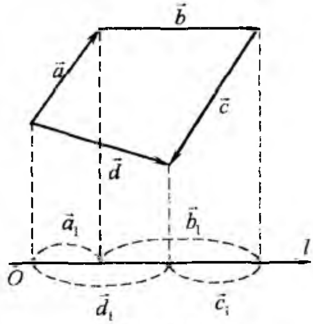
$$\Pr_l \vec{a} = 0 = |\vec{a}|\cos\frac{\pi}{2} = |\vec{a}|\cos\varphi.$$

Bu xossadan quyidagi natijalar kelib chiqadi.

1-natija. Vektorning o‘qdagi proeksiyasi:

1) vektor o‘q bilan o‘tkir burchak tashkil qilsa, musbat bo‘ladi;

2) vektor o‘q bilan o‘tmas burchak tashkil qilsa, manfiy bo‘ladi;



9-shakl.

3) vektor o‘q bilan to‘g‘ri burchak tashkil qilsa, nolga teng bo‘ladi.

2-natija. Teng vektorlarning bitta o‘qdagi proeksiyalari teng bo‘ladi.

2-xossa. Bir nechta vektor yig‘indisining berilgan o‘qdagi proeksiyasi vektorlarning shu o‘qdagi proeksiyalari yig‘indisiga teng, masalan,

$$\Pr_l(\vec{a} + \vec{b} + \vec{c}) = \Pr_l \vec{a} + \Pr_l \vec{b} + \Pr_l \vec{c}.$$

Isboti. $\vec{d} = \vec{a} + \vec{b} + \vec{c}$ bo‘lsin.

U holda (9-shakl)

$$\Pr_l \vec{d} = +|\vec{d}_1| = +|\vec{a}_1| + |\vec{b}_1| - |\vec{c}_1|,$$

ya’ni

$$\Pr_l(\vec{a} + \vec{b} + \vec{c}) = \Pr_l \vec{a} + \Pr_l \vec{b} + \Pr_l \vec{c}.$$

3-xossa. Vektor skalyar songa ko'paytirilsa, uning o'qdagi proeksiyasi ham shu songa ko'payadi, ya'ni

$$\Pr_l(\lambda\vec{a}) = \lambda \cdot \Pr_l \vec{a}.$$

Isboti. Vektorning o'qdagi proeksiyasining 1-xossasiga ko'ra,

$\lambda > 0$ da $\Pr_l(\lambda\vec{a}) = |\lambda\vec{a}|\cos\varphi = \lambda|\vec{a}|\cos\varphi = \lambda \cdot \Pr_l \vec{a}.$

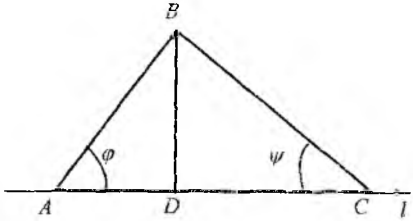
$\lambda < 0$ da $\Pr_l(\lambda\vec{a}) = |\lambda\vec{a}|\cos(\pi - \varphi) = -\lambda|\vec{a}|(-\cos\varphi) = \lambda|\vec{a}|\cos\varphi = \lambda \cdot \Pr_l \vec{a}.$

$\lambda = 0$ da $\Pr_l(0 \cdot \vec{a}) = 0 = 0 \cdot \Pr_l \vec{a} = \lambda \cdot \Pr_l \vec{a}.$

3-natija. Vektorlar chiziqli kombinatsiyasining o'qdagi proeksiyasi bu vektorlar o'qdagi proeksiyasilarining mos chiziqli kombinatsiyasiga teng, masalan,

$$\Pr_l(k\vec{a} + n\vec{b}) = k\Pr_l \vec{a} + n\Pr_l \vec{b}.$$

2-misol. ABC to'g'ri burchakli uchburchakning katetlari $BA = 5$, $BC = 5\sqrt{3}$. ABC uchburchak balandliklari bo'ylab yo'nalgan vektorlarning AC gipotenuza bo'ylab yo'nalgan l o'qdagi proeksiyalarini toping.



10-shakl.

Yechish. $\angle BAC = \varphi$, $\angle BCA = \psi$ bo'lsin deylik.

U holda

$$\cos\varphi = \frac{BA}{AC} = \frac{5}{\sqrt{5^2 + (5\sqrt{3})^2}} = \frac{1}{2}, \quad \cos\psi = \frac{BC}{AC} = \frac{5\sqrt{3}}{\sqrt{5^2 + (5\sqrt{3})^2}} = \frac{\sqrt{3}}{2}.$$

ABC uchburchak balandliklari bo'ylab yo'nalgan vektorlar $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{BD}$ bo'ladi (10-shakl). Bu vektorlarning l o'qdagi proeksiyalarini topamiz:

$$\Pr_l \overrightarrow{AB} = |\overrightarrow{AB}| \cdot \cos\varphi = 5 \cdot \frac{1}{2} = \frac{5}{2},$$

$$\Pr_l \overrightarrow{CB} = -|\overrightarrow{CB}| \cdot \cos\psi = -5\sqrt{3} \cdot \frac{\sqrt{3}}{2} = -\frac{15}{2},$$

$$\Pr_l \overrightarrow{BD} = |\overrightarrow{BD}| \cdot \cos\frac{\pi}{2} = 0.$$

2.1.4. Vektorlarning chiziqli bog'liqligi. Bazis

Uch o'lchovli R^3 fazoda $\vec{a}_1, \vec{a}_2, \vec{a}_3$ vektorlar berilgan bo'lsin. Ixtiyoriy $\alpha_1, \alpha_2, \alpha_3$ sonlarni olamiz va $\vec{a}_1, \vec{a}_2, \vec{a}_3$ vektorlardan

$$\vec{a} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 \qquad (1.1)$$

vektorni tuzamiz. Bunda $\vec{a}$ vektorga $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ *vektorlarning chiziqli kombinatsiyasi,* $\alpha_1, \alpha_2, \alpha_3$ sonlarga bu chiziqli kombinatsiyaning koeffitsiyentlari deyiladi.

6-ta'rif. Agar $\vec{a}_1, \vec{a}_2, \vec{a}_3$ vektorlar uchun kamida bittasi nolga teng bo'lmagan shunday $\alpha_1, \alpha_2, \alpha_3$ sonlar topilsa va bu sonlar uchun

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = 0 \qquad (1.2)$$

tenglik bajarilsa, $\vec{a}_1, \vec{a}_2, \vec{a}_3$ vektorlar sistemasiga *chiziqli bog'liq vektorlar* deyiladi.

7-ta'rif. Agar (1.2) tenglik faqat $\alpha_1 = \alpha_2 = \alpha_3 = 0$ bo'lganda o'rinli bo'lsa, $\vec{a}_1, \vec{a}_2, \vec{a}_3$ vektorlar sistemasiga *chiziqli erkli vektorlar* deyiladi.

1-teorema. $\vec{a}_1, \vec{a}_2, \vec{a}_3$ vektorlar chiziqli bog'liq bo'lishi uchun ulardan hech bo'lmaganida bittasi qolganlarining chiziqli kombinatsiysidan iborat bo'lishi zarur va yetarli.

Isboti. Zarurligi. Berilgan vektorlar chiziqli bog'liq bo'lsin. U holda (1.2) tenglik bajariladi, bunda α_k $(k=1,2,3)$ sonlardan kamida bittasi nolga teng bo'lmaydi. $\alpha_3 \neq 0$ bo'lsin deylik.

U holda

$$\vec{a}_3 = -\frac{\alpha_1}{\alpha_3}\vec{a}_1 - \frac{\alpha_2}{\alpha_3}\vec{a}_2$$

bo'ladi, ya'ni $\vec{a}_3$ vektor qolgan vektorarning chiziqli kombinatsiyasidan iborat.

Yetarliligi. Berilgan vektorlardan bittasi, masalan, $\vec{a}_3$ qolganlarining chiziqli kombinatsiysidan iborat bo'lsin:

$$\vec{a}_3 = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 .$$

Bundan

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + (-1)\vec{a}_3 = 0$$

kelib chiqadi, ya'ni (1.2) tenglik $\alpha_3 = -1$da bajariladi. Demak, berilgan vektorlar chiziqli bog'liq.

7-ta'rifga ko'ra bitta nol bo'lmagan $\vec{a}$ vektordan iborat sistema chiziqli erkli bo'ladi, chunki $\alpha \cdot \vec{a} = 0$ tenglik faqat va faqat $\alpha = 0$ da bajariladi.

Demak, bitta $\vec{a}$ vektordan iborat sistema faqat va faqat $\vec{a} = 0$ bo'lganida chiziqli bog'liq bo'ladi.

Tekislikdagi vektorlarning chiziqli bog'liq yoki chiziqli erkli bo'lishi haqidagi masalani qaraymiz.

2-teorema. Ikkita $\vec{a}_1$ va $\vec{a}_2$ vektor chiziqli bog'liq bo'lishi uchun ular kollinear bo'lishi zarur va yetarli.

Isboti. Zarurligi. $\vec{a}_1$ va $\vec{a}_2$ vektorlar kollinear bo'lsin. U holda $\vec{a}_2 = \lambda\vec{a}_1$ bo'ladi, bundan $\lambda\vec{a}_1 + (-1)\vec{a}_2 = 0$ yoki $\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 = 0$. Demak, vektorlar chiziqli bog'liq.

Yetarliligi. $\vec{a}_1$ va $\vec{a}_2$ vektorlar chiziqli bog'liq bo'lsin. U holda $\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 = 0$ tenglik α_1 va α_2 sonlardan kamida bittasi, masalan, α_2 noldan farqli bo'lganida bajariladi. Bundan $\vec{a}_2 = -\dfrac{\alpha_1}{\alpha_2}\vec{a}_1$ yoki $\vec{a}_2 = \lambda\vec{a}_1$ kelib chiqadi. Demak, vektorlar kollinear.

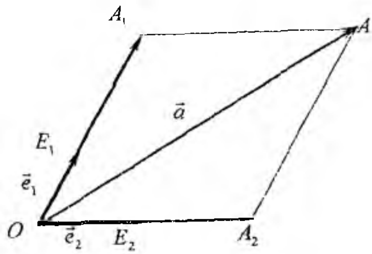
Bu teoremadan quyidagi natija kelib chiqadi.

4-natija. Ikkita $\vec{a}_1$ va $\vec{a}_2$ vektor chiziqli erkli bo'lishi uchun ular kollinear bo'lmasligi zarur va yetarli.

3-teorema. Agar $\vec{e}_1$ va $\vec{e}_2$ biror tekislikning ikkita kollinear bo'lmagan vektorlari bo'lsa, u holda shu tekislikning istalgan uchinchi $\vec{a}$ vektorini bu vektorlar bo'yicha yagona usulda

$$\vec{a} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 \qquad (1.3)$$

ko'rinishda yoyish mumkin.



11-shakl.

Isboti. Bitta O nuqtaga vektorlarni qo'yamiz: $\overrightarrow{OE_1} = \vec{e}_1$, $\overrightarrow{OE_2} = \vec{e}_2$, $\overrightarrow{OA} = \vec{a}$. A nuqtadan $\vec{e}_1$ va $\vec{e}_2$ vektorlarga parallel ikkita to'g'ri chiziq o'tkazamiz. Bu to'g'ri chiziqlar bilan $\vec{e}_1$ va $\vec{e}_2$ vektorlar yotgan to'g'ri chiziqlarning kesishish nuqtalarini mos ravishda A_1 va A_2 bilan belgilaymiz (11-shakl).

Ikki vektor yig'indisi ta'rifiga ko'ra, $\overrightarrow{OA} = \overrightarrow{OA_1} + \overrightarrow{A_1A}$, bu yerda

$\overrightarrow{A_1A}=\overrightarrow{OA_2}$ ekani hisobga olinsa

$$\vec{a}=\overrightarrow{OA_1}+\overrightarrow{OA_2}. \qquad (1.4)$$

$\vec{e}_1$ va $\overrightarrow{OA_1}$ vektorlar kollinear bo'lgani uchun $\overrightarrow{OA_1}=\alpha_1\vec{e}_1$ va shu kabi $\overrightarrow{OA_2}=\alpha_1\vec{e}_2$ bo'ladi. Bu tengliklarga ko'ra, (1.4) tenglikdan (1.3) tenglik kelib chiqadi.

(1.3) yoyilmaning α_1 va α_2 koeffitsiyentlari bir qiymatli aniqlanishini ko'rsatamiz. Buning uchun

$$\vec{a}=\beta_1\vec{e}_1+\beta_2\vec{e}_2 \qquad (1.5)$$

yoyilma mavjud bo'lsin deb faraz qilamiz.

(1.5) tenglikni (1.3) tenglikdan hadma-had ayirib, topamiz:

$$(\alpha_1-\beta_1)\vec{e}_1+(\alpha_2-\beta_2)\vec{e}_2=0.$$

Shartga ko'ra, $\vec{e}_1$ va $\vec{e}_2$ vektorlar kollinear emas. U holda 1-narijaga ko'ra, ular chiziqli erkli. Shu sababli oxirgi tenglik faqat $\alpha_1-\beta_1=0$ va $\alpha_2-\beta_2=0$ bo'lganida bajariladi. Bundan $\alpha_1=\beta_1$, $\alpha_2=\beta_2$. Demak, (1.3) yoyilmaning α_1 va α_2 koeffitsiyentlari bir qiymatli aniqlanadi.

4-teorema. *Tekislikdagi har qanday uchta vektor chiziqli bog'liq bo'ladi.*

Isboti. $\vec{a}_1,\vec{a}_2,\vec{a}_3$ vektorlardan ikkitasi, masalan, $\vec{a}_1$ va $\vec{a}_2$ kollinear vektorlar bo'lsin. U holda $\vec{a}_2=\alpha_1\vec{a}_1$ yoki $\vec{a}_2=\alpha_1\vec{a}_1+0\vec{a}_3$ bo'ladi. Demak, $\vec{a}_1,\vec{a}_2,\vec{a}_3$ vektorlar chiziqli bog'liq.

$\vec{a}_1,\vec{a}_2,\vec{a}_3$ vektorlardan istalgan ikkitasi kollinear bo'lmasin.

U holda 3-teoremaga ko'ra $\vec{a}_3$ vektorni

$$\vec{a}_3=\alpha_1\vec{a}_1+\alpha_2\vec{a}_2,$$

ko'rinishda yozish mumkin. Demak, $\vec{a}_1,\vec{a}_2,\vec{a}_3$ vektorlar chiziqli bog'liq.

2- va 4-teoremalardan quyidagi natijalar kelib chiqadi.

5-natija. Tekislikdagi chiziqli erkli vektorlar soni ko'pi bilan ikkiga teng.

6-natija. Tekislikdagi vektorlar uchun ko'rsatilgandagi kabi fazodagi vektorlar uchun quyidagi xulosalarni ko'rsatish mumkin:

1) *Uchta vektor chiziqli bog'liq bo'lishi uchun ular komplanar bo'lishi* zarur va yetarli;

2) Uchta vektor chiziqli erkli bo'lishi uchun ular komplanar bo'lmasligi zarur va yetarli;

3) Agar $\vec{e}_1, \vec{e}_2, \vec{e}_3$ vektorlar komplanar bo'lmasa, u holda istalgan $\vec{a}$ vektor bu vektorlar bo'yicha yagona ko'rinishda yoyilishi mumkin, ya'ni

$$\vec{a} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3; \qquad (1.6)$$

4) Uch o'lchovli fazodagi har qanday to'rtta vektor chiziqli bog'liq bo'ladi;

5) Fazodagi chiziqli erkli vektorlar soni ko'pi bilan uchga teng.

Tekislikdagi bazis deb, istalgan ikkita chiziqli erkli vektorga aytiladi.

5-natija va 3-teoremaga ko'ra, tekislikdagi istalgan $\vec{a}$ vektorni $\vec{e}_1, \vec{e}_2$ bazis bo'yicha yagona ko'rinishda yoyish mumkin, ya'ni

$$\vec{a} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2. \qquad (1.7)$$

(1.7) tenglikka $\vec{a}$ vektorning $\vec{e}_1, \vec{e}_2$ bazis bo'yicha yoyilmasi, α_1, α_2 sonlarga $\vec{a}$ vektorning $\vec{e}_1, \vec{e}_2$ bazisdagi *affin koordinatalari* deyiladi va $\vec{a} = \{\alpha_1; \alpha_2\}$ deb yoziladi.

Fazodagi bazis deb, istalgan uchta chiziqli erkli vektorga aytiladi.

6-natijada keltirilgan xulosalarga ko'ra, uch o'lchovli fazodagi istalgan $\vec{a}$ vektorni $\vec{e}_1, \vec{e}_2, \vec{e}_3$ bazis bo'yicha yagona ko'rinishda yoyish mumkin, ya'ni

$$\vec{a} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3. \qquad (1.8)$$

Bunda $\alpha_1, \alpha_2, \alpha_3$ sonlar $\vec{a}$ vektorning $\vec{e}_1, \vec{e}_2, \vec{e}_3$ bazisdagi *affin koordinatalari* bo'ladi, ya'ni $\vec{a} = \{\alpha_1; \alpha_2; \alpha_3\}$.

3-misol. Uchburchakli muntazam piramidada $AB, AC, AD - A$ uchning qirralari, $DO - D$ uchdan tushirilgan balandlik (12-shakl). Agar $\vec{e}_1, \vec{e}_2, \vec{e}_3$ mos ravishda AB, AC, AD qirralar bo'ylab yo'nalgan vektorlar bo'lsin. $\overrightarrow{DO}$ vektorning $\vec{e}_1, \vec{e}_2, \vec{e}_3$ bazis bo'yicha yoyilmasini toping.

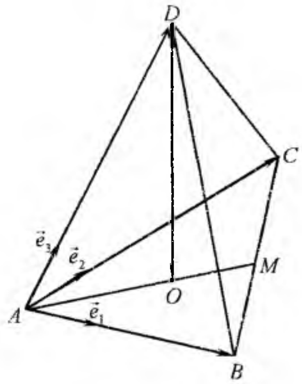
Yechish. Vektorlarni songa ko'paytirish amalining xossasiga ko'ra:

$$\overrightarrow{AB} = \lambda_1\vec{e}_1, \quad \overrightarrow{AC} = \lambda_2\vec{e}_2, \quad \overrightarrow{AD} = \lambda_3\vec{e}_3,$$

bu yerda $\lambda_1, \lambda_2, \lambda_3$ – haqiqiy sonlar.

Piramidaning bir uchidan chiqqan qirralarida yotgan $\vec{e}_1,\vec{e}_2,\vec{e}_3$ vektorlar komplanar emas. Shu sababli $\overline{DO}$ vektorni $\vec{e}_1,\vec{e}_2,\vec{e}_3$ bazis bo'yicha yoyish mumkin.

Piramida muntazam bo'lgani uchun uning balandligi asosining medianalari kesishish nuqtasiga tushadi, ya'ni O–uchburchak medianalarining kesishish nuqtasi bo'ladi.



12-shakl.

Vektorlarni qo'shish qoidasiga ko'ra

$$\overline{DO} = \overline{DA} + \overline{AO}.$$

Bunda

$$\overline{DA} = -\overline{AD} = -\lambda_3\vec{e}_3,$$

$$\overline{AO} = \frac{2}{3}\overline{AM} = \frac{2}{3}\cdot\frac{\overline{AB}+\overline{AC}}{2} = \frac{1}{3}(\lambda_1\vec{e}_1 + \lambda_2\vec{e}_2).$$

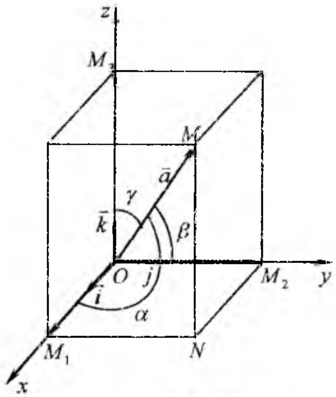
Demak,

$$\overline{DO} = -\lambda_3\vec{e}_3 + \frac{1}{3}(\lambda_1\vec{e}_1 + \lambda_2\vec{e}_2).$$

2.1.5. Dekart koordinatalar sistemasida vektorlar

Fazodagi bazis sifatida o'zaro perpendikular bo'lgan $\vec{i},\vec{j},\vec{k}$ birlik vektorlarni olaylik. Bunday bazis *ortonormallashgan bazis* deyiladi. Bunda $\vec{i},\vec{j},\vec{k}$ vektorlar bazis ortlari deb ataladi.

Koordinatalar boshidan mos ravishda $\vec{i},\vec{j},\vec{k}$ bazis ortlari yo'nalishida o'tkazilgan Ox,Oy,Oz o'qlarga koordinata o'qlari deyiladi. Ox,Oy,Oz o'qlardan tashkil topgan $Oxyz$ koordinatalar sistemaga to'g'ri burchakli (yoki dekart) koordinatalar sistemasi deyiladi.



13-shakl.

Fazoda ixtiyoriy $\vec{a}$ vektorni olib, uning boshini koordinatalar boshiga keltiramiz, ya'ni $\vec{a}=\overrightarrow{OM}$ vektorni hosil qilamiz (13-shakl).

$\vec{a}$ vektorning koordinata oq'laridagi proeksiyalarini topamiz. Buning uchun $\overrightarrow{OM}$ vektorning oxiridan koordinata tekisliklariga parallel tekisliklar o'tkazamiz va ularning koordinata o'qlari bilan kesishish nuqtalarini mos ravishda M_1, M_2, M_3 orqali belgilaymiz. Bu tekisliklar koordinata tekisliklari bilan birgalikda diagonallaridan biri $\overrightarrow{OM}$ vektor bo'lgan to'g'ri burchakli parallelepipedni hosil qiladi.

Bunda

$$\Pr_x \vec{a}=|\overrightarrow{OM_1}|,\ \Pr_y \vec{a}=|\overrightarrow{OM_2}|,\ \Pr_z \vec{a}=|\overrightarrow{OM_3}|.$$

Bir nechta vektorlarni qo'shish qoidasiga ko'ra,

$$\vec{a}=\overrightarrow{OM_1}+\overrightarrow{M_1N}+\overrightarrow{NM}.$$

Yoki $\overrightarrow{M_1N}=\overrightarrow{OM_2}$, $\overrightarrow{NM}=\overrightarrow{OM_3}$ni hisobga olsak,

$$\vec{a}=\overrightarrow{OM_1}+\overrightarrow{OM_2}+\overrightarrow{OM_3}. \tag{1.9}$$

Shunindek,

$$\overrightarrow{OM_1}=|\overrightarrow{OM_1}|\cdot\vec{i},\ \overrightarrow{OM_2}=|\overrightarrow{OM_2}|\cdot\vec{j},\ \overrightarrow{OM_3}=|\overrightarrow{OM_3}|\cdot\vec{k}. \tag{1.10}$$

$\vec{a}=\overrightarrow{OM}$ vektorning koordinata o'qlaridagi proeksiyalarini mos ravishda a_x, a_y va a_z orqali belgilaymiz, ya'ni $|\overrightarrow{OM_1}|=a_x$, $|\overrightarrow{OM_2}|=a_y$, $|\overrightarrow{OM_3}|=a_z$.

U holda (1.9) va (1.10) tengliklardan topamiz:

$$\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}. \tag{1.11}$$

(1.11) tenglik $\vec{a}$ vektorning $\vec{i},\vec{j},\vec{k}$ bazis bo'yicha yoyilmasi deb yuritiladi. a_x, a_y, a_z sonlarga $\vec{a}$ vektorning *dekart koordinatalari* (yoki oddiygina *koordinatalari*) deyiladi va $\vec{a}=\{a_x;a_y;a_z\}$ kabi yoziladi.

$\vec{a}=\{a_x;a_y;a_z\}$ vektor berilgan bo'lsin.

To'g'ri burchakli parallelepipedning diagonali haqidagi teoremani qo'llaymiz:

$$|\overrightarrow{OM}|^2=|\overrightarrow{OM_1}|^2+|\overrightarrow{OM_2}|^2+|\overrightarrow{OM_3}|^2,$$

Bundan

$$|\vec{a}|^2=a_x^2+a_y^2+a_z^2$$

yoki

$$|\vec{a}|=\sqrt{a_x^2+a_y^2+a_z^2}\,, \qquad (1.12)$$

ya'ni vektorning uzunligi uning koordinatalari kvadratlarining yig'indisining kvadrat ildiziga teng.

$\vec{a}$ vektorning Ox, Oy, Oz o'qlar bilan tashkil qilgan burchaklari mos ravishda α, β, γ bo'lsin (13-shakl). $\cos\alpha$, $\cos\beta$, $\cos\gamma$ lar $\vec{a}$ *vektorning yo'naltiruvchi kosinuslari* deb ataladi.

Vektorning o'qdagi proeksiyasi 1-xossasidan topamiz:

$$a_x=|\vec{a}|\cos\alpha,\ a_y=|\vec{a}|\cos\beta,\ a_z=|\vec{a}|\cos\gamma.$$

Bundan $\vec{a}\neq 0$ bo'lganida

$$\cos\alpha=\frac{a_x}{|\vec{a}|},\quad \cos\gamma=\frac{a_z}{|\vec{a}|},\quad \cos\beta=\frac{a_y}{|\vec{a}|}. \qquad (1.13)$$

(1.13) tenglikdan (1.12) formulani hisobga olib, topamiz:

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1, \qquad (1.14)$$

ya'ni nol bo'lmagan vektorning yo'naltiruvchi kosinuslari kvadratlarining yig'indisi birga teng.

Bundan $\vec{a}^0$ birlik vektorning koordinatalari $\cos\alpha$, $\cos\beta$, $\cos\gamma$ bo'lishi kelib chiqadi, ya'ni $\vec{a}^0=\{\cos\alpha;\cos\beta;\cos\gamma\}$.

4-misol. Uzunligi $|\vec{a}|=2$ ga teng vektor Ox, Oy koordinata o'qlari bilan $\alpha=60^\circ, \beta=120^\circ$ li burchaklar tashkil qiladi. $\vec{a}$ vektorning koordinatalarini toping.

Yechish. Vektorning o'qdagi proyeksiyasining 1-xossasisiga ko'ra:

$$a_x=|\vec{a}|\cos\alpha=2\cos 60^\circ=2\cdot\frac{1}{2}=1;$$

$$a_y=|\vec{a}|\cos\beta=2\cos 120^\circ=2\cdot\left(-\frac{1}{2}\right)=-1.$$

Vektorning uzunligini topamiz:

$$2=\sqrt{1+1+a_z^2}\,.$$

Bundan

$$a_z^2=2 \quad \text{yoki } a_z=\sqrt{2},\ a_z=-\sqrt{2}.$$

Demak,

$$\vec{a}=\{1;-1;\sqrt{2}\} \quad \text{yoki} \quad \vec{a}=\{1;-1;-\sqrt{2}\}.$$

$\vec{a}$ va $\vec{b}$ vektorlar koordinatalari bilan berilgan bo'lsin:

$$\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}, \quad \vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}.$$

Vektorning o'qdagi proeksiyasining xossalariga ko'ra:

$$\vec{a}\pm\vec{b}=(a_x\pm b_x)\vec{i}+(a_y\pm b_y)\vec{j}+(a_z\pm b_z)\vec{k}, \tag{1.15}$$

$$\lambda\vec{a}=\lambda a_x\vec{i}+\lambda a_y\vec{j}+\lambda a_z\vec{k}, \tag{1.16}$$

$$\vec{a}=\vec{b} \Leftrightarrow \begin{cases} a_x=b_x, \\ a_y=b_y, \\ a_z=b_z. \end{cases} \tag{1.17}$$

Shunday qilib,

1) vektorlar qo'shilganida (ayrilganida) ularning mos koordinatalari qo'shiladi (ayriladi);

2) vektor songa ko'paytirilganida uning barcha koordinatalari shu songa ko'payadi;

3) teng vektorlarning mos koordinatalari teng bo'ladi va aksincha.

5-misol. $\vec{a}=-4\vec{i}-2\vec{j}+4\vec{k}$ vektor berilgan. Bu vektorga qarama-qarshi yo'nalgan, kollinear va uzunligi $|\vec{b}|=9$ bo'lgan vektorning koordinatalarini toping.

Yechish. $\vec{b}$ vektorning koordinatalari b_x, b_y, b_z, ya'ni $\vec{b}=\{b_x;b_y;b_z\}$ bo'lsin.

$\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa $\vec{a}=\lambda\vec{b}$ bo'ladi, bu yerda λ – biror son.

U holda ikki vektorning tengligi shartidan

$$b_x=\lambda a_x, \quad b_y=\lambda a_y, \quad b_z=\lambda a_z$$

yoki

$$b_x=-4\lambda, \ b_y=-2\lambda, \ b_z=4\lambda.$$

Bu koordinatalarni va $\vec{b}$ vektorning uzunligini hisobga olib, topamiz:

$$\sqrt{16\lambda^2+4\lambda^2+16\lambda^2}=9, \ \pm 6\lambda=9 \quad \text{yoki} \quad \lambda=\pm\frac{3}{2}.$$

$\vec{a}$ va $\vec{b}$ vektorlar qarama-qarshi yo'nalgani uchun $\lambda < 0$, ya'ni $\lambda = -\frac{3}{2}$.

Demak, $\vec{b} = \{6;3;-6\}$.

Koordinatlar boshiga qo'yilgan va oxiri M nuqta bo'lgan $\vec{r} = \overrightarrow{OM}$ vektorga M nuqtaning *radius vektori* deyiladi. $M(x;y;z)$ nuqta radius vektorining koordinatalari $r = \{x;y;z\}$ bo'ladi.

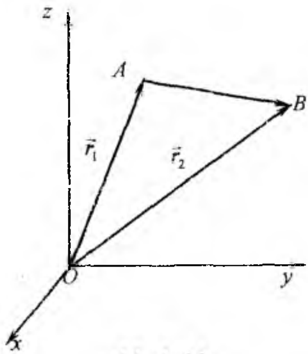
Boshi $A(x_1;y_1;z_1)$ nuqtada va oxiri $B(x_2;y_2;z_2)$ nuqtada bo'lgan $\overrightarrow{AB}$ vektorni qaraymiz. A va B nuqtalarning radius vektorlarini mos ravishda $r_1 = \{(x_1;y_1;z_1\}$ va $r_2 = \{x_2;y_2;z_2\}$ bo'ladi.

14-shaklga ko'ra, $\overrightarrow{AB} = \vec{r}_2 - \vec{r}_1$. Bundan (1.15) tenglikka asosan

$$\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

yoki

$$\overrightarrow{AB} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}. \qquad (1.18)$$



14-shakl.

Shunday qilib, vektorning koordinatalari uning oxiri va boshining mos koordinatalari ayirmasiga teng.

(1.18) tenglikdan $\overrightarrow{AB}$ vektorning modulini topamiz:

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad (1.19)$$

$\overrightarrow{AB}$ vektorning uzunligi A va B nuqtalar orasidagi masofani aniqlaydi. Shu sababli (1.19) tenglikka *ikki nuqta orasidagi masofani topish formulasi* deyiladi.

6-misol. Parallelogrammning uchta ketma-ket uchi berilgan: $A(-1;3;1)$, $B(-2;-5;3)$, $C(0;-1;1)$. BD diagonal uzunligini toping.

Yechish. Parallelogramm D uchning $x;y;z$ koordinatalarini parallelogramm uchun $\overrightarrow{AD} = \overrightarrow{BC}$ ekanidan aniqlaymiz. $\overrightarrow{AD}$ va $\overrightarrow{BC}$ vektorlarni (1.18) formula ko'rinishida yozamiz:

$$\overrightarrow{AD} = \{x+1; y-3; z-1\},$$

$$\overrightarrow{BC} = \{0-(-2); -1-(-5); 1-3\} = \{2;4;-2\}.$$

U holda $\overrightarrow{AD}=\overrightarrow{BC}$ tenglikdan $x+1=2,\ y-3=4,\ z-1=-2$, ya'ni $D(1;7;-1)$ bo'lishi kelib chiqadi. $\overrightarrow{BD}$ vektorning koordinatalarini topamiz:

$$\overrightarrow{BD}=\{1-(-2);7-(-5);-1-3\}=\{3;12;-4\}.$$

U holda

$$|\overrightarrow{BD}|=\sqrt{3^2+12^2+(-4)^2}=\sqrt{9+144+16}=13.$$

$A(x_1;y_1;z_1)$ va $B(x_2;y_2;z_2)$ nuqtalarni tutashtiruvchi kesma berilgan bo'lsin. AB kesmani berilgan $\lambda>0$ nisbatda bo'luvchi, ya'ni $\dfrac{AC}{CB}=\lambda$ tenglikni ta'minlaydigan $C(x;y;z)$ nuqtani topish masalasini yechamiz.

Masalaning shartiga ko'ra, $\overrightarrow{AC}$ va $\overrightarrow{CB}$ vektorlar kollinear, ya'ni $\overrightarrow{AC}=\lambda\overrightarrow{CB}$.

U holda

$$\overrightarrow{AC}=\{x-x_1;y-y_1;z-z_1\},\quad \overrightarrow{CB}=\{x_2-x;y_2-y;z_2-z\}$$

inobatga olinsa,

$$x-x_1=\lambda(x_2-x),\quad y-y_1=\lambda(y_2-y),\quad z-z_1=\lambda(z_2-z).$$

Bu tengliklardan x,y,z larni topamiz:

$$x=\frac{x_1+\lambda x_2}{1+\lambda},\quad y=\frac{y_1+\lambda y_2}{1+\lambda},\quad z=\frac{z_1+\lambda z_2}{1+\lambda}. \tag{1.20}$$

(1.20) formulaga *kesmani berilgan λ nisbatda bo'lish formulasi* deyiladi.

(1.20) tengliklardan $\lambda=1$ da kesma o'rtasining koordinatalarini topish formulalari kelib chiqadi:

$$x=\frac{x_1+x_2}{2},\quad y=\frac{y_1+y_2}{2},\quad z=\frac{z_1+z_2}{2}. \tag{1.21}$$

1-izoh. Tekislikda yotuvchi AB kesma uchun (1.20) va (1.21) formulalar quyidagi ko'rinishlarni oladi:

$$x=\frac{x_1+\lambda x_2}{1+\lambda},\qquad y=\frac{y_1+\lambda y_2}{1+\lambda}, \tag{1.22}$$

$$x=\frac{x_1+x_2}{2},\qquad y=\frac{y_1+y_2}{2}. \tag{1.23}$$

7-misol. $A(2;5)$ va $B(4;9)$ nuqtalarni tutashtiruvchi AB kesmani $AC:CB=1:3$ nisbatda bo‘luvchi C nuqtaning koordinatalarini toping.

Yechish. Masalaning shartiga ko‘ra, $\lambda=\frac{1}{3}$. $C(x;y)$ nuqtani (1.22) formulalar bilan topamiz:

$$x=\frac{2+\frac{1}{3}\cdot 4}{1+\frac{1}{3}}=\frac{5}{2},\qquad y=\frac{5+\frac{1}{3}\cdot 9}{1+\frac{1}{3}}=6,\ \text{yoki}\quad C\left(\frac{5}{2};6\right).$$

2.1.6. Mashqlar

1. Agar $|\vec a+\vec b|=|\vec a-\vec b|$ bo‘lsa, $\vec a$ va $\vec b$ vektorlar qanday shartni qanoatlantiradi?

2. $|\vec a|=13$, $|\vec b|=19$ va $|\vec a+\vec b|=24$ bo‘lsa, $|\vec a-\vec b|$ ni hisoblang.

3. Agar $M-AB$ kesmaning o‘rtasi bo‘lsa, fazoning ixtiyoriy O nuqtasi uchun $\overrightarrow{OM}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$ tenglikni isbotlang.

4. AD, BE, CF kesmalar ABC uchburchakning medianalari bo‘lsa, $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=0$ tenglikni isbotlang.

5. $AN-ABC$ uchburchakning bissiktrisasi. $\overrightarrow{AB}=a$, $\overrightarrow{AC}=\vec b$, $|\vec a|=2$, $|\vec b|=1$ bo‘lsa, $\overrightarrow{AN}$ vektorni toping.

6. $ABCD$ teng yonli trapetsiyada $\angle DAB=60^\circ$, $|AD|=|DC|=|CB|=2$, $M,N-$mos ravishda DC va BC tomonlarning o‘rtalari. $\overrightarrow{NM}$ vektorni mos ravishda $\overrightarrow{AB}$ va $\overrightarrow{AD}$ tomonlar bo‘ylab yo‘nalgan $\vec m$ va $\vec n$ birlik vektorlar orqali ifodalang.

7. O nuqtaga $\overrightarrow{OA}=\vec a$ va $\overrightarrow{OB}=\vec b$ vektorlar qo‘yilgan. AOB burchak bissektrisasida yotuvchi biror $\overrightarrow{OM}$ vektorni toping.

8. $ABCD$ parallelogrammda $\overrightarrow{AB}=\vec a$ va $\overrightarrow{AD}=\vec b$, $M-$ parallelogramm diagonallarining kesishish nuqtasi. $\overrightarrow{MA}$, $\overrightarrow{MB}$ vektorlarni $\vec a$ va $\vec b$ vektorlar orqali ifodalang.

9. m ning qanday qiymatida $\vec c=\vec a-m\vec b$ va $\vec d=-\sqrt{3}\vec a+6\vec b$ vektorlar kollinear bo‘ladi?

10. Biror bazisda $\vec a=\{m;-1;2\}$, $\vec b=\{3;n;6\}$ vektorlar berilgan. $\vec a$ va $\vec b$ vektorlar kollinear bo‘lsa m va n ni toping.

11. $ABCD$ to‘g‘ri burchakli trapetsiya asoslari $|AB|=4$ va $|CD|=2$ va $\angle ABC=45^0$. $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{DC}$, $\overrightarrow{AC}$ vektorlarning $\overrightarrow{CB}$ vektor bilan aniqlanuvchi o‘qqa proyeksiyalarini toping.

12. ABC teng tomonli uchburchakning tomonlari $2\sqrt{3}$ ga teng, AD, BF, CE – uchburchakning balandliklari. $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AD}$, $\overrightarrow{BF}$, $\overrightarrow{CE}$ vektorlarning $\angle BAC$ burchak bissiktrisasi bo‘ylab yo‘nalgan l o‘qqa proyeksiyalarini toping.

13. $ABCD$ parallelogrammda P va Q mos ravishda BC va CD tomonlarning o‘rtalari. Bazis vektorlar $\vec{e}_1=\overrightarrow{AD}$ va $\vec{e}_2=\overrightarrow{AB}$ bo‘lsa, $\overrightarrow{PQ}$ vektorni bu bazis bo‘yicha yoying.

14. $OACB$ to‘g‘ri to‘rtburchakda M va N mos ravishda BC va AC tomonlarning o‘rtasi. $\overrightarrow{OC}$ vektorning $\overrightarrow{OM}=\vec{a}$ va $\overrightarrow{ON}=\vec{b}$ vektorlar bo‘yicha bazisga yoying.

15. $ABCD$ piramidada P va Q mos ravishda AD va BC qirralarning o‘rtasi. Bazis vektorlar $\vec{e}_1=\overrightarrow{AB}$, $\vec{e}_2=\overrightarrow{AC}$ va $\vec{e}_3=\overrightarrow{AD}$ bo‘lsa, $\overrightarrow{PQ}$ vektorni bu bazis bo‘yicha yoying.

16. $OABC$ tetraedr berilgan. $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ vektorlardan iborat bo‘lgan bazisda $\overrightarrow{OF}$ vektorni toping, bu yerda F – asos medianalarining kesishish nuqtasi.

17. Tekislikda uchta $\vec{a}=\{3;-2\}$, $\vec{b}=\{-2;1\}$ va $\vec{c}=\{7;-4\}$ vektorlar berilgan. Har bir vektorning qolgan ikki vektor bo‘yicha yoyilmasini toping.

18. $\vec{a}=\{2;1;0\}$, $\vec{b}=\{1;-1;2\}$, $\vec{c}=\{2;2;-1\}$ vektorlar bazis tashkil qilishini ko‘rsating. $\vec{d}=\{3;7;-7\}$ vektorning shu bazis bo‘yicha yoyilmasini toping.

19. $\vec{a}=\{-1;5;-2\}$ va $\vec{b}=\{2;-1;3\}$ vektor berilgan. $3\vec{a}-2\vec{b}$ va $-\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}$ vektorlarning koordinata o‘qlaridagi proeksiyalarini toping.

20. $\vec{a}=\{1;-1;2\}$, $\vec{b}=\{-2;-3;1\}$ va $\vec{c}=\{0;-3;-2\}$ vektorlar berilgan. $-2\vec{a}+3\vec{b}-\vec{c}$ va $3\vec{a}-2\vec{b}+2\vec{c}$ vektorlarning koordinata o‘qlaridagi proeksiyalarini toping.

21. Tomonlari $\vec{a}=\{-1;0;7\}$ va $\vec{b}=\{5;-4;-5\}$ vektorlarga qurilgan parallelogramm diagonallarining uzunliklarini toping.

22 $\overrightarrow{AB}=\{2;6;4\}$ va $\overrightarrow{AC}=\{4;2;-2\}$ bo‘lsa, ABC uchburchakning CP medianasi uzunligini toping.

23. Fazoda M nuqtaning radius vektori koordinata o‘qlari bilan bir xil burchak tashkil qiladi va uzunligi 3 ga teng. M nuqtaning koordinatalarini toping.

24. $\vec{a}$ vektor OX va OZ o‘qlari bilan mos ravishda 60^0 va 120^0 li burchak tashkil qiladi. Agar $|\vec{a}|=4$ bo‘lsa, bu vektorning koordinatalarini toping.

25. Agar $\vec{a}=\{2;-1;1\}$ vektorning boshi $A(3;-2;-4)$ bo'lsa, uning oxirining koordinatalarini toping.

26. Agar $\vec{a}=\{2;4;-1\}$ vektorning oxiri $B(-1;3;-4)$ bo'lsa, uning boshining koordinatalarini toping.

27. A va B nuqtalar berilgan. $\overrightarrow{AB}$ vektorning ortini toping:
1) $A(-4;-9;6)$, $B(8;6;-10)$; 2) $A(6;-1;9)$, $B(2;-4;-3)$.

28. $A(2;-1;0)$, $B(1;-1;2)$, $C(0,5;3)$ nuqtalar berilgan. $\vec{a}=\overrightarrow{AB}-\overrightarrow{CB}$ vektorning ortini toping.

29. $\vec{a}=\{2;3\}$, $\vec{b}=\{1;-3\}$, $\vec{c}=\{-1;3\}$ vektorlar berilgan. α ning qanday qiymatlarida $\vec{m}=\vec{a}+\alpha\vec{b}$ va $\vec{n}=\vec{a}+3\vec{c}$ vektorlar kollinear bo'ladi.

30. $\vec{a}=16\vec{i}-12\vec{j}+15\vec{k}$ vektor bilan bir xil yo'nalgan va uzunligi $|\vec{b}|=15$ bo'lgan vektorning koordinatalarini toping.

31. Uchlari $A(-1;-3)$, $B(2;-3)$, $C(2;1)$ bo'lgan uchburchakning perimetrini toping.

32. Uchlari $A(-3;-3)$, $B(-1;3)$, $C(1;-1)$ bo'lgan uchburchakning to'g'ri burchakli ekanini ko'rsating.

33. Uchlari $A(2;1)$, $B(-1;1)$, $C(-3;2)$ nuqtalarda bo'lgan uchburchakka tashqi chizilgan aylananing markazini va radiusini toping.

34. $M_1(-1;-2)$ va $M_2(3;4)$ nuqtalar berilgan. M_1M_2 to'g'ri chiziqda yotuvchi va M_2 nuqtaga nisbatan M_1 nuqtaga 3 barobar yaqin bo'lgan $M(x;y)$ nuqtani toping.

35. Uchlari $A(1;4)$, $B(-5;0)$, $C(-2;-1)$ nuqtalarda bo'lgan uchburchak medianalarining kesishish nuqtasini toping.

36. Parallelogrammning uchta ketma-ket $A(-6;-4)$, $B(-4;8)$, $C(-1;5)$ uchlari berilgan. Parallelogrammning to'rtinchi uchini toping.

37. Uchlari $A(4;1;-3)$, $B(1;4;-2)$, $C(1;10;-8)$ nuqtalarda bo'lgan ABC uchburchakning AD medianasi uzunligini toping.

2.2. VEKTORLARNING KO'PAYTMALARI

2.2.1. Ikki vektorning skalyar ko'paytmasi

Skalyar ko'paytmaning ta'rifi

1-ta'rif. *Ikki $\vec{a}$ va $\vec{b}$ vektorning skalyar ko'paytmasi* deb bu vektorlar uzunliklari bilan ular orasidagi burchak kosinusi ko'paytmasiga teng songa aytiladi va u $\vec{a}\vec{b}$ (yoki $\vec{a}\cdot\vec{b}$ yoki $(\vec{a},\vec{b})$) kabi belgilanadi, ya'ni

$$\vec{a}\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi, \tag{2.1}$$

bu yerda $\varphi-\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak (bunda vektorlarning boshi bir nuqtaga qo'yiladi).

(2.1) formulani boshqa ko'rinishda yozish mumkin.

Ma'lumki,

$$\Pr_{\vec{b}}\vec{a}=|\vec{a}|\cos\varphi \quad \text{va} \quad \Pr_{\vec{a}}\vec{b}=|\vec{b}|\cos\varphi.$$

Bundan

$$\vec{a}\vec{b}=|\vec{a}|\cdot\Pr_{\vec{a}}\vec{b} \tag{2.2}$$

yoki

$$\vec{a}\vec{b}=|\vec{b}|\cdot\Pr_{\vec{b}}\vec{a}, \tag{2.3}$$

ya'ni ikki vektorning skalyar ko'paytmasi ulardan birining moduli bilan ikkinchisining birinchi vektor yo'nalishidagi o'qqa proeksiyasining ko'paytmasiga teng.

Skalyar ko'paytmaning xossalari

1-xossa. *Ko'paytuvchilarning o'rin almashtirish* xossasi:

$$\vec{a}\vec{b}=\vec{b}\vec{a}.$$

Isboti. $\vec{a}\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos(\vec{a}\,\hat{,}\,\vec{b})=|\vec{b}|\cdot|\vec{a}|\cos(\vec{b}\,\hat{,}\,\vec{a})=\vec{b}\vec{a}.$

2-xossa. *Skalyar ko'paytuvchiga nisbatan guruhlash* xossasi:

$$(\lambda\vec{a})\vec{b}=\lambda(\vec{a}\vec{b}).$$

Isboti. (2.2) formulaga ko'ra, $(\lambda\vec{a})\vec{b}=|\vec{b}|\cdot\Pr_{\vec{b}}(\lambda\vec{a})$. Vektorning o'qdagi proeksiyasining 3-xossasiga asosan $\Pr_{\vec{b}}(\lambda\vec{a})=\lambda\cdot\Pr_{\vec{b}}|\vec{a}|$.

Bundan

$$(\lambda\vec{a})\vec{b}=|\vec{b}|\cdot\Pr_{\vec{b}}(\lambda\vec{a})=\lambda\cdot|\vec{b}|\cdot\Pr_{\vec{b}}|\vec{a}|=\lambda\cdot(|\vec{b}|\Pr_{\vec{b}}|\vec{a}|)=\lambda(\vec{a}\vec{b}).$$

3-xossa. *Qo'shishga nisbatan taqsimot* xossasi:

$$\vec{a}(\vec{b}+\vec{c})=\vec{a}\vec{b}+\vec{a}\vec{c}.$$

Isboti. Vektorning o'qdagi proeksiyasining 2-xossasiga ko'ra,

$$\Pr_{\vec{a}}(\vec{b}+\vec{c})=\Pr_{\vec{a}}\vec{b}+\Pr_{\vec{a}}\vec{c}.$$

Demak,

$$\vec{a}(\vec{b}+\vec{c})=|\vec{a}|\cdot \Pr_{\vec{a}}(\vec{b}+\vec{c})=|\vec{a}|\cdot(\Pr_{\vec{a}}\vec{b}+\Pr_{\vec{a}}\vec{c})=$$

$$=|\vec{a}|\cdot \Pr_{\vec{a}}\vec{b}+|\vec{a}|\cdot \Pr_{\vec{a}}\vec{c}=\vec{a}\vec{b}+\vec{a}\vec{c}.$$

4-xossa. Agar $\vec{a}$ va $\vec{b}$ vektorlar perpendikular bo'lsa, u holda ularning skalyar ko'paytmasi nolga teng bo'ladi. Shunindek, teskari tasdiq o'rinli: agar $\vec{a}\vec{b}=0$ $(|\vec{a}|\neq 0, |\vec{b}|\neq 0)$ bo'lsa, u holda $\vec{a}\perp\vec{b}$ bo'ladi.

Xususan: $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=\vec{j}\cdot\vec{i}=\vec{k}\cdot\vec{j}=\vec{i}\cdot\vec{k}=0.$

Isboti. $\vec{a}\perp\vec{b}$ bo'lganda $\cos\varphi=0$ bo'ladi. Bundan $\vec{a}\vec{b}=0$.

$\vec{a}\vec{b}=0$ $(|\vec{a}|\neq 0, |\vec{b}|\neq 0)$ bo'lsa, $\cos\varphi=0$ bo'ladi. Bundan $\varphi=\dfrac{\pi}{2}$, ya'ni $\vec{a}\perp\vec{b}$.

5-xossa. Vektorning skalyar kvadrati uning uzunligi kvadratiga teng, ya'ni $\vec{a}^2=|\vec{a}|^2$.

Xususan: $\vec{i}^2=\vec{j}^2=\vec{k}^2=1$.

Isboti. $\vec{a}^2=\vec{a}\cdot\vec{a}=|\vec{a}|\cdot|\vec{a}|\cos 0^\circ=|\vec{a}|\cdot|\vec{a}|=|\vec{a}|^2$.

1-izoh. Agar $\vec{a}$ vektorni skalyar kvadratga oshirib, keyin kvadrat ildiz chiqarilsa, $\vec{a}$ vektorning o'zi emas, balki uning moduli hosil bo'ladi, ya'ni

$$\sqrt{\vec{a}^2}=|\vec{a}|\ (\sqrt{\vec{a}^2}\neq\vec{a}).$$

1-misol. $|\vec{a}|=4$, $|\vec{b}|=6$, $\varphi=(\widehat{\vec{a},\vec{b}})=\dfrac{\pi}{3}$ bo'lsin. $(3\vec{a}-\vec{b})\cdot(2\vec{a}+4\vec{b})$ ko'paytmani toping.

Yechish. Avval 3-xossa yordamida qavslarni ochamiz va keyin skalyar ko'paytmaning ta'rifi va xossalaridan foydalanib, topamiz:

$$(3\vec{a}-\vec{b})\cdot(2\vec{a}+4\vec{b})=3\vec{a}\cdot 2\vec{a}-\vec{b}\cdot 2\vec{a}+3\vec{a}\cdot 4\vec{b}-\vec{b}\cdot 4\vec{b}=6\vec{a}^2+10\vec{a}\vec{b}-4\vec{b}^2=$$

$$=6|\vec{a}|^2+10|\vec{a}|\cdot|\vec{b}|\cos\frac{\pi}{3}-4|\vec{b}|^2=$$

$$=6\cdot 4^2+10\cdot 4\cdot 6\cdot\frac{1}{2}-4\cdot 6^2=96+120-144=72.$$

2-misol. $|\vec{a}|=4$, $|\vec{b}|=3$, $\varphi=(\widehat{\vec{a},\vec{b}})=\dfrac{2\pi}{3}$ bo'lsin. Bu vektorlarga qurilgan parallelogramm diagonallarining uzunliklarini toping.

Yechish. $\vec{a}$ va $\vec{b}$ vektorlarga qurilgan parallelogramm diagonallarini $\vec{a}+\vec{b}$ va $\vec{a}-\vec{b}$ vektorlar orqali ifodalash mumkin.

Skalyar ko'paytmaning xossalaridan foydalanib, topamiz:

$$|\vec{a}+\vec{b}|=\sqrt{(\vec{a}+\vec{b})^2}=\sqrt{\vec{a}^2+2\vec{a}\vec{b}+\vec{b}^2}=\sqrt{|\vec{a}|^2+2|\vec{a}||\vec{b}|\cos\varphi+|\vec{b}|^2}=$$

$$=\sqrt{16+2\cdot4\cdot3\cdot\left(-\frac{1}{2}\right)+9}=\sqrt{13},$$

$$|\vec{a}-\vec{b}|=\sqrt{(\vec{a}-\vec{b})^2}=\sqrt{\vec{a}^2-2\vec{a}\vec{b}+\vec{b}^2}=\sqrt{|\vec{a}|^2-2|\vec{a}||\vec{b}|\cos\varphi+|\vec{b}|^2}=$$

$$=\sqrt{16+2\cdot4\cdot3\cdot\frac{1}{2}+9}=\sqrt{37}.$$

Koordinatalari bilan berilgan vektorlarning skalyar ko'paytmasi

Ikkita $\vec{a}=\{a_x;a_y;a_z\}$ va $\vec{b}=\{b_x;b_y;b_z\}$ vektor berilgan bo'lsin.

U holda bu vektorlarni $\vec{i},\vec{j},\vec{k}$ birlik vektorlar orqali ifodalab, skalyar ko'paytmaning xossalarini va $\vec{i},\vec{j},\vec{k}$ vektorlarning skalyar ko'paytmalarini hisobga olib, topamiz:

$$\vec{a}\vec{b}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\cdot(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})=a_xb_x\vec{i}\vec{i}+a_xb_y\vec{i}\vec{j}+a_xb_z\vec{i}\vec{k}+$$
$$+a_yb_x\vec{j}\vec{i}+a_yb_y\vec{j}\vec{j}+a_yb_z\vec{j}\vec{k}+a_zb_x\vec{k}\vec{i}+a_zb_y\vec{k}\vec{j}+a_zb_z\vec{k}\vec{k}=$$
$$=a_xb_x+a_yb_y+a_zb_z.$$

Demak,

$$\vec{a}\vec{b}=a_xb_x+a_yb_y+a_zb_z, \qquad (2.4)$$

ya'ni koordinatalari bilan berilgan ikki vektorning skalyar ko'paytmasi ularning mos koordinatalari ko'paytmalarining yig'indisiga teng.

3-misol. $\vec{a}=\{4;-2;3\}$, $\vec{b}=\{1;-2;0\}$, $\vec{c}=\{2;1;-3\}$ bo'lsin. $(\vec{a}+3\vec{b})\cdot(\vec{a}-\vec{b}+\vec{c})$ ko'paytmani toping.

Yechish. Avval $\vec{m}=\vec{a}+3\vec{b}$ va $\vec{n}=\vec{a}-\vec{b}+\vec{c}$ vektorlarning koordinatalarini topamiz:

$$\vec{m}=\{4+3\cdot1;\ -2+3\cdot(-2);\ 3+3\cdot0\}=\{7;-8;3\},$$

$$\vec{n}=\{4-1+2;\ -2+2+1;\ 3-0-3\}=\{5;1;0\}.$$

Bundan (2.4) formulaga ko'ra

$$\vec{m}\cdot\vec{n}=7\cdot5+(-8)\cdot1+3\cdot0=27.$$

Skalyar ko'paytmaning ayrim tatbiqlari

1. Ikki vektor orasidagi burchak

$\vec{a}=\{a_x;a_y;a_z\}$ va $\vec{b}=\{b_x;b_y;b_z\}$ vektorlar orasidagi φ burchak kosinusini (2.1) va (2.4) tengliklardan topamiz:

$$\cos\varphi=\frac{\vec{a}\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \tag{2.5}$$

yoki

$$\cos\varphi=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_z^2}}. \tag{2.6}$$

Shu kabi, fazodagi ikki yo'nalish orasidagi burchak kosinusi bu yo'nalishlarning mos (bir nomdagi) yo'naltiruvchi kosinuslari ko'paytmalarining yig'indisiga teng:

$$\cos\varphi=\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2. \tag{2.7}$$

2. Ikki vektorning perpendikulyarlik sharti

$\vec{a}\perp\vec{b}$ bo'lsin. U holda $\cos\varphi=0$ bo'lgani uchun (2.6) tenglikdan

$$a_xb_x+a_yb_y+a_zb_z=0 \tag{2.8}$$

kelib chiqadi.

Fazodagi ikki *yo'nalishlarning perpendikularlik shart*ini (2.7) tenglikdan topamiz:

$$\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0. \tag{2.9}$$

3. Vektorning berilgan yo'nalishdagi proeksiyasi

(2.3) tenglikdan topamiz:

$$\text{Pr}_{\vec{b}}\,\vec{a}=\frac{\vec{a}\vec{b}}{|\vec{b}|}\ \left(\text{Pr}_{\vec{a}}\,\vec{b}=\frac{\vec{a}\vec{b}}{|\vec{a}|}\right) \tag{2.10}$$

yoki

$$\text{Pr}_{\vec{b}}\,\vec{a}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{b_x^2+b_y^2+b_z^2}}\ \left(\text{Pr}_{\vec{a}}\,\vec{b}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}}\right). \tag{2.11}$$

Shu kabi $\vec{a}(x;y;z)$ *vektorning yo'naltiruvchi kosinuslari* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *bo'lgan* l *yo'nalishdagi (o'qdagi) proeksiyasi:*

$$\mathrm{Pr}_l\, \vec{a} = x\cos\alpha + y\cos\beta + z\cos\gamma\,. \tag{2.12}$$

4. Kuchning bajargan ishi

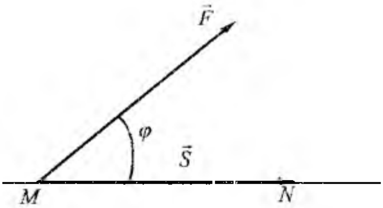
$\overrightarrow{MN}$ vektor bilan φ burchak tashkil etuvchi $\vec{F}$ kuch ta'sirida moddiy nuqta M nuqtadan N nuqtaga to'g'ri chiziq bo'ylab ko'chayotgan bo'lsin (15-shakl).

Fizika kursidan ma'lumki, $\vec{F}$ kuchning $\overrightarrow{MN} = \vec{S}$ ko'chishdagi bajargan ishi

$$A = |\vec{F}|\cdot|\vec{S}|\cdot\cos\varphi \quad \text{yoki} \quad A = \vec{F}\vec{S} \tag{2.13}$$

formula bilan aniqlanadi.

Demak, moddiy nuqtaning to'g'ri chiziqli harakatida o'zgarmas kuchning bajargan ishi kuch vektori va ko'chish vektorining skalyar ko'paytmasiga teng. Bu jumla *skalyar ko'paytmaning mexanik ma'nosini* anglatadi.



15-shakl.

4-misol. Moddiy nuqta $A(1;-2;2)$ nuqtadan $B(5;-5;-3)$ nuqtaga $\vec{F} = \{2;-1;-3\}$ kuch ta'sirida to'g'ri chiziq bo'ylab ko'chgan.

Quyidagilarni toping: 1) $\vec{F}$ kuchning bajargan ishini; 2) $\vec{F}$ kuchning ko'chish yo'nalishidagi proeksiyasini; 3) $\vec{F}$ kuchning ko'chish yo'nalishi bilan tashkil qilgan burchagini.

Yechish. Avval moddiy nuqta ko'chish vektorini, uning va $\vec{F}$ kuchning uzunligini topamiz:

$$\vec{S} = \overrightarrow{AB} = \{4;-3;-5\},\ |\vec{S}| = \sqrt{16+9+25} = 5\sqrt{2},\ |\vec{F}| = \sqrt{4+1+9} = \sqrt{14}\,.$$

U holda:

1) $A = \vec{F}\vec{S} = 2\cdot 4 + (-1)\cdot(-3) + (-3)\cdot(-5) = 26$ (*ish b.*);

2) $\mathrm{Pr}_{\vec{s}}\, \vec{F} = \dfrac{\vec{F}\vec{S}}{|\vec{S}|} = \dfrac{26}{5\sqrt{2}} = \dfrac{13\sqrt{2}}{5}$;

3) $\cos\varphi = \dfrac{\vec{F}\vec{S}}{|\vec{F}|\cdot|\vec{S}|} = \dfrac{26}{5\sqrt{2}\cdot\sqrt{14}} = \dfrac{13\sqrt{7}}{35}, \quad \varphi = \arccos\dfrac{13\sqrt{7}}{35}.$

5-misol. $\vec{m} = \vec{a} + 2\vec{b}$ va $\vec{n} = 5\vec{a} - 4\vec{b}$ o‘zaro perpendikular vektorlar bo‘lsin. $\vec{a}$ va $\vec{b}$ birlik vektorlar orasidagi burchakni toping.

Yechish. $\vec{m} \perp \vec{n}$ bo‘lgani uchun $(\vec{a} + 2\vec{b})\cdot(5\vec{a} - 4\vec{b}) = 0$ bo‘ladi.

Bundan

$$5\vec{a}^2 + 6\vec{a}\vec{b} - 8\vec{b}^2 = 0 \quad \text{yoki} \quad 5|\vec{a}|^2 + 6|\vec{a}|\cdot|\vec{b}|\cos\varphi - 8|\vec{b}|^2 = 0.$$

$\vec{a}$ va $\vec{b}$ birlik vektorlar bo‘lgani sababli: $5 + 6\cos\varphi - 8 = 0.$

U holda

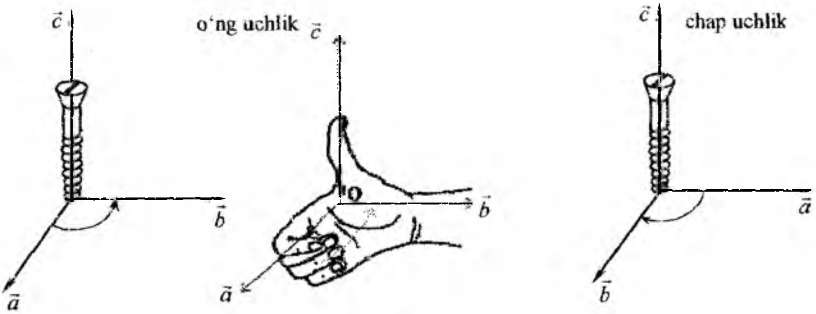
$$\cos\varphi = \frac{1}{2} \text{ yoki } \varphi = \frac{\pi}{3}.$$

2.2.2. Ikki vektorning vektor ko‘paytmasi

Vektor kopaytmaning ta’rifi

Agar uchta vektordan qaysi biri birinchi, qaysi biri ikkinchi va qaysi biri uchinchi ekani ko‘rsatilgan bo‘lsa, bu vektorlarga tartiblangan uchlik deyiladi.

Tartiblangan uchlikda vektorlar joylashish tartibida yoziladi.



16-shakl.

Agar komplanar bo‘lmagan vektorlar tartiblangan uchligining uchinchi vektori uchidan qaralganda birinchi vektordan ikkinchi vektorga qisqa burilish soat strelkasi yo‘nalishiga teskari bo‘lsa, bunday uchlikka *o'ng uchlik*, agar soat strelkasi yo'nalishida bo'lsa *chap uchlik* deyiladi (16-shakl).

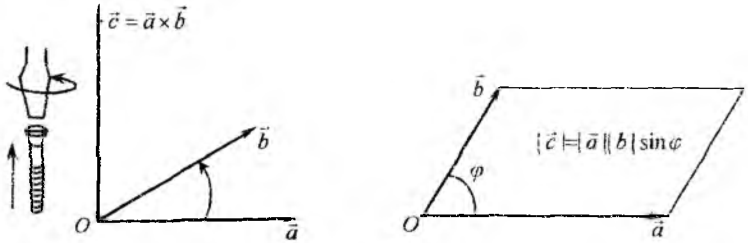
2-ta'rif. $\vec{a}$ *vektorning* $\vec{b}$ *vektorga vektor ko'paytmasi deb* quyidagi shartlar bilan aniqlanadigan $\vec{c}$ vektorga aytiladi (17-shakl):

1) $\vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarga perpendikular, ya'ni $\vec{c}\perp\vec{a}$ va $\vec{c}\perp\vec{b}$;

2) $\vec{c}$ vektorning uzunligi son jihatidan tomonlari $\vec{a}$ va $\vec{b}$ vektorlardan iborat bo'lgan parallelogrammning yuziga teng, ya'ni $|\vec{c}|=|\vec{a}|\cdot|\vec{b}|\sin\varphi$, bu yerda $\varphi=(\vec{a}\,\hat{,}\,\vec{b})$;

3) $\vec{a},\vec{b},\vec{c}$ vektorlar o'ng uchlik tashkil qiladi.

$\vec{a}$ va $\vec{b}$ vektorlarning vektor ko'paytmasi $\vec{a}\times\vec{b}$ yoki $[\vec{a},\vec{b}]$ kabi belgilanadi.



17-shakl.

Vektor ko'paytmaning xossalari

1-xossa. Ko'paytuvchilarning o'rinlari almashtirilsa vektor ko'paytma ishorasini qarama-qarshisiga o'zgartiradi, ya'ni

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}.$$

Isboti. Vektor ko'paytmaning ta'rifiga ko'ra, $\vec{a}\times\vec{b}$ va $\vec{b}\times\vec{a}$ vektorlar bir xil uzunlikka ega (parallelogrammning yuzi o'zgarmaydi), kollinear, ammo qarama-qarshi yo'nalgan, chunki $\vec{a},\vec{b},\vec{a}\times\vec{b}$ vektorlar ham, $\vec{b},\vec{a},\vec{b}\times\vec{a}$ vektorlar ham o'ng uchlik tashkil qiladi.

Demak,

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}.$$

2-xossa. *Skalyar ko'paytuvchiga nisbatan guruhlash* xossasi:

$$(\lambda\vec{a})\times\vec{b}=\lambda(\vec{a}\times\vec{b}).$$

Isboti. $\lambda > 0$ bo'lsin. U holda $(\lambda\vec{a})\times\vec{b}$ va $\lambda(\vec{a}\times\vec{b})$ vektorlar $\vec{a}$ va $\vec{b}$ vektorlarga perpendikular bo'ladi, chunki $\vec{b}$, $\lambda\vec{a}$ va $\vec{a}$ vektorlar bir tekislikda yotadi. Shu sababli $(\lambda\vec{a})\times\vec{b}$ va $\lambda(\vec{a}\times\vec{b})$ vektorlar kollinear. Shuningdek, bu vektorlar yo'nalishdosh ($\lambda\vec{a}$ va $\vec{a}$ vektorlar yo'nalishdosh) hamda ular bir xil uzunlikka ega:

$$|(\lambda\vec{a})\times\vec{b}| = |\lambda\vec{a}|\cdot|\vec{b}|\sin((\lambda\vec{a})\hat{,}\vec{b})) = \lambda|\vec{a}|\cdot|\vec{b}|\sin(\vec{a}\hat{,}\vec{b}),$$

$$|\lambda(\vec{a}\times\vec{b})| = \lambda|\vec{a}\times\vec{b}| = \lambda|\vec{a}|\cdot|\vec{b}|\sin(\vec{a}\hat{,}\vec{b}).$$

Demak,

$$(\lambda\vec{a})\times\vec{b} = \lambda(\vec{a}\times\vec{b}).$$

Xossa $\lambda < 0$ da ham shu kabi isbotlanadi.

3-xossa. *Qo'shishga nisbatan taqsimot* xossasi:

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}.$$

Bu xossaning isbotini keltirmaymiz.

4-xossa. Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa, u holda ularning vektor ko'paytmasi nolga teng bo'ladi. Shunindek, teskari tasdiq o'rinli: agar $\vec{a}\times\vec{b} = 0$ ($|\vec{a}|\neq 0, |\vec{b}|\neq 0$) bo'lsa, u holda $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'ladi.

Isboti. $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa, ular orasidagi burchak $\varphi = 0^\circ$ yoki $\varphi = 180^\circ$ ga teng va $\sin\varphi = 0$ bo'ladi. U holda $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\sin\varphi = 0$. Bundan

$$\vec{a}\times\vec{b} = 0.$$

$\vec{a}\times\vec{b} = 0$ bo'lsa, $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\sin\varphi = 0$ bo'ladi. U holda $|\vec{a}|\cdot|\vec{b}|\neq 0$ bo'lgani uchun $\sin\varphi = 0$. Bundan $\varphi = 0^\circ$ yoki $\varphi = 180^\circ$, ya'ni $\vec{a}$ va $\vec{b}$ vektorlar kollinear.

6-misol. $\vec{i},\vec{j},\vec{k}$ vektorlarning vektor ko'paytmalarini toping.

Yechish. Bunda vektor ko'paytmaning ta'rifidan quyidagi tengliklar bevosita kelib chiqadi:

$$\vec{i}\times\vec{j} = \vec{k},\ \ \vec{j}\times\vec{k} = \vec{i},\ \ \vec{k}\times\vec{i} = \vec{j}.$$

Haqiqatan ham, masalan, $\vec{i}\times\vec{j} = \vec{k}$ tenglik o'rinli, chunki:

1) $\vec{k}\perp\vec{i}$, $\vec{k}\perp\vec{j}$;

2) $|\vec{k}| = |\vec{i}||\vec{j}|\sin 90^\circ = 1$; 3) $\vec{i},\vec{j},\vec{k}$ vektorlar o'ng uchlik tashkil qiladi.

Shuningdek, 1- xossaga ko'ra,

$$\vec{j}\times\vec{i}=-\vec{k},\ \ \vec{k}\times\vec{j}=-\vec{i},\ \ \vec{i}\times\vec{k}=-\vec{j}.$$

Vektor ko'paytmaning 4-xossasidan topamiz:

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0.$$

7-misol. $|\vec{a}|=3,\ |\vec{b}|=2,\ \varphi=(\vec{a}\,\hat{,}\,\vec{b})=\frac{\pi}{6}$ bo'lsin. $|(\vec{a}+2\vec{b})\times(\vec{a}-3\vec{b})|$ni hisoblang.

Yechish. Vektor ko'paytmaning ta'rifi va xossalaridan foydalanib, topamiz:

$$(\vec{a}+2\vec{b})\times(\vec{a}-3\vec{b})=\vec{a}\times\vec{a}+2\vec{b}\times\vec{a}-3\vec{a}\times\vec{b}-6\vec{b}\times\vec{b}=-5\vec{a}\times\vec{b}.$$

Bundan

$$|(\vec{a}+2\vec{b})\times(\vec{a}-3\vec{b})|=|-5\vec{a}\times\vec{b}|=5|\vec{a}|\cdot|\vec{b}|\sin\varphi=5\cdot3\cdot2\cdot\sin\frac{\pi}{6}=15.$$

Koordinatalari bilan berilgan vektorlarning vektor ko'paytmasi

Ikkita $\vec{a}=\{a_x;a_y;a_z\}$ va $\vec{b}=\{b_x;b_y;b_z\}$ vektor berilgan bo'lsin.

$\vec{i},\vec{j},\vec{k}$ vektorlarning vektor ko'paytmalari formulalaridan foydalanib, topamiz:

$$\vec{a}\times\vec{b}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\times(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})=a_xb_x(\vec{i}\times\vec{i})+a_xb_y(\vec{i}\times\vec{j})+a_xb_z(\vec{i}\times\vec{k})+$$

$$+a_yb_x(\vec{j}\times\vec{i})+a_yb_y(\vec{j}\times\vec{j})+a_yb_z(\vec{j}\times\vec{k})+a_zb_x(\vec{k}\times\vec{i})+a_zb_y(\vec{k}\times\vec{j})+a_zb_z(\vec{k}\times\vec{k})=$$

$$=a_xb_y\vec{k}-a_xb_z\vec{j}-a_yb_x\vec{k}+a_yb_z\vec{i}+a_zb_x\vec{j}-a_zb_y\vec{i}=(a_yb_z-a_zb_y)\vec{i}-(a_xb_z-a_zb_x)\vec{j}+$$

$$+(a_xb_y-a_yb_x)\vec{k}=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\vec{i}-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\vec{j}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\vec{k},$$

ya'ni

$$\vec{a}\times\vec{b}=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\vec{i}-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\vec{j}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\vec{k}. \tag{2.14}$$

Oxirgi tenglikni quyidagicha yozish mumkin:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}. \tag{2.15}$$

8-misol. $\vec{a}=\{3;-1;-2\}, \vec{b}=\{0;-2;4\}$ bo'lsin. $(\vec{a}+2\vec{b})\times(2\vec{a}-3\vec{b})$ ko'paytmani toping.

Yechish. Avval $\vec{m}=\vec{a}+2\vec{b}$ va $\vec{n}=2\vec{a}-3b$ vektorlarning koordinatalarini topamiz:

$$\vec{m}=\{1\cdot 3+2\cdot 0;1\cdot(-1)+2\cdot(-2);1\cdot(-2)+2\cdot 4\}=\{3;-5;6\},$$

$$\vec{n}=\{2\cdot 3-3\cdot 0;2\cdot(-1)-3\cdot(-2);2\cdot(-2)-3\cdot 4\}=\{6;-8;-16\}.$$

Bundan (2.14) formulaga ko'ra

$$\vec{m}\times\vec{n}=\begin{vmatrix}-5 & 6\\ -8 & -16\end{vmatrix}\vec{i}-\begin{vmatrix}3 & 6\\ 6 & -16\end{vmatrix}\vec{j}+\begin{vmatrix}3 & -5\\ 6 & -8\end{vmatrix}\vec{k}=128\vec{i}+84\vec{j}+6\vec{k}.$$

Vektor ko'paytmaning ayrim tatbiqlari

1. Ikki vektorning kollinearlik sharti

Vektor ko'paytmaning 4-xossasiga ko'ra, $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa

$$\vec{a}\times\vec{b}=0$$

yoki

$$\vec{a}\times\vec{b}=(a_yb_z-a_zb_y)\vec{i}-(a_xb_z-a_zb_x)\vec{j}+(a_xb_y-a_yb_x)\vec{k}=0$$

bo'ladi.

Bundan

$$a_yb_z-a_zb_y=0,\ a_xb_z-a_zb_x=0,\ a_xb_y-a_yb_x=0$$

yoki

$$\frac{a_x}{b_x}=\frac{a_y}{b_y}=\frac{a_z}{b_z}, \tag{2.16}$$

ya'ni kollinear vektorlarning koordinatalari proporsional bo'ladi va aksincha proporsional koordinatalarga ega vektorlar kollinear bo'ladi.

2. Parallelogramm va uchburchakning yuzlari

Vektor ko'paytmaning ta'rifiga ko'ra $|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\sin\varphi$, ya'ni

$$S_{par}=|\vec{a}\times\vec{b}|.$$

Bundan

$$S_{par}=|\vec{a}\times\vec{b}|,\quad S_{uchb}=\frac{1}{2}|\vec{a}\times\vec{b}|. \tag{2.17}$$

3. Nuqtaga nisbatan kuch momenti

O nuqtasi mahkamlangan qattiq jism A nuqtasiga qo'yilgan $\vec{F}$ kuch ta'sirida O nuqta atrofida aylanma harakat qilayotgan bo'lsin, masalan, bolt kalit yordamida buralayotgan bo'lsin (18-shakl).

Fizika kursidan ma'lumki, $\vec{F}$ kuchning O nuqtaga nisbatan momenti deb O nuqtadan o'tuvchi va quyidagi shartlarni qanoatlantiruvchi $\overrightarrow{M}$ vektorga aytiladi:

1) $\overrightarrow{M} \perp \vec{r}$ va $\overrightarrow{M} \perp \vec{F}$, bu yerda $\vec{r} = \overrightarrow{OA}$ – A nuqtaning radius vektori;

2) $|\overrightarrow{M}| = |\vec{r}| \cdot |\vec{F}| \sin\varphi$, bu yerda $\varphi = (\vec{r}\,\hat{,}\,\vec{F})$;

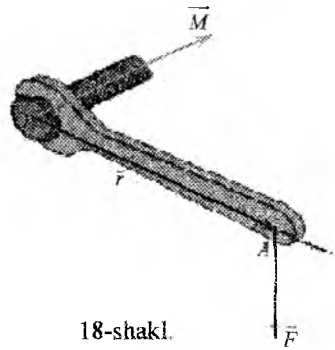
3) $\vec{r}, \vec{F}, \overrightarrow{M}$ vektorlar o'ng uchlik tashkil qiladi.

Shunday qilib,

$$\overrightarrow{M} = \vec{r} \times \vec{F},$$

ya'ni qo'zg'almas nuqtaga nisbatan kuch momenti kuch qo'yilgan nuqta radius vektorining kuch vektoriga vektor ko'paytmasiga teng.

Bu jumla *vektor ko'paytmaning mexanik ma'nosini* anglatadi.



18-shakl.

4. Aylanma harakatning chiziqli tezligi

Yuqorida keltirilgandagi kabi qo'zg'almas O nuqta atrofida $\vec{\omega}$ burchak tezlik bilan aylnma harakat qilayotgan qattiq jism M nuqtasining chiziqli tezligi Eyler formulasi bilan topiladi:

$$\vec{v} = \vec{\omega} \times \vec{r},$$

bu yerda $\vec{r} = \overrightarrow{OM}$ – M nuqtaning radius vektori.

9-misol. m, n ning qanday qiymatlarida $\vec{a} = \{-2; 3; n\}$ va $\vec{b} = \{m; -6; 2\}$ vektorlar kollinear bo'ladi?

Yechish. Ikki vektorning kollinearlik shartiga ko'ra, $\frac{-2}{m} = \frac{3}{-6} = \frac{n}{2}$.

Bundan $m = 4$, $n = -1$.

10-misol. $\vec{a} = 2\vec{j} - 3\vec{k}$ va $\vec{b} = 4\vec{i} + 3\vec{j}$ vektorlarga qurilgan parallelogrammning yuzini hisoblang.

Yechish. Yuzani (2.17) formula bilan hisoblaymiz:

$$S=|\vec{a}\times\vec{b}|=\sqrt{\begin{vmatrix}2 & -3\\ 3 & 0\end{vmatrix}^2+\begin{vmatrix}0 & -3\\ 4 & 0\end{vmatrix}^2+\begin{vmatrix}0 & 2\\ 4 & 3\end{vmatrix}^2}=\sqrt{9^2+12^2+(-8)^2}=17(y.b.).$$

2.2.3. Uchta vektorning aralash ko'paytmasi

Aralash ko'paytmaning ta'rifi va geometrik ma'nosi

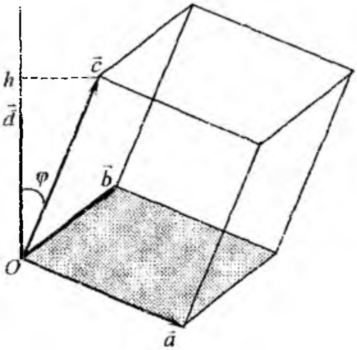
3-ta'rif. *Uchta $\vec{a},\vec{b}$ va $\vec{c}$ vektorning aralash ko'paytmasi* deb $\vec{a}\times\vec{b}$ vektorning $\vec{c}$ vektorga skalyar ko'paytmasiga teng songa aytiladi va $\vec{a}\vec{b}\vec{c}$ kabi belgilanadi.

Uchta komplanar bo'lmagan $\vec{a},\vec{b},\vec{c}$ vek-torlar berilgan bo'lsin. Bu vektorlarga parallelepiped quramiz va $\vec{a}\times\vec{b}=\vec{d}$ vektorni yasaymiz (19-shakl).

Vektor ko'paytmaning ta'rifiga ko'ra,

$$\vec{d}\perp\vec{a},\quad \vec{d}\perp\vec{b},\quad |\vec{d}|=S_{par},$$

bu yerda S_{par} – parallelepiped asosining yuzi.



19-shakl.

Ta'rifga ko'ra $\quad \vec{d}\cdot\vec{c}=|\vec{d}|\cdot|\vec{c}|\cos\varphi,$

bu yerda φ – $\vec{c}$ va $\vec{d}$ vektorlar orasidagi burchak.

19-shaklda $\vec{a},\vec{b},\vec{c}$ vektorlar o'ng uchlik tashkil qiladi va $\varphi<\frac{\pi}{2}$, ya'ni $\cos\varphi>0$. U holda $|\vec{c}|\cos\varphi=h$ va $\vec{d}\cdot\vec{c}=S_{par}\cdot h=V$. Ikkinchi tomondan $\vec{d}\cdot\vec{c}=(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\vec{b}\vec{c}$. Demak, $V=\vec{a}\vec{b}\vec{c}$.

Agar $\vec{a},\vec{b},\vec{c}$ vektorlar chap uchlik tashkil qilsa, $\varphi>\frac{\pi}{2}$ va $\cos\varphi<0$ bo'ladi. U holda $|\vec{c}|\cos\varphi=-h$, $V=-\vec{a}\vec{b}\vec{c}$.

Shunday qilib, komplanar bo'lmagan uchta vektor aralash ko'paytmasining moduli qirralari bu vektorlarning uzunliklaridan iborat bo'lgan parallelepiped hajmiga teng:

$$V=|\vec{a}\vec{b}\vec{c}|. \tag{2.18}$$

Bu jumla *aralash ko'paytmaning geometrik ma'nosini* anglatadi.

Aralash ko'paytmaning xossalari

1-xossa. Amallarining o'rinlari almashtirilsa aralash ko'paytma o'zgarmaydi, ya'ni

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c}).$$

Isboti. Skalyar ko'paytmaning o'rin almashtirish xossasiga ko'ra,

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{c}\cdot(\vec{a}\times\vec{b}).$$

(2.18) formulaga ko'ra,

$$V=|(\vec{a}\times\vec{b})\cdot\vec{c}|,\quad V=|(\vec{b}\times\vec{c})\cdot\vec{a}|.$$

Bunda $\vec{a},\vec{b},\vec{c}$ va $\vec{b},\vec{c},\vec{a}$ uchliklarning har ikkalasi bir vaqtda yoki o'ng uchlik yoki chap uchlik tashkil qiladi. Shu sababli $(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}$.

Bundan

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c}).$$

2-xossa. Ko'paytuvchilarning o'rinlari doiraviy almashtirilsa, aralash ko'paytma o'zgarmaydi, ya'ni

$$\vec{a}\vec{b}\vec{c}=\vec{b}\vec{c}\vec{a}=\vec{c}\vec{a}\vec{b}.$$

Isboti. 1-xossa va skalyar ko'paytmaning o'rin almashtirish xossasidan topamiz:

$$\vec{a}\vec{b}\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{b}\times\vec{c})\cdot\vec{a}=\vec{b}\vec{c}\vec{a},$$

$$\vec{b}\vec{c}\vec{a}=\vec{b}\cdot(\vec{c}\times\vec{a})=(\vec{c}\times\vec{a})\cdot\vec{b}=\vec{c}\vec{a}\vec{b}.$$

3-xossa. Ikkita qo'shni ko'paytuvchining o'rinlari almashtirilsa, aralash ko'paytma ishorasi qarama-qarshisiga almashadi. Masalan, $\vec{a}\vec{b}\vec{c}=-\vec{b}\vec{a}\vec{c}$.

Isboti. $\vec{a}\vec{b}\vec{c}=(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{b}\times\vec{a})\cdot\vec{c}=-\vec{b}\vec{a}\vec{c}.$

4-xossa. Agar nolga teng bo'lmagan $\vec{a},\vec{b},\vec{c}$ vektorlar komplanar bo'lsa, u holda ularning aralash ko'paytmasi nolga teng bo'ladi.

Shunindek, teskari tasdiq o'rinli: agar $\vec{a}\vec{b}\vec{c}=0$ $(|\vec{a}|\neq 0,|\vec{b}|\neq 0,|\vec{c}|\neq 0)$ bo'lsa, u holda $\vec{a},\vec{b},\vec{c}$ vektorlar komplanar bo'ladi.

Isboti. $\vec{a}\vec{b}\vec{c}=0$ $(|\vec{a}|\neq 0,|\vec{b}|\neq 0,|\vec{c}|\neq 0)$ bo'lsin. $\vec{a},\vec{b},\vec{c}$ vektorlar komplanar emas deb faraz qilamiz.

U holda bu vektorlarga hajmi $V \neq 0$ bo'lgan parallelopiped qurish mumkin. $V = \pm \vec{a}\vec{b}\vec{c}$ dan $\vec{a}\vec{b}\vec{c} \neq 0$ kelib chiqadi. Bu $\vec{a}\vec{b}\vec{c} = 0$ shartga zid. Demak, qilingan faraz noto'g'ri va $\vec{a}, \vec{b}, \vec{c}$ vektorlar komplanar.

$\vec{a}, \vec{b}, \vec{c}$ vektorlar komplanar bo'lsin.

U holda $\vec{d} = \vec{a} \times \vec{b}$ vektor $\vec{a}, \vec{b}, \vec{c}$ vek-torlar yotgan tekislikka perpendikulyar bo'ladi.

Bundan $\vec{d} \perp \vec{c}$. Shu sababli $\vec{d} \cdot \vec{c} = 0$ yoki $\vec{a}\vec{b}\vec{c} = 0$.

Koordinatalari bilan berilgan vektorlarning aralash ko'paytmasi

Uchta $\vec{a} = \{a_x; a_y; a_z\}$, $\vec{b} = \{b_x; b_y; b_z\}$ va $\vec{c} = \{c_x; c_y; c_z\}$ vektor berilgan bo'lsin.

U holda

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k},$$

$$\vec{a}\vec{b}\vec{c} = \left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k} \right) \cdot (c_x \vec{i} + c_y \vec{j} + c_z \vec{k}) =$$

$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z$$

yoki

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{2.19}$$

11-misol. $\vec{a} = \{-2;2;1\}$, $\vec{b} = \{3;-2;5\}$, $\vec{c} = \{1;-1;3\}$ vektorlar berilgan. $\vec{a}\vec{b}\vec{c}$ ko'paytmani hisoblang.

Yechish. $\vec{a}\vec{b}\vec{c}$ ni (2.19) formula bilan topamiz:

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} -2 & 2 & 1 \\ 3 & -2 & 5 \\ 1 & -1 & 3 \end{vmatrix} = 12 + 10 - 3 + 2 - 10 - 18 = -7.$$

Aralash ko'paytmaning ayrim tatbiqlari

1. Fazodagi vektorlarning o'zaro joylashishi

$\vec{a},\vec{b},\vec{c}$ vektorlarning fazoda o'zaro joylashishini aniqlash $V=\pm\vec{a}\vec{b}\vec{c}$ bo'lishiga asoslanadi. Bunda agar $\vec{a}\vec{b}\vec{c}>0$ bo'lsa, u holda vektorlar o'ng uchlik tashkil qiladi, agar $\vec{a}\vec{b}\vec{c}<0$ bo'lsa, u holda vektorlar chap uchlik tashkil qiladi.

2. Uchta vektorning komplanarlik sharti

Aralash ko'paytmaning 4-xossasiga ko'ra nolga teng bo'lmagan $\vec{a},\vec{b},\vec{c}$ vektorlar komplanar bo'lsa, u holda $\vec{a}\vec{b}\vec{c}=0$ yoki

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0. \tag{2.20}$$

3. Parallelepiped va piramidaning hajmlari

Aralash ko'paytmaning geometrik ma'nosiga ko'ra, $\vec{a},\vec{b},\vec{c}$ vektorlarga qurilgan parallelopiped hajmini $V_{par}=|\vec{a}\vec{b}\vec{c}|$ bilan va piramida hajmini $V_{pir}=\frac{1}{6}|\vec{a}\vec{b}\vec{c}|$ bilan topish mumkin.

Shunday qilib,

$$V_{par}=\mathrm{mod}\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}, \qquad V_{pir}=\frac{1}{6}\mathrm{mod}\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{2.21}$$

12-misol. Uchlari $A(2;3;1)$, $B(4;1;-2)$, $C(6;3;7)$, $D(-5;-4;8)$ nuqtalarda bo'lgan piramidaning D uchidan tushirilgan h balandligi uzunligini toping.

Yechish. Avval piramida qirralarini ifodalovchi vektorlarni topamiz:

$$\overline{AB}=\{2;-2;-3\},\ \overline{AC}=\{4;0;6\},\ \overline{AD}=\{-7;-7;7\}.$$

Piramida hajmini hisoblaymiz:

$$V=\frac{1}{6}\mathrm{mod}\begin{vmatrix} 2 & -2 & -3 \\ 4 & 0 & 6 \\ -7 & -7 & 7 \end{vmatrix}=\frac{1}{6}|84+84+84+56|=\frac{154}{3}.$$

ABC yoq yuzini hisoblaymiz:

$$S=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{\begin{vmatrix} -2 & -3 \\ 0 & 6 \end{vmatrix}^2+\begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix}^2+\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix}^2}=$$

$$=\frac{1}{2}\sqrt{(-12)^2+24^2+8^2}=14.$$

Piramida uchun $V=\frac{1}{3}hS$.

Bundan

$$h=\frac{3V}{S}=\frac{3\cdot\frac{154}{3}}{14}=\frac{154}{14}=11\ (u.b.).$$

13-misol. Fazoda A,B,C,D nuqtalar koordinatalari bilan berilgan. $A(7;2;2)$, $B(5;7;7)$, $C(4;6;10)$, $D(2;3;7)$. Quyidagilarni toping:

1) $\overrightarrow{AB}$ vektor proeksiyalari va yo'nalishini;
2) $\overrightarrow{AB}\cdot\overrightarrow{AC}$, $\overrightarrow{AB}\times\overrightarrow{AC}$ ko'paytmalarni;
3) ABC uchburchak yuzasini;
4) $ABCD$ piramida hajmini.

Yechish. 1) $\overrightarrow{AB}$ vektor proeksiyasini (1.18) formula bilan topamiz:

$$\overrightarrow{AB}=\{a_x;a_y;a_z\}=\{5-7;7-2;7-2\}=\{-2;5;5\}.$$

Bundan (1.12) formulaga ko'ra,

$$|\overrightarrow{AB}|=|\vec{a}|=\sqrt{(-2)^2+5^2+5^2}=3\sqrt{6}.$$

$\overrightarrow{AB}$ vektor yo'nalishini (1.13) formulalar bilan topamiz:

$$\cos\alpha=\frac{a_x}{|\vec{a}|}=-\frac{\sqrt{6}}{9},\quad \cos\beta=\frac{a_y}{|\vec{a}|}=\frac{5\sqrt{6}}{18},\quad \cos\gamma=\frac{a_z}{|\vec{a}|}=\frac{5\sqrt{6}}{18}.$$

Topilgan yechimlarni *Maple* paketida bajaramiz:

```
> with(geom3d):
> point(A,7,2,2),point(B,5,7,7), point(C,4,6,10), point(D,2,3,7):
> with(LinearAlgebra):
> v := <a,b,c>;
```

$$v := (a)e_x + (b)e_y + (c)e_z$$

```
> VectorNorm(v,2,conjugate=false);
```

$$\sqrt{a^2 + b^2 + c^2}$$

```
> v1 := <5-7,7-2,7-2>;
```

$$v1 := -2e_x + 5e_y + 5e_z$$

```
> VectorNorm(v1,2,conjugate=false);
```

$$3\sqrt{6}$$

```
> Normalize(<a,b,c>,Euclidean,conjugate=false);
```

$$\begin{bmatrix} \dfrac{a}{\sqrt{a^2+b^2+c^2}} \\ \dfrac{b}{\sqrt{a^2+b^2+c^2}} \\ \dfrac{c}{\sqrt{a^2+b^2+c^2}} \end{bmatrix}$$

```
> Normalize(v1,Euclidean,conjugate=false);
```

$$\begin{bmatrix} -\frac{1}{9}\sqrt{6} \\ \frac{5}{18}\sqrt{6} \\ \frac{5}{18}\sqrt{6} \end{bmatrix}$$

2) (2.2.8) formuladan $\overline{AB} = \{-2;5;5\}$, $\overline{AC} = \{-3,4,8\}$ larni hisobga olib, topamiz:

$$\overline{AB} \cdot \overline{AC} = (-2)\cdot(-3) + 5\cdot 4 + 5\cdot 8 = 66$$

```
> vAC := <4-7,6-2,10-2>;
```

$$vAC := -3e_x + 4e_y + 8e_z$$

```
> AB.AC:=Dotroduct( <5-7,7-2,7-2>, <4-7,6-2,10-2> );
```

$$VectorCalculus:-.(AB, AC) := 66$$

$\overrightarrow{AB}\times\overrightarrow{AC}$ vektor ko‘paytmani (2.15) formula bilan topamiz:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 5 & 5\\ -3 & 4 & 8\end{vmatrix}=\begin{vmatrix}5 & 5\\ 4 & 8\end{vmatrix}\vec{i}-\begin{vmatrix}-2 & 5\\ -3 & 8\end{vmatrix}\vec{j}-\begin{vmatrix}-2 & 5\\ -3 & 4\end{vmatrix}\vec{k}=20\vec{i}+\vec{j}+7\vec{k}$$

```
> A := <<i,l,o>|<j,m,>|<k,n,q>>;
```

$$A:=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ l & m & n\\ o & p & q\end{vmatrix}$$

```
> Determinant(A);
```

$$i\,m\,q-i\,n\,p+l\,k\,p-l\,j\,q+o\,j\,n-o\,k\,m$$

```
> axb := <<i,-2,-3>|<j,5,4>|<k,5,8>>;
```

$$axb:=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 5 & 5\\ -3 & 4 & 8\end{vmatrix}$$

```
> Determinant(axb);
```

$$20\,i+7\,k+j$$

3) ABC uchburchak yuzasini (2.2.17) formula bilan hisoblaymiz:

$$S=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{20^2+1^2+7^2}=\frac{15\sqrt{2}}{2}$$

```
> modN:=VectorNorm(N,2,conjugate=false);
```

$$modN:=15\sqrt{2}$$

```
> s:=modN/2;
```

$$s:=\frac{15}{2}\sqrt{2}$$

4) $ABCD$ piramida hajmini topamiz:

$$V=\frac{1}{6}\mathrm{mod}\begin{vmatrix}-2 & 5 & 5\\ -3 & 4 & 8\\ -5 & 1 & 5\end{vmatrix}=\frac{1}{6}\cdot 64=\frac{32}{3}$$

```
> with(geom3d):
> abc := <<-5,-2,-3>|<1,5,4>|<5,5,8>>;
```

$$abc:=\begin{vmatrix}-2 & 5 & 5\\ -3 & 4 & 8\\ -5 & 1 & 5\end{vmatrix}$$

> Determinant(abc);

$$-64$$

> VABCD:={ABC(Determinant(abc))/6};

$$VABCD = \frac{32}{3}.$$

2.2.4. Mashqlar

1. Tomonlari birga teng bo‘lgan teng tomonli ABC uchburchak berilgan. $\overrightarrow{AB}\cdot\overrightarrow{BC}+\overrightarrow{BC}\cdot\overrightarrow{CA}+\overrightarrow{CA}\cdot\overrightarrow{AB}$ ifodaning qiymatini toping.

2. Tomonlari $BC=5,\ CA=6,\ AB=7$ ga teng bo‘lgan ABC uchburchak berilgan. $\overrightarrow{AB}\cdot\overrightarrow{BC}$ skalyar ko‘paytmani toping.

3. Agar $|\vec{a}|=6,\ |\vec{b}|=4,\ \varphi=(\vec{a}\,\hat{,}\,\vec{b})=\frac{2\pi}{3}$ bo‘lsin. Toping:

1) $(2\vec{a}+\vec{b})^2$; 2) $(2\vec{a}-3\vec{b})\cdot(\vec{a}-2\vec{b})$.

4. $\vec{a}=\{1;-2;2\}$ va $\vec{b}=\{2;4;-5\}$ vektorlar berilgan. Toping:

1) $(3\vec{a}-2\vec{b})\cdot(\vec{a}+\vec{b})$; 2) $(\vec{a}-\vec{b})^2$.

5. Berilgan vektorlar m ning qanday qiymatlarida perpendikular bo‘ladi?

1) $\vec{a}=\{1;-2m;0\}$, $\vec{b}=\{4;2;3m\}$; 2) $\vec{a}=\{m;-5;2\}$, $\vec{b}=\{m-2;m;m+3\}$.

6. $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ birlik vektorlar uchun $\vec{e}_1+\vec{e}_2+\vec{e}_3=0$ bo‘lsa, $\vec{e}_1\vec{e}_2+\vec{e}_2\vec{e}_3+\vec{e}_3\vec{e}_1$ ni toping.

7. Tomonlari $\vec{a}=2\vec{i}+\vec{j}$ va $\vec{b}=-\vec{j}+2\vec{k}$ vektorlardan iborat bo‘lgan parallelogrammning diagonallari orasidagi burchakni toping.

8. Uchlari $A(-1;-2;4)$, $B(-4;-2;0)$, $C(3;-2;1)$ bo‘lgan ABC uchburchak berilgan. $\angle B$ ni toping.

9. xOz va yOz burchaklarning bissektrisalari qanday burchak tashkil qiladi?

10. Koordinata o‘qlari bilan tashkil qilgan burchaklari berilgan fazodagi ikki yo‘nalish orasidagi burchakni toping:

1) $l_1:\left(\frac{\pi}{4};\frac{\pi}{2};\frac{\pi}{4}\right)$ va $l_2:\left(\frac{\pi}{4};\frac{\pi}{4};\frac{\pi}{2}\right)$; 2) $l_1:\left(\frac{\pi}{6};\frac{\pi}{3};\frac{\pi}{4}\right)$ va $l_2:\left(\frac{5\pi}{6};\frac{2\pi}{3};\frac{\pi}{2}\right)$.

2.11. $\vec{a}=\{3;-6;-1\}$, $\vec{b}=\{1;4;-5\}$, $\vec{c}=\{3;-4;12\}$ vektorlar berilgan. Quyidagilarni toping:

1) $\mathrm{Pr}_{\vec{c}}\,\vec{a}$; 2) $\mathrm{Pr}_{\vec{c}}(2\vec{a}-3\vec{b})$.

12. $A(1;2;-3)$ nuqtani $B(5;6;-1)$ nuqtaga to‘g‘ri chiziq bo‘ylab ko‘chirishda $\vec{F}=\{2;-1;3\}$ kuchning bajargan ishini toping.

13. $\vec{a}=\{3;-1;5\}$ va $\vec{b}=\{1;2;-3\}$ vektorlar berilgan. Agar $\vec{x}\cdot\vec{a}=9$, $\vec{x}\cdot\vec{b}=-4$ va $\vec{x}$ vektor Oz oqiga perpendikular bo‘lsa, $\vec{x}$ vektorning koordinatalarini toping.

14. $\vec{a}=\{2;-3;1\}$, $\vec{b}=\{1;-2;3\}$ va $\vec{c}=\{1;2;-7\}$ vektorlar berilgan. Agar $\vec{x}\perp\vec{a}$, $\vec{x}\perp\vec{b}$, $\vec{x}\cdot\vec{c}=10$ bo‘lsa, $\vec{x}$ vektorni toping.

15. Agar $|\vec{a}|=4$, $|\vec{b}|=6$, $\varphi=(\vec{a}\,\hat{,}\,\vec{b})=\dfrac{5\pi}{6}$ bo‘lsa, quyidagilarni toping:

1) $\vec{a}\times\vec{b}$; 2) $|(2\vec{a}-3\vec{b})\times(\vec{a}+4\vec{b})|$.

16. Tomonlari $\vec{a}$ va $\vec{b}$ vektorlar uzunliklaridan iborat bo‘lgan parallelogrammning yuzini toping:

1) $\vec{a}=\vec{m}+2\vec{n}$, $\vec{b}=2\vec{m}+\vec{n}$, bu yerda $|\vec{m}|=1$, $|\vec{n}|=1$, $\varphi=(\vec{m}\,\hat{,}\,\vec{n})=\dfrac{\pi}{6}$;

2) $\vec{a}=3\vec{m}-2\vec{n}$, $\vec{b}=5\vec{m}+4\vec{n}$, bu yerda $|\vec{m}|=2$, $|\vec{n}|=3$, $\varphi=(\vec{m}\,\hat{,}\,\vec{n})=\dfrac{\pi}{3}$.

17. Agar $|\vec{a}|=5$, $|\vec{b}|=10$, $\vec{a}\vec{b}=25$ bo‘lsa, $|\vec{a}\times\vec{b}|$ni toping.

18. Agar $|\vec{a}|=3$, $|\vec{b}|=13$, $|\vec{a}\times\vec{b}|=36$ bo‘lsa, $\vec{a}\vec{b}$ ni toping.

19. $\vec{a}=\{-1;2;3\}$ va $\vec{b}=\{2;-1;3\}$ vektorlar berilgan. Vektor ko‘paytmalarini toping:

1) $\vec{a}\times\vec{b}$; 2) $(2\vec{a}+\vec{b})\times(3\vec{b}-\vec{a})$.

20. Tomonlari $\vec{a}$ va $\vec{b}$ vektorlar uzunliklaridan iborat bo‘lgan uchburchakning yuzini toping:

1) $\vec{a}=\{2;-2;1\}$, $\vec{b}=\{8;4;1\}$; 2) $\vec{a}=\{3;5;-8\}$, $\vec{b}=\{6;3;-2\}$.

21. Uchburchak uchlari $A(1;2;0)$, $B(3;0;-3)$, $C(5;2;6)$ berilgan. Uning B uchidan AC tomonga tushirilgan balandlik uzunligini toping.

22. A nuqtaga $\vec{F}$ kuch qo‘yilgan. Bu kuchning B nuqtaga nisbatan momentini
toping:) $\vec{F}=\{2;-4;5\}$, $A(0;2;1)$, $B(-1;2;3)$; 2) $\vec{F}=\{1;2;-1\}$, $A(-1;4;-2)$, $B(2;3;-1)$.

23. $\vec{F}=\{2;2;4\}$ kuch $A(4;2;-3)$ qo‘yilgan. $B(0;2;4)$ nuqtaga nisbatan kuch momentining qiymatini va yo‘naltiruvchi kosinuslarini toping.

24. Kollinear bo‘lmagan $\vec{m}$ va $\vec{n}$ vektorlar berilgan. $\vec{a}=\alpha\cdot\vec{m}+6\vec{n}$ va $\vec{b}=3\vec{m}-2\vec{n}$ vektorlar α ning qanday qiymatida kollinear bo‘ladi?

25. $\vec{a}=\{-1;3;\alpha\}$ va $\vec{b}=\{\beta;-6;-3\}$ vektorlar α va β ning qanday qiymatlarida kollinear bo‘ladi?

26. Ikkita $\vec{a}=\{2;-3\}$, $\vec{b}=\{-1;5\}$ vektorlar berilgan. Quyidagi shartlarni qanoatlantiruvchi $\vec{x}$ vektorning koordinatalarini toping:

1) $\vec{x}\perp\vec{a}$ va $\vec{b}\cdot\vec{x}=7$; 2) $\vec{x}\parallel\vec{a}$ va $\vec{b}\cdot\vec{x}=17$.

27. $\vec{a}=\{4;-2;-3\}$ va $\vec{b}=\{0;1;3\}$ vektorlarga perpendikular, uzunligi 26 ga teng va Oy o‘q bilan o‘tmas burchak tashkil qiluvchi $\vec{x}$ vektorning koordinatalarini toping.

28. Berilgan vektorlarning komplanar yoki komplanar emasligini aniqlang.

1) $\vec{a}=\{3;-2;1\}$, $\vec{b}=\{2;1;2\}$, $\vec{c}=\{3;-1;-2\}$; 2) $\vec{a}=\{2;-1;2\}$, $\vec{b}=\{3;-4;7\}$, $\vec{c}=\{1;2;-3\}$.

29. α ning qanday qiymatlarida $\vec{a},\vec{b},\vec{c}$ vektorlar komplanar bo‘ladi?

1) $\vec{a}=\{1;1;\alpha\}$, $\vec{b}=\{0;1;0\}$, $\vec{c}=\{3;0;1\}$; 2) $\vec{a}=\{\alpha;3;1\}$, $\vec{b}=\{5;-1;2\}$, $\vec{c}=\{-1;5;4\}$.

30. Piramida uchlarining koordinatalari berilgan. Piramidaning hajmini va D uchidan tushirilgan balandligini toping:

1) $A(1;-2;2)$, $B(-1;1;2)$, $C(-1;-2;8)$, $D(1;1;10)$; 2) $A(1;1;1)$, $B(2;0;2)$, $C(2;2;2)$, $D(3;4;-3)$.

31. $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar berilgan. Bu vektorlar qanday uchlik tashkil etishini aniqlang va qirralari bu vektorlardan iborat bo‘lgan parallelepiped hajmini toping:

1) $\vec{a}=\{1;-2;1\}$, $\vec{b}=\{3;2;1\}$, $\vec{c}=\{-1;0;1\}$; 2) $\vec{a}=\{1;3;3\}$, $\vec{b}=\{-1;2;0\}$, $\vec{c}=\{1;2;-3\}$.

3

- Tekislikdagi to‘g‘ri chiziq
- Ikkinchi tartibli chiziqlar
- Tekislik
- Fazodagi to‘g‘ri chiziq
- Ikkinchi tartibli sirtlar

Rene Dekart (1596–1650) – frantsuz faylasufi, matematigi, mexanigi, fizigi va fiziologi.

Rene Dekart zamonaviy analitik geometriya va algebraik ramzlarga asos solgan.

Dekart tomonidan analitik geometriyaning yaratilishi egri chiziqlar tenglamalarini biror koordinatalar sistemasida tahlil etish imkonini berdi va funksiya tushunchasi tomon hal qiluvchi qadam bo‘ldi.

ANALITIK GEOMETRIYA

Analitik geometriya – matematikaning bo‘limlaridan biri bo‘lib, u geometriya bilan algebrani birlashtiradi, ya’ni ayrim geometrik tushunchalarni algebraik tahlil qilish va ayrim algebraik bog‘lanishlarni geometrik izohlash imkonini beradi. Bunda asosiy e’tibor ikkita masalaga, xossalariga ko‘ra geometrik shaklning tenglamasini keltirib chiqarishga va tenglamasiga ko‘ra geometrik shaklning ko‘rinishi va xossalarini o‘rganishga, qaratiladi.

Fransuz matematigi *Rene Dekart analitik geometriyaning asoschisi* hisoblanadi. Dekart tomonidan 1637-yilda kiritilgan koordinatalar usuli nuqtaning o‘rnini biror koordinatalar sistemasiga nisbatan aniqlashga asoslanadi.

3.1. TEKISLIKDAGI TO‘G‘RI CHIZIQ

3.1.1. Tekislikdagi chiziq

Umumiy boshlang‘ich O nuqtaga ega bo‘lgan o‘zaro perpendikulyar Ox va Oy koordinata o‘qlari tekislikda to‘g‘ri burchakli Oxy koordinatalar sistemasini hosil qiladi.

Oxy koordinatalar sistemasida ikkita x va y sonlari tekislikdagi har qanday M nuqtaning o‘rnini to‘liq aniqlaydi. Bunda nuqta $M(x;y)$ kabi belgilanadi: x ga M nuqtaning *absissasi*, y ga M nuqtaning *ordinatasi* deyiladi.

Oxy tekislikdagi chiziq tenglamasi deb aynan shu chiziq nuqtalarining x va y koordinatalari orasidagi bog‘lanishni aniqlovchi ikki noma’lumli

$$F(x,y)=0$$

ko‘rinishdagi tenglamaga aytiladi.

Shu kabi, koordinatalari ikki noma’lumli $F(x,y)=0$ tenglamani qanoatlantiruvchi *Oxy* tekislikning barcha $M(x;y)$ nuqtalari to‘plamiga *tekislikda* shu tenglama bilan aniqlanuvchi *chiziq* deyiladi.

Ayrim hollarda tekislikdagi chiziq $y=f(x)$ tenglama bilan beriladi. Bunda chiziq $y=f(x)$ funksiyaning grafigi deb ataladi.

Tekislikdagi chiziq ikkita $x=x(t),\ y=y(t),\ t\in T$ tenglamalar bilan ham berilishi mumkin. Bunda barcha $M(x(t);y(t)),\ t\in T$ nuqtalar to‘plami tekislikdagi chiziqni ifodalaydi. $x=x(t),\ y=y(t)$ funksiyalarga bu chiziqning parametrik tenglamalari, t o‘zgaruvchiga parametr deyiladi.

Tekislikdagi chiziqning ikkita $x=x(t),\ y=y(t)$ parametrik (skalyar) tenglamalarini bitta $\vec{r}=\vec{r}(t)$ vektor tenglama bilan berish mumkin. Bunda t parametr (vaqt) o‘zgarishi bilan $\vec{r}=\vec{r}(t)$ vektorning oxiri biror chiziqni chizadi. Bu chiziqqa nuqtaning traektoriyasi, $\vec{r}=\vec{r}(t)$ tenglamaga harakat tenglamasi deyiladi. Bu jumla chiziqning vektor va parametrik tenglamalarining *mexanik ma’nosini* bildiradi.

Shunday qilib, tekislikdagi har qanday chiziqqa ikki o‘zgaruvchining biror $F(x,y)=0$ tenglamasi mos keladi va aksincha, ikki o‘zgaruvchining har qanday $F(x,y)=0$ tenglamasiga, umuman olganda, tekislikdakdagi biror chiziq mos keladi. Bunda «umuman olganda» iborasi aytilganlarda mustasnoga yo‘l qo‘yilishi mumkinligini bildiradi. Masalan, $(x-1)^2+(y-4)^2=0$ tenglamaga chiziq emas, balki $M(1;4)$ nuqta mos keladi; $x^2+y^2+3=0$ tenglamaga tekislik nuqtalarining hech bir geometrik o‘rni mos kelmaydi.

3.1.2. Tekislikdagi to‘g‘ri chiziq tenglamalari

To‘g‘ri chiziqning tekislikdagi o‘rni turli parametrlar bilan bir qiymatli aniqlanishi mumkin. *Masalan,* to‘g‘ri chiziqda yotuvchi nuqta va to‘g‘ri chiziqqa perpendikular vektor bilan, to‘g‘ri chiziqning koordinata o‘qlarida ajratgan kesmalari bilan va hokazo. Bunday parametrlar to‘g‘ri chiziqning tenglamalarini keltirib chiqarish uchun asos bo‘ladi. Quyida berilgan parametrlariga ko‘ra to‘g‘ri chiziq tenglamalarini keltirib chiqarish bilan tanishamiz.

I. To‘g‘ri chiziqda yotuvchi $M_0(x_0;y_0)$ nuqta va to‘g‘ri chiziqqa perpendikular bo‘lgan $\vec{n}=\{A;B\}$ vektor berilgan.

l to‘g‘ri chiziqda yotuvchi ixtiyoriy $M(x;y)$ nuqtani olamiz va $\overline{M_0M}=\{x-x_0;y-y_0\}$ vektorni yasaymiz (1-shakl).

Bunda $\vec{n}\perp\overline{M_0M}$ bo‘ladi. Ikki vektorning perpendikulyarlik shartiga asosan to‘g‘ri chiziq tenglamasini topamiz:

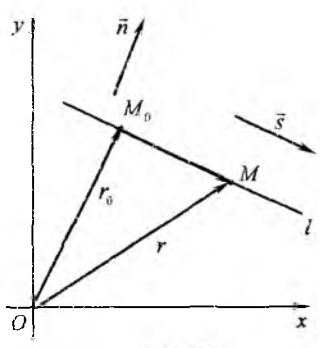
$$A(x-x_0)+B(y-y_0)=0. \qquad (1.1)$$

(1.1) tenglamaga *berilgan nuqtadan o‘tuvchi va berilgan vektorga perpendikular to‘g‘ri chiziq tenglamasi* deyiladi.

To‘g‘ri chiziqqa perpendikular bo‘lgan har qanday vektorga *to‘g‘ri chiziqning normal vektori* deyiladi.

Demak, $\vec{n}=\{A;B\}$ vektor (1.1) tenglama bilan aniqlanuvchi to‘g‘ri chiziqning normal vektori bo‘ladi.

1-misol. $M_1(2;3)$ va $M_2(-1;0)$ nuqtalar berilgan. M_2 nuqtadan o‘tuvchi va $\overline{M_1M_2}$ vektorga perpendikular to‘g‘ri chiziq tenglamasini tuzing.



1-shakl.

Yechish. Avval $\overline{M_1M_2}$ vektorini topamiz:

$$\overline{M_1M_2}=\{-1-2;0-3\}=\{-3;-3\}.$$

Bundan $A=-3,\ B=-3.$

Izlanayotgan to‘g‘ri chiziq tenglamasini (1.1) formula bilan tuzamiz:

$$-3(x-(-1))-3(y-0)=0$$

yoki

$$x+y+1=0.$$

II. To‘g‘ri chiziqda yotuvchi $M_0(x_0;y_0)$ nuqta va to‘g‘ri chiziqqa parallel bo‘lgan $\vec{s}=\{p;q\}$ vektor berilgan.

l to‘g‘ri chiziqda yotuvchi $M_0(x_0;y_0)$ va $M(x;y)$ nuqtalardan $\overline{M_0M}=\{x-x_0;y-y_0\}$ vektorni yasaymiz (1-shakl).

Bunda $\vec{s}$ va $\overline{M_0M}$ vektorlar kollinear bo‘ladi. Ikki vektorning kollinearlik shartidan quyidagini topamiz:

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}. \tag{1.2}$$

(1.2) tenglamaga *berilgan nuqtadan o‘tuvchi va berilgan vektorga parallel to‘g‘ri chiziq tenglamasi* deyiladi.

Shunindek, bu tenglama *to‘g‘ri chiziqning kanonik tenglamasi* deb ataladi.

To‘g‘ri chiziqqa parallel bo‘lgan (yoki to‘g‘ri chiziqda yotuvchi) nolga teng bo‘lmagan har qanday vektorga to‘g‘ri chiziqning *yo‘naltiruvchi vektori* deyiladi.

Demak, $\vec{s}=\{p;q\}$ vektor (1.2) tenglama bilan aniqlanuvchi to‘g‘ri chiziqning yo‘naltiruvchi vektori bo‘ladi.

1-izoh. (1.2) tenglamadan to‘g‘ri chiziqning keltirilgam II shartni qanoatlantiruvchi boshqa tenglamalarini hosil qilish mumkin.

1. (1.2) tenglamada

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}=t,\ t\in(-\infty;+\infty)$$

belgilash kiritamiz.

Bundan

$$x=x_0+tp,\ \ y=y_0+tq \tag{1.3}$$

tenglamalar kelib chiqadi, bu yerda t – parametr.

(1.3) tenglamalarga *to‘g‘ri chiziqning parametrik tenglamalari* deyiladi.

2. Ma'lumki, tekislikdagi chiziqning ikkita parametrik (skalyar) tenglamalarini bitta vektor tenglama bilan berish mumkin, ya'ni (1.3) tenglamalarni

$$\vec{r} = \vec{r}_0 + t\vec{s} \tag{1.4}$$

ko'rinishda yozish mumkin, bu yerda $\vec{r} = \{x; y\}$, $\vec{r}_0 = \{x_0; y_0\}$ – mos ravishda $M(x; y), M_0(x_0; y_0)$ nuqtalarning radius vektorlari; $\vec{s} = \{p; q\}$ – to'g'ri chiziqning yo'naltiruvchi vektori (1-shakl).

(1.4) tenglamaga *to'g'ri chiziqning vektor tenglamasi* deyiladi.

2-misol. $M(-2;4)$ nuqtadan o'tuvchi va $\vec{s} = \{1;-3\}$ vektorga parallel to'g'ri chiziqning kanonik, parametrik va vektor tenglamalarini tuzing.

Yechish. To'g'ri chiziqning kanonik, parametrik va vektor tenglamalarini (1.2), (1.3) va (1.4) formulalar bilan topamiz:

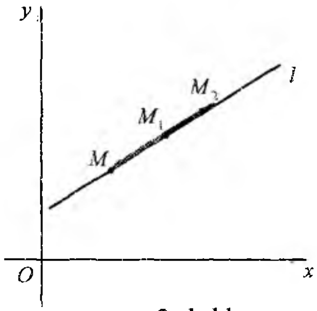
$$\frac{x+2}{1} = \frac{y-4}{-3};$$

$$x = -2 + t, \quad y = 4 - 3t, \quad t \in T;$$

$$\vec{r} = \vec{r}_0 + t\vec{s}, \ r_0 = \{-2 : 4\}.$$

III. To'g'ri chiziqda yotuvchi ikkita $M_1(x_1; y_1)$ va $M_2(x_2; y_2)$ nuqta berilgan.

l to'g'ri chiziqda yotuvchi ixtiyoriy $M(x; y)$ nuqtani olib, $\overline{M_1M} = \{x - x_1; y - y_1\}$ va $\overline{M_1M_2} = \{x_2 - x_1; y_2 - y_1\}$ vektorlarni yasaymiz (2-shakl). Bunda $\overline{M_1M}$ va $\overline{M_1M_2}$ vektorlar kollinear bo'ladi.



2-shakl.

Ikki vektorning kollinearlik shartidan topamiz:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \tag{1.5}$$

bo'ladi.

(1.5) tenglamaga *berilgan ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi* deyiladi.

IV. To'g'ri chiziqning Ox va Oy o'qlaridan ajratgan kesmalari a va b berilgan.

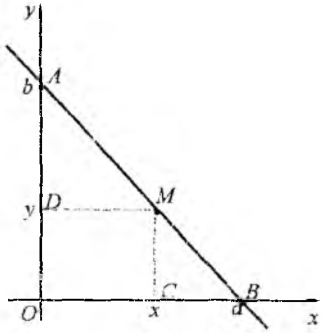
l to'g'ri chiziqda yotuvchi ixtiyoriy $M(x; y)$ nuqtani olamiz (3-shakl).

ΔCBM va ΔOBA o‘xshash. U holda uchburchaklarning o‘xshashlik alomatiga ko‘ra,

$$\frac{CB}{OB}=\frac{CM}{OA}\Rightarrow \qquad \frac{OB-OC}{OB}=\frac{OD}{OA}\Rightarrow \qquad \frac{OC}{OB}+\frac{OD}{OA}=1.$$

Bundan $OC=x,\ OB=a,\ OD=y,\ OA=b$ o‘rniga qo‘yish bajarib, topamiz:

$$\frac{x}{a}+\frac{y}{b}=1. \qquad (1.6)$$



3-shakl.

(1.6) tenglamaga *to‘gri chiziqning kesmalarga nisbatan tenglamasi* deyiladi.

3- misol. $4x+3y-12=0$ tenglama bilan berilgan to‘g‘ri chiziqni chizmada tasvirlang.

Yechish. Tekislikdagi to‘g‘ri chiziqni chizish uchun uning ikkita nuqtasini bilish yetarli bo‘ladi.

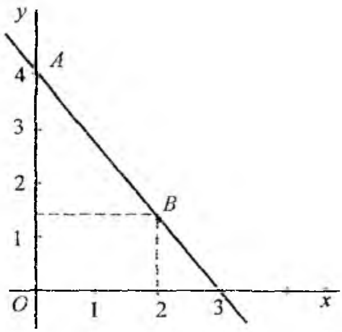
To‘g‘ri chiziq tenglamasida, masalan $x=0$ deb, $y=4$ ni, ya’ni $A(0;4)$ nuqtani va shu kabi $B\left(2;\frac{4}{3}\right)$ nuqtani topamiz. Bu nuqtalarni tutashtirib, berilgan tenglamaga mos to‘g‘ri chiziqni chizamiz (4-shakl).

Bu masalani boshqacha, ya’ni to‘g‘ri chiziq tenglamasini kesmalarga nisbatan tenglamaga keltirib yechish mumkin. Buning uchun tenglamaning ozod hadi (-12)ni o‘ng tomonga o‘tkazamiz va hosil bo‘lgan tenglikning har ikkala tomonini 12 ga bo‘lamiz:

$$4x+3y=12,\ \ \frac{4x}{12}+\frac{3y}{12}=1$$

yoki

$$\frac{x}{3}+\frac{y}{4}=1.$$



4-shakl.

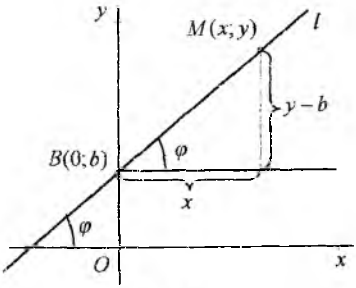
Bu tenglama bilan aniqlanuvchi to‘g‘ri chiziq Ox o‘qidan koordinatalar boshiga nisbatan o‘ng tomonga 3 ga teng kesma, Oy o'qidan esa koordinatalar boshiga nisbatan yuqoriga 4 ga teng kesma ajratadi (4-shakl).

V. To'g'ri chiziqning og'ish burchagi φ va Oy o'qidan ajratgan kesmasi b berilgan.

Ox o'qning musbat yo'nalishidan berilgan to'g'ri chiziqqa soat strelkasiga teskari yo'nalishda hisoblangan φ burchakka *to'g'ri chiziqning og'ish burchagi* deyiladi.

Og'ish burchagining tangensi, ya'ni $k = tg\varphi$ son *to'g'ri chiziqning burchak koeffitsiyenti* deb ataladi.

l to'g'ri chiziqda yotuvchi ixtiyoriy $M(x;y)$ nuqtani olamiz va burchak tangensi ta'rifidan foydalanamiz (5-shakl):



5-shakl.

$$\frac{y-b}{x} = tg\varphi$$

yoki

$$y = tg\varphi x + b.$$

Bundan

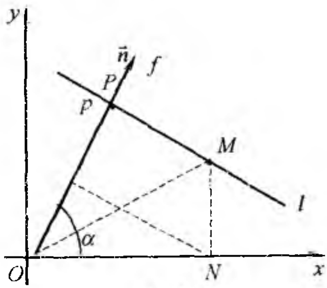
$$y = kx + b. \qquad (1.7)$$

Bu tenglamaga *to'g'ri chiziqning burchak koeffitsiyentli tenglamasi* deyiladi.

2-izoh. (1.7) tenglamadan to'g'ri chiziqning k burchak koeffitsiyentga ega bo'lgan yana bir tenglamasini keltirib chiqaramiz. Bu to'g'ri chiziq $M_1(x_1;y_1)$ nuqtadan o'tsin. U holda bu nuqtaning koordinatalari (1.7) tenglamani qanoatlantiradi: $y_1 = kx_1 + b$.

Bundan $b = y_1 - kx_1$.

U holda (1.7) tenglamadan topamiz:



6-shakl.

$$y = kx - kx_1 + y_1$$

yoki

$$y - y_1 = k(x - x_1). \qquad (1.8)$$

(1.8) tenglamaga *berilgan nuqtadan berilgan yo'nalish bo'yicha o'tuvchi to'g'ri chiziq tenglamasi* deyiladi.

Suningdek, bu tenglama *to'g'ri chiziqlar dastasi tenglamasi* deb ataladi.

VI. To'g'ri chiziq $\bar{n}=\overline{OP}$ normalining yo'nalishi α va uzunligi p berilgan.

l to'g'ri chiziqda yotuvchi ixtiyoriy $M(x;y)$ nuqtani olamiz.
6-shaklga asosan:

$$\Pr_l \overline{OP} = \Pr_l \overline{ON} + \Pr_l \overline{NM} + \Pr_l \overline{MP},$$

bu yerda $\Pr_l \overline{OP} = p$, $\Pr_l \overline{ON} = x\cos\alpha$, $\Pr_l \overline{NM} = y\sin\alpha$, $\Pr_l \overline{MP} = 0$.

Bundan,

$$p = x\cos\alpha + y\sin\alpha$$

yoki

$$x\cos\alpha + y\sin\alpha - p = 0. \qquad (1.9)$$

(1.9) tenglamaga *to'g'ri chiziqning normal tenglamasi* deyiladi.

Keltirib chiqarilgan (1.1)-(1.9) tenglamalar asosida ushbu xulosa kelib chaqadi:

x,y o'zgaruvchilarning har qanday birinchi darajali tenglamasi tekislikdagi biror to'g'ri chiziqni ifodalaydi va aksincha, tekislikdagi har qanday to'g'ri chiziq x,y o'zgaruvchilarning biror birinchi darajali tenglamasi bilan aniqlanadi.

Demak, tekislikdagi har bir l to'g'ri chiziq tenglamasini

$$Ax + By + C = 0 \qquad (1.10)$$

ko'rinishda yozish mumkin, bu yerda C – ozod had; $A^2 + B^2 \neq 0$.

(1.10) tenglamada A va B sonlar to'g'ri chiziq normal vektorining koordinatalari bo'lishini (1.1) tenglama yordamida ko'rsatish mumkin:

$$A(x - x_0) + B(y - y_0) = 0,$$

$$Ax + By - (Ax_0 + By_0) = 0,$$

$$Ax + By + C = 0, \; C = -(Ax_0 + By_0).$$

Demak, $\bar{n} = \{A;B\}$.

(1.10) tenglamaga *to'g'ri chiziqning umumiy tenglamasi* deyiladi.

(1.10) tenglamada:

1) $A=0$ bo'lsa, tenglama $By+C=0$ ko'rinishga keladi. Bunda to'g'ri chiziqning normal vektori Ox o'qqa perpendikular bo'ladi. Shu sababli to'g'ri chiziq Ox o'qqa parallel, Oy o'qqa perpendikular bo'ladi. Shu kabi $B=0$ da kelib chiqadigan $Ax+C=0$ to'g'ri chiziq Oy o'qqa parallel, Ox o'qqa perpendikular bo'ladi;

2) $C=0$ bo'lsa, tenglama $Ax+By=0$ ko'rinishni oladi. Bu tenglamani $O(0;0)$ nuqtaning koordinatalari qanoatlantiradi. Demak, to'g'ri chiziq koordinatalar boshidan o'tadi;

3) $A=0$ va $C=0$ bo'lsa, tenglamadan $y=0$ kelib chiqadi. Bu to'g'ri chiziq Ox o'qda yotadi. Shu kabi $B=0$ va $C=0$ da hosil bo'ladigan $x=0$ to'g'ri chiziq Oy o'qda yotadi.

4- misol. $a$ning qanday qiymatlarida $(a^2+4a)x+(a-5)y-2a+4=0$ to'g'ri chiziq: 1) Ox o'qqa parallel bo'ladi; 2) Ox o'qqa perpendikular bo'ladi; 3) koordinatalar boshidan o'tadi.

Yechish. Misolning shartiga ko'ra: $A=a^2+4a$, $B=a-5$, $C=-2a+4$.

U holda:

1) $a^2+4a=0$ yoki $a=-4$, $a=0$ da $A=0$ bo'ladi. Shu sababli berilgan to'g'ri chiziq Ox o'qqa parallel bo'ladi.

2) $a-5=0$ yoki $a=5$ da $B=0$ va berilgan to'g'ri chiziq Ox o'qqa perpendikular bo'ladi.

3) $-2a+4=0$ yoki $a=2$ da $C=0$ bo'ladi. Demak, $a=2$ da to'g'ri chiziq koordinatalar boshidan o'tadi.

To'g'ri chiziqning (1.1)-(1.10) tenglamalaridan har birini boshqalaridan keltirib chiqarish mumkin.

Misol tariqasida (1.10) tenglamadan (1.9) tenglamani keltirib chiqaramiz. Buning uchun (1.10) tenglikning chap va o'ng tomonini *normallovchi ko'paytuvchi* deb ataluvchi $M=\pm\frac{1}{\sqrt{A^2+B^2}}$ songa ko'paytiramiz.

Hosil bo'lgan $\frac{Ax+By+C}{\sqrt{A^2+B^2}}=0$ tenglamada

$$\cos\alpha=\pm\frac{A}{\sqrt{A^2+B^2}},\quad \sin\alpha=\pm\frac{B}{\sqrt{A^2+B^2}},\quad p=\pm\frac{C}{\sqrt{A^2+B^2}}$$

belgilashlar kiritsak, (1.9) tenglama kelib chiqadi.

Bunda M ko‘paytuvchining ishorasi C koeffitsiyentning ishorasiga qarama-qarshi qilib tanlanadi.

5-misol. To‘g‘ri chiziqning $5x-12y+8=0$ tenglamasini normal ko‘rinishga keltiring.

Yechish.

Tenglamaning chap va o‘ng tomonini $M=-\frac{1}{\sqrt{5^2+(-12)^2}}=-\frac{1}{13}$

(chunki $C>0$) soniga ko‘paytiramiz.

Bundan

$$-\frac{5x}{13}+\frac{12y}{13}-\frac{8}{13}=0$$

yoki

$$x\cos\alpha+y\sin\alpha-p=0,$$

bu yerda $\cos\alpha=-\frac{5}{13}$, $\sin\alpha=\frac{12}{13}$, $p=\frac{8}{13}$.

3.1.3. Tekislikda ikki to‘g‘ri chiziqning o‘zaro joylashishi

Ikki to‘g‘ri chiziq orasidagi burchak

Tekislikdagi ikki l_1 va l_2 to‘g‘ri chiziqlar orasidagi burchak φ bo‘lsin.

Bu burchak to‘g‘ri chiziq tenglamalarining berilishiga ko‘ra, turli formulalar bilan aniqlanishi mumkin.

I. To‘g‘ri chiziqlar umumiy tenglamalari

$$A_1x+B_1y+C_1=0 \text{ va } A_2x+B_2y+C_2=0$$

bilan berilgan bo‘lsin.

Bunda to‘g‘ri chiziqlarning $\vec{n}_1=\{A_1;B_1\}$, $\vec{n}_2=\{A_2;B_2\}$ normal vektorlari orasidagi burchak to‘g‘ri chiziqlar orasidagi burchakka teng, ya’ni $\varphi=(l_1\hat{,}l_2)=(\vec{n}_1\hat{,}\vec{n}_2)$ bo‘ladi (7-shakl).

Ikki vektor orasidagi burchak kosinusi formulasidan topamiz:

$$\cos\varphi=\frac{\vec{n}_1\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}=\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}. \tag{1.11}$$

II. To‘g‘ri chiziqlar kanonik tenglamalari

$$\frac{x-x_0}{p_1}=\frac{y-y_0}{q_1} \text{ va } \frac{x-x_0}{p_2}=\frac{y-y_0}{q_2}$$

bilan berilgan bo‘lsin.

Bunda $\vec{s}_1=\{p_1;q_1\}$, $\vec{s}_2=\{p_2;q_2\}$ bo‘ladi.

U holda $\varphi=(l_1\hat{,}l_2)=(\vec{s}_1\hat{,}\vec{s}_2)$ (7-shakl) ekanini hisobga olib, topamiz:

$$\cos\varphi=\frac{\vec{s}_1\vec{s}_2}{|\vec{s}_1|\cdot|\vec{s}_2|}=\frac{p_1p_2+q_1q_2}{\sqrt{p_1^2+q_1^2}\sqrt{p_2^2+q_2^2}}. \tag{1.12}$$

III. To‘g‘ri chiziqlar burchak koeffitsiyentli

$$y=k_1x+b_1 \text{ va } y=k_2x+b_2$$

tenglamalari bilan berilgan bo‘lsin.

7-shaklga ko‘ra, $\varphi=\varphi_2-\varphi_1$.

Bundan

$$tg\varphi=tg(\varphi_2-\varphi_1),\ tg\varphi=\frac{tg\varphi_2-tg\varphi_1}{1+tg\varphi_2 tg\varphi_1}$$

yoki

$$tg\varphi=\frac{k_2-k_1}{1+k_1k_2} \tag{1.13}$$

kelib chiqadi.

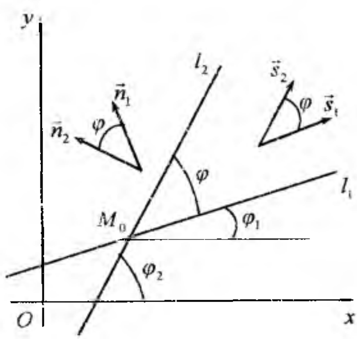
Agar bunda to‘g‘ri chiziqlardan qaysi biri birinchi va qaysi biri ikkinchi ekani ko‘rsatilmasdan ular orasidagi o‘tkir burchakni topish talab qilinsa, u holda (1.13) formulaning o‘ng tomoni modulga olinadi, ya’ni

$$tg\varphi=\left|\frac{k_2-k_1}{1+k_1k_2}\right|. \tag{1.14}$$

Shunday qilib, to‘g‘ri chiziqlar tenglamalarining ko‘rinishiga qarab, ular orasidagi burchak (1.11) - (1.14) formulalardan biri bilan topiladi.

6-misol. $y=-4x+1$ va $5x-3y-7=0$ to‘g‘ri chiziqlar orasidagi burchakni toping.

Yechish. Birinchi tenglamaga ko‘ra,



7-shakl.

$k_1=-4$. Ikkinchi tenglamadan topamiz:

$$5x-3y-7=0,\ y=\frac{5}{3}x-\frac{7}{3},\ \text{bunda}\ k_2=\frac{5}{3}.$$

U holda

$$tg\varphi=\frac{\frac{5}{3}-(-4)}{1+(-4)\cdot\frac{5}{3}}=-1.$$

Demak, $\varphi=\frac{3\pi}{4}.$

Ikki to'g'ri chiziqning perpendikularlik sharti

Tekislikdagi ikki to'g'ri chiziqning perpendikularlik shartlarini ikki to'g'ri chiziq orasidagi burchakni topish formulalaridan keltirib chiqaramiz.

$l_1\perp l_2$ bo'lsin. U holda $\cos\varphi=0$ va (1.11) tenglikdan topamiz:

$$A_1A_2+B_1B_2=0. \tag{1.15}$$

Shu kabi (1.12) tenglikdan

$$p_1p_2+q_1q_2=0 \tag{1.16}$$

kelib chiqadi.

(1.13) tenglikdan

$$ctg\varphi=\frac{1+k_1k_2}{k_1-k_2}.$$

U holda $l_1\perp l_2$da $ctg\varphi=0$ yoki

$$1+k_1k_2=0 \tag{1.17}$$

bo'ladi.

Demak, to'g'ri chiziqlar tenglamalarining ko'rinishiga qarab, ularning
perpendikular bo'lishi (1.15)-(1.17) shartlardan biri bilan aniqlanadi.

Ikki to'g'ri chiziqning parallellik sharti

I. l_1 va l_2 to'g'ri chiziqlar parallel bo'lsin. U holda ularning normal vektorlari $\vec{n}_1=\{A_1;B_1\}$va $\vec{n}_2=\{A_2;B_2\}$ kollinear bo'ladi. Ikki vektorning kollinearlik shartidan ikki to‘g‘ri chiziqning parallellik shartini topamiz:

$$\frac{A_1}{A_2}=\frac{B_1}{B_2} \qquad (1.18)$$

II. Agar l_1 va l_2 to‘g‘ri chiziqlar parallel bo‘lsa, u holda ularning yo‘naltiruvchi vektorlari $\vec{s}_1=\{p_1;q_1\}$ va $\vec{s}_2=\{p_2;q_2\}$ kollinear bo‘ladi. Bundan

$$\frac{p_1}{p_2}=\frac{q_1}{q_2}. \qquad (1.19)$$

III. $l_1 \parallel l_2$ bo‘lganida ular orasidagi burchak uchun $tg\varphi=0$ bo‘ladi. U holda (1.14) tenglikdan topamiz:

$$k_1=k_2\ . \qquad (1.20)$$

Shunday qilib, (1.18)-(1.20) shartlardan biri to‘g‘ri chiziqlar tenglamalarining berilishiga ko‘ra, ularning parallel bo‘lishini aniqlaydi.

7-misol. $M_0(2;1)$ nuqtadan o‘tuvchi va $2x+3y+4=0$ to‘g‘ri chiziqqa perpendikular to‘g‘ri chiziq tenglamasini tuzing.

Yechish. To‘g‘ri chiziq tenglamasini $Ax+By+C=0$ ko‘rinishda izlaymiz.

To‘g‘ri chiziq $M_0(2;1)$ nuqtadan o‘tgani sababli $2A+B+C=0$ va $2x+3y+4=0$ to‘g‘ri chiziqqa perpendikular bo‘lgani uchun $2A+3B=0$ bo‘ladi.

Bu tenglamalarni birgalikda yechib topamiz: $A=-\frac{3}{4}C,\ B=\frac{1}{2}C.$

A va B koeffitsiyentlarni izlanayotgan tenglamaga qo‘yamiz:

$$-\frac{3}{4}Cx+\frac{1}{2}Cy+C=0.$$

Bundan

$$(-3x+2y+4)C=0 \quad \text{yoki} \quad 3x-2y-4=0.$$

Ikki to‘g‘ri chiziqning kesishishi

To‘g‘ri chiziqlar umumiy tenglamalari

$$A_1x+B_1y+C_1=0 \ \text{ va } \ A_2x+B_2y+C_2=0$$

bilan berilgan bo‘lsin va $M_0(x_0;y_0)$ nuqtada kesishsin (7-shakl).

U holda $M_0(x_0;y_0)$ nuqtaning koordinatalari har ikkala tenglamani qanoatlantiradi. Shu sababli ikki to‘g‘ri chiziqning kesishish nuqtasi koordinatalari

$$\begin{cases} A_1x_0 + B_1y_0 + C_1 = 0, \\ A_2x_0 + B_2y_0 + C_2 = 0 \end{cases} \tag{1.21}$$

sistemadan topiladi.

Bunda $M_0(x_0;y_0)$ kesishish nuqtasi orqali o‘tuvchi to‘g‘ri chiziqlar dastasi

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0 \tag{1.22}$$

tenglama bilan aniqlanadi, bu yerda λ – sonli ko‘paytuvchi.

8-misol. $2x - y - 2 = 0$ *a* to‘g‘ri chiziq bo‘ylab yo‘naltirilgan yorug‘lik nuri $x - 2y + 2 = 0$ to‘g‘ri chiziqda sinadi va qaytadi. Qaytuvchi nur yo‘nalgan to‘g‘ri chiziq tenglamasini tuzing.

Yechish. Yorug‘lik nurining qaytish nuqtasi $2x - y - 2 = 0$ va $x - 2y + 2 = 0$ to‘g‘ri chiziqlarning kesishish nuqtasi bo‘ladi.

Bu nuqta $M(x;y)$ bo‘lsin. Uni quyidagi sistemadan topamiz:

$$\begin{cases} 2x - y - 2 = 0, \\ x - 2y + 2 = 0. \end{cases}$$

Bundan $M(2;2)$.

Yorug‘lik nuri sinadigan va yo‘naltirilgan to‘g‘ri chiziqlar orasidagi burchak tangensini topamiz.

Berilgan to‘g‘ri chiziqlarning burchak koeffitsiyentlari $k_1 = \frac{1}{2}$, $k_2 = 2$ bo‘ladi.

Bundan

$$tg\varphi = \frac{\frac{1}{2} - 2}{1 + \frac{1}{2} \cdot 2} = -\frac{3}{4}.$$

Bu son yorug‘lik nuri qaytuvchi va sinuvchi to‘g‘ri chiziqlar orasidagi burchak tangensiga teng bo‘ladi.

U holda

$$-\frac{3}{4} = \frac{k - \frac{1}{2}}{1 + \frac{1}{2} \cdot k},$$

bu yerda k – nur qaytuvchi to‘g‘ri chiziqning burchak koeffitsiyenti.

Bundan $k=-\frac{2}{11}$.

Demak, izlanayotgan to‘g‘ri chiziq uchun: $M(2;2)$, $k=-\frac{2}{11}$.

Bu parametrlar bilan aniqlanuvchi to‘g‘ri chiziq tenglamasini tuzaniz:

$$y-2=-\frac{2}{11}(x-2)$$

yoki

$$2x+11y-18=0.$$

Ikki to‘g‘ri chiziqning ustma-ust tushishi

l_1 va l_2 to‘g‘ri chiziqlar umumiy tenglamalari

$$A_1x+B_1y+C_1=0, \quad A_2x+B_2y+C_2=0$$

bilan berilgan bo‘lsin va ustma-ust tushsin.

Bunda:

– birinchidan $l_1 \parallel l_2$ bo‘ladi va $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\lambda$ tengliklardan $A_1-\lambda A_2=0$, $B_1-\lambda B_2=0$ kelib chiqadi;

– ikkinchidan l_1 to‘g‘ri chiziqning har bir nuqtasi, jumladan, $M_0(x_0;y_0)$ nuqtasi, l_2 to‘g‘ri chiziqda ham yotadi, ya’ni

$$A_1x_0+B_1y_0+C_1=0, \quad A_2x_0+B_2y_0+C_2=0$$

bo‘ladi.

Bu tengliklarning ikkinchisini λ ga ko‘paytiramiz va birinchidan ayiramiz:

$$(A_1-\lambda A_2)x_0+(B_1-\lambda B_2)+(C_1-\lambda C_2)=0.$$

Bundan $C_1=\lambda C_2$ kelib chiqadi.

Demak, to‘g‘ri chiziqlarning ustma-ust tushush sharti

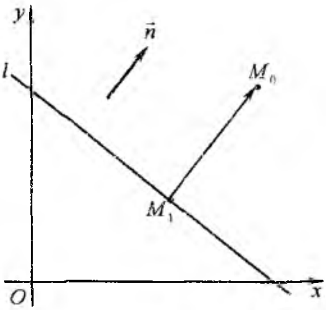
$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}. \qquad (1.23)$$

tengliklar bilan ifodalanadi.

3.1.4. Nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofa

Nuqtadan to‘g‘ri chiziqqa tushirilgan perpendikularning uzunligiga *nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofa* deyiladi.

$M_0(x_0;y_0)$ nuqta va $Ax+By+C=0$ tenglama bilan l to‘g‘ri chiziq berilgan bo‘lsin. M_0 nuqtadan l to‘g‘ri chiziqqa tushiurilgan perpendikularning asosini $M_1(x_1;y_1)$ bilan belgilaymiz (8-shakl).



8-shakl.

U holda $\overline{M_1M_0}=\{x_0-x_1;y_0-y_1\}$ va $M_1(x_1;y_1)$ nuqta l to‘g‘ri chiziqda yotgani sababli $Ax_1+By_1+C=0$, ya’ni $C=-Ax_1-By_1$ bo‘ladi.

$\vec{n}=\{A;B\}$ vektorning l to‘g‘ri chiziqqa perpendikular bo‘lishi ma’lum. Shu sababli M_0 nuqtadan l to‘g‘ri chiziqqacha bo‘lgan masofani vektorning o‘qdagi proeksiyasining xossalaridan foydalanib topamiz:

$$d=\left|\Pr_{\vec{n}}\overline{M_1M_0}\right|=\frac{\left|\overline{M_1M_0}\cdot\vec{n}\right|}{|\vec{n}|}=\frac{|(x_0-x_1)A+(y_0-y_1)B|}{\sqrt{A^2+B^2}}=$$

$$=\frac{|Ax_0+By_0-Ax_1-By_1|}{\sqrt{A^2+B^2}}=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}.$$

Shunday qilib, *nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofa*

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}} \tag{1.24}$$

formula bilan topiladi.

9-misol. $3x+4y-4=0$ va $6x+8y+5=0$ parallel to‘g‘ri chiziqlar orasidagi masofani toping.

Yechish. $3x+4y-4=0$ to‘g‘ri chiziqda ixtiyoriy, masalan $M(0;1)$ nuqtani olamiz. U holda berilgan parallel to‘g‘ri chiziqlar orasidagi d masofa $M(0;1)$ nuqtadan $6x+8y+5=0$ to‘g‘ri chiziqqacha bo‘lgan masofaga teng bo‘ladi. Uni (1.24) formula bilan hisoblaymiz:

$$d=\frac{|6\cdot0+8\cdot1+5|}{\sqrt{6^2+8^2}}=\frac{13}{10}(u.b).$$

3.1.5. Mashqlar

1. To‘g‘ri chiziqlarning burchak koeffitsiyentini va koordinata o‘qlarida ajratgan kesmalarini toping:

1) $3x+4y-12=0$; 2) $x=3y-2$; 3) $\frac{y+1}{2}=\frac{x-3}{4}$; 4) $\frac{x}{5}+\frac{y}{3}=\frac{1}{2}$.

2. To‘g‘ri chiziqning tenglamasini tuzing: 1) $M_1(2;-3)$ nuqtadan o‘tuvchi va $\vec{n}=\{3;4\}$ normal vektorga ega bo‘lgan; 2) $M_2(-2;-3)$ nuqtadan o‘tuvchi va $\vec{s}=\{-1;3\}$ yo‘naltiruvchi vektorlarga ega bo‘lgan; 3) $M_3(-2;3)$ nuqtadan o‘tuvchi Ox o‘qqa perpendikular bo‘lgan; 4) $M_4(3;2)$ nuqtadan o‘tuvchi Oy o‘qda $b=5$ ga teng kesma ajratuvchi.

3. Tenglamalardan qaysilari to‘g‘ri chiziqning normal tenglamasini ifodalaydi?

1) $y+2=0$; 2) $x-2,5=0$; 3) $\frac{3}{5}x-\frac{4}{5}y-3=0$; 4) $\frac{12}{13}x+\frac{5}{13}y+2=0$.

4. To‘g‘ri chiziqlarning kesishish nuqtalarini va ular orasidagi burchakni toping:

1) $5x-y-3=0$, $2x-3y+4=0$; 2) $y=\frac{3}{4}x-\frac{5}{2}$, $4x+3y-5=0$;

3) $\frac{x+1}{3}=\frac{y-1}{1}$, $x-3y+9=0$; 4) $\frac{x-1}{1}=\frac{y+3}{5}$, $\frac{x-2}{-2}=\frac{y-2}{3}$.

5. m va n ning qanday qiymatlarida $mx+9y+n=0$ va $4x+my-2=0$ to‘g‘ri chiziqlar: 1) parallel bo‘ladi; 2) ustma-ust tushadi; 3) perpendikular bo‘ladi?

6. m ning qanday qiymatlarida to‘g‘ri chiziqlar: 1) parallel bo‘ladi; 2) perpendikular bo‘ladi?

1) $x-my+5=0$, $2x+3y+3=0$; 2) $2x-3y+4=0$, $mx-6y+7=0$.

7. $x+y-7=0$ to‘g‘ri chiziqda koordinatalari $2x-y+4=0$ tenglik bilan bog‘langan nuqtani toping.

8. $A(4,2)$ nuqtadan o‘tuvchi va koordinata o‘qlari bilan yuzi 2 kvadrat birlikka teng uchburchak ajratuvchi to‘g‘ri chiziq tenglamasini tuzing.

9. $A(-3;2), B(5;-2), C(0;4)$ bo‘lsa, ABC uchburchakda BD balandlik tenglamasini tuzing.

10. $A(-2;0), B(5;3), C(1;-1)$ bo‘lsa, ABC uchburchakda AD mediana tenglamasini tuzing.

11. $2x-y+3=0$ va $x+y-2=0$ to‘g‘ri chiziqlarning kesishish nuqtasidan o‘tuvchi va $3x-4y-7=0$ to‘g‘ri chiziqqa perpendikular to‘g‘ri chiziq tenglamasini tuzing.

12. To‘g‘ri burchakli teng yonli uchburchak gipotenuzasi orqali o‘tgan to‘g‘ri chiziq tenglamasi $3x+2y-6=0$ dan va uchlaridan biri $A(-1;-2)$ nuqtadan iborat. Uchburchakning katetlari tenglamalarini tuzing.

13. Parallelogrammning ikki uchi $A(1;1)$ va $B(2;-2)$ nuqtalarda yotadi va diagonallari $(-1;0)$ nuqtada kesishadi. Parallelogrammning tomonlari tenglamalarini tuzing.

14. Agar $A(5;3)$, $B(1;1)$, $C(3;5)$, $D(6;6)$ to‘rtburchakning uchlari bo‘lsa, uning diagonallari kesishish nuqtasini va diagonallari orasidagi burchakni toping.

15. Uchburchakning uchlari berilgan: $A(8;3), B(2;5), C(5;-1)$. Uchburchak medianalarining kesishish nuqtasidan o‘tuvchi va $x+y-2=0$ to‘g‘ri chiziqqa perpendikular to‘g‘ri chiziq tenglamasini tuzing.

16. Burchak tomonlaridan birining tenglamasi $4x-3y+9=0$ dan va bissektrisasining tenglamasi $x-7y+21=0$ dan iborat. Burchak ikkinchi tomonining tenglamasini tuzing.

17. Uchburchakning ikki uchi $A(5;1), B(1;3)$ va medianalari kesishish nuqtasi $M(3;4)$ berilgan. Uchburchak tomonlarining tenglamalarini tuzing.

18. Uchburchakning ikki uchi $A(2;-2), B(-6;2)$ va balandliklari kesishish nuqtasi $M(1;2)$ berilgan. Uchburchakning C uchidan tushirilgan balandlik tenglamasini tuzing.

19. Uchburchak tomonlarining o‘rtalari berilgan: $M_1(1;-3), M_2(2;-2), M_3(-3;4)$. Uchburchak tomonlarining tenglamalarini tuzing.

20. Parallelogrammning ikki tomoni $2x+y-2=0$, $x-y+17=0$ to‘g‘ri chiziqlarda yotadi va diagonallari $M(-3,5;3,5)$ nuqtada kesishadi. Parallelogramm qolgan tomonlari yotgan to‘g‘ri chiziqlarning tenglamalarini tuzing.

21. $x-2y+5=0$ to‘g‘ri chiziq bo‘ylab yo‘nalgan yorug‘lik nuri $3x-2y+7=0$ to‘g‘ri chiziqda sinadi va qaytadi. Qaytuvchi nur yo‘nalgan to‘g‘ri chiziq tenglamasini tuzing.

22. Uchlari $A(2;3), B(-1;4), C(5;5)$ nuqtalarda bo‘lgan uchburchak og‘irlik markazidan o‘tuvchi va AC tomonga parallel to‘g‘ri chiziq tenglamasini tuzing.

23. Bir uchi $A(3;4)$ nuqtada bo‘lgan va bir tomoni $2x+5y+3=0$ to‘g‘ri tenglama bilan berilgan to‘g‘ri chiziqda yotgan kvadratning yuzini toping.

24. $4x-3y+8=0$ va $8x-6y-7=0$ to‘g‘ri chiziqlar orasidagi masofani toping.

25. Kvadratning ikki tomoni $5x+12y-61=0$ va $5x+12y+17=0$ tenglamalar bilan berilgan to‘g‘ri chiziqlarda yotadi. Kvadrat diagonalining uzunligini toping.

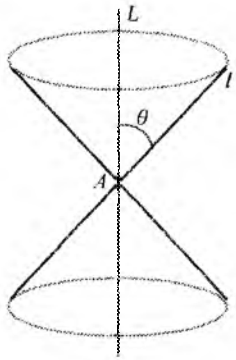
26. $A(2;4)$ nuqtadan o'tuvchi va koordinatalar boshidan 2 birlik masofada yotuvchi to'g'ri chiziq tenglamasini tuzing.

27. $A(-2;3)$ nuqtadan o'tuvchi va $B(5;-1), C(3;7)$ nuqtalardan teng uzoqlikda yotuvchi to'g'ri chiziq tenglamasini tuzing.

28. $M(-8;12)$ nuqtaning $A(-5;1)$ va $B(2;-3)$ nuqtalardan o'tuvchi to'g'ri chiziqdagi proeksiyasini toping.

3.2. IKKINCHI TARTIBLI CHIZIQLAR

Oxy koordinatalar sistemasida x, y o'zgaruvchilarning ikkinchi darajali tenglamasi bilan aniqlanuvchi chiziq tekislikdagi *ikkinchi tartibli chiziq* deyiladi.



9-shakl.

Har qanday ikkinchi tartibli chiziqni doiraviy konusning tekislik bilan kesishish chizig'i sifatida hosil qilish mumkin. Shu sababli ikkinchi tartibli chiziqlar *konus kesimlar* deb ham ataladi. Berilgan l to'g'ri chiziqni uni A nuqtaga kesuvchi boshqa bir fiksirlangan L to'g'ri chiziq atrofida o'zgarmas θ burchak ostida aylantirish natijasida hosil qilingan sirt *doiraviy konus* deyiladi (9-shakl).

Bunda l to'g'ri chiziqqa *konusning yasovchisi*, L to'g'ri chiziqqa *konusning o'qi*, A nuqtaga *konusning uchi*, konusning A nuqta bilan ajratilgan qismlariga *konusning pallalari* deyiladi.

Agar konus tekislik bilan kesilganida (10-shakl):

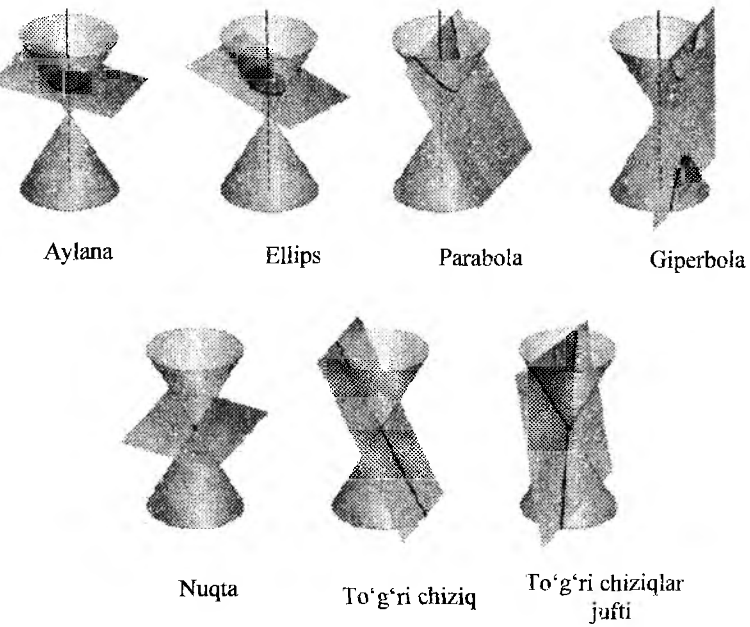
– tekislik konusning A uchidan o'tmasa va konus o'qiga perpendikular bo'lsa, kesimda *aylana* hosil bo'ladi;

– tekislik konus o'qiga perpendikular bo'lmay, konusning faqat bitta pallasini kessa va uning yasovchilaridan birortasiga parallel bo'lmasa, kesimda *ellips* hosil bo'ladi;

– tekislik konus yasovchilaridan biriga parallel ravishda uning pallalaridan birini kessa, kesimda *parabola* hosil bo'ladi;

– tekislik konusning ikkala pallasini kessa, kesimda *giperbola* hosil bo'ladi;

– tekislik konusning A uchidan o'tsa, kesimda *nuqta, to'g'ri chiziq, to'g'ri chiziqlar jufti* hosil bo'ladi.



10-shakl.

Ikkinchi tartibli chiziqlar fan va texnikaning ko‘p sohalarida keng qo‘llaniladi.

Bunga misollar keltiramiz.

1. Ma’lumki, avtomobil g‘ildiraklari aylana shaklida yasaladi.

2. Quyosh sistemasining planetalari quyosh joylashgan umumiy fokusga ega ellipslar bo‘yicha harakat qiladi.

3. Agar parabola fokusiga yorug‘lik manbayi joylashtirilsa, nurlar uning o‘qiga parallel ravishda qaytadi. Projektorning tuzilishi bu xossaga asoslangan.

4. Mexanikada isbot qilinganidek, yer yuzidan gorizontga qarab burchak ostida $v_0 = 11{,}2\ km/c$ (ikkinchi kosmik tezlik) boshlang‘ich tezlik bilan chiqarilgan raketa parabola bo‘ylab yer yuzidan cheksiz uzoqlashsa, $v_0 > 11{,}2\ km/c$ boshlang‘ich tezlik bilan chiqarilgan raketa giperbola bo‘ylab yer yuzidan cheksiz uzoqlashadi, $v_0 < 11{,}2\ km/c$ boshlang‘ich tezlik bilan chiqarilgan raketa esa yerga qaytib tushadi yoki yerning sun’iy yo‘ldoshi bo‘lib qoladi.

3.2.1. Aylana

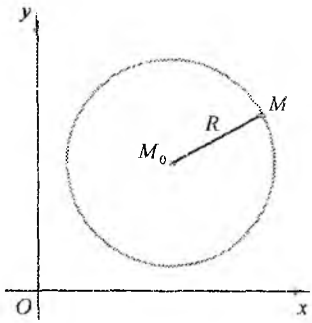
1-ta'rif. Tekislikda markaz deb ataluvchi berilgan nuqtadan teng uzoqlikda yotuvchi nuqtalarning geometrik o'rniga *aylana* deyiladi.

Tekislikda $M_0(x_0;y_0)$ nuqtadan R masofada yotuvchi nuqtalarni qaraymiz. Bu nuqtalardan biri $M(x;y)$ nuqta bo'lsin (11-shakl).

Aylana ta'rifiga ko'ra, $|M_0M|=R$.

Bu tenglikka ikki nuqta orasidagi masofa formulasini qo'llaymiz:

$$\sqrt{(x-x_0)^2+(y-y_0)^2}=R.$$



11-shakl.

Bundan

$$(x-x_0)^2+(y-y_0)^2=R^2. \tag{2.1}$$

(2.1) tenglamaga *aylananing kanonik tenglamasi* deyiladi. Bunda $M_0(x_0;y_0)$ nuqta *aylana markazi*, R masofa *aylana radiusi* deb ataladi.

$x_0=0,\ y_0=0$ da (2.1) tenglamadan topamiz:

$$x^2+y^2=R^2. \tag{2.2}$$

(2.2) tenglama markazi koordintalar boshidan o'tuvchi va radiusi R ga teng aylanani aniqlaydi.

1-misol. Koordinatalari $x=R\cos t,\ y=R\sin t$ tenglamalar bilan aniqlanuvchi $M(x;y)$ nuqta aylana nuqtasi bo'lishini ko'rsating.

Yechish. $M(x;y)$ nuqta koordinatalarining har ikkala tomonini kvadratga ko'taramiz va hadlab qo'shamiz:

$$x^2+y^2=R^2\cos^2 t+R^2\sin^2 t=R^2(\sin^2 t+\cos^2 t)=R^2$$

yoki

$$x^2+y^2=R^2.$$

Demak, koordinatalari $x=R\cos t,\ y=R\sin t,\ t\in R$ tenglamalar bilan aniqlanuvchi $M(x;y)$ nuqta markazi koordintalar boshida yotuvchi va radiusi R ga teng aylanada yotadi.

Aylanani aniqlovchi ushbu

$$\begin{cases} x = R\cos t, \\ y = R\sin t, \quad t \in [0;2\pi] \end{cases} \tag{2.3}$$

tenglamalar sistemasiga *aylananing parametrik tenglamalari* deyiladi.

3.2.2. Ellips

2-ta'rif. Tekislikda fokuslar deb ataluvchi berilgan ikki nuqtagacha bo'lgan masofalarning yig'indisi o'zgarmas kattalikka teng bo'lgan nuqtalarning geometrik o'rniga *ellips* deyiladi.

F_1 va F_2 ellipsning fokuslari, M ellipsning ixtiyoriy nuqtasi bo'lsin. $F_1F_2 = 2c$, $F_1M = r_1$, $F_2M = r_2$ belgilashlar kiritamiz.

Ellipsning ta'rifiga ko'ra, $F_1M + F_2M = 2a$, ya'ni

$$r_1 + r_2 = 2a, \tag{2.4}$$

bu yerda a – o'zgarmas son bo'lib, $a > c$.

Oxy koordinatalar sistemasini Ox o'q fokuslardan, Oy o'q F_1F_2 kesmaning o'rtasidan o'tadigan qilib tanlaymiz (12-shakl).

U holda $F_2(-c;0)$ va $F_1(c;0)$ bo'ladi.

M nuqtaning koordinatalari x va y bo'lsin deylik, ya'ni $M(x;y)$.

Ikki nuqta orasidagi masofa formulasiga ko'ra

$$r_1 = \sqrt{(x-c)^2 + y^2}, \quad r_2 = \sqrt{(x+c)^2 + y^2}.$$

r_1 va r_2 ning bu ifodalarini (2.4) tenglikka qo'yib, almashtirishlar bajaramiz:

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a,$$

$$(x-c)^2 + y^2 = (2a - \sqrt{(x+c)^2 + y^2})^2,$$

$$x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2,$$

$$a\sqrt{(x+c)^2 + y^2} = a^2 + xc,$$

$$a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2,$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

$b^2 = a^2 - c^2$ (chunki $a > c$) belgilash kiritib, topamiz:

$$b^2x^2 + a^2y^2 = a^2b^2,$$

yoki

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.5}$$

(2.5) tenglamaga *ellipsning kanonik tenglamasi* deyiladi.

$x = a\cos t$, $y = b\sin t$ tenglamalar bilan aniqlanuvchi $M(x;y)$ nuqta ellips nuqtasi bo'lishini 1-misoldagi kabi ko'rsatish mumkin.

Ellipsni aniqlovchi ushbu

$$\begin{cases} x = a\cos t, \\ y = b\sin t, \ t \in [0;2\pi] \end{cases} \tag{2.6}$$

tenglamalar sistemasiga *ellipsning parametrik tenglamalari* deyiladi.

Ellipsning shaklini uning kanonik tenglamasidan foydalanib aniqlaymiz.

(2.5) tenglikda x va y ning faqat juft darajalari qatnashgani uchun ellips Ox, Oy o'qlarga va $O(0;0)$ nuqtaga nisbatan simmetrik bo'ladi. Shu sababli (2.5) tenglamani $x \geq 0$, $y \geq 0$ da (I-chorakda) tekshirish yetarli bo'ladi.

I-chorakda (2.5) tenglamadan $y = \frac{b}{a}\sqrt{a^2 - x^2}$ kelib chiqadi. Bunda x koordinata 0 dan a gacha o'sganida y koordinata b dan 0 gacha kamayadi. Ellipsning qolgan choraklardagi shaklini koordinata o'qlariga nisbatan simmetrik qilib chizamiz (12-shakl).

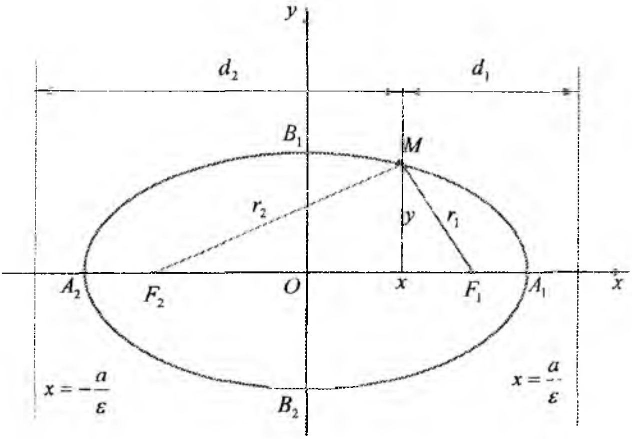
Ellipsda $O(0;0)$ nuqtaga markaz, $A_1(a;0)$, $A_2(-a;0)$, $B_1(0;b)$, $B_2(0;-b)$ nuqtalarga uchlar, A_1A_2, B_1B_2 kesmalarning $2a$, $2b$ uzunliklariga mos ravishda katta va kichik o'qlar, a, b sonlarga mos ravishda katta va kichik yarim o'qlar, F_1M, F_2M kesmalarning r_1, r_2 uzunliklariga fokal radiuslar deyiladi.

Ellipsning shakli $\frac{b}{a}$ nisbatga bog'liq bo'ladi, ammo ellipsning shaklini $\frac{c}{a}$ nisbat yordamida tekshirish qulaylikka ega.

$\varepsilon = \frac{c}{a}$ kattalikka *ellipsning ekstsentritsiteti* deyiladi. Bunda $0 < \varepsilon < 1$, chunki $0<c<a$. $b^2=a^2-c^2$ dan $\frac{b}{a}=\sqrt{1-\left(\frac{c}{a}\right)^2}$, ya'ni $\frac{b}{a}=\sqrt{1-\varepsilon^2}$.

Demak, $\varepsilon\to 1$ da $\frac{b}{a}\to 0$, ya'ni b kichiklashib, ellips Oy o'qiga parallel ravishda Ox o'qqa tomon siqilib boradi, aksincha $\varepsilon\to 0$ da $\frac{b}{a}\to 1$, ya'ni ellips aylanaga yaqinlashib boradi.

$x=\pm\frac{a}{\varepsilon}$ to'g'ri chiziqlar *ellipsning direktrisalari* deb ataladi.



12-shakl.

Ellipsning M nuqtasidan direktrisalargacha bo'lgan d_1 va d_2 masofalar uchun ushbu

$$\frac{r_1}{d_1}=\frac{r_2}{d_2}=\varepsilon$$

tengliklar bajariladi (12-shakl).

Bu tengliklardan ellipsning fokal radiuslari uchun

$$r_1=a-\varepsilon x,\ r_2=a+\varepsilon x$$

formulalar hosil qilinadi.

Fokuslari Oy o'qida va markazi koordinatalar boshda yotuvchi ellipsning kanonik tenglamalari shu kabi aniqlanadi.

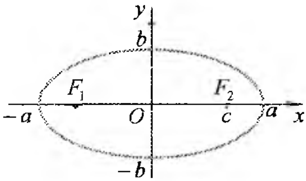
Har ikkala hol uchun ellipsning tenglamalarini va asosiy xossalarini keltiramiz.

Markazi $O(0;0)$ nuqtada bo'lgan ellipsning kanonik tenglamalari va asosiy xossalari

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad a>b>0$

– katta o'q Ox da yotadi va $2a$ ga teng;
– kichik o'q Oy da yotadi va $2b$ ga teng;
– fokuslar: $F_1(-c;0)$, $F_2(c;0)$

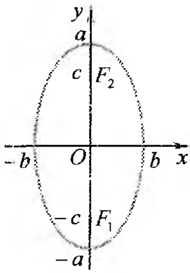
$$c^2=a^2-b^2.$$



2. $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad a>b>0$

– katta o'q Oy da yotadi va $2a$ ga teng;
– kichik o'q Ox da yotadi va $2b$ ga teng;
– fokuslar: $F_1(0;-c)$, $F_2(0;c)$

$$c^2=a^2-b^2$$



Agar $a=b$ bo'lsa, u holda (2.5) tenglamadan $x^2+y^2=a^2$ tenglama, ya'ni markazi koordinata boshida yotuvchi va radiusi a ga teng aylana tenglamasi kelib chiqadi. Demak, aylana ellipsning xususiy holi hisoblanadi.

2-misol. $4x^2+9y^2=144$ ellipsning o'qlari uzunliklarini, fokuslarining koordinatalarini va ekssentrisitetini toping.

Yechish. Ellipsning tenglamasini kanonik ko'rinishga keltiramiz:

$$\frac{x^2}{36}+\frac{y^2}{16}=1.$$

Bundan $a^2=36,\ b^2=16.$

Demak, $a=6,\ b=4,\ 2a=12,\ 2b=8.$

Shunday qilib, ellips o'qlarining uzunliklari mos ravishda 12 va 8 ga teng.

a va b ni bilgan holda c ni aniqlaymiz:

$$c=\sqrt{a^2-b^2}=\sqrt{36-16}=2\sqrt{5}.$$

Bundan fokuslarning koordinatalarini va ekssentrisitetni topamiz:

$$F_{21}(2\sqrt{5};0),\ \ F_2(-2\sqrt{5};0);$$

$$\varepsilon=\frac{c}{a}=\frac{2\sqrt{5}}{6}=\frac{\sqrt{5}}{3}.$$

3.2.3. Giperbola

3-ta'rif. Tekislikda fokuslar deb ataluvchi berilgan ikki nuqtagacha bo'lgan masofalar ayirmasining moduli o'zgarmas kattalikka teng bo'lgan nuqtalarning geometrik o'rniga *giperbola d*eyiladi.

Oxy koordinatalar sistemasini Ox o'q F_1 va F_2 fokuslardan, Oy o'q F_1F_2 kesmaning o'rtasidan o'tadigan qilib tanlaymiz (13-shakl).

$M(x;y)$ giperbolaning ixtiyoriy nuqtasi bo'lsin. $F_1F_2=2c$, $F_1M=r_1$, $F_2M=r_2$ belgilashlar kiritamiz. Ciperbolaning ta'rifiga ko'ra ,

$$|r_1-r_2|=2a, \tag{2.7}$$

bu yerda $a-$ o'zgarmas son bo'lib, $a<c$.

(2.7) ifodada (2.4) ifodada bajarilgan almashtirishlar kabi almashtirishlar bajarib, quyidagi tenglamani keltirib chiqaramiz:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{2.8}$$

bu yerda $b^2=c^2-a^2$.

(2.8) tenglamaga *giperbolaning kanonik tenglamasi* deyiladi.

Giperbolaning shaklini uning kanonik tenglamasidan foydalanib aniqlaymiz.

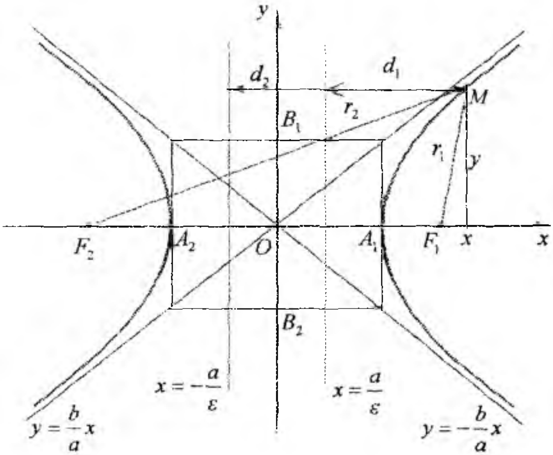
(2.8) tenglikda x va y ning faqat juft darajalari qatnashgani uchun giperbola ellips kabi Ox, Oy o'qlarga va $O(0;0)$ nuqtaga nisbatan simmetrik bo'ladi. Shu sababli (2.8) tenglamani $x\geq 0$, $y\geq 0$ da (I-chorakda) tekshiramiz.

I-chorakda (2.8) tenglamadan $y=\frac{b}{a}\sqrt{x^2-a^2}$ kelib chiqadi. Bunda $x\geq a$ va x koordinata a dan boshlab o'sishi bilan y koordinata ham o‘sib boradi, ya’ni $M(x;y)$ nuqta cheksizlikka intiladi. Bu intilish qanday yuz berishini ko‘rsatish uchun koordinatalar boshidan o‘tuvchi va burchak koeffitsiyenti $k=\frac{b}{a}$ ga teng bo‘lgan $y=\frac{b}{a}x$ to‘g‘ri chiziqni qaraymiz. Bu chiziq ushbu xossaga ega: M nuqta giperbola bo‘ylab harakat qilib koordinata boshidan cheksiz uzoqlashgani sari bu to‘g‘ri chiziqqa juda yaqinlashib boradi, lekin uni kesib o‘tmaydi, ya’ni asimptotik yaqinlashadi.

Shunday qilib, giperbola I-chorakda $A_1(a;0)$ nuqtadan o‘tib, $y=\frac{b}{a}x$ to‘g‘ri chiziqqa asimptotik yaqinlashgani holda o‘ngga va yuqoriga qarab o‘sib boradi.

Giperbolaning qolgan choraklardagi shaklini koordinata o‘qlariga nisbatan simmetrik qilib chizamiz (13-shakl).

Shunday qilib, giperbola ikki qismdan iborat bo‘ladi. Bu qismlarga giperbolaning tarmoqlari deyiladi.



13-shakl.

$y=\pm\frac{b}{a}x$ tenglama bilan aniqlanuvchi to‘g‘ri chiziqlarga *giperbolaning asimptotalari* deyiladi.

Giperbolada $A_1(a;0)$, $A_2(-a;0)$, $B_1(0;b)$, $B_2(0;-b)$ nuqtalarga uchlar, A_1A_2 kesmaning $2a$ uzunligiga haqiqiy o‘q, B_1B_2 kesmaning $2b$ uzunligiga mavhum o‘q, a, b sonlarga mos ravishda haqiqiy va mavhum yarim o'qlar, F_1M, F_2M kesmalarning r_1, r_2 uzunliklariga fokal radiuslar deyiladi.

$\varepsilon = \frac{c}{a}$ kattalikka *giperbolaning ekssentrisiteti* deyiladi.

Bunda $\varepsilon > 1$, chunki $c > a$.

$b^2 = c^2 - a^2$ dan $\frac{b}{a} = \sqrt{\left(\frac{c}{a}\right)^2 - 1}$, ya'ni $\frac{b}{a} = \sqrt{\varepsilon^2 - 1}$.

Demak, ekstsentrisitet birga qanchalik yaqin bo'lsa, $\frac{b}{a}$ shunchalik kichik bo'ladi, ya'ni $\varepsilon \to 1$ da $\frac{b}{a} \to 0$ va giperbola haqiqiy o'qi tomon siqilib boradi, aksincha ε kattalashgan sayin $\frac{b}{a}$ ham kattalashadi va giperbolaning tarmoqlari kengayib boradi.

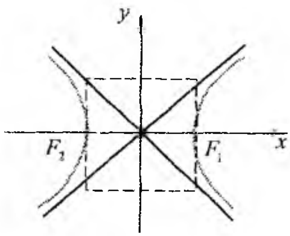
$x = \pm\frac{a}{\varepsilon}$ to'g'ri chiziqlar *giperbolaning direktrisalari* deb ataladi.

Fokuslari Oy o'qida va markazi koordinatalar boshda yotuvchi giperbolaning kanonik tenglamasi shu kabi aniqlanadi.

Markazi $O(0;0)$ nuqtada bo'lgan giperbolaning kanonik tenglamalari va asosiy xossalari

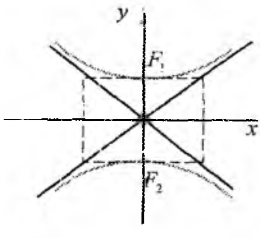
1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

– haqiqiy o'q Ox da yotadi va $2a$ ga teng;
– mavhum o'q Oy da yotadi va $2b$ ga teng;
– fokuslar: $F_1(-c;0)$, $F_2(c;0)$;
$c^2 = a^2 + b^2$;
– asimptotalari: $y = \pm\frac{b}{a}x$



2. $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

– haqiqiy o'q Oy da yotadi va $2a$ ga teng;
– mavhum o'q Ox da yotadi va $2b$ ga teng;
– fokuslar: $F_1(0;-c)$, $F_2(0;c)$;
$c^2 = a^2 + b^2$;
– asimptotalari: $y = \pm\frac{a}{b}x$.



Giperbolaning M nuqtasidan direktrisalargacha bo'lgan d_1 va d_2 masofalar uchun ushbu

$$\frac{r_1}{d_1}=\frac{r_2}{d_2}=\varepsilon$$

tengliklar bajariladi (13-shakl).

Bu tengliklardan giperbolaning fokal radiuslari uchun ushbu

$$x>0 \quad \text{bo'lganda} \quad r_1=\varepsilon x-a,\ r_2=\varepsilon x+a;$$

$$x<0 \quad \text{bo'lganda} \quad r_1=-a-\varepsilon x,\ r_2=a-\varepsilon x$$

formulalar hosil qilinadi.

Yarim o'qlari teng bo'lgan giperbolaga *teng tomonli giperbola* deyiladi.

Teng tomonli giperbola

$$x^2-y^2=a^2 \tag{2.9}$$

tenglama bilan aniqlanadi.

3-misol. Ekssentrisiteti $\sqrt{2}$ ga teng va $M(\sqrt{3};\sqrt{2})$ nuqtadan o'tuvchi giperbolaning kanonik tenglamasini tuzing. Uning yarim o'qlari uzunligini, fokuslari koordinatalarini toping va asimptotalarining, direktrisalarining tenglamalarini tuzing.

Yechish. Ma'lumki, $\varepsilon=\dfrac{c}{a}=\sqrt{2}$ yoki $c^2=2a^2$. Ikkinchi tomondan $c^2=a^2+b^2$. Bundan $a^2=b^2$. Demak, izlanayotgan giperbola teng tomonli. $M(\sqrt{3};\sqrt{2})$ nuqta giperbolada yotgani uchun $\dfrac{(\sqrt{3})^2}{a^2}-\dfrac{(\sqrt{2})^2}{a^2}=1$, ya'ni $a^2=1$.

Demak, izlanayotgan giperbolaning kanonik tenglamasi

$$x^2-y^2=1$$

ko'rinishni oladi.

Bu tenglama bilan aniqlanuvchi giperbolaning yarim o'qlari $a=b=1$ uzunlikka, fokuslari $F_1(\sqrt{2};0)$, $F_2(-\sqrt{2};0)$ koordinatalarga ega bo'ladi, asimptotalari $y=\pm x$ tenglamalar bilan, direktrisalari $x=\pm\dfrac{1}{\sqrt{2}}$ tenglamalar bilan topiladi.

3.2.4. Parabola

4-ta'rif. Tekislikda fokus deb ataluvchi berilgan nuqtadan va direktrisa deb ataluvchi berilgan to'g'ri chiziqdan teng uzoqlikda yotuvchi nuqtalarning geometrik o'rniga *parabola* deyiladi.

Parabolaning fokusidan direktrisasigacha bo'lgan masofani p $(p>0)$ bilan belgilaymiz.

p kattalikka *parabolaning parametri* deyiladi.

Oxy koordinatalar sistemasini Ox o'q direktrisaga perpendikular va fokusdan o'tadigan, $O(0;0)$ nuqta fokus va direktrisaning o'rtasida yotadigan qilib tanlaymiz. Tanlangan koordinatalar sistemasida $F\left(\frac{p}{2};0\right)$ nuqta fokus, $x=-\frac{p}{2}$ to'g'ri chiziq direktrisa bo'ladi (17-shakl).

$M(x;y)$ parabolaning ixtiyoriy nuqtasi bo'lsin.

M nuqtaning direktrisadagi proyektsiyasini N bilan belgilaymiz. Parabolaning ta'rifiga ko'ra $NM=MF$.

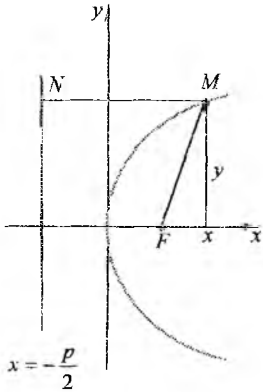
Bundan

$$\left|x+\frac{p}{2}\right|=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2},$$

$$x^2+px+\frac{p^2}{4}=x^2-px+\frac{p^2}{4}+y^2$$

yoki

$$y^2=2px. \qquad (2.10)$$



14-shakl.

(2.10) tenglamaga *parabolaning kanonik tenglamasi* deyiladi.

Parabolaning shaklini uning kanonik tenglamasidan foydalanib aniqlaymiz.

(2.10) tenglikda $y$ning juft darajasi qatnashgani uchun parabola Ox o'qqa nisbatan simmetrik bo'ladi.

Shu sababli (2.10) tenglamani $x\geq 0$, $y\geq 0$da tekshiramiz.

I chorakda (2.10) tenglamadan $y=\sqrt{2px}$ kelib chiqadi. Bunda $x\geq 0$ va x koordinata 0 dan boshlab o'sishi bilan y koordinata ham o'sib boradi. Shunday qilib, $y\geq 0$ bo'lganda $M(x;y)$ nuqta $O(0;0)$ nuqtadan chiqadi va x o'sishi bilan o'ngga va yuqoriga qarab bu nuqtadan cheksiz uzoqlashadi. Parabolaning $y\leq 0$ dagi shaklini Ox o'qqa nisbatan simmetrik qilib chizamiz (14-shakl). Bunda $O(0;0)$ nuqta parabolaning *uchi*, Ox o‘q parabolaning *o‘qi* deb ataladi.

Parabolaning *ekstsentrisiteti* $\varepsilon = \frac{NM}{MF} = 1$ ga teng bo‘ladi, *direktrisasi* $x = -\frac{p}{2}$ tenglama bilan aniqlanadi.

4-misol. $y^2 = 6x$ parabola berilgan. Uning direkrtisasi tenglamasini tuzing va fokusini toping.

Yechish. Berilgan tenglamani parabolaning kanonik tenglamasi (2.10) bilan taqqoslab, ko‘ramizki, $2p = 6$ yoki $p = 3$.

U holda berilgan parabola uchun direktrisasi tenglamasi $x = -\frac{p}{2} = -\frac{3}{2}$ va fokusi $F\left(\frac{p}{2};0\right) = \left(\frac{3}{2};0\right)$ bo‘ladi.

Parabolaning boshqa kanonik tenglamalari shu kabi aniqlanadi.

Uchi $O(0;0)$ nuqtada bo‘lgan parabolaning kanonik tenglamalari va asosiy xossalari

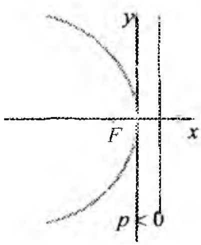
1. $y^2 = 2px$

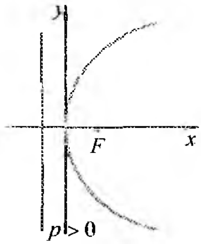
– fokus: $F\left(\frac{p}{2};0\right)$;

– direktritsa: $x = -\frac{p}{2}$

va Ox oqqa simmetrik;

– simmetriya o‘qi- Ox





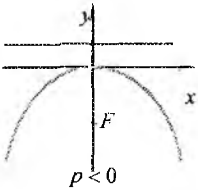
2. $x^2 = 2py$

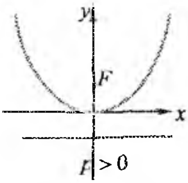
– fokus: $F\left(0;\frac{p}{2}\right)$;

– direktritsa: $y = -\frac{p}{2}$

va Oy oqqa simmetrik;

– simmetriya o‘qi – Oy





3.2.5. Ikkinchi tartibli chiziqlarning umumiy tenglamasi

Ikkita x va y o‘zgaruvchining ikkinchi darajali tenglamasi umumiy ko‘rinishda

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0,\ \ A^2+B^2+C^2\neq 0 \qquad (2.11)$$

kabi yoziladi, bu yerda $A,B,C,D,E,F-$ koeffitsiyentlar.

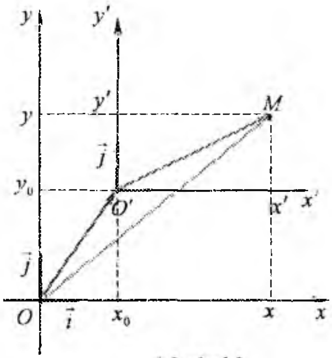
Oldingi bandlarda ta’riflari asosida ellips, giperbola va parabolalarning kanonik tenglamalarini keltirib chiqardik va xossalarini o‘rgandik. Bunda chiziqlarning markazlarini koordinatalar boshiga joylashtirdik va ularning o‘qlarini koordinata o‘qlari bo‘ylab yo‘naltirdik.

Ushbu bandda (2.11) tenglama koeffitsiyentlarining mos qiymatlarida konus kesimlardan birini, yoki mavhum konus kesimlardan birini, yoki bo‘sh to‘plamni aniqlashini ko‘rsatamiz. Bunda konus kesimning markazi koordinatalar boshida yotmasligi va o‘qlari koordinata o‘qlariga nisbatan og‘ishga ega bo‘lishi qiyinchilik tug‘dirishi mumkin. Bu qiyinchilikni bartaraf qilish uchun koordinatalar usulining ikki qurolidan – koordinatalar o‘qlarini parallel ko‘chirish va burishdan foydalanamiz.

Koordinata o‘qlarini parallel ko‘chirish

Tekislikda Oxy to‘g‘ri burchakli koordinatalar sistemasi berilgan bo‘lsin.

Koordinata o‘qlarini parallel ko‘chirish – bu Oxy sistemadan uning o‘qlari yo‘nalishlarini va masshtablarini o‘zgartirmasdan faqat koordinatalar boshining joylashishini o‘zgartirish orqali yangi $O'x'y'$ sistemaga o‘tishdir.



15-shakl.

Yangi $O'x'y'$ sistemaning koordinatalar boshi O' eski Oxy sistemada $(x_0;y_0)$ koordinatalarga ega bo‘lsin, ya’ni $O'(x_0;y_0)$.

Tekislik ixtiyoriy M nuqtasining Oxy sistemadagi koordinatalarini

$(x;y)$ bilan va $O'x'y'$ sistemadagi koordinatalarini $(x';y')$ bilan belgilaymiz (15-shakl).

U holda

$$\overline{OM} = x\vec{i} + y\vec{j}, \quad \overline{OO'} = x_0\vec{i} + y_0\vec{j}, \quad \overline{O'M} = x'\vec{i} + y'\vec{j}.$$

15-shakldan topamiz: $\overline{OM} = \overline{OO'} + \overline{O'M}$.

Bundan

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + x'\vec{i} + y'\vec{j}$$

yoki

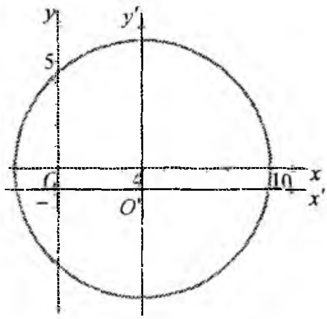
$$x = x_0 + x', \ y = y_0 + y'. \tag{2.12}$$

(2.12) formulalar M nuqtaning Oxy sistemadagi $(x;y)$ koordinatalarini $O'x'y'$ sistemadagi $(x';y')$ koordinatalar orqali topish imkonini beradi va aksincha.

5-misol. $(x-4)^2 + (y+1)^2 = 36$ tenglamani Oxy koordinatalar sistemasini parallel ko'chirish orqali soddalashtiring.

Yechish. Berilgan tenglama Oxy koordinatalar sistemasida markazi (4;–1) nuqtada yotuvchi va radiusi $R = 6$ ga teng aylanani ifodalaydi.

Oxy koordinatalar sistemasi $O'(x_0;y_0) = O'(4;-1)$ nuqtaga parallel ko'chirilsa, berilgan tenglama yangi $O'x'y'$ sistemada ham aylana tenglamasini beradi.



16-shakl.

(2.12) formulalarni qo'llab, topamiz:

$$x' = x - x_0 = x - 4, \quad y' = y - y_0 = y + 1.$$

U holda berilgan tenglama $O'x'y'$ sis-temada

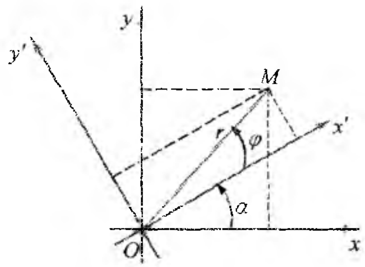
$$x'^2 + y'^2 = 36$$

ko'rinishni oladi, ya'ni markazi koordinatalar boshida bo'lgan va radiusi $R = 6$ ga teng aylanani ifodalaydi.

Aylana grafigini Oxy va $O'x'y'$ sistemalarda chizamiz (16-shakl).

Tekislikda Oxy to'g'ri burchakli koordinatalar sistemasi berilgan bo'lsin.

Koordinata o'qlarini burish – bu Oxy sistemadan uning koordinatalar boshini va o'qlari masshtablarini o'zgartirmasdan faqat koordinata o'qlarini biror burchakka burish orqali yangi $Ox'y'$ sistemaga o'tishdir.



17-shakl.

Oxy sistemani O nuqta atrofida soat strelkasi yo'nalishiga teskari yo'nalishda α burchakka burib, $Ox'y'$ sistemaga o'tamiz. Tekislik ixtiyoriy M nuqtasining Oxy sistemadagi koordinatalarini $(x;y)$ bilan va $O'x'y'$ sistemadagi koordinatalarini $(x';y')$ bilan belgilaymiz. M nuqta radius vektorining uzunligi r ga, uning Ox' o'q bilan tashkil qilgan burchagi φ ga teng bo'lsin (17-shakl).

17-shakldan topamiz:

$$x' = r\cos\varphi, \quad y' = r\sin\varphi, \tag{2.13}$$

$$x = r\cos(\varphi + \alpha), \quad y' = r\sin(\varphi + \alpha). \tag{2.14}$$

(2.14) tengliklar ustida almashtirishlar bajaramiz va (2.13) tengliklarni hisobga olib, topamiz:

$$x = r\cos(\varphi + \alpha) = r(\cos\varphi\cos\alpha - \sin\varphi\sin\alpha) =$$

$$= (r\cos\varphi)\cos\alpha - (r\sin\varphi)\sin\alpha = x'\cos\alpha - y'\sin\alpha,$$

$$y = r\sin(\varphi + \alpha) = r(\sin\varphi\cos\alpha + \cos\varphi\sin\alpha) =$$

$$= r(\cos\varphi)\sin\alpha + r(\sin\varphi)\cos\alpha = x'\sin\alpha + y'\cos\alpha.$$

Demak,

$$x = x'\cos\alpha - y'\sin\alpha, \quad y = x'\sin\alpha + y'\cos\alpha. \tag{2.15}$$

(2.15) formulalarga koordinata o'qlarini burish formulalari deyiladi. Bu formulalar M nuqtaning Oxy sistemadagi $(x;y)$ koordinatalarini

$Ox'y'$ sistemadagi $(x';y')$ koordinatalar orqali topish imkonini beradi va aksincha.

6-misol. $xy=-2$ tenglamani koordinata o'qlarini 45°ga burish orqali kanonik shaklga keltiring.

Yechish. (2.15) tengliklardan $\alpha=45^\circ$da topamiz:

$$x=x'\cos45^\circ-y'\sin45^\circ=\frac{\sqrt{2}}{2}(x'-y'),$$

$$y=x'\sin45^\circ+y'\cos45^\circ=\frac{\sqrt{2}}{2}(x'+y').$$

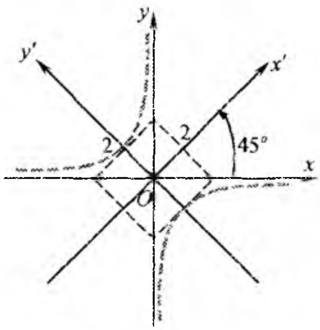
x va y ni $xy=-2$ tenglamaga qo'yamiz:

$$\frac{\sqrt{2}}{2}(x'-y')\cdot\frac{\sqrt{2}}{2}(x'+y')=-2,$$

$$\frac{1}{2}(x'^2-y'^2)=-2,$$

$$\frac{y'^2}{4}-\frac{x'^2}{4}=1.$$

Bu tenglama giperbolaning kanonik tenglamasi hisoblanadi, ya'ni $xy=-2$ tenglama bilan aniqlanuvchi chiziq $Ox'y'$ sistemada $\frac{y'^2}{4}-\frac{x'^2}{4}=1$ tenglama bilan aniqlanuvchi giperbolani ifodalaydi. Demak, $y=-\frac{2}{x}$ funksiyaning grafigi asimptotalari koordinata o'qlari bilan ustma-ust tushadigan teng tomonli giperboladan iborat (18-shakl).



18-shakl.

Ikkinchi tartibli umumiy tenglamani soddalashtirish

Ikkinchi tartibli umumiy tenglamani qaraymiz:

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0. \qquad (2.11)$$

Bunda $B\neq0$ bo'lsin. Koordinata o'qlarini α burchakka buramiz, ya'ni (2.15) formulalar yordamida eski koordinatalarni yangi koordinatalar orqali ifodalaymiz:

$$A(x'\cos\alpha - y'\sin\alpha)^2 + 2B(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha) +$$

$$+C(x'\sin\alpha + y'\cos\alpha)^2 + 2D(x'\cos\alpha - y'\sin\alpha) + 2E(x'\sin\alpha + y'\cos\alpha) + F = 0$$

yoki

$$A_1x'^2 + 2B_1x'y' + C_1y'^2 + 2D_1x' + 2E_1y' + F_1 = 0,$$

bu yerda

$A_1 = A\cos^2\alpha + 2B\cos\alpha\sin\alpha + C\sin^2\alpha;$

$B_1 = (C-A)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha);$

$C_1 = A\sin^2\alpha - 2B\cos\alpha\sin\alpha + C\cos^2\alpha;$

$D_1 = D\cos\alpha + E\sin\alpha; \qquad E_1 = E\sin\alpha - D\cos\alpha; \qquad F_1 = F.$

α burchakni shunday tanlaymizki, yuqoridagi tenglamada $x'y'$ oldidagi koeffitsiyent nolga aylansin, ya'ni

$$B_1 = (C-A)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha) = 0$$

tenglik bajarilsin.

Bundan

$$ctg2\alpha = \frac{A-C}{2B}. \tag{2.16}$$

Shunday qilib, koordinata o‘qlarini (2.16) shartni qanoatlantiruvchi α burchakka burish (2.11) tenglamani quyidagi tenglamaga keltiradi:

$$A_1x'^2 + C_1y'^2 + 2D_1x' + 2E_1y' + F_1 = 0, \qquad A_1^2 + B_1^2 \neq 0. \tag{2.17}$$

1-teorema. (2.17) tenglama hamma vaqt yoki aylanani ($A_1 = C_1$da), yoki ellipsni ($A_1 \cdot C_1 > 0$da), yoki giperbolani ($A_1 \cdot C_1 < 0$da), yoki parabolani ($A_1 \cdot C_1 = 0$da) aniqlaydi. Bunda ellips (aylana) uchun – nuqta yoki mavhum ellips (aylana), giperbola uchun – kesishuvchi to‘g‘ri chiziqlar juftligi, parabola uchun – parallel to‘g‘ri chiziqlar juftligi kabi buzilishlar bo‘lishi mumkin.

Isboti. $A_1 = C_1$ bo‘lgan holni batafsil tahlil qilamiz.

$A_1 = C_1$ da (2.17) tenglik ustida almashtirishlar bajaramiz:

$$A_1x'^2 + A_1y'^2 + 2D_1x' + 2E_1y' + F_1 = 0,$$

$$x'^2+y'^2+2\frac{D_1}{A_1}x'+2\frac{E_1}{A_1}y'+\frac{F_1}{A_1}=0,$$

$$x'^2+2\frac{D_1}{A_1}x'+\left(\frac{D_1}{A_1}\right)^2+y'^2+2\frac{E_1}{A_1}y'+\left(\frac{E_1}{A_1}\right)^2+\frac{F_1}{A_1}-\left(\frac{D_1}{A_1}\right)^2-\left(\frac{E_1}{A_1}\right)^2=0,$$

$$\left(x'+\frac{D_1}{A_1}\right)^2+\left(y'+\frac{E_1}{A_1}\right)^2=\left(\frac{D_1}{A_1}\right)^2+\left(\frac{E_1}{A_1}\right)^2-\frac{F_1}{A_1}. \tag{2.18}$$

Bunda, (2.18) tenglama va mos ravishda (2.17) tenglama:

1) $\left(\frac{D_1}{A_1}\right)^2+\left(\frac{E_1}{A_1}\right)^2-\frac{F_1}{A_1}>0$ bo‘lganda markazi $O_1\left(-\frac{D_1}{A_1},-\frac{E_1}{A_1}\right)$ nuqtada joylashgan va radiusi $R=\sqrt{\left(\frac{D_1}{A_1}\right)^2+\left(\frac{E_1}{A_1}\right)^2-\frac{F_1}{A_1}}$ ga teng aylanani aniqlaydi;

2) $\left(\frac{D_1}{A_1}\right)^2+\left(\frac{E_1}{A_1}\right)^2-\frac{F_1}{A_1}=0$ bo‘lganda $\left(x+\frac{D}{A}\right)^2+\left(y+\frac{E}{A}\right)^2=0$ ko‘rinishga keladi. Bu tenglikni yagona $O_1\left(-\frac{D_1}{A_1},-\frac{E_1}{A_1}\right)$ nuqta koordinatalari qanoatlantiradi. Bunda «aylana nuqtaga buzilgan» deyiladi;

3) $\left(\frac{D_1}{A_1}\right)^2+\left(\frac{E_1}{A_1}\right)^2-\frac{F_1}{A_1}<0$ bo‘lganda hech bir chiziqni aniqlamaydi.

Bunda «aylana mavhum aylanaga buzilgan» deyiladi.

Qolgan hollarda teorema shu kabi tahlil qilinadi.

Shunday qilib, (2.17) tenglama (mos ravishda (2.11) tenglama) ikkinchi tartibli chiziqlardan birini aniqlaydi.

7-misol. $4x^2-25y^2-24x+50y-89=0$ tenglama bilan berilgan ikkinchi tartibli chiziq ko‘rinishini aniqlang.

Yechish. Berilgan tenglamada $A=4,\ C=-25$.

Bundan $A\cdot C=4\cdot(-25)<0$.

Teoremaga ko‘ra, berilgan tenglama giperbolani ifodalaydi.

Tenglamada almashtirishlar bajaramiz:

$$4(x^2-6x+9)-25(y^2-2y+1)-36+25-89=0,$$

$$4(x-3)^2-25(y-1)^2=100,$$

$$\frac{(x-3)^2}{25}-\frac{(y-1)^2}{4}=1.$$

Demak, berilgan tenglama simmetriya markazi $O(3;1)$ nuqtada joylashgan va yarim o'qlari $a=5$, $b=2$ ga teng bo'lgan giperbolani aniqlaydi.

3.2.6. Mashqlar

1. Aylananing tenglamasini tuzing: 1) markazi $M_1(-1;3)$ nuqtada joylashgan va radiusi $R=6$ ga teng bo'lgan; 2) markazi $M_2(-3;5)$ nuqtada joylashgan va $A(4;4)$ nuqtadan o'tgan; 3) diametrlaridan birining uchlari $B(-1;3)$ va $C(-3;5)$ nuqtalarda bo'lgan; 4) $D(8;-4)$ nuqtadan o'tgan va koordinata o'qlariga uringan; 5) markazi $M(2;-1)$ nuqtada joylashgan va urinmalaridan biri $3x+4y+3=0$ to'g'ri chiziqda bo'lgan.

2. Aylanalarning markazi va radiusini toping:

1) $x^2+y^2+8x-14y+16=0$; 2) $x^2+y^2+4x-6y-3=0$;

3) $x^2+y^2-x+2y-1=0$; 4) $x^2+y^2+3x-7y-\frac{3}{2}=0$.

3. $A(1;-2)$ nuqta berilgan. Uning aylanalarning ichkarisida, tashqarisida yoki ustida yotishini aniqlang.

1) $x^2+y^2=5$; 2) $x^2+y^2=9$;

3) $x^2+y^2-8x-4y-5=0$; 4) $x^2+y^2-10x+8y=0$.

4. $x^2+y^2-2x+4y-20=0$ va $x^2+y^2-10y+20=0$ tenglamalar bilan berilgan aylanalar markazlari orasidagi masofani toping.

5. $\frac{x}{4}+\frac{y}{3}=1$ to'g'ri chiziqdan koordinata o'qlari kesgan kesmani diametr qilib aylana chizilgan. Bu aylana tenglamasini tuzing.

6. $A(2;-1)$, $B(3;4)$ nuqtalardan o'tgan va markazi $x-y-4=0$ to'g'ri chiziqda joylashgan aylana tenglamasini tuzing.

7. Uchlari $A(-2;2), B(0;-2), C(-1;-1)$ bo'lgan ABC uchburchakka tashqi chizilgan aylananing markazi va radiusini toping.

8. k ning qanday qiymatlarida $y=kx$ to'g'ri chiziq $x^2+y^2-8x-2y+16=0$ aylanani kesadi, bu aylanaga urinadi?

9. $(x-4)^2+(y-2)^2=4$ aylanaga uringan va koordinatalar boshidan o'tgan to'g'ri chiziqlar tenglamalarini tuzing.

10. Aylana tenglamalarini parametrik ko'rinishga keltiring:

1) $x^2+y^2=16x$; 2) $x^2+y^2=4y$; 3) $x^2+y^2=2x+2y$.

11. Fokuslari Oy o'qda $O(0;0)$ nuqtaga nisbatan simmetrik joylashgan va quyidagi shartlarni qanoatlantiruvchi ellipsning kanonik tenglamasini tuzing: 1) kichik o'qi 12 ga va ekssentrisiteti $\frac{4}{5}$ ga teng; 2) fokuslari orasidagi masofa 10 ga va ekssentrisiteti $\frac{5}{7}$ ga teng; 3) $M_1(6;0)$ va $M_2(0;9)$ nuqtalardan o'tgan; 4) direktrisalari orasidagi masofa $\frac{50}{3}$ ga va ekssentrisiteti $\varepsilon=\frac{3}{5}$ ga teng.

12. $\frac{x^2}{12}+\frac{y^2}{4}=1$ ellipsga tomonlari ellips o'qlariga parallel qilib kvadrat ichki chizilgan. Kvadratning yuzini toping.

13. $5x^2+20y^2-100=0$ ellipsning $x+y-20=0$ to'g'ri chiziqqa parallel bo'lgan urinmasi tenglamasini tuzing.

14. $16x^2+25y^2-400=0$ ellipsning bir fokusidan uning kichik o'qiga parallel o'tgan vatari uzunligini toping.

15. $\frac{x^2}{50}+\frac{y^2}{18}=1$ ellipsning $M(x;y)$ nuqtasidan uning o'ng fokusigacha bo'lgan masofa chap fokusigacha bo'lgan masofadan to'rt marta katta. $M(x;y)$ nuqtani toping.

16. $\frac{x^2}{9}+\frac{y^2}{8}=1$ ellipsning $M(x;y)$ nuqtasidan uning chap fokusigacha bo'lgan masofa o'ng fokusigacha bo'lgan masofadan ikki marta katta. $M(x;y)$ nuqtani toping.

17. Ellipsning bir fokusidan uning katta o'qi oxirlarigacha bo'lgan masofalar 2 va 8 ga teng. Ellipsning kanonik tenglamasini tuzing.

18. Ellipsning tenglamalarini parametrik ko'rinishga keltiring:

1) $16x^2+25y^2-400=0$; 2) $144x^2+25y^2-3600=0$.

19. $x+2y-7=0$ to'g'ri chiziq bilan $x^2+4y^2=25$ ellipsning kesishish nuqtalarini toping.

20. Fokuslari ordinatalar o'qida joylashgan va quyidagi shartlarni qanoatlantiruvchi giperbolaning kanonik tenglamasini tuzing: 1) direktrisalari orasidagi masofa $\frac{18}{5}$ ga va ekssentrisiteti $\frac{5}{3}$ ga teng; 2) direktrisalari orasidagi masofa $\frac{288}{13}$ ga teng va asimptotalari tenglamalari $y=\pm\frac{12}{5}x$; 3) direktrisalari orasidagi masofa $\frac{32}{5}$ ga va haqiqiy o'qi 8 ga teng; 4) direktrisalari orasidagi masofa $\frac{50}{7}$ ga va fokuslari orasidagi masofa 14 ga teng.

21. Giperbolaning nuqtalaridan biri va asimptotalarining tenglamalari berilgan. Giperbolaning kanonik tenglamasini tuzing:

1) $M(6;2)$, $y=\pm\frac{\sqrt{3}}{3}x$; 2) $M(4;2)$, $y=\pm\frac{\sqrt{2}}{2}x$;

3) $M(4;3)$, $y=\pm\frac{3}{2}x$; 4) $M(6;3)$, $y=\pm\frac{\sqrt{3}}{2}x$.

22. Giperbolaning ekssentrisiteti 2 ga teng. Uning asimptotalari orasidagi burchakni toping.

23. Giperbolaning asimptotasi haqiqiy o'q bilan $\frac{\pi}{4}$ ga teng burchak tashkil qiladi. Giperbolaning ekssentrisitetini toping.

24. $5x^2+17y^2-85=0$ ellips berilgan. Ellips bilan bir xil fokuslarga ega bo'lgan teng tomonli giperbolaning kanonik tenglamasini tuzing.

25. Giperbola $25x^2+9y^2=225$ ellips bilan bir xil fokuslarga ega. Giperbolaning ekssentrisiteti 2 ga teng bo'lsa, uning kanonik tenglamasini tuzing.

26. Assimptotalari $O'(2;-3)$ nuqtada kesishuvchi va $B(4;-1)$ nuqtadan o'tuvchi giperbola tenglamasini tuzing.

27. Giperbolaning $y=\frac{4x-3}{3x+5}$ tenglamasini sodda ko'rinishga keltiring.

28. Berilgan fokusi va direktrisasi tenglamasiga ko'ra parabolaning kanonik tenglamasini tuzing:

1) $F(-3;4)$, $x-5=0$; 2) $F(5;3)$, $y+2=0$.

29. Berilgan tenglamasiga ko'ra parabolaning uchini va simmetriya o'qining tenglamasini aniqlang:

1) $y^2-2y+16x+65=0$; 2) $2x^2+y-8x+5=0$.

30. Berilgan tenglamalar qanday chiziqlarni aniqlaydi?

1) $\begin{cases} x=\frac{1}{2}(e^t+e^{-t}), \\ y=\frac{1}{2}(e^t-e^{-t}) \end{cases}$; 2) $\begin{cases} x=\frac{2}{t^2}, \\ y=\frac{3}{t} \end{cases}$;

3) $y=-2\sqrt{x^2+1}$; 4) $x=-\sqrt{y^2+4}$.

31. $x=\frac{a}{\sin t}$, $y=\frac{b\cos t}{\sin t}$ giperbolaning parametrik tenglamalari bo'lishini ko'rsating.

32. $x=2pt^2$, $y=2pt$, $t\in R$ parabolaning parametrik tenglamalari bo'lishini ko'rsating.

33. $13x^2+10xy+13y^2-72=0$ tenglamani kanonik shaklga keltiring.

34. $x^2-6\sqrt{3}xy-5y^2-8=0$ tenglamani kanonik shaklga keltiring.

35. Egri chiziqning tenglamasini soddalashtiring, chiziqning turini aniqlang va shaklini chizing:

1) $5x^2+9y^2-30x+18y+9=0$; 2) $2x^2-12x+y+13=0$;

3) $5x^2-4y^2+30x+8y+21=0$; 4) $2y^2-x-12y+14=0$;

5) $x^2-6x+y^2-8=0$; 6) $x^2+y+y^2-1=0$.

36. $\frac{x^2}{36}+\frac{y^2}{12}=1$ ellipsga $M(-3;3)$ nuqtada o'tkazilgan urinma tenglamasini tuzing.

37. $\frac{x^2}{2}-\frac{y^2}{16}=1$ giperbolaga $M(2;-4)$ nuqtada o'tkazilgan urinma tenglamasini tuzing.

38. $\frac{y^2}{4}-x^2=1$ giperbolaga $M\left(\frac{\sqrt{5}}{2};3\right)$ nuqtada o'tkazilgan urinma tenglamasini tuzing.

39. $x^2=16y$ parabolaning $2x+4y+7=0$ to'g'ri chiziqqa perpendikular bo'lgan urinmasini toping.

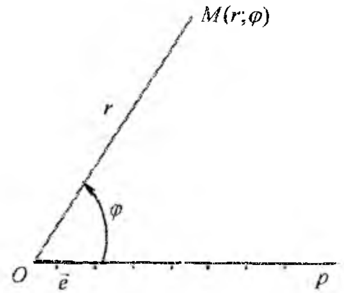
3.3. QUTB KOORDINATALARDA CHIZIQLAR

3.3.1. Qutb koordinatalari

Tekislikda sanoq boshiga, musbat yo'nalishga va masshtab birligiga ega bo'lgan Op nur *qutb o'qi*, uning $O-$ sanoq boshi *qutb* deb ataladi.

M tekislikning qutb bilan ustma-ust tushmaydigan ixtiyoriy nuqtasi bo'lsin. Bunda M nuqtaning holati ikkita son, O qutbdan M nuqtagacha bo‘lgan r masofa va Op qutb o‘qi bilan $\overline{OM}$ yo‘nalgan kesma orasidagi φ burchak bilan aniqlanadi (Op nurdan boshlab burchak yo‘nalishi soat strelkasi yo‘alishiga teskari tanlanadi).

r va φ sonlariga M nuqtaning *qutb koordinatalari* deyiladi va $M(r;\varphi)$ deb yoziladi. Bunda r masofa *qutb radiusi*, φ burchak *qutb burchagi* deb ataladi (19-shakl).

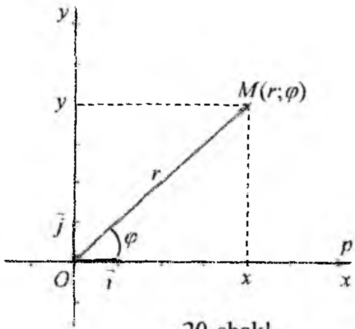


19-shakl.

Tekislikning barcha nuqtalarini aniqlash uchun r va φ kattaliklarni $0 \le r < +\infty$, $-\pi < \varphi \le \pi$ chegaralarda olish yetarli bo‘ladi. Bunda tekislikning har bir nuqtasiga yagona r va φ sonlar jufti mos keladi, va aksincha, har bir $(r;\varphi)$ sonlar juftiga tekislikdagi yagona nuqta mos keladi.

Nuqtaning qutb va to‘g‘ri burchakli koordinatalari orasidagi bog‘lanishni topamiz. Bunda to‘g‘ri burchakli koordinatalar sistemasining koordinatalari boshini qutb bilan va abssissalar o‘qini qutb o‘qi bilan ustma-ust tushadigan qilib tanlaymiz (20-shakl).

M nuqta x va y to‘g‘ri burchakli koordinatalarga, r va φ qutb koordinatalarga ega bo‘lsin.



20-shakl.

20-shakldan topamiz:

$$x = r\cos\varphi,\ \ y = r\sin\varphi. \qquad (3.1)$$

Bu tengliklar nuqtaning to‘g‘ri burchakli koordinatalarini uning qutb koordinatalari bilan bog‘laydi.

(3.1) tengliklardan nuqtaning qutb koordinatalari bilan uning to‘g‘ri burchakli koordinatalari o‘rtasida quyidagi bog‘lanish hosil qilinadi:

$$r = \sqrt{x^2 + y^2},\ tg\varphi = \frac{y}{x}. \qquad (3.2)$$

Bunda φ burchakning qiymati nuqtaning joylashgan choragiga (x, y larning ishoralari asosida) qarab, $-\pi < \varphi \le \pi$ oraliqda tanlanadi.

1 -misol. $M(-1;-\sqrt{3})$ nuqtaning qutb koordinatalarini toping.

Yechish. Berilgan nuqtaning qutb koordinatalarini (3.2) formulalar bilan aniqlaymiz:

$$r=\sqrt{(-1)^2+(-\sqrt{3})^2}=2, \qquad tg\varphi=\frac{-\sqrt{3}}{-1}=\sqrt{3}.$$

M nuqtan III chorakda yotadi.

U holda $\varphi=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$ bo'ladi. Demak, $M\left(2;-\frac{2\pi}{3}\right)$.

3.3.2. Qutb koordinatalar sistemasida chiziqlar

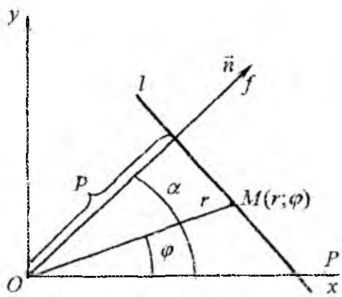
Qutb koordinatalar sistemasida chiziq tenglamasi deb aynan shu chiziq barcha nuqtalarining r va φ koordinatalari orasidagi bog'lanishni aniqlovchi ikki noma'lumli

$$F(r;\varphi)=0 \tag{3.3}$$

ko'rinishdagi tenglamaga aytiladi.

Mumkin bo'lganda (3.3) tenglama r ga nisbatan yechiladi va $r=r(\varphi)$ ko'rinishga keltiriladi.

Chiziqning $F(x;y)=0$ tenglamasidan qutb tenglamasiga o'tish uchun x va y lar o'rniga ularning (3.1) formulalardagi qiymatlari qo'yiladi va aksincha chiziqning qutb tenglamasidan $F(x;y)=0$ tenglamasiga o'tish (3.2) formulalar bilan amalga oshiriladi.



21-shakl.

To'g'ri chiziqning qutb tenglamasi

To'g'ri chiziq tenglamasini qutb koordinatalarida topamiz.

Bunda O qutbdan l to'g'ri chiziqqacha bo'lgan p masofa va Op qutb o'qi bilan berilgan to'g'ri chiziqqa perpendikular bo'lgan f o'q orasidagi α burchak berilgan bo'lsin (21-shakl).

l chiziqning ixtiyoriy $M(r;\varphi)$ nuqtasi uchun $\Pr_f\overrightarrow{OM}=p$ bo'ladi.

Ikkinchi tomondan

$$\Pr_f\overrightarrow{OM}=|\overrightarrow{OM}|\cdot\cos(\alpha-\varphi)=r\cos(\alpha-\varphi).$$

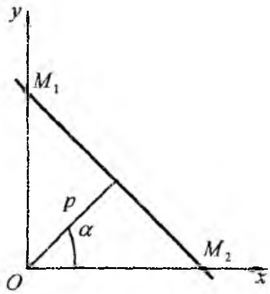
U holda

$$r\cos(\alpha-\varphi)=p. \qquad (3.4)$$

(3.4) tenglamaga *to'g'ri chiziqrning qutb tenglamasi* deyiladi.

2-misol. $M_1\left(5;\frac{\pi}{2}\right)$ va $M_2(5;0)$ nuqtalardan o'tuvchi to'g'ri chiziqning qutb tenglamasini tuzing.

Yechish. To'g'ri chiziqning M_1 va M_2 nuqtalar orasidagi kesmasi katetlari 5 ga teng bo'lgan to'g'ri burchakli uchburchakning gipotenuzasi bo'ladi. Bunda qutbdan to'g'ri chiziqqacha bo'lgan masofa to'g'ri burchak uchidan gipotenuzaga tushirilgan balandlikdan iborat (22-shakl).



22-shakl.

Uning uzunligini (p ni) va yo'nalshini (α ni) topamiz:

$$p=\frac{OM_1\cdot OM_2}{\sqrt{OM_1^2+OM_2^2}}=\frac{5\cdot 5}{\sqrt{5^2+5^2}}=\frac{5\sqrt{2}}{2},\ \alpha=\frac{\pi}{4}.$$

U holda (3.4) formulaga ko'ra,

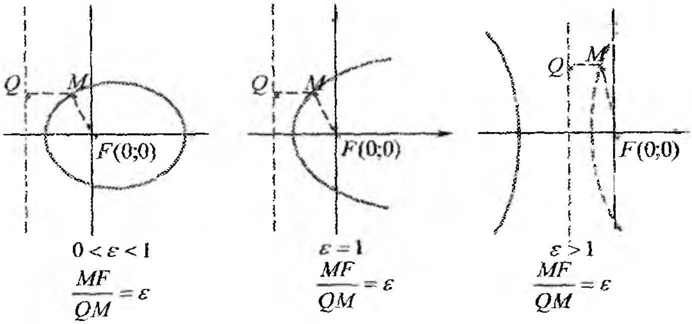
$$r\cos\left(\varphi-\frac{\pi}{4}\right)=\frac{5\sqrt{2}}{2}.$$

Konus kesimlarining qutb tenglamalari

Konus kesimlarning (ellipsning, giperbolaning, parabolaning) qutb tenglamalarini keltirib chiqaramiz. Bunda chiziqlarning fokuslaridan biri qutbda yotsa, uning tenglamasi sodda ko'rinishni oladi.

Konus kesimlarning tenglamalarini keltirib chiqarishda ularning har bir nuqtasidan fiksirlangan nuqtagacha (fokusgacha) bo'lgan masofaning fiksirlangan to'g'ri chiziqqacha (direktrisagacha) bo'lgan masofaga nisbati o'zgarmas kattalikka – ekstsentrisitetga teng bo'lishi xossasidan foydalaniladi. Bunda konus kesim $\varepsilon<1$ bo'lganda ellipsdan, $\varepsilon=1$ bo'lganda paraboladan va $\varepsilon>1$ bo'lganda giperboladan iborat bo'ladi (23-shakl).

l – konus kesimlardan birining tarmog'i, F – bu tarmoqning fokusi, d – qutbdan chapda joylashgan vertikal direktrisasi, ε – ekstsentisitet bo'lsin. Fokusdan direktrisagacha bo'lgan masofani p bilan belgilaymiz (24-shakl).



23-shakl.

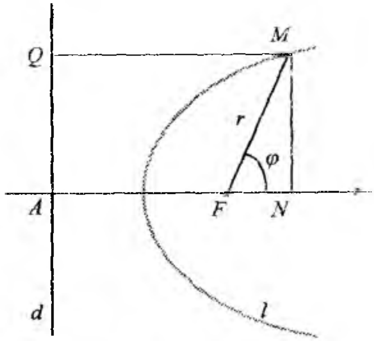
l chiziqda ixtiyoriy $M(r,\varphi)$nuqtani olamiz. U holda konus kesimlarning xossasiga ko‘ra, $\frac{FM}{QM}=\varepsilon$ bo‘ladi.

Bundan $FM=\varepsilon QM$ yoki

$$r=\varepsilon AN=\varepsilon(AF+FN)=\varepsilon(p+r\cos\varphi).$$

Oxirgi tenglikni r ga nisbatan yechamiz:

$$r=\frac{\varepsilon\cdot p}{1-\varepsilon\cos\varphi}. \qquad (3.5)$$



24-shakl.

(3.5) tenglamaga *konus kesimlarning qutb tenglamasi* deyiladi. Bu tenglama $\varepsilon<1$ da ellipsni, $\varepsilon>1$ giperbolaning bir tarmog‘ini va $\varepsilon=1$ da parabolani aniqlaydi.

1-izoh. Konus kesimlarning qutb tenglamalari direktritsaning joylashishiga bog‘liq bo‘ladi:

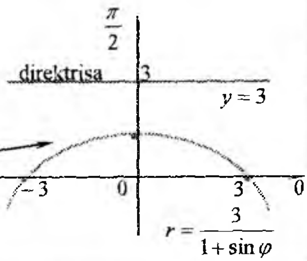
Direktrisaning holati – tenglama

vertikal va qutbdan chapda – $r=\frac{\varepsilon\cdot p}{1-\varepsilon\cos\varphi}$;

vertikal va qutbdan o‘ngda – $r=\frac{\varepsilon\cdot p}{1+\varepsilon\cos\varphi}$;

gorizontal va qutbdan yuqorida – $r=\frac{\varepsilon\cdot p}{1+\varepsilon\sin\varphi}$;

gorizontal va qutbdan pastda – $r=\frac{\varepsilon\cdot p}{1-\varepsilon\sin\varphi}$.



Boshqa chiziqlarning qutb tenglamalari

Qutb tenglama bilan modellashtiriladigan masalalardan birini qaraymiz.

Uzunligi $2a\,(a>0)$ga teng bolgan AB kesma uchlari bilan biror to‘g‘ri burchakning tomonlari bo‘ylab sirpanib harakatlanayotgan bo‘lsin. To‘g‘ri burchakning O uchidan bu kesmaga OM perpendikular tushurilgan (25-shakl). AB kesmaning harakati vaqtida OM perpendikular M asosining traektoriyasini topaylik.

M asosning (nuqtaning) traektoriyasini qutb koordinatalar sistemasida tuzish uchun to‘g‘ri burchakning O uchini qutb, OA o‘qni qutb o‘qi deb olamiz. M nuqtaning qutb koordinatalarini r, φ deb belgilaymiz, ya’ni $M(r;\varphi)$ deymiz.

AOM uchburchakdan topamiz:

$$OM = OA\cos\varphi,$$
$$r = OA\cos\varphi.$$

AOB uchburchakdan topamiz:

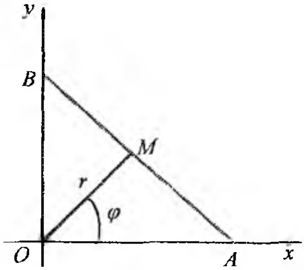
$$OA = AB\sin\varphi,$$
$$OA = 2a\sin\varphi.$$

Bundan M asos izlanayotgan traektoriyasining qutb koordinatalar sistemasidagi tenglamasini hosil qilamiz:

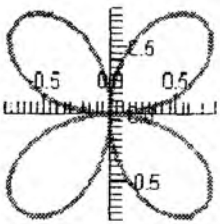
$$r = a\sin 2\varphi.$$

Bu tenglama bilan aniqlanuvchi chiziq to‘rt yaproqli gul deb ataladi. Uning grafigini $a=1$ da *Maple* paketi yordamida chizamiz (26-shakl).

```
> with(plots):
> animatecurve([sin(2*t),t,t=-1*i..1*i], coords=olar,
frames=60,numpoints=100);
```



25-shakl.



26-shakl.

Matematikaning keyingi bo'limlarining misol va masalalarini yechishda foydalaniladigan chiziqlarning grafiklari va ularning qutb yoki parametrik tenglamalari 1-ilovada keltirilgan.

3.3.3. Mashqlar

1. $A(\sqrt{3};1)$ va $B(-\sqrt{3};-1)$ nuqtalarning qutb koordinatalarini toping.

2. $A\left(2;-\frac{\pi}{3}\right)$ va $B\left(1;\frac{2\pi}{3}\right)$ nuqtalarning to'g'ri burchakli koordinatalarini toping.

3. $A\left(5;\frac{\pi}{4}\right)$ va $B\left(8;-\frac{\pi}{12}\right)$ nuqtalar orasidagi masofani toping.

4. Uchlari O qutbda va $A(r_1;\varphi_1)$, $B(r_2;\varphi_2)$ nuqtalarda joylashgan OAB uchburchak-ning yuzini toping, bu yerda $\varphi_2 > \varphi_1$.

5. Ikkita qarama-qarshi uchlari $A\left(2;-\frac{\pi}{6}\right)$, $B\left(2;-\frac{2\pi}{3}\right)$ nuqtalarda bo'lgan kvadratning yuzini toping.

6. Kvadratning ikkita qo'shni uchlari berilgan: $A\left(6;\frac{\pi}{3}\right)$, $B\left(2;\frac{4\pi}{3}\right)$. Kvadratning yuzini toping.

7. Berilgan tenglamalarni dekart koordinatalarida yozing:

1) $r = 3\sin\varphi$;
2) $r = 5\cos\varphi$;
3) $r^2 = \sin 2\varphi$;
4) $r = 4\cos 2\varphi$;
5) $r = \frac{2}{1+\sin\varphi}$;
6) $r = \frac{3}{1-\cos\varphi}$;
7) $r = \frac{2}{5+3\cos\varphi}$;
8) $r = \frac{32}{3+5\sin\varphi}$.

8. Berilgan tenglamalarni qutb koordinatalarida yozing:

1) $y = 5$;
2) $y = 2x - 1$;
3) $x^2 = 4y$;
4) $x^2 - y^2 = a^2$;
5) $x^2 + y^2 - 2ax = 0$;
6) $xy = 4$;
7) $\frac{x^2}{25} + \frac{y^2}{16} = 1$;
8) $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

9. Berilgan tenglamasiga ko'ra chiziqlarning turini aniqlang:

1) $r = \dfrac{6}{2+\sin\varphi}$;   2) $r = \dfrac{5}{-1+2\cos\varphi}$;

3) $r = \dfrac{3}{1+\cos\varphi}$;   4) $r = \dfrac{2}{2-\cos\varphi}$.

10. Berilgan chiziqlarning grafiklarini $\varepsilon = 1$, $\varepsilon = \dfrac{1}{2}$, $\varepsilon = \dfrac{3}{4}$ larda chizing:

1) $r = \dfrac{4\varepsilon}{1+\varepsilon\cos\varphi}$;   2) $r = \dfrac{4\varepsilon}{1+\varepsilon\sin\varphi}$;

3) $r = \dfrac{4\varepsilon}{1-\varepsilon\sin\varphi}$;   4) $r = \dfrac{4\varepsilon}{1-\varepsilon\cos\varphi}$.

3.4. TEKISLIK

3.4.1. Fazoda sirt va chiziq

Umumiy boshlang'ich O nuqtaga va bir xil masshtab birligiga ega bo'lgan o'zaro perpendikulyar Ox, Oy va Oz o'qlar fazoda to'g'ri burchakli $Oxyz$ koordinatalar sistemasini hosil qiladi.

$Oxyz$ koordinatalar sistemasida uchta x, y va z sonlari fazodagi har qanday M nuqtaning o'rnini to'liq aniqlaydi. Bunda nuqta $M(x;y;z)$ kabi belgilanadi, x ga M nuqtaning *abssissasi*, y ga M nuqtaning *ordinatasi*, z ga M nuqtaning *applikatasi* deyiladi.

$Oxyz$ fazodagi sirt tenglamasi deb aynan shu sirt nuqtalarining x, y, z koordinatalari orasidagi bog'lanishni
aniqlovchi uch noma'lumli

$$F(x, y, z) = 0$$

tenglamaga aytiladi.

Shu kabi, koordinatalari uch noma'lumli $F(x, y, z) = 0$ tenglamani qanoatlantiruvchi $Oxyz$ fazoning barcha $M(x;y;z)$ nuqtalari to'plamiga *fazoda* shu tenglama bilan aniqlanuvchi *sirt* deyiladi.

Fazodagi sirt $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $(u;v) \in D$ parametrik tenglamalar bilan ham berilishi mumkin, bu yerda $x(u,v), y(u,v), z(u,v)$ – D sohada berilgan sirt barcha nuqtalarining va faqat shu nuqtalarning koordinatalarini beruvchi ikki o'zgaruvchili funksiyalar.

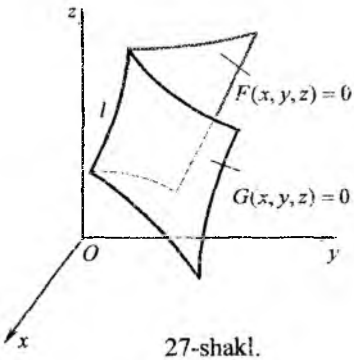
Masalan,

$$x = R\sin u\cos v,\ \ y = R\sin u\sin v,\ \ z = R\cos u,\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$$

parametrik tenglamalar *sferani* ifodalaydi.

Fazodagi chiziqni ikki sirtning kesishish chizig'i yoki ikki sirt umumiy nuqtalarining gometrik o'rni deb qarash mumkin (27-shakl).

l chiziqni aniqlovchi ikki sirt $F(x,y,z)=0$ va $G(x,y,z)=0$ tenglamalar bilan berilgan bo'lsin (27-shakl). U holda l chiziq ikkala tenglamani ham qanoatlantiruvchi $M(x;y;z)$ nuqtalar to'plamidan tashkil topadi.



27-shakl.

Koordinatalari

$$\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0 \end{cases}$$

Tenglamalar sistemasini qanoatlantiruvchi $Oxyz$ fazoning barcha $M(x;y;z)$ nuqtalari to'plamiga *fazodagi* shu tenglamalar sistemasi bilan aniqlanuvchi *chiziq* deyiladi.

Shu kabi, *Oxyz fazodagi chiziq tenglamasi* deb aynan shu chiziq barcha nuqtalarining x, y, z koordinatalarini aniqlovchi

$$\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0 \end{cases}$$

tenglamalar sistemasiga aytiladi.

Fazodagi chiziqni nuqtaning trayektoriyasi deb qarash mumkin. Bunda chiziq $\vec{r} = \vec{r}(t)$ vektor tenglama bilan yoki $x = x(t),\ y = y(t),\ z = z(t),\ t \in T$ parametrik tenglamalar bilan beriladi.

Masalan,

$$x = R\cos at,\ \ y = R\sin at,\ \ z = \frac{h}{2\pi}t$$

parametrik tenglamalar *vint chizig'ini* ifodalaydi.

Fazodagi analitik geometriyada sirtni (yoki to'g'ri chiziqni) o'rganishda ikkita masala ko'riladi: geometrik xossalariga ko'ra, sirtning (yoki to'g'ri chiziqning) tenglamasini keltirib chiqarish; tenglamasiga asosan sirtning (yoki to'g'ri chiziqning) ko'rinishi va xossalarini tekshirish.

3.4.2. Tekislik tenglamalari

Tekislikning fazodagi o'rni turli parametrlar bilan (masalan, tekislikning koordinata o'qlarida ajratgan kesmalari bilan) bir qiymatli aniqlanishi mumkin. Shu sababli parametrlariga ko'ra tekislikning turli tenglamalari keltirib chiqariladi.

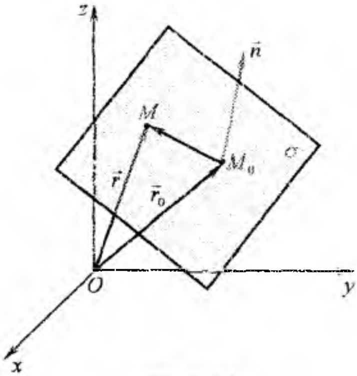
I. Tekislikda yotuvchi $M_0(x_0;y_0;z_0)$ nuqta va to'g'ri chiziqqa perpendikular bo'lgan $\vec{n}=\{A;B;C\}$ vektor berilgan.

Tekislikka perpendikular bo'lgan har qanday vektorga *tekislikning normal vektori* deyiladi.

σ tekislikning ixtiyoriy $M(x;y;z)$ nuqtasini olamiz. M va M_0 nuqtalarning radius vektorlari mos ravishda $\vec{r}$ va $\vec{r}_0$ bo'lsin.

U holda $\overline{M_0M}=\vec{r}-\vec{r}_0$ bo'ladi.

M va M_0 tekislik nuqtalari bo'lgani uchun $\overline{M_0M}$ vektor tekislikda yotadi va tekislikning normal vektoriga perpendikular bo'ladi, ya'ni $\vec{n}\perp\overline{M_0M}$ (28-shakl). Ikki vektorning perpendikularlik shartiga asosan tekislik tenglamasini topamiz:



28-shakl.

$$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0. \tag{4.1}$$

Bu tenglamaga *tekislikning vektor tenglamasi* deyiladi.

(4.1) tenglamaga normal vektor va radius vektorlarining koordinatalarini qo'yib, topamiz:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{4.2}$$

Bu tenglamaga *tekislikning skalyar tenglamasi* deyiladi.

Shuningdek, (4.2) tenglamaga *berilgan nuqtadan o'tuvchi va berilgan vektorga perpendikular tekislik tenglamasi* deyiladi.

1-misol. $M_0(3;4;5)$ nuqtadan o'tuvchi va normal vektori $\vec{n}=\{-1;-3;2\}$ bo'lgan tekislik tenglamasini tuzing.

Yechish. Masalaning shartiga ko'ra,

$$x_0=3, y_0=4, z_0=5,\ A=-1, B=-3, C=2.$$

U holda (4.2) tenglamadan topamiz:

$$(-1)\cdot(x-3)+(-3)\cdot(y-4)+2\cdot(z-5)=0$$

yoki

$$x+3y-2z-5=0.$$

II. Tekislikda yotuvchi uchta $M_1(x_1;y_1;z_1)$, $M_2(x_2;y_2;z_2)$, $M_3(x_3;y_3;z_3)$ *nuqta berilgan.*

σ tekislikda yotuvchi ixtiyoriy $M(x;y;z)$ nuqtani olamiz va

$$\overline{M_1M}=\{x-x_1;y-y_1;z-z_1\},$$

$$\overline{M_1M_2}=\{x_2-x_1;y_2-y_1;z_2-z_1\},$$

$$\overline{M_1M_3}=\{x_3-x_1;y_3-y_1;z_3-z_1\}$$

vektorlarni yasaymiz.

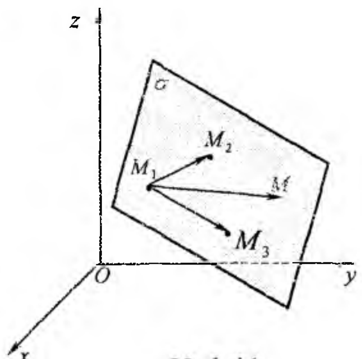
Bunda $\overline{M_1M}, \overline{M_1M_2}, \overline{M_1M_3}$ vektorlar komplanar bo'ladi (29-shakl). Vektorlarning komplanarlik shartidan topamiz:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0. \quad (4.3)$$

(4.3) tenglamaga *berilgan uchta nuqtadan o'tuvchi tekislik tenglamasi* deyiladi.

(4.3) tenglamada

$$\vec{s}=\overline{M_1M_3}=\{p;q;r\}$$



29-shakl.

belgilash kiritib, topamiz:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ p & q & r \end{vmatrix} = 0. \tag{4.4}$$

(4.4) tenglamaga *berilgan ikkita nuqtadan o'tuvchi va berilgan vektorga parallel tekislik tenglamasi* deyiladi.

Shu kabi

$$\vec{s}_1 = \overline{M_1M_2} = \{p_1; q_1; r_1\},$$

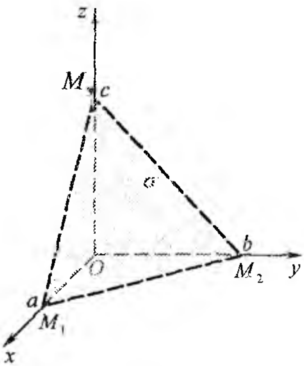
$$\vec{s}_2 = \overline{M_1M_3} = \{p_2; q_2; r_2\}$$

belgilashlarda (4.3) tenglamadan topamiz:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0. \tag{4.5}$$

(4.5) tenglamaga *berilgan nuqtadan o'tuvchi va berilgan ikki vektorga parallel tekislik tenglamasi* deyiladi.

$M_1(x_1; y_1; z_1)$, $M_2(x_2; y_2; z_2)$ va $M_3(x_3; y_3; z_3)$ nuqtalar σ tekislikning mos ravishda Ox, Oy va Oz o'qlarda yotuvchi nuqtalari, ya'ni $M_1(a;0;0)$, $M_2(0;b;0)$ va $M_3(0;0;c)$ bo'lsin (30-shakl).



30-shakl.

U holda (4.3) formulaga ko'ra,

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

bo'ladi.

Bundan

$$bcx - abc + abz + acy = 0, \; bcx + acy + abz = abc$$

yoki

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{4.6}$$

30-shakldan ko'inadiki, a, b, c mos ravishda σ tekislikning Ox, Oy va Oz o'qlarda ajratgan kesmalarini ifodalaydi.

Shu sababli (4.6) tenglamaga *tekislikning kesmalarga nisbatan tenglamasi* deyiladi.

2-misol. $M_0(2;-1;3)$ nuqtadan o'tuvchi, $\vec{a}=\{3,0,-1\}$ va $\vec{b}=\{-3;2;2\}$ vektorlarga parallel tekislik tenglamasini tuzing.

Yechish. Izlanayotgan tekislik tenglamasini (4.5) formula bilan topamiz:

$$\begin{vmatrix} x-2 & y+1 & z-3 \\ 3 & 0 & -1 \\ -3 & 2 & 2 \end{vmatrix}=0,$$

$$(x-2)\cdot 2-(y+1)\cdot(6-3)+(z-3)\cdot 6=0,$$

$$2x-3y+6z-25=0.$$

3-misol. Ox, Oy va Oz o'qlarda mos ravishda 2, (-4) va 6 ga teng kesmalar ajratuvchi tekislik tenglamasini tuzing.

Yechish. Masalaning shartiga ko'ra: $a=2$; $b=-4$; $c=6$.

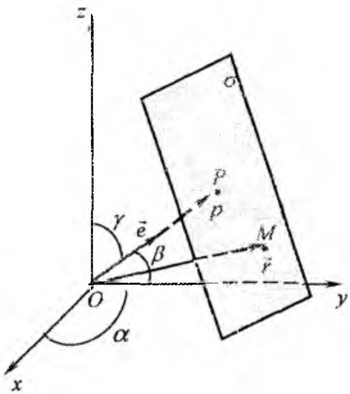
Tekislikning kesmalarga nisbatan tenglamasidan topamiz:

$$\frac{x}{2}+\frac{y}{(-4)}+\frac{z}{6}=1,$$

$$6x-3y+2z-12=0.$$

III. Tekislik $\vec{n}=\overrightarrow{OP}$ normalining uzunligi p va birlik vektori $\vec{e}=\{\cos\alpha;\cos\beta;\cos\gamma\}$ berilgan.

σ tekislikda yotuvchi ixtiyoriy $M(x;y;z)$ nuqtani olamiz. Bu nuqtaning radius vektori $\vec{r}=\overrightarrow{OM}=\{x;y;z\}$ bo'lsin (31-shakl). Bunda $\vec{r}$ radius vektorning $\vec{e}$ vektor yo'nalishidagi proeksiyasi p ga teng bo'ladi, ya'ni



31-shakl.

$$\Pi p_{\vec{e}}\vec{r}=p.$$

Bundan

$$\vec{r}\vec{e}=p, \quad \vec{r}\vec{e}-p=0$$

yoki

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0. \tag{4.7}$$

(4.7) tenglamaga *tekislikning normal tenglamasi* deyiladi.

Keltirib chiqarilgan (4.1)-(4.7) formulalar asosida ushbu xulosa kelib chaqadi:

x, y, z o'zgaruvchilarning har qanday birinchi darajali tenglamasi fazodagi biror tekislikni ifodalaydi va aksincha, fazodagi har qanday tekislik x, y, z o'zgaruvchilarning biror birinchi darajali tenglamasi bilan aniqlanadi.

Demak, har bir σ tekislik tenglamasini

$$Ax + By + Cz + D = 0 \tag{4.8}$$

ko'rinishda yozish mumkin, bu yerda D–ozod had; $A^2 + B^2 + C^2 \neq 0$.

(4.8) tenglamada $\vec{n} = \{A; B; C\}$ bo'lishini (4.2) tenglama yordamida (to'g'ri chiziqdagi kabi) ko'rsatish mumkin.

(4.8) tenglamaga *tekislikning umumiy tenglamasi* deyiladi.

(4.8) tenglamada: 1) $A = 0$ bo'lsa, tenglama $By + Cz + D = 0$ ko'rinishga keladi. Bunda tekislikning $\vec{n} = \{0; B; C\}$ normal vektori Ox o'qqa perpendikular bo'ladi. Shu sababli tekislik Ox o'qqa parallel bo'ladi. Shu kabi $B = 0$ da $Ax + Cz + D = 0$ tenglama Oy o'qqa parallel tekislikni, $C = 0$ da $Ax + By + D = 0$ tenglama Oz o'qqa parallel tekislikni ifodalaydi; $Ax + By + D = 0$ tenglama Oz o'qqa parallel tekislikni ifodalaydi;

2) $D = 0$ bo'lsa, tenglama $Ax + By + Cz = 0$ ko'rinishni oladi. Uni $O(0;0;0)$ nuqta koordinatalari qanoatlantiradi va tekislik koordinatalar boshidan o'tadi;

3) $A = 0$, $D = 0$ bo'lsa, tenglamadan $By + Cz = 0$ kelib chiqadi. Bu tekislik Ox o'qdan o'tadi. Shu kabi $Ax + Cz = 0$tenglama Oy o'qdan o'tuvchi tekislikni, $Ax + By = 0$ tenglama Oz o'qdan o'tuvchi tekislikni ifodalaydi;

4) $A = 0$, $B = 0$ bo'lsa, tenglama $Cz + D = 0$ yoki $z = -\dfrac{D}{C}$ ko'rinishni oladi. Bu tekislik Oxy tekislikka parallel bo'ladi. Shu kabi $By + D = 0$

tenglama Oxz tekislikka parallel tekislikni, $Ax+D=0$ tenglama Oyz tekislikka parallel tekislikni ifodalaydi;

5) $A=0$, $B=0$, $D=0$ bo'lsa, tenglama $Cz=0$ yoki $z=0$ ko'rinishga keladi. Bu tenglama Oxy tekislikni ifodalaydi. Shu kabi Oyz tekislik $x=0$ tenglama bilan, Oxz tekislik $y=0$ tenglama bilan aniqlanadi.

4-misol. Tekislik tenglamalarini tuzing: 1) Ox o'qdan va $M_0(0;-2;3)$ nuqtadan o'tuvchi; 2) Oy o'qqa parallel bo'lgan va $M_1(3;0;-4), M_2(5;-2;3)$ nuqtalardan o'tuvchi; 3) Oxz tekislikka parallel bo'lgan va $M_0(1;-2;3)$ nuqtadan o'tuvchi.

Yechish. 1) Ox o'qdan o'tuvchi tekislik tenglamasi $By+Cz=0$ bo'ladi. Bu tenglamani $M_0(0;-2;3)$ nuqtaning koordinatalari qanoatlantiradi, chunki bu nuqta tekislikda yotadi.

Demak, $(-2)\cdot B+3C=0$ yoki $B=\frac{3}{2}C$.

Bundan

$$\frac{3}{2}Cy+Cz=0$$

yoki

$$3y+2z=0.$$

2) Oy o'qqa parallel tekislik tenglamasi $Ax+Cz+D=0$ bo'ladi. Uni $M_1(3;0;-4), M_2(5;-2;3)$ nuqtalarning koordinatalari qanoatlantiradi, ya'ni

$$\begin{cases} 3A-4C+D=0, \\ 5A+3C+D=0. \end{cases}$$

Bundan

$$A=-\frac{7}{29}D \text{ va } C=\frac{2}{29}D.$$

U holda

$$-\frac{7}{29}Dx+\frac{2}{29}Dz+D=0$$

yoki

$$7x-2z-29=0.$$

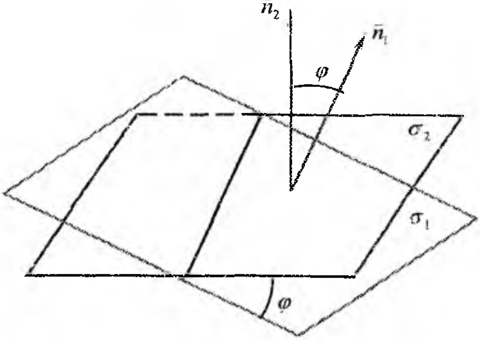
3) Oxz tekislikka parallel tekislik tenglamasi $By+D=0$ bo'ladi. Bundan $M_0(1;-2;3)$ nuqtada $-2B+D=0$ yoki $D=2B$ kelib chiqadi.

U holda

$$By+2B=0$$

yoki

$$y+2=0.$$



32-shakl.

Tekislikning (4.1)-(4.8) tenglamalaridan har birini boshqalaridan keltirib chiqarish mumkin. Masalan, (4.8) tenglamani (4.7) tenglamaga o'tkazish uchun (4.8) tenglikning chap va o'ng tomonini *normallovchi ko'paytuvchi* deb ataluvchi $M=\pm\frac{1}{\sqrt{A^2+B^2+C^2}}$ songa ko'paytiriladi. Bunda M ko'paytuvchining ishorasi D koeffitsiyentning ishorasiga qarama-qarshi qilib tanlanadi.

3.4.3. Fazoda ikki tekislikning o'zaro joylashishi

Ikki tekislik orasidagi burchak

Ikki tekislikning normal vektorlari orasidagi burchakka *ikki tekislik orasidagi burchak* deyiladi.

$$A_1x+B_1y+C_1z+D_1=0,\ A_2x+B_2y+C_2z+D_2=0$$

tenglamalar bilan berilgan σ_1, σ_2 tekisliklar orasidagi burchak φ ga teng bo'lsin(32-shakl).

Bunda $\vec{n}_1=\{A_1;B_1;C_1\}, \vec{n}_2=\{A_2;B_2;C_2\}$ va $\varphi=(\sigma_1\hat{,}\sigma_2)=(\vec{n}_1\hat{,}\vec{n}_2)$.

Ikki vektor orasidagi burchak kosinusi formulasidan topamiz:

$$\cos\varphi=\frac{\vec{n}_1\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}.$$

Odatda, ikki tekislik orasidagi burchak deyilganida $\frac{\pi}{2}$ dan oshmagan burchk tushuniladi.

Shu sababli

$$\cos\varphi = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \tag{4.9}$$

5-misol. $x + y + z - 1 = 0$, $x - 2y + 3z - 1 = 0$ tekisliklar orasidagi burchakni toping.

Yechish. Masalaning shartiga ko‘ra: $\vec{n}_1 = \{1;1;1\}$, $\vec{n}_2 = \{1;-2;3\}$.

U holda

$$\cos\varphi = \frac{|1\cdot1 + 1(-2) + 1\cdot3|}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{1^2+(-2)^2+3^2}} = \frac{2}{\sqrt{42}}.$$

Bundan

$$\varphi = \arccos\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ.$$

Ikki tekislikning perpendikularlik sharti

$\sigma_1 \perp \sigma_2$ bo‘lsin. U holda $\cos\varphi = 0$ va (4.9) tenglikdan topamiz:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \tag{4.10}$$

6-misol. $M_1(1;2;1)$, $M_2(0;3;4)$ nuqtalardan o‘tuvchi va $x + 2y - z = 0$ tekislikka perpendikular tekislik tenglamasini tuzing.

Yechish. Tekislik tenglamasini $Ax + By + Cz + D = 0$ ko‘rinishida izlaymiz.

Misolning shartiga ko‘ra:

$$\begin{cases} A + 2B - C = 0 \ (\textit{tekislik } x + 2y - z = 0 \textit{ tekislikka } \perp), \\ A + 2B + C = -D \ (\textit{tekislik } M_1(1;2;1) \textit{ nuqtadan o'tadi}), \\ 3B + 4C = -D \ (\textit{tekislik } M_2(0;3;4) \textit{ nuqtadan o'tadi}. \end{cases}$$

Sistemaning yechimi:

$$A = -\frac{7}{6}D,\ B = \frac{1}{3}D,\ C = -\frac{1}{2}D.$$

A, B, C koeffitsiyentlarni izlanayotgan tenglamaga qo'yamiz:

$$-\frac{7}{6}Dx+\frac{1}{3}Dy-\frac{1}{2}Dz+D=0.$$

Bundan

$$7x-2y+3z-6=0.$$

Ikki tekislikning parallellik sharti

σ_1 va σ_2 tekisliklar parallel bo'lsin. U holda $\vec{n}_1=\{A_1;B_1;C_1\}$ va $\vec{n}_2=\{A_2;B_2;C_2\}$ vektorlar kollinear bo'ladi. Ikki vektorning kollinearlik shartidan ikki tekislikning parallellik shartini topamiz:

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}\,. \qquad (4.11)$$

Ikki tekislikning ustma-ust tushishi

σ_1 va σ_2 tekisliklar ustma-ust tushsin. U holda birinchidan ular parallel bo'ladi. Ikki tekislikning parallellik shartidan topamiz:

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=\lambda$$

yoki

$$A_1-\lambda A_2=0,\ B_1-\lambda B_2=0,\ C_1-\lambda C_2=0. \qquad (4.12)$$

Ikkinchidan σ_1 tekislikning har bir nuqtasi, jumladan $M_0(x_0;y_0;z_0)$ nuqta σ_2 tekislikda yotadi, ya'ni

$$A_1x_0+B_1y_0+C_1z_0+D_1=0,$$

$$A_2x_0+B_2y_0+C_2z_0+D_2=0.$$

Bu tengliklardan ikkinchisini λ ga ko'paytiramiz va birinchi tenglikdan ayiramiz:

$$(A_1-\lambda A_2)x_0+(B_1-\lambda B_2)y_0+(C_1-\lambda C_2)z_0+(D_1-\lambda D_2)=0.$$

Bundan (4.12) tengliklarni hisobga olsak $D_1-\lambda D_2=0$ yoki $\dfrac{D_1}{D_2}=\lambda$ bo'ladi.

Demak,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}. \qquad (4.13)$$

(4.13) tengliklar tekisliklarning ustma-ust tushish shartini ifodalaydi.

3.4.4. Nuqtadan tekislikkacha bo'lgan masofa

Nuqtadan tekislikka tushirilgan perpendikularning uzunligiga *nuqtadan tekislikkacha bo'lgan masofa* deyiladi.

$M_0(x_0;y_0;z_0)$ nuqta va $Ax+By+Cz+D=0$ tenglama bilan σ tekislik berilgan bo'lsin. M_0 nuqtadan σ tekislikka tushirilgan perpendikularning asosini $M_1(x_1;y_1;z_1)$ bilan belgilaymiz (33-shakl).

U holda M_0 nuqtadan σ tekislikkacha bo'lgan masofa $d=\left|\Pi p_{\vec{n}}\overline{M_1M_0}\right|$ bo'ladi, bu yerda $\overline{M_1M_0}=\{x_0-x_1;y_0-y_1;z_0-z_1\}$.

Ikki vektor skalyar ko'paytmasining xossasiga ko'ra,

$$d=\frac{\left|\overline{M_1M_0}\cdot\vec{n}\right|}{|\vec{n}|}=\frac{|(x_0-x_1)A+(y_0-y_1)B+(z_0-z_1)C|}{\sqrt{A^2+B^2+C^2}}=$$

$$=\frac{|Ax_0+By_0+Cz_0-Ax_1-By_1-Cz_1|}{\sqrt{A^2+B^2+C^2}}.$$

$M_1(x_1;y_1;z_1)$ nuqta σ tekislikda yotgani sababli

$$Ax_1+By_1+Cz_1+D=0,$$

ya'ni

$$D=-Ax_1-By_1-Cz_1.$$

Bundan

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}. \qquad (4.14)$$

Shunday qilib, *nuqtadan tekislikkacha bo'lgan masofa* (4.14) formula bilan topiladi.

7-misol. $M_0(5;4;-1)$ nuqtadan $M_1(3;0;3)$, $M_2(0;4;0)$ va $M_3(0;4;-3)$ nuqtalardan o'tuvchi tekislikkacha bo'lgan masofani toping.

Yechish. Avval berilgan uchta nuqtadan o'tuvchi tekislik tenglamasini tuzamiz:

$$\begin{vmatrix} x-3 & y & z-3 \\ 0-3 & 4 & 0-3 \\ 0-3 & 4 & -3-3 \end{vmatrix} = 0.$$

Bundan

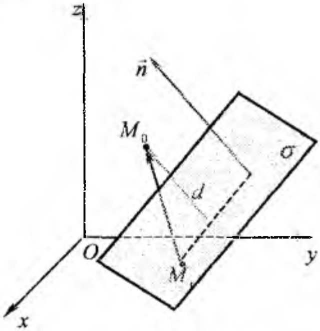
$$-12\cdot(x-3)-9\cdot y+0\cdot(z-3)=0$$

yoki

$$4x+3y-12=0.$$

$M_0(5;4;-1)$ nuqtadan $4x+3y-12=0$ tekislikkacha bo‘lgan masofani (4.14) formula bilan hisoblaymiz:

$$d=\frac{|4\cdot 5+3\cdot 4-12|}{\sqrt{4^2+3^2+0^2}}=4(b).$$



33-shakl.

3.4.5. Mashqlar

1. $M_0(2;-1;3)$ nuqtadan o‘tuvchi va shu nuqtaning radius vektoriga perpendikular bo‘lgan tekislik tenglamasini tuzing.

2. $\vec{n}=\{2;-3;4\}$ vektorga perpendikular bo‘lgan va Oz manfiy yarim o‘qda 5 ga teng kesma ajratuvchi tekislik tenglamasini tuzing.

3. $M(2;3;-1)$ nuqtadan o‘tuvchi $2x-3y+5z-4=0$ tekislikka parallel tekislik tenglamasini tuzing.

4. $M(2;5;-1)$ nuqtadan o‘tuvchi $2x+3y-4z+5=0$ tekislikka parallel tekislik tenglamasini tuzing.

5. Tekislik tenglamalarini tuzing: 1) $M_0(1,3;-2)$ nuqtadan va Ox o‘qdan o‘tuvchi; 2) $M_0(2;-1;3)$ nuqtadan o‘tuvchi va Oy o‘qqa perpendikular; 3) $M_0(3;-2;4)$ nuqtadan o‘tuvchi va Oxy tekislikka parallel; 4) $M_1(2;-3;1)$, $M_2(3;4;0)$ nuqtalardan o‘tuvchi va Oy o‘qqa parallel; 5) koordinatalar boshidan va $M_1(3;-4;2)$, $M_2(-1;3;4)$ nuqtalardan o‘tuvchi.

6. Tekislik tenglamalarini tuzing: 1) $M_0(2;-5;4)$ nuqtadan Oy o'qdan o'tuvchi; 2) $M_0(3;7;-1)$ nuqtadan o'tuvchi va Ox o'qqa perpendikular; 3) $M_0(2;-3;4)$ nuqtadan o'tuvchi va Oxz tekislikka parallel; 4) $M_1(2;1;-2)$, $M_2(-7;-2;1)$ nuqtalardan o'tuvchi va Oy o'qqa parallel; 5) koordinatalar boshidan va $M_1(1;2;-1)$, $M_2(-3;0;4)$ nuqtalardan o'tuvchi.

7. $2x+y-3z+6=0$ tekislikning koordinata o'qlari bilan kesishish nuqtalarini toping.

8. $2x+3y-5z+30=0$ tekislik koordinata o'qlarida qanday kesmalar ajratadi?

9. $M_1(2;-1;3)$, $M_2(-1;3;2)$ nuqtalardan o'tuvchi va Ox, Oz o'qlarida teng musbat kesmalar ajratuvchi tekislik tenglamasini tuzing.

10. $M_0(2;5;-2)$ nuqtadan o'tuvchi va $Ox.Oz$ o'qlarida Oy o'qqa nisbatan uch barobar uzun kesma ajratuvchi tekislik tenglamasini tuzing.

11. Berilgan uchta nuqtadan o'tuvchi tekislik tenglamasini tuzing:
1) $M_1(2;1;-1)$, $M_2(3;1;0)$, $M_3(-1;2;-1)$; 2) $M_1(1;-2;3)$, $M_2(4;1;3)$, $M_3(1;2;-1)$.

12. $M_0(3;3;3)$ nuqtadan koordinata tekisliklariga tushirilgan perpendikular asoslari orqali o'tgan tekislik tenglamasini tuzing.

13. $M_1(1;1;1)$, $M_2(0;2;1)$ nuqtalardan o'tuvchi va $\vec{a}=\{2;0;1\}$ vektorga parallel tekislik tenglamasini tuzing.

14. $M_1(1;2;0)$, $M_2(2;1;1)$ nuqtalardan o'tuvchi va $\vec{a}=\{3;0;1\}$ vektorga parallel tekislik tenglamasini tuzing.

15. $M_0(1;-2;3)$ nuqtadan o'tuvchi va $\vec{a}=\{2;1;1\}$, $\vec{b}=\{3;1;-1\}$ vektorlarga parallel tekislik tenglamasini tuzing.

16. $M_0(0;1;2)$ nuqtadan o'tuvchi va $\vec{a}=\{2;0;1\}$, $\vec{b}=\{1;1;0\}$ vektorlarga parallel tekislik tenglamasini tuzing.

17. $9x-2y+6z-11=0$ tekislik tenglamasining kesmalarga nisbatan va normal ko'rinishlarini yozing.

18. $5x+7y-34z+5=0$ tekislik tenglamasining kesmalarga nisbatan va normal ko'rinishlarini yozing.

19. Tekisliklar orasidagi burchakni toping:

1) $x-2y+2z+5=0$ va $x-y-3=0$;

2) $3x-y+2z+12=0$ va $5x+9y-3z-1=0$;

3) $2x-3y-4z+4=0$ va $5x+2y+z-3=0$;

4) $x+2y+3=0$ va $y+2z-5=0$.

20. m va n ning qanday qiymatlarida tekisliklar parallel bo'ladi:

1) $3x-5y-nz-2=0$, $mx+2y-3z+11=0$; 2) $nx-6y-6z+4=0$, $2x+my+3z-8=0$.

21. m ning qanday qiymatlarida tekisliklar perpendikular bo'ladi:

1) $4x-7y+2z-3=0, -3x+2y+mz+5=0$; 2) $x-my+z=0, 2x+3y+mz-4=0$.

22. Tekislik tenglamalarini tuzing:

1) $M_0(2;2;-2)$ nuqtadan o'tuvchi va berilgan tekislikka parallel:

a) $x-2y-3z=0$; b) $2x+3y+z-1=0$;

2) $M_0(-1;-1;2)$ nuqtadan o'tuvchi va berilgan ikki tekislikka perpendikular:

a) $x+2y-2z+6=0$, $x-2y+z+4=0$; b) $x+3y+z-1=0$, $2x-y+z-2=0$.

3) $M_1(5;-4;3)$, $M_2(-2;1;8)$ nuqtalardan o'tuvchi va berilgan tekislikka perpendikular: a) Oxy; b) Oyz; c) Oxz.

23. $M(-2;1;3)$ nuqtadan va $x-2y-2z+6=0$, $2x+3y-z+3=0$ tekisliklarning kesishish chizig'idan o'tuvchi tekislik tenglamasini tuzing.

24. $M(2;1;-2)$ nuqtadan o'tuvchi va $x+3y+2z+1=0$, $3x+2y-z+8=0$ tekisliklar kesishish chizig'iga perpendikular tekislik tenglamasini tuzing.

25. $M_1(2;0;0)$, $M_2(0;1;0)$ nuqtalardan o'tuvchi va Oxy tekislik bilan 45° li burchak tashkil qiluvchi tekislik tenglamasini tuzing.

26. Tekisliklarning kesishish nuqtasini toping:

1) $x+2y-z+2=0$, $x-y-2z+7=0$, $3x-y-2z+11=0$;

2) $x-2y-4z=0$, $x+2y-4z+4=0$, $3x+y-z-4=0$.

27. $M_0(5;-1;4)$ nuqtadan $M_1(3;3;0)$, $M_2(0;-3;4)$, $M_3(0;0;4)$ nuqtalar yotuvchi tekislikkacha bo'lgan masofani toping.

28. Ox oqning $2x+y-2z+6=0$, $x+2y+2z-9=0$ tekisliklardan teng uzoqlikda yotuvchi nuqtasini toping.

29. $2x-y-2z-5=0$ tekislikka parallel bo'lgan va $M_0(4;3;-2)$ nuqtadan $d=3$ masofadan o'tuvchi tekislik tenglamasini tuzing.

30. Ikki yog'i $12x+3y-4z-4=0$ va $12x+3y-4z+22=0$ tekisliklarda yotuvchi kubning hajmini toping.

31. $M(4;3;1)$ nuqtaning $3x-4y+12z+14=0$ tekislikdan chetlashishini toping.

32. $M(3;0;1)$ nuqtaning $2x+9y-6z+33=0$ tekislikdan chetlashishini toping.

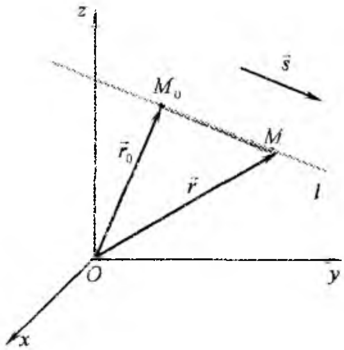
33. $10x+2y-2z-5=0$, $5x+y-z-1=0$ parallel tekisliklar orasidagi masofani toping.

3.5. FAZODAGI TO'G'RI CHIZIQ

3.5.1. Fazodagi to'g'ri chiziq tenglamalari

To'g'ri chiziqning kanonik tenglamasi

Analitik geometriyaning bir qancha masalalarini yechishda fazodagi to'g'ri chiziqning kanonik tenglamasi deb ataluvchi maxsus tenglama keng qo'llaniladi. Bu tenglamani keltirib chiqaramiz.



34-shakl.

Fazoda biror to'g'ri chiziq berilgan bo'lsin. Bu to'g'ri chiziqqa parallel bo'lgan (yoki bu to'g'ri chiziqda yotuvchi) nolga teng bo'lmagan har qanday vektorga bu to'g'ri chiziqning *yo'naltiruvchi vektori* deyiladi.

Berilgan $M_0(x_0;y_0;z_0)$ nuqtadan o'tuvchi va yo'naltiruvchi vektori $\vec{s}=\{p;q;r\}$ bo'lgan l to'g'ri chiziq tenglamasini tuzamiz. Buning uchun l to'g'ri chiziqning ixtiyoriy $M(x;y;z)$ nuqtasini olamiz va $\overline{M_0M}=\{x-x_0;y-y_0;z-z_0\}$ vektorni yasaymiz (34-shakl).

Bunda $\vec{s}$ va $\overline{M_0M}$ vektorlar kollinear bo'ladi. Ikki vektorning kollinearlik shartidan topamiz:

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}. \tag{5.1}$$

Bu tengliklarni l to'g'ri chiziqda yotuvchi har bir $M(x;y;z)$ nuqtaning koordinatalari qanoatlantiradi, va aksincha agar $M(x;y;z)$ nuqta l to'g'ri chiziqda yotmasa, u holda uning koordinatalari (5.1) tengliklarni qanoatlantirmaydi, chunki bunda $\bar{s}$ va $\overline{M_0M}$ vektorlar kollinear bo'lmaydi.

(5.1) tengliklarga *to'g'ri chiziqning kanonik tenglamasi* deyiladi.

Bunda ixtiyoriy $\vec{s}$ yo'naltiruvchi vektorning p, q, r koordinatalari bu to'g'ri chiziqning yo'naltiruvchi parametrlari va $\vec{s}$ vektorning yo'naltiruvchi kosinuslari bu to'g'ri chiziqning yo'naltiruvchi kosinuslari deb ataladi.

To'g'ri chiziqning kanonik tenglamasidan uning boshqa tenglamalarini keltirib chiqaramiz.

To'g'ri chiziqning kanonik tenglamasini

$$\begin{cases} \dfrac{x-x_0}{p}=\dfrac{y-y_0}{q}, \\ \dfrac{x-x_0}{p}=\dfrac{z-z_0}{r} \end{cases}$$

tenglamalar sistemasi deb qarash mumkin.

Bu tenglamalarning har ikkalasi birinchi darajali tenglamalar hisoblanadi, ya'ni tekislik tenglamalari bo'ladi. Bundan *fazodagi to'g'ri chiziq ikkita parallel bo'lmagan tekislikning kesishisidan hosil bo'ladi* degan xulosaga kelish mumkin.

Masalan, $\frac{x-1}{5}=\frac{y}{-2}=\frac{z}{-3}$ to'g'ri chiziq $2x+5y-2=0$ va $3y-2z=0$ tekisliklarning kesisish chizig'i bo'ladi.

Shunday qilib, agar σ_1 va σ_2 tekisliklarning $\vec{n}_1=\{A_1;B_1;C_1\}$ va $\vec{n}_2=\{A_2;B_2;C_2\}$ normal vektorlari kollinear bo'lmasa, bu tekisliklarning kesishishidan hosil bo'lgan l to'g'ri chiziq quyidagi tenglamalar sistemasi bilan ifodalanadi:

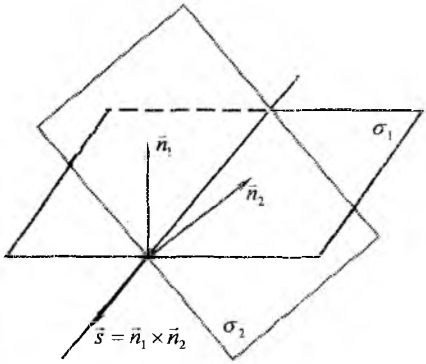
$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, \\ A_2x+B_2y+C_2Z+D_2=0. \end{cases} \tag{5.2}$$

Bu tenglamalar sistemaga *to'g'ri chiziqning umumiy tenglamalari* deyiladi.

(5.2) umumiy tenglamalari bilan berilgan to'g'ri chiziqning kanonik tenglamasi quyidagi tartibda topiladi.

1. To'g'ri chiziqning biror $M_0(x_0;y_0;z_0)$ nuqtasi topiladi. Buning uchun avval noma'lum x_0, y_0, z_0 koordinatalardan biriga qiymat beriladi va bu qiymat (5.2) tenglamalardagi mos o'zgaruvchi o'rniga qo'yiladi, keyin boshqa koordinatalar (5.2) sistemani yechish orqali aniqlanadi.

2. To'g'ri chiziqning $\vec{s}$ yo'naltiruvchi vektori topiladi. l to'g'ri chiziq $\vec{n}_1$ va $\vec{n}_2$ vektorlarga perpendekular bo'lgani uchun $\vec{s}\perp\vec{n}_1$, $\vec{s}\perp\vec{n}_2$ bo'ladi (35-shakl). Bundan



35-shakl.

$$\vec{s}=\vec{n}_1\times\vec{n}_2=\left\{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix};\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix};\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\right\} \tag{5.3}$$

vektor aniqlanadi.

3. Topilgan M_0 nuqta va $\vec{s}$ vektor asosida kanonik tenglama tuziladi.

1-misol. $\begin{cases} x-2y+3z+1=0, \\ 2x+y-4z-8=0 \end{cases}$ tenglamani kanonik ko'rinishga keltiring.

Yechih. Misol shartiga ko'ra:

$$A_1=1,\ B_1=-2,\ C_1=3,\ A_2=2,\ B_2=1,\ C_2=-4.$$

To'g'ri chiziqning $M_0(x_0;y_0;z_0)$ nuqtasini topish uchun $z_0=0$ deb olamiz.

U holda

$$\begin{cases} x_0-2y_0=-1, \\ 2x_0+y_0=8 \end{cases}$$

sistemadan $x_0=3,\ y_0=2$ ekanini topamiz.

To'g'ri chiziqning yo'naltiruvchi vektorini (5.3) formuladan topamiz:

$$\vec{s}=\left\{\begin{vmatrix}-2 & 3\\ 1 & -4\end{vmatrix}; \begin{vmatrix}3 & 1\\ -4 & 2\end{vmatrix}; \begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}\right\}=\{5;10;5\}.$$

M_0 nuqta va $\vec{s}$ vektorning koordinatalarini (5.1) tenglamaga qo‘yamiz:

$$\frac{x-3}{5}=\frac{y-2}{10}=\frac{z}{5}$$

yoki

$$\frac{x-3}{1}=\frac{y-2}{2}=\frac{z}{1}.$$

(5.1) tenglamada

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}=t,\ t\in R$$

belgilash kiritamiz.

Bundan

$$\begin{aligned}x&=x_0+pt,\\ y&=y_0+qt,\\ z&=z_0+rt\end{aligned} \tag{5.4}$$

tenglamalar kelib chiqadi, bu yerda $t\in R$ – parametr.

(5.4) tenglamalarga *to‘g‘ri chiziqning parametrik tenglamalari* deyiladi.

Ma’lumki, fazodagi chiziqning uchta parametrik (skalyar) tenglamalarini bitta vektor tenglama bilan ifodalash mumkin.

Demak, (5.4) tenglamalarni

$$\vec{r}=\vec{r}_0+t\vec{s} \tag{5.5}$$

ko‘rinishda yozish mumkin, bu yerda $\vec{r}=\{x;y;z\}$, $\vec{r}_0=\{x_0;y_0;z_0\}$ – mos ravishda $M(x;y;z)$, $M_0(x_0;y_0;z_0)$ nuqtalarning radius vektorlari; $\vec{s}=\{p;q;r\}$ – to‘g‘ri chiziqning yo‘naltiruvchi vektori (34-shakl).

(5.5) tenglamaga *to‘g‘ri chiziqning vektor tenglamasi* deyiladi.

Berilgan $M_1(x_1;y_1;z_1)$ va $M_2(x_2;y_2;z_2)$ nuqtalardan o‘tuvchi l to‘g‘ri chiziq tenglamasini tuzamiz. Buning uchun l to‘g‘ri chiziqning yo'naltiruvchi vektori sifatida

$$\vec{s}=\overline{M_1M_2}=\{x_2-x_1;y_2-y_1;z_2-z_1\}$$

vektorni olamiz va (5.1) tengliklardan

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \tag{5.6}$$

tengliklarni keltirib chiqaramiz.

Bu tengliklarga *berilgan ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi* deyiladi.

3.5.2. Fazoda ikki to'g'ri chiziqning o'zaro joylashishi

Ikki to'g'ri chiziq orasidagi burchak

$\dfrac{x-x_1}{p_1}=\dfrac{y-y_1}{q_1}=\dfrac{z-z_1}{r_1}$ va $\dfrac{x-x_2}{p_2}=\dfrac{y-y_2}{q_2}=\dfrac{z-z_2}{r_2}$ tenglamalari bilan berilgan ikki l_1 va l_2 to'g'ri chiziqlar orasidagi burchak φ bo'lsin.

Bunda to'g'ri chiziqlarning yo'naltiruvchi vektorlari $\vec{s}_1=\{p_1;q_1;r_1\}$, $\vec{s}_2=\{p_2;q_2;r_2\}$ ga va ular orasidagi burchak to'g'ri chiziqlar orasidagi burchaklardan biriga teng, ya'ni $\varphi=(\widehat{l_1,l_2})=(\widehat{\vec{s}_1,\vec{s}_2})$ bo'ladi.

φ burchak kosinusini topamiz:

$$\cos\varphi=\frac{\vec{s}_1\vec{s}_2}{|\vec{s}_1||\vec{s}_2|}=\frac{p_1p_2+q_1q_2+r_1r_2}{\sqrt{p_1^2+q_1^2+r_1^2}\sqrt{p_2^2+q_2^2+r_2^2}}. \tag{5.7}$$

2-misol. $x=3t-2,\ y=0,\ z=-t+3$ va $x=2t-1,\ y=0,\ z=t-3$ to'g'ri chiziqlar orasidagi o'tmas burchakni toping.

Yechish. To'g'ri chiziqlar tenglamalarini kanonik shaklga keltiramiz:

$$\frac{x+2}{3}=\frac{y}{0}=\frac{z-3}{-1}, \quad \frac{x+1}{2}=\frac{y}{0}=\frac{z+3}{1}.$$

U holda (5.7) formuladan topamiz:

$$\cos\varphi=\frac{3\cdot2+0\cdot0+(-1)\cdot1}{\sqrt{3^2+0^2+(-1)^2}\cdot\sqrt{2^2+0^2+1^2}}=\pm\frac{\sqrt{2}}{2}.$$

O‘tmas burchak uchun $\cos\varphi = -\frac{\sqrt{2}}{2}$ bo‘ladi. Bundan $\varphi = 135^0$.

Ikki to‘g‘ri chiziqning perpendikularlik sharti

$l_1 \perp l_2$ bo‘lsin. U holda $\cos\varphi = 0$ va (5.7) tenglikdan topamiz:

$$p_1 p_2 + q_1 q_2 + r_1 r_2 = 0. \qquad (5.8)$$

Bu tenglik ikki to‘g‘ri chiziqning perpendikularlik shartini ifodalaydi.

Ikki to‘g‘ri chiziqning parallellik sharti

l_1 va l_2 to‘g‘ri chiziqlar parallel bo‘lsin. U holda $\vec{s}_1 = \{p_1; q_1; r_1\}$ va $\vec{s}_2 = \{p_2; q_2; r_2\}$ vektorlar kollinear bo‘ladi. Ikki vektorning kollinearlik shartidan ikki to‘g‘ri chiziqning parallellik shartini keltirib chiqaraamiz:

$$\frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}. \qquad (5.9)$$

Ikki to‘g‘ri chiziqning bir tekislikda yotishi

l_1 va l_2 to‘g‘ri chiziqlar bitta σ tekislikda yotsin. $M_1(x_1; y_1; z_1)$ – l_1 to‘g‘ri chiziqning nuqtasi va $M_2(x_2; y_2; z_2) - l_2$ to‘g‘ri chiziqning nuqtasi bo‘lsin.

U holda $\vec{s}_1 = \{p_1; q_1; r_1\}$, $\vec{s}_2 = \{p_2; q_2; r_2\}$, $\overline{M_1 M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$ vektorlar σ tekislikda yotadi, ya’ni bu vektorlar komplanar bo‘ladi.

Vektorlarning komplanarlik shartiga ko‘ra topamiz:

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0. \qquad (5.10)$$

Bu tenglik ikki to‘g‘ri chiziqning bir tekislikda yotishi shartini ifodalaydi.

Ikki to‘g‘ri chiziqning ayqash bo‘lishi

l_1 va l_2 to‘g‘ri chiziqlar ayqash bo‘lsa, $\vec{s}_1 = \{p_1; q_1; r_1\}, \vec{s}_2 = \{p_2; q_2; r_2\}$, $\overline{M_1 M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$ vektorlar bir tekislikda yotmaydi, ya’ni komplanar bo‘lmaydi.

Shu sababli

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \neq 0. \qquad (5.11)$$

bo‘ladi. Bu shart ikki to‘g‘ri chiziqning ayqash bo‘lishini belgilaydi.

Ikki to‘g‘ri chiziqning ustma-ust tushishi

l_1 va l_2 to‘g‘ri chiziqlar ustma-ust tushsin. U holda bu to‘g‘ri chiziqlar birinchidan, parallel bo‘ladi va ikkinchidan, $\overline{M_1M_2} = \{x_2 - x_1; y_2 - y_1; z_2 - z_1\}$ vektor bu to‘g‘ri chiziqlardan birida, masalan l_1 da yotadi.

Shu sababli

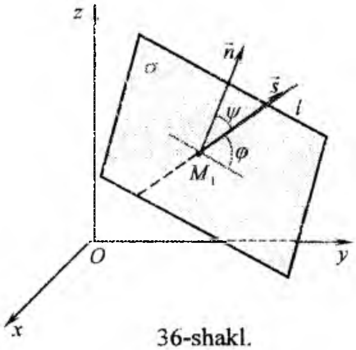
$$\begin{cases} \dfrac{p_1}{p_2} = \dfrac{q_1}{q_2} = \dfrac{r_1}{r_2}, \\ \dfrac{x_2 - x_1}{p_1} = \dfrac{y_2 - y_1}{q_1} = \dfrac{z_2 - z_1}{r_1}. \end{cases} \qquad (5.12)$$

(5.12) tengliklar ikki to‘g‘ri chiziqning ustma-ust tushish shartini ifodalaydi.

3.5.3. Fazoda to‘g‘ri chiziq bilan tekislikning o‘zaro joylashishi

To‘g‘ri chiziq bilan tekislik orasidagi burchak

To‘g‘ri chiziq bilan uning tekislikdagi proeksiyasi orasidagi burchakka *to‘g‘ri chiziq bilan tekislik orasidagi burchak* deyiladi. $l: \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$ to‘g‘ri chiziq bilan $\sigma: Ax + By + Cz + D = 0$ tekislik orasidagi burchak φ bo‘lsin. U holda to‘g‘ri chiziqning yo‘naltiruvchi vektori $\vec{s} = \{p; q; r\}$ bilan tekislikning normal vektori



36-shakl.

$\vec{n}=\{A;B;C\}$ orasidagi burchak $\psi=90^0-\varphi$ bo'ladi (36-shakl).

$\cos\psi=\sin\varphi$ tenglikni hisobga olib, topamiz:

$$\sin\varphi=\frac{|Ap+Bq+Cz|}{\sqrt{A^2+B^2+C^2}\sqrt{p^2+q^2+r^2}} \tag{5.13}$$

3-misol. $\frac{x+1}{1}=\frac{y-2}{1}=\frac{z-5}{-2}$ to'g'ri chiziq bilan $2x-y-z+9=0$ tekislik orasidagi o'tkir burchakni toping.

Yechish. Izlanayotgan burchakni (5.13) formula bilan topamiz:

$$\sin\varphi=\frac{|2\cdot1+(-1)\cdot1+(-1)\cdot(-2)|}{\sqrt{2^2+(-1)^2+(-1)^2}\cdot\sqrt{1^2+1^2+(-2)^2}}=\frac{1}{2}.$$

Bundan $\varphi=35^{o}$.

To'g'ri chiziq bilan tekislikning perpendikularlik sharti

$l\perp\sigma$ bo'lsin. U holda to'g'ri chiziqning yo'naltiruvchi vektori $\vec{s}=\{p;q;r\}$ va tekislikning normal vektori $\vec{n}=\{A;B;C\}$ kollinear bo'ladi.

Bundan

$$\frac{A}{p}=\frac{B}{q}=\frac{C}{r} \tag{5.14}$$

to'g'ri chiziq bilan tekislikning perpendikularlik sharti kelib chiqadi.

To'g'ri chiziq bilan tekislikning paralellik sharti

$l\|\sigma$ bo'lsin. U holda, $\vec{s}\perp\vec{n}$ bo'ladi.

Bundan

$$Ap+Bq+Cr=0. \tag{5.15}$$

Bu tenglik to'g'ri chiziq bilan tekislikning parallellik shartini ifodalaydi.

4-misol. $M_0(-1;2;-3)$ nuqtadan o'tuvchi va $2x-3y+6z-1=0$ tekislikka perpendikular to'g'ri chiziq tenglamasini tuzing.

Yechish. To'g'ri chiziq bilan tekislikning perpendikularlik shartiga ko'ra,

$$\frac{2}{p}=\frac{-3}{q}=\frac{6}{r}.$$

Bundan $q=-\frac{3}{2}p,\ r=3p.$

Demak, $M_0(-1;2;-3)$, $\vec{s}=\left\{p;-\frac{3}{2}p;3p\right\}$.

U holda (5.1) tenglamaga ko‘ra:

$$\frac{x+1}{p}=\frac{y-2}{-\frac{3}{2}p}=\frac{z+3}{3p}$$

yoki

$$\frac{x+1}{2}=\frac{y-2}{-3}=\frac{z+3}{6}.$$

Bu masalani boshqacha yechish mumkin. To‘g‘ri chiziq tekislikka perpendikular bo‘lgani sababli tekislikning normal vektori to‘g‘ri chiziqning
yo‘naltiruvchi vektori bo‘ladi, ya’ni $\vec{s}=\{2;-3;6\}$.

U holda $M_0(-1;2;-3)$ nuqtadan o‘tuvchi to‘g‘ri chiziqning kanonik tenglamasi:

$$\frac{x+1}{2}=\frac{y-2}{-3}=\frac{z+3}{6}$$

5-misol. m ning qanday qiymatida $\frac{x+2}{3}=\frac{y-1}{m}=\frac{z+3}{m+1}$ to‘g‘ri chiziq va $3x+y-3z-1=0$ tekislik parallel bo‘ladi?

Yechish. m ning izlanayotgan qiymatini to‘g‘ri chiziq va tekislikning parallellik shartidan topamiz: $3\cdot 3+1\cdot m+(-3)\cdot(m+1)=0$. Bundan $m=3$.

To‘g‘ri chiziq bilan tekislikning kesishishi

Agar $l\|\sigma$ bo‘lmasa, u holda to‘g‘ri chiziq va tekislik kesishadi. Shu sababli

$$Ap+Bq+Cr\neq 0 \tag{5.16}$$

bo‘ladi.

Bu shart to‘g‘ri chiziq bilan tekislikning keshishishini belgilaydi.

Shunday qilib, (5.16) shart bajarilsa, to‘g‘ri chiziq bilan tekislik qandaydir nuqtada kesishadi. Bu nuqta $M_1(x_1,y_1,z_1)$ bo‘lsin. U holda $M_1(x_1,y_1,z_1)$ nuqtaning koordinatalari to‘g‘ri chiziq va tekislikning tenglamalarini qanoatlantiradi:

$$\frac{x_1-x_0}{p}=\frac{y_1-y_0}{q}=\frac{z_1-z_0}{r}, \tag{5.17}$$

$$Ax_1 + By_1 + Cz_1 + D = 0. \qquad (5.18)$$

Bu tenglamalardan $M_1(x_1, y_1, z_1)$ nuqtani topish quyidagi tartibda amalga oshiriladi:

1°. (5.17) tenglama parametrik ko'rinishga keltiriladi:

$$x_1 = x_0 + pt, \quad y_1 = y_0 + qt, \quad z_1 = z_0 + rt; \qquad (5.19)$$

2°. x_1, y_1 va z_1 lar (5.18) tenglamaga qo'yiladi va u t ga nisbatan yechiladi;

3°. t ning topilgan qiymati (5.19) tenglamalarga qo'yiladi va $M_1(x_1, y_1, z_1)$ nuqta aniqlanadi.

6-misol. . $\frac{x+2}{-1} = \frac{y+1}{-2} = \frac{z-1}{3}$ to'g'ri chiziq bilan $2x + 3y - z - 3 = 0$ tekislikning kesishish nuqtasini toping.

Yechish.

1°. $\frac{x_1+2}{-1} = \frac{y_1+1}{-2} = \frac{z_1-1}{3} = t, x_1 = -2 - t, \quad y_1 = -1 - 2t, \quad z_1 = 1 + 3t;$

2°. $2(-2-t) + 3(-1-2t) - (1+3t) - 3 = 0 \Rightarrow t = -1;$

3°. $x_1 = -2 - (-1) = -1, \quad y_1 = -1 - 2\cdot(-1) = 1, \quad z_1 = 1 + 3\cdot(-1) = -2.$

Demak, $M_1(-1;1;-2)$.

To'g'ri chiziqning tekislikda yotishi

l to'g'ri chiziq σ tekislikda yotsin.

U holda birinchidan, $\vec{s} \perp \vec{n}$ bo'ladi va ikkinchidan, to'g'ri chiziqning $M_0(x_0; y_0; z_0)$ nuqtasi tekislikda ham yotadi.

Shu sababli

$$\begin{cases} Ap + Bq + Cr = 0, \\ Ax_0 + By_0 + Cz_0 + D = 0. \end{cases} \qquad (5.20)$$

(5.20) shart to'g'ri chiziqning tekislikda yotishini belgilaydi.

3.5.4. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa

$M_1(x_1; y_1; z_1)$ nuqta va $\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$ tenglama bilan l to'g'ri chiziq berilgan bo'lsin. l to'g'ri chiziq $M_0(x_0; y_0; z_0)$ nuqtadan o'tadi va

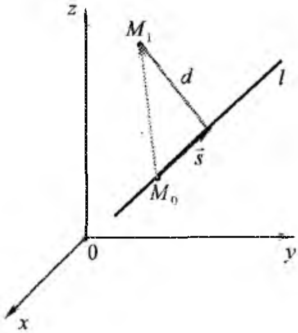
$\vec{s}=\{p;q;r\}$ yo'naltiruvchi vektorga ega bo'ladi. $M_1(x_1;y_1;z_1)$ nuqtadan to'g'ri chiziqqacha bo'lgan masofa d bo'lsin.

Izlanayotgn d masofa $\overline{M_0M_1}$ va $\vec{s}$ vektorlarga qurilgan parallelogramm balandligining uzunligiga teng bo'ladi (37-shakl).

Bu parallelogrammning yuzi $|\overline{M_0M_1}\times\vec{s}|$ ga teng. Bundan

$$d=\frac{|\overline{M_0M_1}\times\vec{s}|}{|\vec{s}|} \qquad (5.21)$$

kelib chiqadi.



37-shakl.

7-misol. $M_1(1;-1;-2)$ nuqtadan $\frac{x+3}{3}=\frac{y+2}{2}=\frac{z-8}{-2}$ to'g'ri chiziqqacha bo'lgan masofani toping.

Yechish. Misolning shartiga ko'ra: $M_1(1;-1;-2)$, $M_0(-3;-2;8)$, $\vec{s}=\{3;2;-2\}$.

Bundan

$$\overrightarrow{M_0M_1}=\{1-(-3);-1-(-2);-2-8\}=\{4;1;-10\}.$$

U holda

$$\overline{M_0M_1}\times\vec{s}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 1 & -10\\ 3 & 2 & -2\end{vmatrix}=$$

$$=(-2+20)\vec{i}-(-8+30)\vec{j}+(8-3)\vec{k}=18\vec{i}-22\vec{j}+5\vec{k},$$

$$|\overline{M_0M_1}\times\vec{s}|=\sqrt{18^2+(-22)^2+5^2}=7\sqrt{17},\ |\vec{s}|=\sqrt{3^2+2^2+(-2)^2}=\sqrt{17}.$$

(5.21) formula bilan topamiz:

$$d=\frac{7\sqrt{17}}{\sqrt{17}}=7(u.b).$$

3.5.5. Mashqlar

1. Berilgan to'g'ri chiziqning kanonik tenglamasini tuzing: 1) $M_1(1;1;-2)$ nuqtadan o'tuvchi va $\vec{s}=\{2;3;-1\}$ vektorga parallel; 2) $M_2(2;-3;-1)$ nuqtadan o'tuvchi va Oy o'qqa parallel; 3) $M_3(2;-1;-2)$ nuqtadan o'tuvchi va $6x+2y-4z-5=0$ tekislikka perpendikular.

2. $M_0(2;-3;5)$ nuqtadan o'tuvchi berilgan to'g'ri chiziqlarga parallel to'g'ri chiziq tenglamasini tuzing:

1) $\frac{x+1}{4}=\frac{y+1}{1}=\frac{z-2}{3}$; 2) $x=3+2t, y=-1+3t, z=1-t$; 3) $\begin{cases} x+3y+\ z+6=0, \\ 2x-\ y-4z+3=0. \end{cases}$

3. $M(-3;6;2)$ nuqtadan o'tuvchi va Oz o'qni to'g'ri burchak ostida kesuvchi to'g'ri chiziq tenglamasini tuzing.

4. $M(-1;2;-3)$ nuqtadan o'tuvchi va koordinata o'qlari bilan $\alpha=\frac{\pi}{3}$, $\beta=\frac{\pi}{4}$, $\gamma=\frac{2\pi}{3}$ burchaklar tashkil qiluvchi to'g'ri chiziq tenglamalarini tuzing.

5. Berilgan nuqtalardan o'tuvchi to'g'ri chiziqning umumiy tenglamasini tuzing:

1) $M_1(-1;2;2), M_2(3;1;-2)$; 2) $M_1(1;-2;1), M_2(3;1;-1)$.

6. $M(2;2;-1)$ nuqtadan o'tuvchi va $\vec{a}=\{1;1;2\}$, $\vec{b}=\{-1;3;1\}$ vektorlarga perpendikular to'g'ri chiziqning umumiy tenglamasini tuzing.

7. To'g'ri chiziq tenglamasini parametrik ko'rinishga keltiring:

1) $\begin{cases} 5x+y-3z+5=0, \\ 8x-4y-z+6=0; \end{cases}$ 2) $\begin{cases} x+y-z-1=0, \\ x-y+2z+1=0. \end{cases}$

8. To'g'ri chiziq tenglamasini kanonik ko'rinishga keltiring:

1) $\begin{cases} 4x-y-z+12=0, \\ \quad\ y-z-\ 2=0; \end{cases}$ 2) $\begin{cases} \ x+y-z+3=0, \\ 2x-y\qquad -1=0. \end{cases}$

9. Uchburchakning uchlari berilgan: $A(-1;2;3), B(-1;-2;1), C(3;4;5)$. A uchdan o'tkazilgan mediana tenglamasini tuzing.

10. $ABCD$ parallelogrammning ikki uchi $A(-1;2;0), B(4;1;3)$ va diagonallari kesishish nuqtasi $O(-2;1;2)$ berilgan. Parallelogramm CD tomonining tenglamasini tuzing.

11. To'g'ri chiziqlar orasidagi o'tkir burchakni toping:

1) $x=-2+3t, y=0, z=3-t$ va $x=-1+2t, y=0, z=-3+t$;

2) $\frac{x}{2}=\frac{y-2}{-1}=\frac{z+2}{3}$ va $\begin{cases} 2x+y-\ z-1=0, \\ 2x-y+3z+5=0. \end{cases}$

12. $M(-2;3;-1)$ nuqtadan o'tuvchi va berilgan to'g'ri chiziqlarga perpendikular to'g'ri chiziq tenglamasini tuzing:

1) $\frac{x}{2}=\frac{y}{1}=\frac{z-2}{3}$, $\frac{x+1}{1}=\frac{y+1}{-1}=\frac{z-2}{2}$; 2) $\frac{x-5}{3}=\frac{y+1}{1}=\frac{z-3}{-2}$, $\frac{x+2}{2}=\frac{y}{-5}=\frac{z+1}{4}$.

13. $M(1;-1;2)$ nuqtadan o'tuvchi va $\frac{x-2}{1}=\frac{y+1}{3}=\frac{z-2}{2}$ to'g'ri chiziqqa parallel to'g'ri chiziq tenglamasini tuzing.

14. To'g'ri chiziqlarning o'zaro joylashishini aniqlang:

1) $\frac{x-5}{-4}=\frac{y-4}{-3}=\frac{z-3}{2}$, $x=2+8t, y=6t, z=-3-4t$;

2) $\frac{x+4}{3}=\frac{y+3}{2}=\frac{z-1}{1}$, $\frac{x}{-2}=\frac{y-1}{3}=\frac{z+2}{-1}$.

15. To'g'ri chiziq bilan tekislik orasidagi burchakni toping:

1) $\frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{-2}$, $2x+2y-9=0$;

2) $\begin{cases} x-2y-1=0, \\ y-z-2=0, \end{cases}$ $x+2y-z+6=0$.

16. To'g'ri chiziq bilan tekislikning o'zaro joylashishini aniqlang:

1) $\begin{cases} x-y+4z-6=0, \\ 2x+y-z+3=0, \end{cases}$ $3x-y+6z-12=0$;

2) $\frac{x+1}{2}=\frac{y-2}{8}=\frac{z+2}{3}$, $2x+y-4z-8=0$.

17. To'g'ri chiziq bilan tekislikning kesishish nuqtasini toping:

1) $\frac{x-4}{1}=\frac{y-7}{5}=\frac{z-5}{4}$, $x-3y-2z+5=0$;

2) $\frac{x}{2}=\frac{y+13}{17}=\frac{z+7}{13}$, $5x-z-4=0$.

18. $M(4;5;-6)$ nuqtadan berilgan tekislikka tushirilgan perpendikular tenglamasini tuzing:

1) $x-2y-3=0$;

2) $x-y+z-5=0$.

19. m va n ning qanday qiymatlarida $\frac{x-3}{-4}=\frac{y-1}{4}=\frac{z+3}{-1}$ to'g'ri chiziq:

1) $mx+2y-4z+n=0$ tekislikda yotadi; 2) $mx+ny+3z-5=0$ tekislikka perpendikular bo'ladi; 3) $2x+3y+2mz-n=0$ tekislikka parallel bo'ladi.

20. $M(1;-1;-1)$ nuqtadan o'tuvchi va berilgan to'g'ri chiziqqa perpendikular tekislik tenglamasini tuzing:

1) $\frac{x+1}{2}=\frac{y+2}{-3}=\frac{z+2}{4}$;

2) $\frac{x+3}{4}=\frac{y-1}{-1}=\frac{z-5}{-2}$.

21. $M(0;1;2)$ nuqtadan va $\begin{cases} x-3y+5=0 \\ 2x+y+z-2=0 \end{cases}$ to'g'ri chiziqdan o'tuvchi tekislik tenglamasini tuzing.

22. $\frac{x-1}{3}=\frac{y+1}{-1}=\frac{z-2}{5}$ to'g'ri chiziq bilan $x+y-2z-4=0$ tekislikning kesishish nuqtasini toping.

23. $M(2;3;4)$ nuqtaning $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$ to‘g‘ri chiziqdagi proeksiyasini toping.

24. $\begin{cases} 8x+2y+3z+6=0, \\ 2x+4y+\ \ z+1=0 \end{cases}$ to‘g‘ri chiziqdan o‘tuvchi va $\frac{x+1}{2}=\frac{y-4}{3}=\frac{z-1}{-2}$ to‘g‘ri chiziqqa parallel tekislik tenglamasini tuzing.

25. $\begin{cases} 3x+y-4z+5=0, \\ x-y+2z-1=0 \end{cases}$ to‘g‘ri chiziqdan va $M(1;-1;2)$ nuqtadan o‘tgan tekislik tenglamasini tuzing.

26. $M(2;-3;-1)$ nuqtadan berilgan to‘g‘ri chiziqqacha bo‘lgan masofani toping:

1) $\frac{x-3}{4}=\frac{y+2}{3}=\frac{z+1}{5}$;   2) $\frac{x+1}{2}=\frac{y+2}{-1}=\frac{z+1}{2}$.

3.6. IKKINCHI TARTIBLI SIRTLAR

Oxyz koordinatalar sistemasida x, y, z o‘zgaruvchilarning ikkinchi darajali tenglamasi bilan aniqlanuvchi sirt *ikkinchi tartibli sirt* deyiladi.

Uchta x, y va z o‘zgaruvchining ikkinchi darajali tenglamasi umumiy ko‘rinishda

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Kz+L=0, \qquad (6.1)$$

kabi yoziladi, bu yerda A,B,C,D,E,F,G,H,K,L – o‘zgarmaslar; $A^2+B^2+C^2\neq 0$.

Har qanday (6.1) ko‘rinishdagi tenglamani koordinata o‘qlarini parallel ko‘chirish va burish orqali *kanonik ko‘rinishga* keltirish mumkin. Kanonik tenglamada har bir o‘zgaruvchi faqat bir marta, bitta (yo nolinchi, yo birinchi, yo ikkinchi) darajada qatnashadi. (6.1) tenglama koordinatalar sistemasining o‘qlari sirtning simmetriya o‘qlari bilan ustma-ust tushganida va koordinatalar boshi maxsus tanlanganida (masalan, markaziy-simmetrik sirtlarda simmetriya markazi tanlanadi) kanonik ko‘rinishni oladi.

Shu bilan birga ikkinchi tartibli sirt

$$F(x,y)=0 \ \ (G(x,y)=0,\ H(x,z)=0) \qquad (6.2)$$

tenglama bilan berilishi mumkin. Bunday tenglama bilan aniqlanuvchi sirtlar silindrik sirtlar deyiladi.

Sirtlarning shaklini tasavvur qilish va chizish uchun «*parallel kesimlar usuli*» deb ataluvchi usulni qo'llaymiz. Bunda sirtning shakli uning koordinata tekisliklari yoki bu tekisliklarga parallel tekisliklar bilan keshishish chiziqlarini (kesimlarini) tekshirish yordamida o'rganiladi.

3.6.1. Sfera

Fazoda markaz deb ataluvchi nuqtadan teng uzoqlikda yotuvchi nuqtalarning geometrik o'rniga *sfera* deyiladi.

$M_0(x_0;y_0;z)$ nuqtadan R masofada yotuvchi fazodagi nuqtalarni qaraymiz. Bu nuqtalardan biri $M(x;y;z)$ nuqta bo'lsin.

Sferaning ta'rifiga ko'ra, $|M_0M|=R$.

Bundan

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=R$$

yoki

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2. \qquad (6.3)$$

(6.3) tenglamaga *sferaning kanonik tenglamasi* deyiladi. Bunda $M_0(x_0;y_0;z_0)$ nuqta *sfera markazi*, R masofa *sfera radiusi* deb ataladi.

1-misol. Markazi $M_0(-2;2;1)$ nuqtada yotgan va $2x+y-2z-5=0$ tekislikka uringan sfera tenglamasini tuzing.

Yechish. Sfera tekislikka uringani sababli uning $M_0(-2;2;1)$ markazidan $2x+y-2z-5=0$ tekislikkacha bo'lgan masofa sferaning radiusiga teng bo'ladi.

Nuqtadan tekislikkacha bo'lgan masofa formulasidan topamiz:

$$R=\frac{|2\cdot(-2)+1\cdot 2+(-2)\cdot 1-5|}{\sqrt{2^2+1^2+(-2)^2}}=\frac{9}{3}=3.$$

U holda (6.3) formulaga ko'ra,

$$(x+2)^2+(y-2)^2+(z-1)^2=9.$$

3.6.2. Ellipsoid

$Oxyz$ koordinatalar sistemasida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (6.4)$$

kanonik tenglama bilan aniqlanuvchi sirtga *ellipsoid* deyiladi.

Ellipsoidning Oxy tekislikka parallel tekisliklar bilan kesimlarini qaraymiz. Bu tekisliklarning har biri $z=h$ tenglamaga ega bo‘ladi. Bunda h – birorta son.

Kesimda hosil bo‘lgan chiziq

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1-\dfrac{h^2}{c^2}, \\ z=h \end{cases} \tag{6.5}$$

tenglamalar sistemasi bilan aniqlanadi.

(6.5) sistema tenglamalarini tekshiramiz.

$|h|>c$ bo‘lganda $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}<0$ bo‘ladi va (6.5) sirtning $z=h$ tekislik bilan kesishish nuqtasi mavjud bo‘lmaydi.

$|h|=c$, ya’ni $h=\pm c$ bo‘lganda $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=0$ bo‘ladi. Bunda sirtlar $(0;0;c)$ va $(0;0;-c)$ nuqtalarga kesishadi va $z=c$ va $z=-c$ tekisliklar berilgan sirtga urinadi.

$|h|<c$ bo‘lganda (6.5) tenglamalarni quyidagicha yozish mumkin:

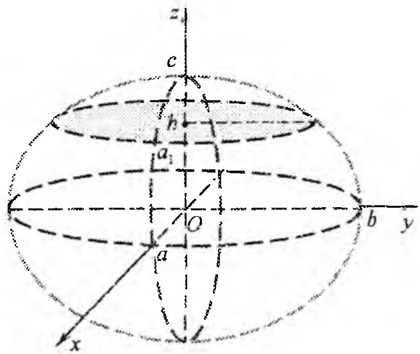
$$\begin{cases} \dfrac{x^2}{a_1^2}+\dfrac{y^2}{b_1^2}=1, \\ z=h, \end{cases}$$

bu yerda $a_1=a\sqrt{1-\dfrac{h^2}{c^2}}$, $b_1=b\sqrt{1-\dfrac{h^2}{c^2}}$.

Demak, kesimda yarim o‘qlari a_1 va b_1 bo‘lgan ellips hosil bo‘ladi (38-shakl). Bunda $|h|$ qancha kichik bo‘lsa, yarim o‘qlar shuncha katta bo‘ladi. $h=0$ da ular o‘zlarining eng katta qiymatlariga erishadi: $a_1=a$, $b_1=b$.

(6.4) sirtning $x=h$ va $y=h$ tekisliklar bilan kesimlari ham ellipslardan iborat bo‘ladi.

Shunday qilib, qaralgan kesimlar (6.4) tenglama bilan aniqlanuvchi sirt 38-shaklda keltirilgan ellipsoiddan iborat bo‘lishini ko‘rsatadi.



38-shakl.

a, b, c kattaliklarga ellipsoidning *yarim o'qlari* deyiladi. Yarim o'qlar har xil bo'lganda ellipsoid *uch o'qli ellipsoid* bo'ladi. Yarim o'qlardan istalgan ikkitasi bir-biriga teng bo'lganda ellipsoid *aylanish ellipsoidi* bo'ladi.

Yarim o'qlarning uchalasi teng bo'lganda ellipsoid tenglamasi

$$x^2 + y^2 + z^2 = R^2, \ R = a = b = c \tag{6.6}$$

bo'lgan sferaga aylanadi.

2-misol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning Ox va Oy oqlari atrofida aylanishidan hosil bo'lgan sirtlarning tenglamalarini toping.

Yechish. Agar ikkinchi tartibli chiziq $F(x, y) = 0$ tenglama bilan berilgan bo'lsa, u holda bu sirtning Ox oqi atrofida aylanishidan hosil bo'lgan sirt $F(x; \pm\sqrt{y^2 + z^2}) = 0$ tenglama bilan, Oy oqi atrofida aylanishidan hosil bo'lgan sirt esa $F(\pm\sqrt{x^2 + z^2}; y) = 0$ tenglama bilan aniqlanadi.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning Ox oqi atrofida aylanishidan hosil bo'lgan sirt tenglamasini topamiz:

$$\frac{x^2}{a^2} + \frac{(\pm\sqrt{y^2 + z^2})^2}{b^2} = 1 \ \text{yoki} \ \frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$

Ellipsning Oy oqi atrofida aylanishidan hosil bo'lgan sirt tenglamasini shu kabi topiladi:

$$\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hosil bo'lgan tenglamalarning har ikkalasi aylanish ellipsoidini aniqlaydi.

3.6.3. Giperboloidlar

$Oxyz$ koordinatalar sistemasida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{6.7}$$

kanonik tenglama bilan aniqlanuvchi sirtga *bir pallali giperboloid* deyiladi.

Bu sirtni Oxy tekislikka parallel $z=h$ tekisliklar bilan kesamiz. Kesimda

$$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{z^2}{c^2}, \\ z=h \end{cases}$$

yoki

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1+\frac{h^2}{c^2}}\right)^2}+\frac{y^2}{\left(b\sqrt{1+\frac{h^2}{c^2}}\right)^2}=1, \\ z=h, \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi chiziq hosil boʻladi. Bu chiziq yarim oʻqlari $a_1=a\sqrt{1+\frac{h^2}{c^2}}$ va $b_1=b\sqrt{1+\frac{h^2}{c^2}}$ boʻlgan ellipsdan iborat. Yarim oʻqlar $h=0$ da eng kichik qiymatlariga erishadi: $a_1=a, b_1=b$. $|h|$ ning oʻsishi bilan ular oʻsib boradi.

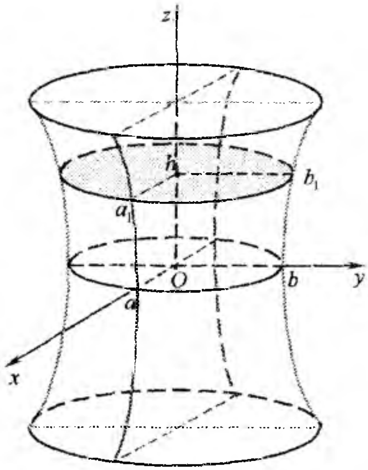
Sirtning Oxz va Oyz tekisliklar bilan kesimlarni

$$\begin{cases} \frac{x^2}{a^2}-\frac{z^2}{c^2}=1, \\ y=0 \end{cases} \qquad \begin{cases} \frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \\ x=0 \end{cases}$$

tenglamalar sistemalari bilan aniqlanuvchi giperbolalardan iborat boʻladi.

Kesimlarning tahlili shuni koʻrsatadiki, (6.7) tenglama bilan aniqlanuvchi giperboloid musbat va manfiy yoʻnalishlarida chegaralanmagan holda kengayuvchi «trubka» koʻrinishdagi sirtdan iborat boʻladi (39-shakl).

$a=b$ boʻlganda (6.7) tenglama *bir pallali aylanish giperboloid*ni ifodalaydi.



39-shakl.

$Oxyz$ kordinatlar sistemasida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \tag{6.8}$$

kanonik tenglama bilan aniqlanuvchi sirtga *ikki pallali giperboloid* deyiladi.

Bu sirtning Oxy tekislikka parallel tekisliklar bilan kesisish chizig'i

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{h^2}{c^2}-1, \\ z=h \end{cases} \tag{6.9}$$

tenglamalar sistemasi bilan aniqlanadi. Bunda $|h|<c$ bo'lganda $z=h$ tekislik sirtni kesmaydi, $|h|=c$ bo'lganda $z=c$ va $z=-c$ tekisliklar sirtga $(0;0;c)$ va
$(0;0;-c)$ nuqtalarga urinadi, $|h|>c$ bo'lganda $z=h$ tekislik sirtni kesadi.

$|h|>c$ bo'lganda (6.9) tenglamalarni quyidagicha yozish mumkin:

$$\begin{cases} \dfrac{x^2}{a_1^2}+\dfrac{y^2}{b_1^2}=1, \\ z=h, \end{cases}$$

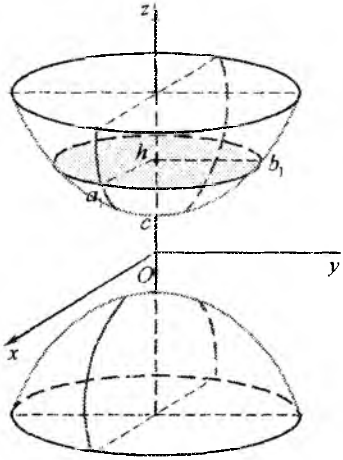
bu yerda $a_1=a\sqrt{\dfrac{h^2}{c^2}-1}$, $b_1=b\sqrt{\dfrac{h^2}{c^2}-1}$.

Bu chiziq $|h|$ ning o'sishi bilan yarim o'qlari o'suvchi ellipsni beradi.

Sirtning Oxz va Oyz tekisliklar bilan kesimlari

$$\begin{cases} \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=-1, \\ y=0 \end{cases} \qquad \begin{cases} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1, \\ x=0 \end{cases}$$

tenglamalar sistemalari bilan aniqlanuvchi giperbolalar bo'ladi.



40-shakl.

Bu kesimlar (40-shakl) (6.8) sirtning ikki pallali giperboloid deb atalishiga sabab bo'ladi. $a=b$ bo'lganda (6.8) tenglama *ikki pallali aylanish giperboloid*ni aniqlaydi.

3-misol. $x^2-4y^2+4z^2+2x+8y-7=0$ tenglama bilan aniqlanuvchi sirt turini toping.

Yechish. Tenglamaning chap tomonini to'la kvadratlarga ajratamiz:

$$x^2+2x+1-4(y^2+2y+1)+4z^2-1+4-7=0,$$

$$(x+1)^2-4(y-1)^2+4z^2=4.$$

Bundan

$$\frac{(x+1)^2}{2^2}+\frac{z^2}{1^2}-\frac{(y-1)^2}{1^2}=1.$$

$x' = x+1,\ y' = y-1,\ z' = z$ deb, $Oxyz$ sistema markazini $O'(-1;1;0)$ nuqtaga parallel ko‘chirish orqali $O'x'y'z'$ sistemaga o‘tamiz.

Bu sistemada tenglama

$$\frac{x'^2}{2^2} + \frac{z'^2}{1^2} - \frac{y'^2}{1^2} = 1$$

ko‘rinishni oladi.

Bu tenglama $O'y'$ oq bo‘ylab yo‘nalgan bir pallali giperboloidni aniqlaydi.

3.6.4. Konuslar

$Oxyz$ koordinatalar sistemasida

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{6.10}$$

kanonik tenglama bilan aniqlanuvchi sirt *ikkinchi tartibli konus* deyiladi.

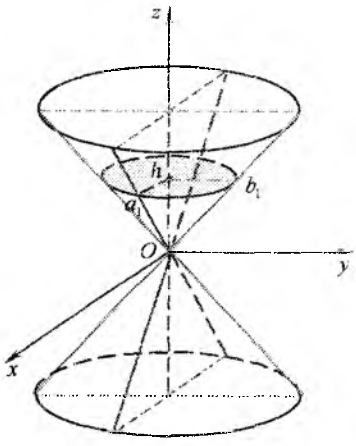
(6.10) sirtning Oxy tekislikka parallel tekisliklar bilan kesisish chizig‘i $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{h^2}{c^2}, z = h$ bo‘ladi.

U $h = 0$ da $O(0;0;0)$ nuqtaga aylanadi.

$h \neq 0$ bo‘lsa, kesimda

$$\begin{cases} \dfrac{x^2}{\left(\dfrac{ah}{c}\right)^2} + \dfrac{y^2}{\left(\dfrac{bh}{c}\right)^2} = 1, \\ z = h, \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi ellips hosil bo‘ladi. Bu ellipsning yarim o‘qlari $|h|$ ning o‘sishi bilan o‘sadi.



41-shakl.

Sirtning Oxz va Oyz tekisliklar bilan kesimlari

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 0, \\ y = 0 \end{cases} \qquad \begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0, \\ x = 0 \end{cases}$$

sistemalar bilan aniqlanuvchi ikkita kesishuvchi to‘g‘ri chiziqlardan iborat bo‘ladi (41-shakl).

3.6.5. Paraboloidlar

Oxyz koordinatalar sistemasida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c},\quad a>0, b>0, c>0 \tag{6.11}$$

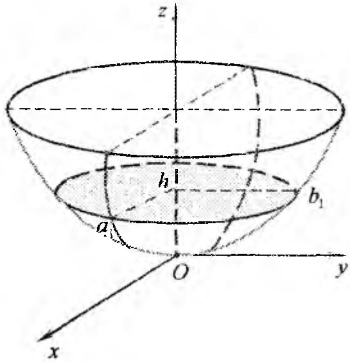
kanonik tenglama bilan aniqlanuvchi sirt *elliptik paraboloid* deyiladi.

Bu sirtning *Oxy* tekislikka parallel tekisliklar bilan kesimi ushbu

$$\begin{cases}\dfrac{x^2}{(ah)^2}+\dfrac{y^2}{(bh)^2}=1,\\ z=h, h>0\end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi ellips bo'ladi. Uning yarim o'qlari $|h|$ ning o'sishi bilan o'sadi.

Sirtning *Oxz* va *Oyz* tekisliklar bilan kesimlarida $z=\frac{x^2}{a^2}$ va $z=\frac{y^2}{b^2}$ parabolalar hosil bo'ladi (42-shakl). Shu sababli (6.11) tenglama bilan aniqlanuvchi sirt elliptik paraboloid deyiladi.



42-shakl.

$a=b$ bo'lganda (6.11) tenglama *aylanish paraloidi*ni aniqlaydi.

4-misol. $M_1(0;b;0)$ nuqtadan va $y=-b$ tekislikdan teng uzoqlikda yotuvchi nuqtalarning geometrik o'rnini toping.

Yechish. $M(x;y;z)$ izlanayotgan nuqta bo'lsin.

Masala shartiga ko'ra

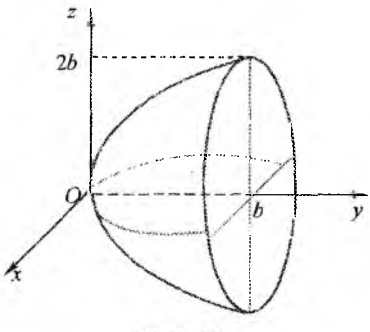
$$|M_1M|=|y+b|$$

yoki

$$\sqrt{x^2+(y-b)^2+z^2}=|y+b|.$$

Bundan

$$x^2+y^2-2yb+b^2+z^2=y^2+2yb+b^2,$$
$$x^2+z^2=4by$$



43-shakl.

yoki

$$\frac{x^2}{4b}+\frac{z^2}{4b}=y.$$

Sirtning Oxz tekislikka parallel tekislik bilan kesimi ushbu

$$\begin{cases} x^2+y^2=4bh, \\ y=h, \ h>0 \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi aylanalardan iborat. Sirtning Oxy va Oyz tekisliklar bilan kesimlarida $y=\frac{x^2}{4b}$ va $y=\frac{z^2}{4b}$ parabolalar hosil bo‘ladi.

Bu sirt aylanish paraboloidini aniqlaydi (43-shakl).

$Oxyz$ koordinatalar sistemasida

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z, \ a>0, \ b>0 \qquad (6.12)$$

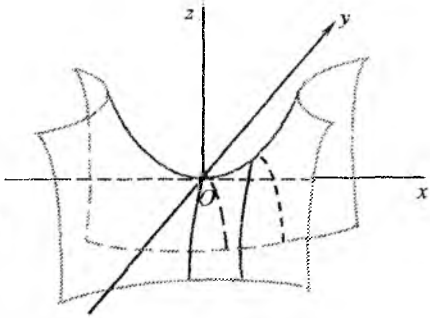
kanonik tenglama bilan aniqlanuvchi sirt *giperbolik paraboloid* deyiladi.

Sirtning Oxy tekislikka parallel tekisliklar bilan kesimi

$$\begin{cases} \frac{x^2}{(ah)^2}-\frac{y^2}{(bh)^2}=1, \\ z=h, h>0 \end{cases}$$

tenglamalar sistemasi bilan aniqlanuvchi giperboladan, Oxz va Oyz tekisliklar bilan kesimlari $z=\frac{x^2}{a^2}$ va $z=\frac{y^2}{b^2}$ parabolalardan iborat bo‘ladi.

Shunday qilib, (6.12) tenglama bilan aniqlanuvchi sirtning ko‘rinishi «egar» shaklida bo‘ladi (44-shakl). Bu sirt giberbolik paraloboid deb ataladi.

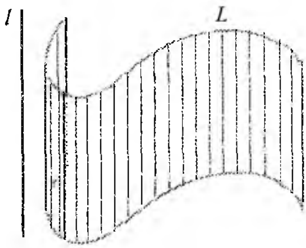


44-shakl.

3.6.6. Silindrik sirtlar

Tekislikda L chiziq va bu tekislikka perpendikular l to‘g‘ri chiziq berilgan bo‘lsin.

L chiziqning har bir nuqtasi orqali l to‘g‘ri chiziqqa parallel qilib o‘tkazilgan to‘g‘ri chiziqlar to‘plamidan hosil bo‘lgan sirtga *silindrik sirt* deyiladi. Bunda L chiziq *silindrik sirtning yo‘naltiruvchisi*, l to‘g‘ri chiziqqa parallel bo‘lgan to‘g‘ri chiziqlar *silindrik sirtning yasovchilari* deb ataladi (45-shakl).



45-shakl.

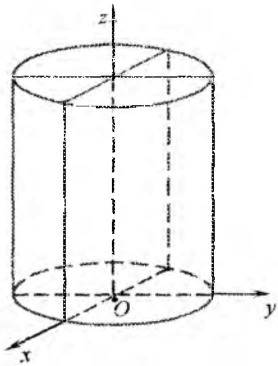
$Oxyz$ koordinatalar sistemasini Oz o‘q l yasovchiga parallel va L yo‘naltiruvchi Oxy tekislikda yotadigan qilib tanlaymiz.

L yo‘naltiruvchining Oxy tekislikdagi tenglamasi $F(x,y)=0$ bo‘lsin. U holda $F(x,y)=0$ tenglama yosovchilari Oz o‘qqa parallel bo‘lgan silindrik sirtni ifodalaydi.

Silindrik sirtning nomlanishi va tenglamasi L yo‘naltiruvchining shakli asosida aniqlanadi.

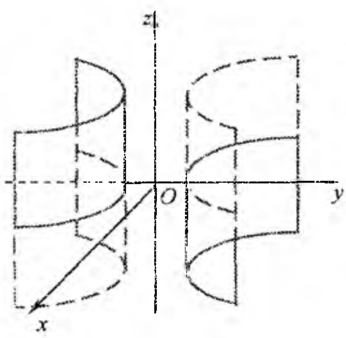
Agar Oxy tekislikdagi yo‘naltiruvchi ellipsdan iborat bo‘lganda $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tenglama *elliptik silindrni* ifodalaydi (46-shakl).

$x^2+y^2=R^2$ tenglama bilan aniqlanuvchi *doiraviy silindr* elliptik silindrning xususiy holi bo‘ladi.

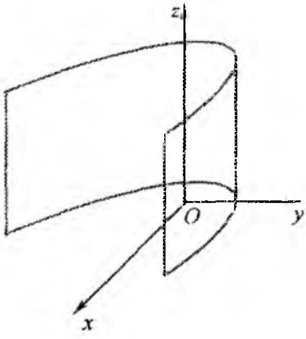


46-shakl.

Shu kabi $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tenglama *giperbolik silindrni* (47-shakl), $y^2=2px$ tenglama *parabolik silindrni* ifodalaydi (48-shakl).



47-shakl.



48-shakl.

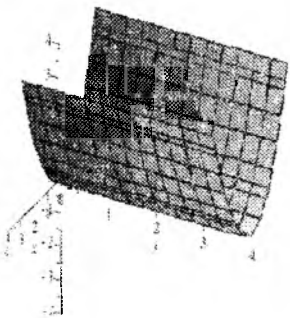
5-misol. $x^2 = 2z$ tenglama bilan aniqlanuvchi sirt shaklini chizing.

Yechish. Berilgan tenglamada y qatnashmaydi va $x^2 = 2z$ chiziq Oxz
tekislikda yotuvchi parabolani ifodalaydi.

Shu sababli $\begin{cases} x^2 = 2z, \\ y = 0 \end{cases}$ tenglama yosovchilari Oy o‘qqa parallel bo‘lgan parobolik silindrni ifodalaydi.

Parabola $y = 0$ tekislikda Oz o‘qqa nisbatan simmetrik bo‘ladi, uchi $O(0;0;0)$ nuqtada yotadi va $M_1(-2;0;2)$, $M_2(2;0;2)$ nuqtalardan o‘tadi. Chiziq shaklini *Maple* paketida chizamiz (49-shakl):

```
> with(plots):
> imlicitplot3d([x^2=2z], x=-4..4, y=-0..4,
z=-4..4, grid=[13,13,13]);
```



49-shakl.

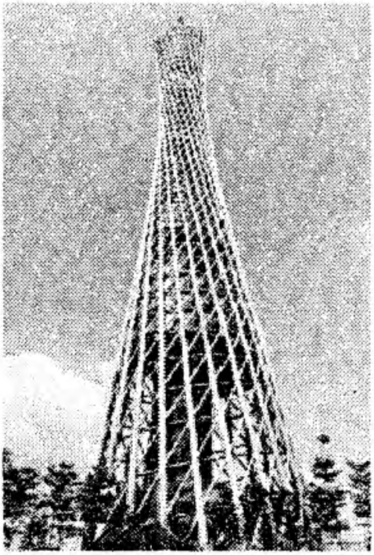
3.6.7. Ikkinchi tartibli sirtlarning to‘g‘ri chiziqli yasovchilari

To‘g‘ri chiziqlarning harakatidan hosil bo‘ladigan sirtlarga *to‘g‘ri chiziqli sirtlar* deyiladi. Bu sirtlarning yasovchilari *to‘g‘ri chiziqli yasovchilar* deb ataladi.

To‘g‘ri chiziqli sirtlarga konus sirtlar va silindrik sirtlarlar misol bo‘la oladi. Bundan tashqari, bir pallali giperboloid va giperbolik paraboloid ham to‘g‘ri chiziqli sirtlar bo‘lishi isbotlangan.

Bir pallali giperboloidning har bir nuqtasi orqali

$$\begin{cases} \frac{x}{a} + \frac{z}{c} = k\left(1 + \frac{y}{b}\right), \\ \frac{x}{a} - \frac{z}{c} = \frac{1}{k}\left(1 - \frac{y}{b}\right) \end{cases}, \quad \begin{cases} \frac{x}{a} + \frac{z}{c} = l\left(1 - \frac{y}{b}\right), \\ \frac{x}{a} - \frac{z}{c} = \frac{1}{l}\left(1 + \frac{y}{b}\right) \end{cases} \quad (6.13)$$



50-shakl.

tenglamalar bilan berilgan to‘g‘ri chiziqlar oilasidan faqat bitta chiziq o‘tishi ko‘rsatilgan, bu yerda a,b,c – bir pallali giperboloidning yarim o‘qlari, k,l – nolga teng bo‘lmagan sonlar.

Shunday qilib, (6.13) tenglamalar k va l ning turli qiymatlarida bir pallali giperboloidda yotuvchi va uni to‘liq qoplovchi cheksiz to‘g‘ri chiziqlar sistemasini tashkil qiladi. Bu to‘g‘ri chiziqlarga bir pallali giperboloidning *to‘g‘ri chiziqli yasovchilari* deyiladi.

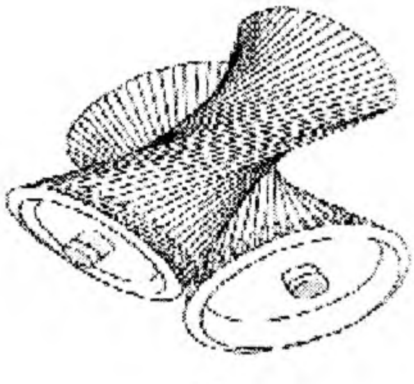
Shu kabi

$$\begin{cases} \frac{x}{a}+\frac{z}{b}=kz, \\ \frac{x}{a}-\frac{z}{b}=\frac{1}{k} \end{cases} \quad \text{va} \quad \begin{cases} \frac{x}{a}+\frac{z}{b}=\frac{1}{l}, \\ \frac{x}{a}-\frac{z}{b}=lz \end{cases} \tag{6.14}$$

tenglamalar bilan berilgan to‘g‘ri chiziqlar giperbolik paraboloidning to‘g‘ri chiziqli yasovchilari bo‘ladi.

To‘g‘ri chiziqli yasovchilardan bir pallali giperboloid sirtlarning yasovchilari qurilish va texnikaning konstruksiyalarida keng foydalaniladi.

Masalan, bir pallali giperboloidning to‘g‘ri chiziqli yasovchilari bo‘yicha metall balkalar joylashtirilgan radiomachta, suv va yadro qurilmalarining minoralari, teleminoralar (masalan, Guanjou (Xitoy) teleminorasi, 50-shakl), tirgaklar va uzatmalar (51-shakl) kabi konstruksiyalar ishlab chiqilgan. Bunday konstruksiyalar yengil va mustahkam bo‘lgani sababli keng tatbiq etiladi.



51-

3.6.8. Mashqlar

1. Sferaning tenglamasini tuzing: 1) diametrlaridan birining uchlari $M_1(4;1;-3)$ va $M_2(2;-3;5)$ nuqtalarda yotgan; 2) markazi $M_0(3;-5;-2)$ nuqtada yotgan va $2x-y-3z+11=0$ tekislikka uringan.

2. Sfera markazining koordinatalarini va radiusini toping:

1) $x^2+y^2+z^2+x-y+z=0$; 2) $x^2+y^2+z^2-6x+8y+10z+25=0$.

3. Har bir nuqtasidan $M_1(-3;0;0)$ va $M_2(3;0;0)$ nuqtalargacha bo'lgan masofalar kvadratlarining yig'indisi 36 ga teng bo'lgan fazoviy nuqtalarning geometrik o'rnini toping.

4. $M\left(0;\frac{5}{2};0\right)$ nuqtadan va $y=-\frac{5}{2}$ tekislikdan teng uzoqlikda yotgan fazoviy nuqtalarining geometrik o'rnini toping.

5. Har bir nuqtasidan $M_1(0;0;-4)$ va $M_2(0;0;4)$ nuqtalargacha bo'lgan masofalar yig'indisi 10 ga teng bo'lgan sirtning tenglamasini tuzing.

6. Har bir nuqtasidan $M_1(-5;0;0)$ va $M_2(5;0;0)$ nuqtalargacha bo'lgan masofalar ayirmasining moduli 6 ga teng bo'lgan sirtning tenglamasini tuzing.

7. Berilgan sirtning ko'rsatilgan o'qlar atrofida aylanishidan hosil bo'lgan sirt tenglamasini tuzing:

1) $z=-\frac{x^2}{2}$, Ox va Oz; 2) $\frac{x^2}{16}-\frac{y^2}{25}=1$, Ox va Oy; 3) $\frac{y^2}{64}+\frac{z^2}{16}=1$, Oy va Oz.

8. Markazi koordinatalar boshida yotgan va yo'naltiruvchilari $x^2-2z+1=0$, $y-z+1=0$ tenglamalar bilan berilgan konus tenglamasini tuzing.

9. m ning qanday qiymatlarida $x+my-2=0$ tekislik $\frac{x^2}{2}+\frac{z^2}{3}=y$ elliptik parabaloidni 1) ellips bo'yicha; 2) parabola bo'yicha kesadi?

10. Berilgan sirtlarning kesishish chizig'ini aniqlang:

1) $\frac{x^2}{3}+\frac{y^2}{6}=2z,\ 3x-y+6z-14=0;$ 2) $\frac{x^2}{4}-\frac{y^2}{3}=2z,\ 3x-y+6z-14=0;$

3) $\frac{(x-1)^2}{4}-\frac{(y+1)^2}{3}=2z,\ x-2y-1=0;$ 4) $\frac{x^2}{3}+\frac{y^2}{9}-\frac{z^2}{25}=-1,\ 5x+2z+5=0.$

11. Har bir nuqtasidan $M(3;0;0)$ nuqtagacha va $x=1$ tekislikkacha bo'lgan masofalar nisbati $\sqrt{3}$ ga teng bo'lgan fazoviy nuqtalarning geometrik o'rnini toping.

12. Berilgan sirt va to'g'ri chiziqning kesishish nuqtasini toping:

1) $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1,\ \frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4};$ 2) $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{4}=1,\ \frac{x}{4}=\frac{y}{-3}=\frac{z+2}{4}.$

13. Berilgan tenglama bilan aniqlanuvchi sirt turini aniqlang:

1) $36x^2+64y^2-144z^2+576=0;$ 2) $x^2+y^2+z^2-2(x+y+z)-22=0;$

3) $3x^2+2y^2-12z=0;$ 4) $16x^2+3y^2+16z^2-64x-6y+19=0;$

5) $25x^2-9y^2-225=0;$ 6) $9x^2-4y^2-36z=0;$

7) $4x^2+3y^2-5z^2+60=0;$ 8) $x^2+y^2-2x-3=0.$

4

MATEMATIK ANALIZGA KIRISH

- Haqiqiy sonlar
- Sonli ketma-ketliklar
- Bir o'zgaruvchining funksiyasi
- Funksiyaning limiti
- Funksiyaning uzluksizligi

Ogyusten Lui Koshi (1789–1857) – fransuz matematigi va mexanigi.

Matematik analiz asoslarini ishlab chiqqan, matematikaning matematik analiz, algebra, matematik fizika va boshqa bo'limlarining rivojiga katta hissa qo'shgan.

O.L. Koshi birinchi bo'lib limit, uzluksizlik, hosila, differensial, integral, qatorning yaqinlashishi tushunchalariga qat'iy ta'rif bergan.

Matematik analiz tarixdan «Cheksiz kichiklar tahlili»ga mos bo'limlar majmuasi bo'lib, u differensial va integral hisobni birlashtiradi. Matematik analiz sistemali bo'lim sifatida XVII–XVIII asrlarda I.Nuyton, G.Leybnis, L.Eyler, J.Lagranj va boshqa olimlar asarlarida yuzaga keldi. Uning asosi bo'lgan limitlar nazariyasi XIX asrda O.Koshi tomonidan ishlab chiqildi. Matematik analizning boshlang'ich tushunchalari XIX–XX asrlarda to'plamlar nazariyasi, o'lchamlar nazariyasi, haqiqiy o'zgaruvchi funksiyalari nazariyalarining rivojiga asoslanib chuqur tahlil qilindi va umumlashtirildi.

Matematikaning «Matematik analizga kirish» bo'limida matematik analizning asosiy tushunchalari bo'lgan bir o'zgaruvchining funksiyasi, sonli ketma-ketliklar, limitlar, cheksiz kichik funksiyalar va uzluksizlik tushunchalari o'rganiladi.

4.1. HAQIQIY SONLAR

4.1.1. To'plam

To'plam matematikaning boshlang'ich (ta'riflanmaydigan) va muhim tushunchalaridan biri hisoblanadi. To'plam deganda tayin xossaga ega bo'lgan ixtiyoriy tabiatli obyektlar majmuasi tushuniladi. Masalan, guruhdagi talabalar to'plami, butun sonlar to'plami, berilgan nuqtadan o'tuvchi to'g'ri chiziqlar to'plami.

To'plamni tashkil etuvchi obyektlarga to'plamning *elementlari* deyiladi. To'plam, odatda, lotin alifbosining bosh harflari bilan, uning elementlari esa shu alifboning kichik harflari bilan belgilanadi.

A to'plamning a,b,c,d elementlardan tashkil topganligi $A=\{a,b,c,d\}$ kabi yoziladi. Ba'zan to'plam sonlar, belgilar, so'zlar yoki formulalar yordamida beriladi.

a elementning A to'plamga tegishli ekanligi $a\in A$ deb yoziladi. b elementning A to'plamga tegishli emasligi $b\overline{\in} A$ (yoki $b\notin A$) kabi belgilanadi. Masalan, $A=\{2,4,6,8\}$ to'plam uchun $4\in A$ va $5\overline{\in} A$.

A to'plam chekli sondagi elementlardan tashkil topgan bo'lsa A to'plamga *chekli to'plam*, aks holda *cheksiz to'plam* deyiladi. Masalan, $A=\{x:6<x<20, x\in N\}$ chekli to'plam, $B=\{x:x>15, x\in N\}$ cheksiz to'plam bo'ladi.

Bitta ham elementga ega bo'lmagan to'plam *bo'sh to'plam* deb ataladi va Ø kabi belgilanadi. Masalan, $A=\{x:x^2+1=0, x\in R\}$ bo'sh to'plam, chunki $x^2+1=0$ tenglama haqiqiy sonlar to'plami R da yechimga ega emas.

Agar A to'plamning har bir elementi B to'plamning ham elementi bo'lsa A to'plamga B *to'plamning qismi* (*qismiy to'plami*) deyiladi va $A\subset B$ (yoki $B\supset A$) kabi belgilanadi. Masalan, $A=\{2,3,4\}$ va $B=\{1,2,3,4,5\}$ bo'lsa $A\subset B$ bo'ladi.

Agar $A\subset B$ va $B\subset A$ bo'lsa A va B to'plamlarga *teng to'plamlar* deyiladi va $A=B$ kabi yoziladi. Demak, $A=B$ tenglik A va B to'plamlarning bir xil elementlardan tashkil topganini bildiradi.

A va B to'plamlarning har ikkalasiga tegishli bo'lgan element bu to'plamlarnng *umumiy elementi* deyiladi.

1-ta'rif. A va B *to'plamlarning birlashmasi* (yoki *yig'indisi*) deb ularning kamida bittasiga tegishli bo'lgan elementlardan tashkil topgan to'plamga aytiladi va $A\cup B$ (yoki $A+B$) kabi belgilanadi.

Demak, $A\cup B=\{x: x\in A \text{ } yoki \text{ } x\in B\}$.

2-ta'rif. A va B *to'plamlarning kesishmasi* (yoki *ko'paytmasi*) deb ularning barcha umumiy elementlaridan tashkil topgan to'plamga aytiladi va $A\cap B$ (yoki $A\cdot B$) kabi belgilanadi.

Demak, $A\cap B=\{x: x\in A \text{ } va \text{ } x\in B\}$.

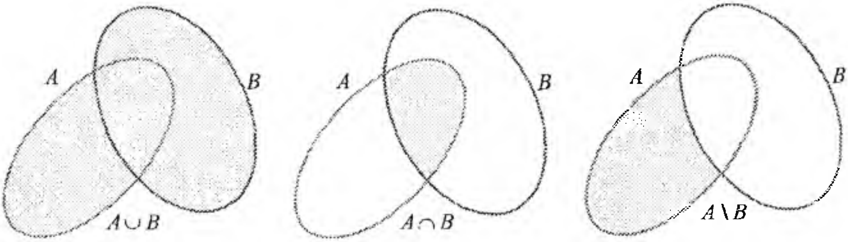
3-ta'rif. A to'plamdan B *to'plamning ayirmasi* deb A to'plamning B to'plamga kirmagan elementlaridan tashkil topgan to'plamga aytiladi va $A\backslash B$ kabi belgilanadi.

Demak, $A \setminus B = \{x : x \in A \; va \; x \overline{\in} B\}$.

Masalan, $A = \{1,3,5,7\}$ va $B = \{2,5,7,9\}$ to'plamlar uchun $A \cup B = \{1,2,3,5,7,9\}$, $A \cap B = \{5,7\}$, $A \setminus B = \{1,3\}$, $B \setminus A = \{2,9\}$ bo'ladi.

B to'plam A to'plamning qismiy to'plami bo'lsa, $A \setminus B$ ayirma B to'plamning A *to'plamga to'ldiruvchisi* deyiladi.

1-3 ta'riflarning chizmadagi ifodasi 1-shaklda keltirilgan. Bunda $A \cap B$, $A \cup B$, $A \setminus B$ bo'yab ko'rsatilgan.



1-shakl.

4.1.2. Sonli to'plamlar

Haqiqiy sonlar va ularning asosiy xossalari

Elementlari sonlardan iborat bo'lgan to'plam *sonli to'plam* deyiladi.

Son matematik analizning asosiy tushunchalaridan biri bo'lib, uzoq tarixiy rivojlanish yo'liga ega. Narsalarni, buyumlarni sanash zaruriyati tufayli natural sonlar paydo bo'lgan. Natural sonlar to'plamiga ularga qarama-qarshi sonlarni va nol sonini qo'shish bilan butun sonlar to'plami hosil qilingan. Matematikaning taraqqiyoti ratsional sonlarning va keyinchalik irratsional sonlarning kiritilishini taqozo etgan. Ratsional sonlar to'plami va irratsional sonlar to'plami haqiqiy sonlar to'plami deb atalgan.

Shunday qilib, $N \subset Z \subset Q \subset R$ sonli to'plamlar hosil qilingan, bu yerda $N = \{1,2,3,\ldots,n,\ldots\}$ – barcha natural sonlar to'plami; $Z = \{0,\pm1,\pm2,\ldots,\pm n,\ldots\}$ – barcha butun sonlar to'plami; $Q = \left\{\frac{p}{q}, p \in Z, q \in N\right\}$ – barcha ratsional sonlar to'plami; R – barcha haqiqiy sonlar to'plami.

Har qanday ratsional son yoki chekli o'nli kasr bilan yoki cheksiz davriy o'nli kasr bilan ifodalanadi. Masalan, $\frac{3}{2} = 1{,}5 = (1{,}500...)$, $\frac{1}{3} = 0{,}333...$ – ratsional sonlar.

Ratsional bo'lmagan haqiqiy sonlarga irratsional sonlar deyiladi. Irratsional son cheksiz davriy bo'lmagan o'nli kasr bilan ifodalanadi. Masalan, $\sqrt{2} = 1{,}4142356...$, $\pi = 3{,}1415926...$ – irratsional sonlar.

Shunday qilib, *haqiqiy sonlar* to'plamini barcha cheksiz o'nli kasrlar to'plami deyish va $R = \{x:\ x = a,\alpha_1\alpha_2\alpha_3...\}$ kabi yozish mumkin, bu yerda $a \in Z, \alpha_i \in \{0,1,2,...9\}, i = 1,2,...$.

Haqiqiy sonlar to'plami R quyidagi asosiy xossalarga ega bo'ladi.

1°. R to'plam tartiblangan to'plamdir, ya'ni istalgan ikkita har xil a va b sonlar uchun $a < b$ (yoki $b < a$) tengsizlik bajariladi.

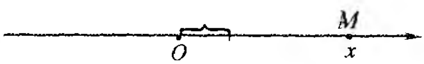
2°. R to'plam zichdir, ya'ni istalgan ikkita har xil a va b sonlar orasida $a < x < b$ tengsizlikni qanoatlantiruvchi cheksiz ko'p x haqiqiy sonlar mavjud bo'ladi;

3°. R to'plam uzluksizdir.

Son o'qi. Sonlarning sodda to'plamlari

Haqiqiy sonlarning uzluksizligi xossasi asosida barcha haqiqiy sonlar to'plami bilan to'g'ri chiziq nuqtalari to'plami orasida bir qiymatli moslik o'rnatiladi.

Buning uchun biror to'g'ri chiziqda (u gorizontal yo'nalgan bo'lsin (2-shakl)) musbat yo'nalishni, O hisob boshini va masshtab birligini tanlaymiz. Musbat x sonini ifodalash uchun bu to'g'ri chiziqda O hisob boshidan o'ng tomonda tanlangan masshtab birligida berilgan x songa teng masofada yotuvchi M nuqtani olamiz; manfiy x sonini ifodalash uchun esa bu to'g'ri chiziqda O hisob boshidan chap tomonda $|x|$ songa (bu son haqida keyingi bandda tushuncha beriladi) teng masofada yotuvchi M nuqtani olamiz; $x = 0$ soniga O hisob boshi mos keladi.



2-shakl.

Barcha nuqtalari uchun barcha haqiqiy sonlar to'plami bilan ko'rsatilgan bir qiymatli moslik o'rnatilgan to'g'ri chiziqqa *son o'qi* (yoki *sonli to'g'ri chiziq*) deyiladi.

Shunday qilib, har bir haqiqiy songa son o‘qining yagona M nuqtasi mos qo‘yiladi va aksincha, bu son o‘qining har bir M nuqtasiga yagona x haqiqiy son mos keladi. Bunda haqiqiy son va son o‘qining nuqtasi bitta x belgi bilan ifodalanadi. Shu sababli «x son» so‘zi o‘rniga ko‘p hollarda «x nuqta» so‘zi ishlatiladi.

Son o‘qi haqiqiy sonlarning joylashishi to‘g‘risida ko‘rgazmali ma’lumot beradi. $x_1 < x_2$ tengsizlik x_1 nuqta x_2 nuqtaga nisbatan chapda yotishini anglatadi, $x_1 < x_2 < x_3$ tengsizlik x_2 nuqta x_1 va x_3 nuqtalar orasida yotishini bildiradi.

$a \in R,\ b \in R,\ a < b$ bo‘lsin. Haqiqiy sonlar to‘plamining quyidagi qism to‘plamlariga *oraliqlar* (*intervallar*) deyiladi:

$[a;b] = \{x : a \le x \le b\}$ – kesma (yopiq oraliq, sigment);

$(a;b) = \{x : a < x < b\}$ – interval (ochiq oraliq);

$[a;b) = \{x : a \le x < b\}, (a;b] = \{x : a < x \le b\}$ – yarim ochiq intervallar;

$(-\infty;b] = \{x : x \le b\},\ (-\infty;b) = \{x : x < b\},\ [a;+\infty) = \{x : x \ge a\},$

$(a;+\infty) = \{x : x > a\},\ (-\infty;+\infty) = \{x : -\infty < x < +\infty\}$ – cheksiz intervallar.

Bunda a va b sonlar mos ravishda bu oraliqlarning chap va o‘ng chegaralarini aniqlaydi, $-\infty$ va $+\infty$ belgilar son o‘qi nuqtalarining O nuqtadan chapga va o‘ngga qarab cheksiz uzoqlashishini simvolik tasvirlaydi.

$x_0 (x_0 \in R)$ nuqtani o‘z ichiga olgan har qanday $(a;b)$ intervalga x_0 *nuqtaning atrofi* deyiladi. Xususan, $(x_0 - \varepsilon; x_0 + \varepsilon)$ interval x_0 *nuqtaning* ε *atrofi* deb ataladi. Bunda x_0 soniga bu atrofning markazi, ε $(\varepsilon > 0)$ soniga bu atrofning radiusi deyiladi.

Agar $x_0 \in (x_0 - \varepsilon; x_0 + \varepsilon)$ bo‘lsa, u holda $x_0 - \varepsilon < x < x_0 + \varepsilon$ yoki $|x - x_0| < \varepsilon$ tengsizlik bajariladi. Bu tengsizlikning bajarilishi x nuqta x_0 nuqtaning ε atrofiga tushishini bildiradi.

Sonning absolut qiymati

4-ta’rif. $x\ (x \in R)$ sonining *absolut qiymati* (yoki *moduli*) deb $x \ge 0$ bo‘lganida x sonining o‘ziga, x manfiy bo‘lganida $(-x)$ soniga aytiladi.

x sonining absolut qiymati $|x|$ belgi bilan belgilanadi.

Demak, ta’rifga ko‘ra,

$$|x| = \begin{cases} x, & agar\ x \ge 0, \\ -x, & agar\ x < 0. \end{cases}$$

Sonning absolut qiymati quyidagi xossalarga ega.

1°. $x\in R$ da $|x|\geq 0$, $|-x|=|x|$, $-|x|\leq x\leq |x|$;

2°. $a>0$ da $|x|\leq a$ tengsizlik $-a\leq x\leq a$ tengsizlikka ekvivalent bo‘ladi;

3°. $x\in R, y\in R$ da

$$|x+y|\leq |x|+|y|,\ |x-y|\geq |x|-|y|,\ |x\cdot y|=|x|\cdot|y|,\ \left|\frac{x}{y}\right|=\frac{|x|}{|y|}, (y\neq 0).$$

Bu xossalarning isboti bevosita sonning absolut qiymati ta’rifidan kelib chiqadi. Ulardan birini, masalan, $|x+y|\leq |x|+|y|$ bo‘lishini isbotlaymiz.

Isboti. $x+y>0$ bo‘lsin.

U holda $|x+y|=x+y$, $x\leq |x|$, $y\leq |y|$ bo‘ladi.

Bundan

$$|x+y|=x+y\leq |x|+|y|.$$

$x+y<0$ bo‘lsin.

U holda

$$|x+y|=-(x+y)=(-x)+(-y),\ -x\leq |x|,\ -y\leq |y|$$

bo‘ladi.

Bundan

$$|x+y|=(-x)+(-y)\leq |x|+|y|.$$

Sonli to‘plamning aniq chegaralari

5-ta’rif. Agar shunday M soni topilsa va istalgan $x\in X$ uchun $x\leq M$ tengsizlik bajarilsa. X haqiqiy sonlar to‘plami *yuqoridan chegaralangan* deyiladi.

Masalan, $X=(-\infty,1]$ to‘plam yuqoridan chegaralangan.

6-ta’rif. Agar shunday m soni topilsa va istalgan $x\in X$ uchun $x\geq m$ tengsizlik bajarilsa, X haqiqiy sonlar to‘plami *quyidan chegaralangan* deyiladi.

Masalan, barcha natural sonlar to‘plami quyidan chegaralangan.

7-ta'rif. Agar X to'plam ham quyidan ham yuqoridan chegaralangan bo'lsa, y'ani shunday m va M sonlari topilsa va istalgan $x \in X$ uchun $m \leq X \leq M$ tengsizlik bajarilsa, X haqiqiy sonlar to'plamiga *chegaralangan* deyiladi.

Bu ta'rifdan agar X to'plamning elementlari biror kesmada joylashsa, u holda bu to'plam chegaralangan bo'ladi degan xulosa kelib chiqadi.

Yuqoridan (quyidan) chegaralanmagan to'plamga *yuqoridan* (*quyidan*) *chegaralanmagan* deyiladi. Masalan, barcha natural sonlar to'plami yuqoridan chegaralanmagan (ammo quyidan chegaralangan) bo'lsa, barcha manfiy sonlar to'plami quyidan chegaralanmagan (ammo yuqoridan chegaralangan). Barcha butun sonlar to'plami, barcha ratsional sonlar to'plami, shuningdek, barcha haqiqiy sonlar to'plami ham quyidan, ham yuqoridan chegaralanmagan.

Agar X to'plam yuqoridan M soni bilan chegaralangan bo'lsa, bu songa X *to'plamning yuqori chegarasi* deyiladi. Bunda M sonidan katta bo'lgan ixtiyoriy M' son ham X to'plamning yuqori chegarasi bo'ladi.

8-ta'rif. Agar istalgan $x \in X$ uchun $x \leq M$ bo'lsa va yetarlicha kichik ixtiyoriy $\varepsilon > 0$ musbat son uchun shunday $x^* \in X$ soni topilsa va $M - \varepsilon < x^* < M$ tengsizlik bajarilsa, M soniga X *to'plamning aniq yuqori chegarasi* deyiladi.

Boshqacha aytganda, X to'plamning aniq yuqori chegarasi X ning barcha yuqori chegaralarining eng kichigi bo'ladi.

X to'plamning aniq yuqori chegarasi $M = \sup X$ (yoki $M = \sup\limits_{x \in X}\{x\}$) bilan belgilanadi (lotin tilida supremum – eng yuqori so'zidan olingan).

Yuqoridan chegaralanmagan X to'plam uchun ta'rif asosida $\sup X = +\infty$ deb olinadi.

Agar X to'plam quyidan m soni bilan chegaralangan bo'lsa, bu songa X *to'plamning quyi chegarasi* deyiladi. Bunda m sonidan kichik bo'lgan ixtiyoriy m' son ham X to'plamning quyi chegarasi bo'ladi.

9-ta'rif. Agar istalgan $x \in X$ uchun $x \geq m$ bo'lsa va yetarlicha kichik ixtiyoriy $\varepsilon > 0$ musbat son uchun shunday $x^* \in X$ soni topilsa va $m < x^* < m + \varepsilon$ tengsizlik bajarilsa, m soniga X *to'plamning aniq quyi chegarasi* deyiladi.

Shunday qilib, X to'plamning aniq quyi chegarasi X ning barcha quyi chegaralarining eng kattasi bo'ladi.

X to'plamning aniq quyi chegarasi $M = \inf X$ (yoki $M = \inf\limits_{x \in X}\{x\}$)

bilan belgilanadi (lotin tilida infimum – eng quyi so'zidan olingan).

Quyidan chegaralanmagan X to'plam uchun ta'rif asosida $\inf X = -\infty$ deb olinadi. Masalan, $X = \left\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\}$ to'plam uchun $\inf X = 0$, $\sup X = 1$.

X to'plamning aniq chegaralari uchun quyidagiu tasdiq o'rinli bo'ladi.

1-teorema. Har qanday yuqoridan (quyidan) chegaralangan bo'sh bo'lmagan haqiqiy sonlar to'plami aniq yuqori (aniq quyi) chegaraga ega bo'ladi.

4.1.3. Matematik mantiq elementlari

Mantiqiy belgilar

Ta'riflarning, teoremalarning ifodalanishida va bosqa matemtik tasdiqlarda ko'pincha ayrim so'zlar va butun ifodalar takrorlanib keladi. Bunday hollarda ularning yozilishida mantiqiy belgilarni qo'llash qulay bo'ladi.

Matematik mantiqda *mulohaza* deb rost yoкi yolg'onligi bir qiymatli aniqlanadigan darak gaplarga aytiladi. Masalan, «Yer quyosh atrofida aylanadi», «6 oddiy son» gaplari mulohaza bo'lsa, «kech kirmoqda», «matematika qiyin fan»
gaplari mulohaza bo'lmaydi.

α mulohazaning *inkori* $\overline{\alpha}$ – «α emas» yoki «α bo'lishi to'g'ri emas» deb o'qiladi.

α va β mulohazalarning *konyunksiyasi* $\alpha \wedge \beta$ – «α va β» deb o'qiladi.

α va β mulohazalarning *dizyunksiyasi* $\alpha \vee \beta$ – «α yoki β» deb o'qiladi.

α va β mulohazalarning *implikatsiyasi* $\alpha \Rightarrow \beta$ – «agar α bo'lsa, u holda β bo'ladi» (yoki «α dan β kelib chiqadi») mulohazasini bildiradi.

α va β mulohazalarning *ekvivalensiyasi* $\alpha \Leftrightarrow \beta$ – « α va β ekvivalent» (yoki «α dan β kelib chiqadi va β dan α kelib chiqadi») mulohazasini bildiradi.

Mavjudlik kvantori $\exists$ – «mavjudki», «topiladiki» so'zlarini bildiradi.

Umumiylik kvantori $\forall$ – «har qanday», «ixtiyoriy», «barcha» so'zlarini ifodalaydi.

$:$ – «o'rinli bo'ladi», «bajariladi» so'zlarini anglatadi.

$\mapsto$ – «moslik» ni bildiradi.

Mantiqiy belgilar yordamida yozilgan tasdiqlarni tushunish va o'qishni osonlashtirish uchun belgilarning har biriga tegishli bo'lganlari alohida qavslarga olinadi.

Masalan,

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \neq x_0, |x - x_0| < \delta) : |f(x) - A| < \varepsilon)$$

yozuv «ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topiladiki, $x$ning x_0 ga teng bo'lmagan va $|x - x_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - A| < \varepsilon$ tengsizlik bagariladi» deb o'qiladi.

Zarur va yetarli shartlar

β – birorta mulohaza bo'lsin. β mulohaza kelib chiqadigan har qanday α mulohazaga β mulohaza uchun *yetarli shart* deyiladi.

β mulohazadan kelib chiqadigan har qanday α mulohazaga β mulohaza uchun *zarur shart* deyiladi.

Masalan, α: «x soni nolga teng» va β: «xy ko'paytma nolga teng»
mulahazalari bo'lsin. Bunda α mulohaza β mulohaza uchun yetarli shart bo'ladi. Haqiqatan ham, xy ko'paytma nolga teng bo'lishi uchun x nolga teng bo'lishi yetarli. x nolga teng bo'lishi uchun xy ko'paytma nolga teng bo'lishi zarur. Ammo, β mulohaza α mulohaza uchun yetarli shart bo'lmaydi, chunki xy ko'paytma nolga teng bo'lishidan x sonining, albatta, nolga teng bo'lishi kelib chiqmaydi.

«Agar α mulohaza rost bo'lsa, u holda β mulohaza rost bo'ladi» teoremani $\alpha \Rightarrow \beta$ ko'rinishda yozish va quyidagi ifodalardan biri bilan berish mumkin: «α mulohaza β mulohaza uchun yetarli shart bo'ladi»; «β mulohaza α mulohaza uchun zarur shart bo'ladi».

Agar α va β mulohazalarning har biridan ikkinchisi kelib chiqsa, ya'ni $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$ bo'lsa, u holda α va β mulohazalarning har biri ikkinchisi uchun zarur va yetarli shart bo'ladi va $\alpha \Leftrightarrow \beta$ deb yoziladi.

Bu yozuv boshqacha quyidagicha o'qilishi mumkin:

1) α o'rinli bo'lishi uchun β o'rinli bo'lishi zarur va yetarli;

2) α faqat va faqat β bajarilsa o'rinli bo'ladi;

3) α faqat va faqat β rost bo'lganida rost bo'ladi.

Matematik induksiya metodi

Matematik induksiya metodi muhim matematik isbotlash usullaridan biri hisoblanadi. Bu usul n natural songa bog'liq tasdiqlarni isbotlash uchun qo'llaniladi.

Uni umumiy holda ifodalymiz: n natural songa bog'liq biror tasdiqni (masalan, formulani) isbotlash quyidagi tartibda amalga oshiriladi:

1) tasdiqning to'g'riligi $n=1$ da tekshiriladi (agar $n=1$ da tasdiq ma'noga ega bo'lmasa, tekshirish tasdiq ma'noga ega bo'ladigan eng kichik n dan boshlanadi);

2) tasdiq biror n $(n>1)$ da to'g'ri deb faraz qilinadi va uning $n+1$ da to'g'ri bo'lishi isbotlanadi. Keyin bu tasdiqning istalgan n natural son bajarilishi haqida xulosa chiqariladi.

Matematik induksiya metodi bilan *Nuyton binomi formulasi* deb ataluvchi

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^n b^n \tag{1.1}$$

formulani isbotlaymiz, bu yerda n – natural son; $0 \le k \le n$.

(1.1) formulada qatnashayotgan

$$C_n^k = \frac{n!}{k!(n-k)!}$$

koeffitsiyentlarga *binominal koeffitsiyentlar* (yoki n ta elementdan k tadan guruhlashlar soni) deyiladi, bu yerda $n!$ (*en foktorial*) belgi orqali birinchi n ta natural son ko'paytmasi belgilanadi. Binominal koeffitsiyentlar va faktorial uchun quyidagi bog'lanishlar o'rinli bo'ladi:

$$C_n^{k+1} + C_n^k = C_{n+1}^{k+1};$$

$$(n+1)! = n!(n+1) \text{ (bunda } 0! = 1 \text{ deb olinadi).}$$

(1.1) formulani isbotlaymiz. Buning uchun: 1) formula to'g'ri bo'lishini $n=1$ da tekshiramiz:

$$(a+b)^1 = C_1^0 a^1 + C_1^1 a^0 b = \frac{1!}{0! \cdot 1!} a + \frac{1!}{1! \cdot 0!} b = a + b;$$

2) (1.1) formula biror n da to'g'ri bo'ladi deb faraz qilamiz va

$(n+1)$ da shu kabi formula o'rinli bo'lishini ko'rsatamiz, ya'ni

$$(a+b)^{n+1} = C_{n+1}^0 a^{n+1} + C_{n+1}^1 a^n b + \ldots + C_{n+1}^{k+1} a^{n-k} b^{k+1} + \ldots + C_{n+1}^n ab^n + C_{n+1}^{n+1} b^{n+1} \quad (1.2)$$

formulani isbotlaymiz.

Haqiqatan ham,

$$(a+b)^{n+1} = (C_n^0 a^n + C_n^1 a^{n-1} b + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^n b^n)(a+b) =$$

$$= C_n^0 a^{n+1} + C_n^1 a^n b + \ldots + C_n^{k+1} a^{n-k} b^{k+1} + \ldots + C_n^n ab^n + C_n^0 a^n b + \ldots +$$

$$+ C_n^k a^{n-k} b^{k+1} + \ldots + C_n^{n-1} ab^n + C_n^n b^{n+1} = C_n^0 a^{n+1} + (C_n^0 + C_n^1) a^n b + \ldots +$$

$$+ (C_n^k + C_n^{k+1}) a^{n-k} b^{k+1} + \ldots + (C_n^{n-1} + C_n^n) ab^n + C_n^n b^{n+1}.$$

Binominal koeffitsiyentlar uchun

$$C_n^0 = 1 = C_{n+1}^0, \quad C_n^0 + C_n^1 = C_{n+1}^1, \quad C_n^k + C_n^{k+1} = C_{n+1}^{k+1},$$

$$C_n^{n-1} + C_n^n = C_{n+1}^n, \quad C_n^n = 1 = C_{n+1}^{n+1}$$

bo'lishi inobatga olinsa, oxirgi tenglikdan (1.2) tenglik kelib chiqadi.

Demak, matematik induksiya metodining 1) va 2) bandlari bajarilgani uchun
Nyuton binomi formulasi istalgan n natural soni uchun to'g'ri bo'ladi.

(1.1) formula qisqacha

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$$

ko'rinishda yoziladi.

Xususan, (1.1) formuladan $n=2$ va $n=3$ da tanish qisqa ko'paytirish formulalari kelib chiqadi:

$$(a+b)^2 = C_2^0 a^2 + C_2^1 ab + C_2^2 b^2 = a^2 + 2ab + b^2;$$

$$(a+b)^3 = C_3^0 a^3 + C_3^1 a^2 b + C_3^2 ab^2 + C_3^3 b^3 = a^3 + 3a^2 b + 3ab^2 + b^3.$$

4.1.4. Mashqlar

1. A va B to'plamlar berilgan. $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A$ to'plamlarni toping.

1) $A = \{1,2,3,4\}$, $B = \{4,5,6\}$; 2) $A = \{1,3,4,8\}$, $B = \{1,2,4,5,7,8,9\}$;

3) $A = \{x \in R : x^2 + x - 20 = 0\}$, $B = \{x \in R : x^2 - x + 12 = 0\}$.

2. A – musbat juft sonlar to'plami va B – musbat toq sonlar to'plami bo'lsa, $A \cap B$ va $A \cup B$ to'plamlarni toping.

3. A – barcha 2 ga bo'linadigan sonlar to'plami va B – barcha 5 ga bo'linadigan sonlar to'plami bo'lsa, $A \cap B$ to'plamni toping.

4. $\lg 5$ irratsional son ekanini ko'rsating.

5. $\sqrt{2}-\sqrt{5}<\sqrt{3}-2$ ekanini ko'rsating.

6. Berilgan to'plam elementlarini toping.

1) $A=\{x \in N : x^2-3x-4 \le 0\}$; 2) $A=\left\{x \in N : \log_{\frac{1}{2}} \frac{1}{x} < 2\right\}$.

7. Berilgan tenglamalarni yeching.

1) $|3x-4|=\frac{1}{2}$; 2) $|-x^2+2x-3|=1$;

3) $\sqrt{x^2}+x^3=0$; 4) $\sqrt{(x-2)^2}=-x+2$.

8. Berilgan tengsizliklarni yeching.

1) $|x-2|\ge 1$; 2) $|x^2-7x+12|>x^2-7x+12$;

3) $x^2+2\sqrt{(x+3)^2}-10 \le 0$; 4) $\sqrt{(x+1)^2} \le -x-1$.

9. Berilgan X to'plam uchun $\sup X$ va $\inf X$ larni toping.

1) $X=\{x \in Z : -5 \le x < 0\}$; 2) $X=\{x \in R : x<0\}$.

10. Tengliklarni matematik induksiya metodi bilan isbotlang:

1) $1+3+6+...+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$, $\forall n \in N$; 2) $1+3+5+...+(2n-1)=n^2$.

4.2. SONLI KETMA-KETLIKLAR

4.2.1. Sonli ketma-ketliklar

1-ta'rif. $1,2,3,...,n,...$ natural sonlar qatorining har bir n natural soniga mos qo'yilgan $x_1, x_2, x_3, ..., x_n ...$ haqiqiy sonlar to'plamiga *sonli ketma-ketlik* (yoki *ketma-ketlik*) deyiladi va $\{x_n\}$ bilan belgilanadi.

Bunda $x_1, x_2, x_3, ..., x_n ...$ sonlar $\{x_n\}$ ketma-ketlikning hadlari, x_n bu ketma-ketlikning umumiy hadi, n uning nomeri deb ataladi.

Agar ketma-ketlikning har bir hadini topish mumkin bo'lsa, ketma-ketlik berilgan deyiladi. Ketma-ketlik analitik yoki rekurrent usullarda berilishi mumkin.

*Analitik usul*da ketma-ketlikning umumiy hadi formula ko'rinishida beriladi. Bunda n ga 1,2,3,4,... qiymatlar beriladi va ketma-ketlikning mos hadlari topiladi.

Masalan, $x_n = (-1)^n \cdot n$ formula $\{x_n\} = \{-1, 2, -3, ..., (-1)^n \cdot n, ...\}$ ketma-ketlikni beradi.

*Rekurrent usul*da ketma-ketlikning birinchi (yoki bir nechta birinchi) hadi beriladi va keyingi hadni (yoki bir nechta keyingi hadlarni) birinchi (yoki bir nechta birinchi) had (hadlar) asosida topish formulasi beriladi. Masalan, $x_1 = a_1$, $x_{n+1} = x_n + d$ rekurrent formula arifmetik progressiyani, $x_1 = b_1$, $x_{n+1} = x_n q$ rekurrent formula geometrik progressiyani beradi.

Agar $\forall n \in N$ uchun $x_n = c (c \in R)$ bo'lsa, $\{x_n\}$ ketma-ketlikka *o'zgarmas ketma-ketlik* deyiladi.

2-ta'rif. Agar shunday M soni (m soni) topilsa va $\forall n \in N$ uchun $x_n \le M$ ($x_n \ge m$) tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlikka *yuqoridan chegaralangan* (*quyidan chegaralangan*) deyiladi.

3-ta'rif. Agar $\{x_n\}$ ketma-ketlik ham quyidan ham yuqoridan chegaralangan bo'lsa, ya'ni shunday m, M sonlari topilsa va $\forall n \in N$ uchun $m \le x_n \le M$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlikka *chegaralangan* deyiladi.

$A = \max\{|m|, |M|\}$ bo'lsin. U holda ketma-ketlikning chegaralanganlik shartini $|x_n| \le A$ ko'rinishda yozish mumkin.

4-ta'rif. Agar $\forall A > 0$ son uchun $\{x_n\}$ ketma-ketlikning $|x_n| > A$ ($x_n > A$ yoki $x_n < -A$) tengsizlikni qanoatlantiruvchi hadi topilsa, $\{x_n\}$ ketma-ketlikka *chegaralanmagan* deyiladi.

Ta'riflardan ko'rinadiki, ketma-ketlikning barcha elementlari u: yuqoridan chegaralangan bo'lsa, $(-\infty; M]$ oraliqqa, quyidan chegaralangan bo'lsa $[m; +\infty)$ oraliqqa, ham quyidan ham yuqoridan chegaralangan bo'lsa $[m; M]$ oraliqqa tegishli bo'ladi. Chegaralanmagan ketma-ketlik yuqoridan yoki quyidan chegaralangan bo'lishi mumkin.

Chegaralangan va chegaralanmagan ketma-ketliklarga misollar keltiramiz.

1. $\{x_n\} = \{n^2 + 1\} = \{2, 5, 10, ..., n^2 + 1, ...\}$ – quyidan chegaralangan ($m = 2$), ammo yuqoridan chegaralanmagan.

2. $\{y_n\} = \{-n^2\} = \{-1, -4, -9, ..., -n^2, ...\}$ – yuqoridan chegaralangan ($M = -1$), ammo quyidan chegaralanmagan.

3. $\{z_n\}=\left\{\frac{n-1}{n}\right\}=\left\{0,\frac{1}{2},\frac{2}{3},...,\frac{n-1}{n},...\right\}$ – chegaralangan $(m=0, M=1)$.

4. $\{u_n\}=\{(-1)^n n\}=\{-1,2,-3,4,...,(-1)^n n,...\}$ – chegaralanmagan.

5-ta'rif. Agar $\forall n\in N$ uchun: $x_n<x_{n+1}$ bo'lsa, $\{x_n\}$ ketma-ketlikka *qat'iy o'suvchi* deyiladi; $x_n>x_{n+1}$ bo'lsa, $\{x_n\}$ ketma-ketlikka *qat'iy kamayuvchi* deyiladi; $x_n\le x_{n+1}$ bo'lsa, $\{x_n\}$ ketma-ketlikka *kamaymaydigan* deyiladi; $x_n\ge x_{n+1}$ bo'lsa, $\{x_n\}$ ketma-ketlikka *o'smaydigan* deyiladi.

Barcha bunday ketma-ketliklar umumiy bitta *monoton ketma-ketlik* nomi bilan birlashtiriladi. Bunda o'suvchi va kamayuvchi ketma-ketliklarga *qat'iy monoton* ketma-ketliklar deyiladi.

Monoton ketma-ketliklarga misollar keltiramiz.

1. $\{x_n\}=\left\{\frac{n}{n+1}\right\}=\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},...,\frac{n}{n+1},...\right\}$ – o'suvchi va chegaralangan ketma-ketlik.

2. $\{y_n\}=\{n,n\}=\{1,1,2,2,...,n,n,...\}$ – kamaymaydigan va chegaralanmagan ketma-ketlik.

3. $\{z_n\}=\left\{\frac{1}{n^2}\right\}=\left\{1,\frac{1}{4},\frac{1}{9},...,\frac{1}{n^2},...\right\}$ – kamayuvchi va chegaralangan ketma-ketlik.

4. $\{u_n\}=\left\{\frac{1}{n},\frac{1}{n}\right\}=\left\{1,1,\frac{1}{2},\frac{1}{2},...,\frac{1}{n},\frac{1}{n},...\right\}$ – o'smaydigan va chegaralangan ketma-ketlik.

Ikkita $\{x_n\}$ va $\{y_n\}$ ketma-ketlikning *yig'indisi, ayirmasi, ko'paytmasi, bo'linmasi* (bunda $y_n\neq 0$) deb, har bir hadi bu ketma-ketliklar mos hadlarining yig'indisidan, ayirmasidan, ko'paytmasidan va bo'linmasidan iborat bo'lgan ketma-ketlikka aytiladi.

Ko'rsatilgan amallar simvolik tarzda quyidagicha yoziladi:

$$\{x_n\}+\{y_n\}=\{x_n+y_n\},\quad \{x_n\}-\{y_n\}=\{x_n-y_n\},$$

$$\{x_n\}\cdot\{y_n\}=\{x_n\cdot y_n\},\quad \frac{\{x_n\}}{\{y_n\}}=\left\{\frac{x_n}{y_n}\right\}, y_n\neq 0.$$

Xususan, $\{x_n\}$ ketma-ketlikning m songa ko'paytmasi $m\cdot\{x_n\}$ deb, har bir hadi $\{x_n\}$ ketma-ketlik mos hadining shu songa ko'paytmasidan iborat bo'lgan $\{m\cdot x_n\}$ ketma-ketlikka aytiladi.

4.2.2. Cheksiz katta va cheksiz kichik ketma-ketliklar

6-ta'rif. Agar $\forall A>0$ son uchun shunday N nomer topilsa va $\forall n>N$ uchun $|x_n|>A$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlikka *cheksiz katta* ketma-ketlik deyiladi.

Har qanday cheksiz katta ketma-ketlik chegaralanmagan bo'ladi. Ammo chegaralanmagan ketma-ketlik cheksiz katta bo'lmasligi mumkin. Masalan, $\left\{n\sin\frac{n\pi}{2}\right\}$ shunday ketma-ketliklardan biridir.

7-ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $N=N(\varepsilon)$ nomer topilsa va $\forall n>N$ uchun $|x_n|<\varepsilon$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlikka *cheksiz kichik* ketma-ketlik deyiladi.

1-misol. $\{\alpha_n\}=\left\{\frac{1}{n}\right\}$ ketma-ketlik cheksiz kichik ekanini ko'rsating.

Yechish. $\forall\varepsilon>0$ son olamiz. $|\alpha_n|=\left|\frac{1}{n}\right|<\varepsilon$ tengsizlikdan $n>\frac{1}{\varepsilon}$ tengsizlik kelib chiqadi. N ni $\frac{1}{\varepsilon}$ ning butun qismi, ya'ni $N=\left[\frac{1}{\varepsilon}\right]$ desak, u holda $\forall n>N$ uchun $|\alpha_n|<\varepsilon$ bo'ladi. Demak, ta'rifiga ko'ra, $\left\{\frac{1}{n}\right\}$ ketma-ketlik cheksiz kichik bo'ladi.

Keltirilgan misolda $N=\left[\frac{1}{\varepsilon}\right]$ nomer ε ga bog'liq bo'ladi, ya'ni ε ning turli qiymatida har xil qiymatlarni qabul qiladi. Masalan, $\varepsilon=0{,}1$ da $N=10$, $\varepsilon=0{,}01$ da $N=100$. Shu sababli cheksiz kichik ketma-ketlikning ta'rifida $N=N(\varepsilon)$ deb yozilgan.

1-teorema. Agar $\{x_n\}$ ketma-ketlik cheksiz katta va $x_n\neq 0,\ n\in N$ bo'lsa, u holda $\left\{\frac{1}{x_n}\right\}$ ketma-ketlik cheksiz kichik bo'ladi, va aksincha $\{\alpha_n\}-$ cheksiz kichik va $\alpha_n\neq 0,\ n\in N$ bo'lsa, $\left\{\frac{1}{\alpha_n}\right\}-$ cheksiz katta bo'ladi.

Isboti. $\{x_n\}$ cheksiz katta ketma-ketlik, $x_n \neq 0$ bo'lsin. $\forall \varepsilon > 0$ son olamiz va $A = \frac{1}{\varepsilon}$ deymiz. 6-ta'rifga ko'ra bu A soni uchun shunday N nomer topiladi va $\forall n > N$ uchun $|x_n| > A$ tengsizlik bajariladi. Bundan $\forall n > N$ uchun $\left|\frac{1}{x_n}\right| = \frac{1}{|x_n|} < \frac{1}{A} = \varepsilon$ bo'ladi. Bu esa $\left\{\frac{1}{x_n}\right\}$ ketma-ketlikning cheksiz kichik bo'lishini bildiradi. Teoremaning ikkinchi qismi ham shu kabi isbotlanadi.

Cheksiz kichik ketma-ketliklar quyidagi xossalarga ega.

1^o. Ikkita (chekli sondagi) cheksiz kichik ketma-ketlikning algebraik yig'indisi cheksiz kichik ketma-ketlik bo'ladi.

2^o. Ikkita (chekli sondagi) cheksiz kichik ketma-ketlikning ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

3^o. Cheksiz kichik ketma-ketlikning chegaralangan ketma-ketlikka ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

4^o. Cheksiz kichik ketma-ketlikning chekli songa ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

Xossalardan birining, masalan, 3-xossaning isbotini keltirish bilan chegaralanamiz.

$\{x_n\}$ chegaralangan ketma-ketlik, $\{\alpha_n\}$ cheksiz kichik ketma-ketlik bo'lsin. $\{x_n \cdot \alpha_n\}$ ketma-ketlik cheksiz kichik bo'lishini isbotlash kerak.

$\{x_n\}$ ketma-ketlik chegaralangan. Shu sababli biror $A > 0$ son va $\forall m$ uchun $|x_n| \leq A$ bo'ladi.

$\forall \varepsilon > 0$ son olamiz. $\{\alpha_n\}$ cheksiz kichik bo'lgani sababli $\frac{\varepsilon}{A} > 0$ son uchun shunday N nomer topiladi va $\forall n > N$ uchun $|\alpha_n| < \frac{\varepsilon}{A}$ bo'ladi.

U holda $\forall n > N$ da $|x_n \cdot \alpha_n| = |x_n| \cdot |\alpha_n| < A \cdot \frac{\varepsilon}{A} = \varepsilon$ bo'ladi.

Demak, $\{x_n \cdot \alpha_n\}-$ cheksiz kichik ketma-ketlik.

4.2.3. Ketma-ketlikning limiti

8-ta'rif. Agar $\forall\varepsilon > 0$ son uchun shunday $N = N(\varepsilon)$ nomer topilsa va $\forall n > N$ uchun $|x_n - a| < \varepsilon$ tengsizlik bajarilsa, a soniga $\{x_n\}$ *ketma-ketlikning limiti* deyiladi va bu $\lim\limits_{n\to\infty} x_n = a$ kabi yoziladi.

2-misol. Limit ta'rifidan foydalanib, $\lim\limits_{n\to\infty}\dfrac{3n-2}{3n+1} = 1$ bo'lishini ko'rsating.

Yechish. $\forall\varepsilon > 0$ olamiz. Misolning shartidan topamiz:

$$|x_n - 1| = \left|\frac{3n-2}{3n+1} - 1\right| = \left|\frac{-3}{3n+1}\right| = \frac{3}{3n+1}.$$

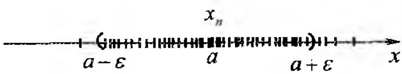
$|x_n - a| < \varepsilon$ tengsizlikni qanoatlantiruvchi n ning qiymatlarini topish uchun $\dfrac{3}{3n+1} < \varepsilon$ tengsizlikni yechamiz: $n > \dfrac{3-\varepsilon}{3\varepsilon}$.

N nomer sifatida $\dfrac{3-\varepsilon}{3\varepsilon}$ sonining butun qismini, ya'ni $N(\varepsilon) = \left[\dfrac{3-\varepsilon}{3\varepsilon}\right]$ sonini olish mumkin. Bunda $\forall\varepsilon > 0$ son olinganda ham $\forall n > N$ uchun $|x_n - 1| < \varepsilon$ bo'ladi.

Shu sababli ketma-ketlik limitining ta'rifiga asosan,

$$\lim_{n\to\infty}\frac{2n-1}{2n+1} = 1.$$

Ma'lumki, $|x_n - a| < \varepsilon$ tengsizlik x_n had a nuqtaning ε atrofiga tushushini bildiradi. Shu sababli ketma-ketlikning limiti ta'rifini quyidagicha talqin qilish mumkin: agar $\lim\limits_{n\to\infty} x_n = a$ bo'lsa, u holda a nuqtaning istalgan ε atrofi uchun shunday N nomer topiladi va $n > N$ nomerli barcha x_n nuqtalar a nuqtaning ε atrofiga, ya'ni $(a-\varepsilon; a+\varepsilon)$ intervalda yotadi va bu intervaldan tashqarida berilgan ketma-ketlikning chekli sondagi nuqtalari joylashadi (3-shakl). Bu jumla ketma-ketlik *limitining geometrik ma'nosini* anglatadi.



3-shakl.

4.2.4. Yaqinlashuvchi ketma-ketliklar

8-ta'rif. Chekli limitga ega bolgan ketma-ketlikka *yaqinlashuvchi* ketma-ketlik deyiladi.

Yaqinlashuvchi bo‘lmagan ketma-ketlikka *uzoqlashuvchi* ketma-ketlik deyiladi.

$\{x_n\}$ ketma-ketlik yaqinlashuvchi va a limitga ega bo‘lsin. U holda $\{x_n - a\} = \{\alpha_n\}$ cheksiz kichik ketma-ketlik bo‘ladi, chunki $\forall\varepsilon > 0$ son uchun shunday $N(\varepsilon)$ nomer topiladi va $\forall n > N$ uchun $|\alpha_n| = |x_n - a| < \varepsilon$ bo‘ladi. Shu sababli yaqinlashuvchi va a limitga ega bo‘lgan ixtiyoriy $\{x_n\}$ ketma-ketlikning umumiy hadini $x_n = a + \alpha_n$ ko‘rinishda yozish mumkin bo‘ladi va aksincha, $\{x_n\}$ ketma-ketlikning istalgan hadini $x_n = a + \alpha_n$ ko‘rinishida ifodalash mumkin bo‘lsa, u holda $\lim\limits_{n\to\infty} x_n = a$ bo‘ladi, bu yerda $\{\alpha_n\}$ – cheksiz kichik ketma-ketlik.

Shunday qilib, cheksiz kichik ketma-ketlikning limiti nolga teng bo‘ladi. Cheksiz katta ketma-ketlik limitga ega bo‘lmaydi. Uning limiti ∞ deb belgilanadi.

Yaqinlashuvchi ketma-ketlikdar quyidagi xossalarga ega.

1^o. Yaqinlashuvchi ketma-ketlik yagona limitga ega bo‘ladi.

2^o. Yaqinlashuvchi ketma-ketlik chegaralangan bo‘ladi.

3^o. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lsa, u holda $\{x_n \pm y_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n$ bo‘ladi.

4^o. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lsa, u holda $\{x_n \cdot y_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty} x_n \cdot y_n = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$ bo‘ladi.

5^o. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi va $\lim\limits_{n\to\infty} y_n \neq 0$ bo‘lsa, u holda $\left\{\dfrac{x_n}{y_n}\right\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ bo‘ladi.

6^o. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, u holda $\lim\limits_{n\to\infty} c \cdot x_n = c \cdot \lim\limits_{n\to\infty} x_n$ $(c \in R)$ bo‘ladi.

7^o. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi va $\forall n \in N$ uchun $x_n \le y_n$ bo‘lsa, u holda $\lim\limits_{n\to\infty} x_n \le \lim\limits_{n\to\infty} y_n$ bo‘ladi.

8^o. Agar $\{x_n\}$ va $\{z_n\}$ ketma-ketliklar yaqinlashuvchi, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a$ va $\forall n \in N$ uchun $x_n \le y_n \le z_n$ bo‘lsa, u holda $\lim\limits_{n\to\infty} y_n = a$ bo‘ladi.

Xossalardan birining, masalan, 3-xossaning isbotini keltirish bilan chegaralanamiz.

$\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$ bo'lsin. U holda $x_n = a + \alpha_n$ va $y_n = b + \beta_n$ bo'ladi, bu yerda $\{\alpha_n\}$, $\{\beta_n\}$ cheksiz kichik ketma-ketliklar.
Bundan

$$x_n \pm y_n = (a \pm b) + (\alpha_n \pm \beta_n)$$

kelib chiqadi.

Cheksiz kichik ketma-ketlikning 1-xossasiga asosan $\{\alpha_n \pm \beta_n\}$ cheksiz kichik ketma-ketlik. Shu sababli $\{x_n \pm y_n\}$ ketma-ketlik $a \pm b$ limitga ega bo'ladi, ya'ni

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n .$$

3-misol. $\{x_n\} = \left\{\left(\dfrac{n+2}{n^2}\right)^n\right\}$ ketma-ketlikning yaqinlashuvchi ekanini ko'rsating.

Yechish. Birinchidan, $x_n = \dfrac{n+2}{n^2} \le \dfrac{n+2n}{n^2} = \dfrac{3n}{n^2} = \dfrac{3}{n}$.

Bunda $\forall n \ge 6$ uchun $x_n \le \dfrac{1}{2}$ bo'ladi. Ikkinchidan, $\forall n \in N$ uchun $x_n = \dfrac{n+2}{n^2} \ge \dfrac{1+2}{n^2} = \dfrac{3}{n^2} > 0$ bo'ladi. Agar $y_n = 0$, $z_n = \dfrac{1}{2^n}$ belgilash kiritsak, $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = 0$ va $\forall n \ge 6$ uchun $y_n \le x_n \le z_n$ bo'ladi. U holda 8-xossaga ko'ra,
$\lim\limits_{n\to\infty} x_n = 0$, ya'ni berilgan ketma-ketlik yaqinlashuvchi bo'ladi.

Har qanday ketma-ketlik ham limitga ega bo'lmaydi. Ketma-ketlik limiti mavjud bo'lishi haqidagi teorema bilan tanishamiz.

2-teorema (*Veershtrass teoremasi*). Har qanday chegaralangan monoton ketma-ketlik limitiga ega bo'ladi.

Isboti. Monoton kamaymaydigan ketma-ketlikni qaraymiz.
$x_1 \le x_2 \le x_3 \le \ldots \le x_n \le x_{n+1} \le \ldots$ va shunday M soni topilsin va $x_n \le M$ bo'lsin. Elementlari bu ketma-ketlikning hadlaridan tashkil topgan X to'plamni qaraymiz.

Shartga ko'ra, bu to'plam bo'shmas va yuqoridan chegaralangan. U holda
4.2 banddagi 1-teoremaga asosan, X to'plam aniq yuqori chegaraga ega bo'ladi. Bu chegarani a bilan belgilaymiz va a soni berilgan ketma-ketlikning limiti bo'lishini ko'rsatamiz.

a soni x_n ketma-ketlik elementlarining aniq yuqori chegarasi bo'lgani uchun uning xossasiga ko'ra, $\forall\varepsilon>0$ son uchun shunday N nomer topiladi va $n>N$ da $x_n>a-\varepsilon$ bo'ladi. Ikkinchidan, aniq yuqori chegaraning ta'rifiga asosan, barcha n lar uchun $x_n\le a<a+\varepsilon$ bo'ladi. Shunday qilib, $n>N$ uchun $a-\varepsilon<x_n<a+\varepsilon$, ya'ni $|x_n-a|<\varepsilon$ tengsizlik kelib chiqdi. Bu tengsizlik a soni $\{x_n\}$ ketma-ketlikning limiti bo'lishini bildiradi.

Monoton o'smaydigan ketma-ketlik uchun teorema shu kabi isbotlanadi.

Izoh. Monoton ketma-ketlikning chegaralanganligi uning yaqinlashuvchi bo'lishining zarur va yetarli shartini beradi.

Haqiqatan ham, agar monoton ketma-ketlik chegaralangan bo'lsa, u holda 2-teoremaga ko'ra, u yaqinlashadi; agar monoton ketma-ketlik yaqinlashuvchi bo'lsa, yaqinlashuvchi ketma-ketlikning 2-xossasiga asosan, u chegaralangan bo'ladi.

2- teoremadan ichma-ich qo'yilgan kesmalar haqidagi lemma kelib chiqadi.

$[a_1;b_1],[a_2;b_2],\ldots[a_n;b_n],\ldots$ kesmalar ketma-ketligi berilgan bo'lib, bunda $[a_1;b_1]\supset[a_2;b_2]\supset\ldots\supset[a_n;b_n]\supset\ldots$, ya'ni har bir kesma o'zidan oldingi kesmaning ichiga joylashgan va $\lim\limits_{n\to\infty}(b_n-a_n)=0$ bo'lsin. Bu kesmalarga ichma-ich qo'yilgan kesmalar deyiladi.

Lemma (*Kantor lemmasi*). Ixtiyoriy ichma-ich qo'yilgan kesmalar ketma-ketligi uchun bu ketma-ketlikning har biriga tegishli bo'lgan c nuqta mavjud va yagona bo'ladi.

Bu lemma haqiqiy sonlar to'plamining uzluksizligi yoki son o'qining to'laligi haqidagi muhim xossani ifodalaydi.

Yaqinlashuvchi ketma-ketlikning xossalari va limitlar haqidagi teoremalar nafaqat nazariy, balki amaliy jihatdan ham muhim ahamiyatga ega. Ulardan,
masalan, ketma-ketliklarning limitini hisoblashda keng foydalaniladi.

4-misol. $\lim\limits_{n\to\infty}\dfrac{3n+1}{8n-1}$ limitni toping.

Yechish. Kasrning surati va maxrajidagi ketma-ketliklar limitga ega emas, chunki ular chegaralanmagan. Shu sababli bo'linmaning limiti haqidagi 5-xossani to'g'ridan-to'g'ri qo'llab bo'lmaydi. Bu xossani qo'llash uchun avval ketma-ketlikning surat va maxrajini n ga bo'lamiz va keyin yaqinlashuvchi ketma-ketlikning xossalarini qo‘llaymiz:

$$\lim_{n\to\infty}\frac{3n+1}{8n-1}=\lim_{n\to\infty}\frac{3+\frac{1}{n}}{8-\frac{1}{n}}=\frac{\lim\limits_{n\to\infty}\left(3+\frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(8-\frac{1}{n}\right)}=\frac{\lim\limits_{n\to\infty}3+\lim\limits_{n\to\infty}\frac{1}{n}}{\lim\limits_{n\to\infty}8-\lim\limits_{n\to\infty}\frac{1}{n}}=\frac{3+0}{8-0}=\frac{3}{8}.$$

4.2.5. e soni

$\{x_n\}=\left\{\left(1+\frac{1}{n}\right)^n\right\}$ ketma-ketlik berilgan bo‘lsin. Bu ketma-ketlik uchun Nyuton binomi formulasini $a=1,\ b=\frac{1}{n}$ da qo‘llaymiz:

$$x_n=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\ldots+\frac{n(n-1)(n-2)\ldots(n-(n-1))}{n!}\cdot\frac{1}{n^n}$$

yoki

$$\left(1+\frac{1}{n}\right)^n=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right). \qquad (2.1)$$

(2.1) tenglikdan ko‘rinadiki, n ning o‘sishi bilan uning o‘ng tomonida musbat qo‘shiluvchilar soni ortib boradi. Bundan tashqari, n ning o‘sishi bilan $\frac{1}{n}$ kamayadi va $\left(1-\frac{1}{n}\right),\left(1-\frac{2}{n}\right),\ldots$ kattaliklar ortadi.

Shu sababli $\{x_n\}=\left\{\left(1+\frac{1}{n}\right)^n\right\}$ ketma-ketlik monoton o‘suvchi bo‘ladi.

(2.1) tenglikka ko‘ra,

$$\left(1+\frac{1}{n}\right)^n>2. \qquad (2.2)$$

(2.1) tenglikda qavs ichidagi har bir ifoda birdan kichik va shu bilan birga, $n>2$ da $\frac{1}{n!}<\frac{1}{2^{n-1}}$ bo‘ladi.

Bundan

$$x_n<1+1+\frac{1}{2!}+\ldots+\frac{1}{n!}<1+1+\frac{1}{2}+\ldots+\frac{1}{2^{n-1}}=1+\frac{1}{1-\frac{1}{2}}=3 \qquad (2.3)$$

kelib chiqadi. Demak, (2.2) va (2.3) tengsizliklarga ko‘ra, $2 < x_n < 3$, ya’ni $\{x_n\}$ – chgaralangan.

Shunday qilib, $\{x_n\}$ ketma-ketlik monoton o‘suvchi va chegaralangan. U holda Veershtrass teoremagasiga ko‘ra u yaqinlashadi, ya’ni chekli limitga ega bo‘ladi. Bu limitni e harfi bilan belgilaymiz.

Demak,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{2.4}$$

e soniga *Neper soni* deyiladi. e soni irratsional son. e soni matematikaning bir qancha masalalarida muhim rol o‘ynaydi. e soni, masalan, natural logarifmning asosi bo‘ladi: $x>0$ sonining natural logarifmi $\ln x$ bilan belgilanadi. Bu paragrafda e soniga faqat ta’rif berildi va u $2<e<3$ tengsizlikni qanoatlantirishi ko‘rsatildi. Keyinchlik e sonining qiymatini istalgan aniqlikda topish usullari ko‘rsatiladi.

4.2.6. Mashqlar

1. Ketma-ketlikning dastlabki to‘rtta hadi berilgan. Uning umumiy hadini toping:

1) $\frac{1}{2},\frac{1}{5},\frac{1}{8},\frac{1}{11},...;$ 2) $5,\frac{25}{2},\frac{125}{6},\frac{625}{24},...;$

3) $-1,1,-1,1,...;$ 4) $1,5,1,5,...$.

2. Chegaralangan ketma-ketliklarni ko‘rsating:

1) $x_n=\frac{n}{2+n^2};$ 2) $x_n=\cos n\pi+2tgn\pi;$

3) $x_n=\frac{1-n}{\sqrt{n}};$ 4) $x_n=\sqrt{n^2+1}-n;$

5) $x_n=(-1)^n\cdot n;$ 6) $x_n=\ln(n+1)-\ln n.$

3. Ketma-ketliklardan qaysilari monoton va qaysilari qat’iy monoton?

1) $x_n=\frac{n}{3n-2};$ 2) $x_1=1, x_n=\frac{2}{x_{n-1}+1};$

3) $x_n=\frac{3^n}{n};$ 4) $x_n=\frac{n}{5^n}.$

5) $x_n=\lfloor\sqrt{n}\rfloor$ 6) $x_n=\frac{3^n}{n!}.$

4. $1, \frac{1}{7}, \frac{1}{17}, \dots, \frac{1}{2n^2-1}$ ketma-ketlik cheksiz kichik ekanligini isbotlang.

5. $\frac{17}{14}, \frac{37}{29}, \frac{65}{50}, \dots, \frac{4n^2+1}{3n^2+2}$ ketma-ketlik $\frac{4}{3}$ ga teng limitga ega ekanligini ketma-ketlikning limiti ta'rifidan foydalanib isbotlang.

6. Ketma-ketliklarning limitini toping:

1) $x_n = \frac{5-n^2}{3+2n^2}$;

2) $x_n = \frac{3n^2+2}{4-n^3}$;

3) $x_n = \frac{3n+n^3}{2n^2+3n+7}$;

4) $x_n = \left(\frac{2n^2+3n-1}{n^2-2n+1}\right)^3$;

5) $x_n = \frac{(n+2)^2-(2-n)^2}{2n+7}$;

6) $x_n = \frac{(n+1)^3-(n-1)^3}{3n^2+2}$;

7) $x_n = \frac{3n^3}{1+3n^2} + \frac{1-5n^2}{5n+1}$;

8) $x_n = \frac{3}{n+2} - \frac{5n}{2n+1}$;

9) $x_n = \sqrt{n+2} - \sqrt{n-2}$;

10) $x_n = \sqrt{n^2+n} - \sqrt{n^2-n}$;

11) $x_n = \sqrt{n(n-5)} - n$;

12) $x_n = \sqrt[3]{n^3-4n^2} - n$;

13) $x_n = \frac{2n+1}{\sqrt[3]{n^2+n+5}}$;

14) $x_n = \frac{\sqrt[3]{n^4-1}}{\sqrt{n+1}}$;

15) $x_n = \frac{n!+(n+1)!}{(n+1)!-2n!}$;

16) $x_n = \frac{(2n+1)!+(2n+2)!}{(2n+3)!-(2n+2)!}$;

17) $x_n = \frac{2-5+4-7+\dots+2n-(2n+3)}{n+5}$;

18) $x_n = \frac{1+2+3+\dots+n}{n^2-2n+1}$;

19) $x_n = \frac{1}{1\cdot 7} + \frac{1}{7\cdot 13} + \dots + \frac{1}{(6n-5)(6n+1)}$;

20) $x_n = \frac{1}{2\cdot 4} + \frac{1}{4\cdot 6} + \dots + \frac{1}{2n(2n+2)}$;

21) $x_n = \frac{3^{\frac{1}{n}}-1}{3^{\frac{1}{n}}+1}$.

22) $x_n = \frac{6\cdot 6^n+5}{2\cdot 3^n+1} - 3^{n+1}$;

23) $x_n = \frac{3}{4} + \frac{5}{16} + \frac{9}{64} + \dots + \frac{1+2^n}{4^n}$;

24) $x_n = \frac{1+3+9+\dots+3^{n-1}}{2\cdot 3^{n+2}+5}$;

25) $x_n = \frac{1}{n}\cos n^2 - \frac{3n}{6n+1}$;

26) $x_n = \frac{1}{n}\sin n^3 + \frac{2n^2}{n^2-1}$;

27) $x_n = \left(1-\frac{1}{n}\right)^n$;

28) $x_n = \left(\frac{n-1}{1+n}\right)^{2n-5}$;

29) $x_n = \left(\frac{2n+1}{2n-1}\right)^{3n-4}$;

30) $x_n = \left(\frac{n^2-1}{n^2+1}\right)^{3n-n^2}$.

4.3. BIR O'ZGARUVCHINING FUNKSIYASI

4.3.1. Funksiya

Ikki to'plam elmentlari orasidagi bog'lanishni o'rnatishga asoslangan funksiya tushunchasi matematik analiz kursida o'rganilsada, nafaqat bu kursning, balki butun matematikaning asosiy tushunchalaridan biri hisoblanadi.

Funksiya tushunchasi

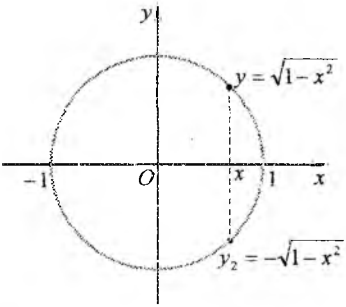
1-ta'rif. Agar X to'plamning har bir x soniga biror f qoidaga ko'ra, Y to'plamning bitta y soni mos qo'yilgan bo'lsa, X to'plamda *funksiya* berilgan deyiladi va $f: x \to y$ yoki $y = f(x)$ kabi belgilanadi.

Bunda f funksiya X to'plamni Y to'plamga akslantiradi deb aytiladi. X to'plam f *funksiyaning aniqlanish sohasi* deb ataladi va $D(f)$ bilan belgilanadi, $y \in Y$ to'plam f *funksiyaning qiymatlar sohasi* deb ataladi va $E(f)$ bilan belgilanadi.

Bunda x *funksiyaning argumenti* yoki *erkli o'zgaruvchi*, y *funksiya* yoki x ga *bog'liq o'zgaruvchi* deb ataladi.

$y = f(x)$ *funksiyaning* $x = x_0 (x_0 \in X)$ *nuqtadagi xususiy qiymati* $f(x_0) = y_0$ yoki $y|_{x=x_0} = y_0$ kabi belgilanadi. Masalan, $f(x) = 3x^2 - 2$ bo'lsa, $f(0) = -2$, $f(1) = 1$.

$y = f(x)$ *funksiyaning grafigi* deb Oxy koordinatalar tekisligining abssissasi x argumentning qiymatlaridan va ordinatasi y funksiyaning mos qiymatlaridan tashkil topgan barcha $(x; f(x))$, $x \in D(f)$ nuqtalari to'plamiga aytiladi. Funksiyaning grafigi tutash chiziqdan (egri chiziqdan yoki to'g'ri chiziqdan) iborat bo'lishi yoki ayrim nuqtalardan tashkil topishi mumkin, masalan, $y = n!$, $n \in N$ funksiyaning grafigi $1, 2, 6, \ldots$ nuqtalardan iborat bo'ladi



4-shakl.

Har qanday chiziq ham biror funksiyaning grafigi bo'lavermaydi, masalan, $x^2 + y^2 = 1$ aylana funksiyaning grafigi bo'lmaydi, chunki har bir $x \in (-1;1)$ uchun y ning bitta emas balki ikkita qiymati mos keladi: $y_1 = \sqrt{1-x^2}$ va $y_2 = -\sqrt{1-x^2}$ (4-shakl). Bunda funksiya ta'rifining bir qiymatlilik sharti buziladi.

Ammo aylananing quyi yarim tekislikdagi bo'lagi $y = -\sqrt{1-x^2}$ funk-siyaning, yuqori yarim tekislikdagi bo'lagi esa $y = \sqrt{1-x^2}$ funksiyaning grafigi bo'ladi.

Funksiyaning berilish usullari

Funksiyaning berilishi, ya'ni x ning har bir qiymatiga y ning yagona qiymatini topish turli usulda berilgan bo'lishi mumkin. Amalda funksiya berilishining analitik, jadval va grafik usullari ko'p qo'llaniladi.

*Analitik usul*da x va y o'zgaruvchilar orasidagi bog'lanish bir yoki bir nechta formula orqali beriladi. Masalan,

$$y = x^3, \quad y = \sin 2x, \quad y = \begin{cases} x-5, & agar\ x<3, \\ x^2+2, & agar\ x \geq 3. \end{cases}$$

Funksiya $y = f(x)$ ko'rinishda yozilganda, ayrim hollarda funksiyaning aniqlanish sohasi $D(f)$ ko'rsatilmaydi. Bunda, funksiyaning aniqlanish sohasi x ning $f(x)$ funksiya ma'noga ega bo'ladigan barcha qiymatlari
to'plamidan iborat deb qaraladi.

*Jadval usuli*da x va y o'zgaruvchilar orasidagi bog'lanish jadval orqali beriladi. Masalan, logorifmik fuksiyalarning, trigonometrik fuksiyalarning jadvallari.

Amalda jadval orqali funksiyani kuzatish natijalari yoki uning tajribada olingan qiymatlari beriladi.

*Grafik usuli*da fuksiya-ning grafigi beriladi. Bunda funksiyaning argumentning u yoki bu qiymatlariga mos qiymatlari bevosita shu grafikdan topiladi.

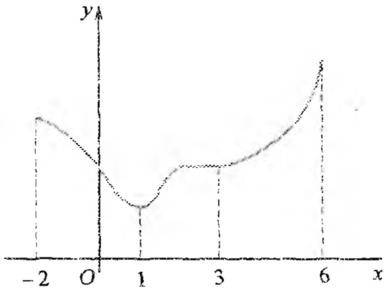
x va y o'zgaruvchilar orasidagi bog'lanish yuqorida keltirilgan uch usul bilan chegaralanib qolmasdan, boshqa shakllarda berilishi ham mumkin. Masalan, EHMning hisoblash programmasi shaklida, tavsiflardangina iborat holda.

Funksiyaning monotonligi

$y = f(x)$ funksiya X to'plamda aniqlangan va $I = (a;b) \subset X$ bo'lsin.

2-ta'rif. Agar $\forall x_1, x_2 \in I$ uchun $x_1 < x_2$ bo'lganda $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$ tengsizlik bajarilsa, $y = f(x)$ funksiyaga I intervalda *o'suvchi* (*kamayuvchi*) deyiladi.

3-ta'rif. Agar $\forall x_1, x_2 \in I$ uchun $x_1 < x_2$ bo'lganda $f(x_1) \le f(x_2)$ $(f(x_1) \ge f(x_2))$ tengsizlik bajarilsa, $y = f(x)$ funksiyaga I intervalda *kamaymaydigan* (*o'smaydigan*) deyiladi.



5-shakl.

Masalan, grafigi 5-shaklda berilgan funksiya (–2;1) intervalda kamayuvchi, (1;6) intervalda kamaymaydigan, (3;6) intervalda o'suvchi.

Barcha bunday funksiyalar *I intervalda monoton funksiya* nomi bilan umumlashtiriladi. Bunda o'suvchi va kamayuvchi funksiyalarga *I intervalda qat'iy monoton* funksiyalar deyiladi. Funksiya monoton bo'lgan intervallar *monotonlik intervallari* deb ataladi.

Funksiyaning juft va toqligi

$y = f(x)$ funksiya X to'plamda aniqlangan bo'lsin.

Agar $\forall x \in X$ uchun $-x \in X$ va $f(-x) = f(x)$ bo'lsa, $f(x)$ funksiyaga *juft funksiya* deyiladi. Masalan, $y = x^2$, $y = \cos x$, $y = \sqrt{1 + x^2}$ –juft funksiyalar.

Juft funksiyaning grafigi ordinata o'qiga nisbatan simmetrik bo'ladi.

Agar $\forall x \in X$ uchun $-x \in X$ va $f(-x) = -f(x)$ bo'lsa $f(x)$ funksiyaga *toq funksiya* deyiladi. Masalan, $y = x^3$, $y = \sin x$ –toq funksiyalar.

Toq funksiyaning grafigi koordinata boshiga nisbatan simmetrik bo'ladi.

Juft ham, toq ham bo'lmagan funksiya umumiy ko'rinishdagi funksiya deb ataladi. Masalan, $y = x - 2$, $y = \sqrt{x}$ – umumiy ko'rinishdagi funksiyalar.

$f(x)$ va $g(x)$ funksiyalar juft funksiyalar bo'lsa,

$$f(x) + g(x),\ f(x) - g(x),\ f(x) \cdot g(x),\ \frac{f(x)}{g(x)}, (g(x) \neq 0)$$

funksiyalar ham juft bo'ladi.

$f(x)$ va $g(x)$ funksiyalar toq funksiyalar bo'lsa,

$$f(x)+g(x),\ f(x)-g(x)$$

funksiyalar toq bo'ladi,

$$f(x)\cdot g(x),\ \frac{f(x)}{g(x)},(g(x)\neq 0)$$

funksiyalar esa juft bo'ladi.

1-misol. $f(x)=\ln(2x+\sqrt{1+4x^2})$ funksiyalarning toq ekanini ko'rsating.

Yechish. Toq funksiya uchun

$$f(-x)=-f(x) \text{ yoki } f(x)+f(-x)=0$$

bo'ladi.

Tekshirib ko'ramiz:

$$f(x)+f(-x)=\ln(2x+\sqrt{1+4x^2})+\ln(-2x+\sqrt{1+4x^2})=$$

$$=\ln(1+4x^2-4x^2)=\ln 1=0.$$

Bu munosabatdan $x\in D(f)$ bo'lsa, $-x\in D(f)$ bo'lishligi kelib chiqadi.

Demak, funksiya toq.

Funksiyaning chegaralanganligi

$y=f(x)$ funksiya X to'plamda aniqlangan bo'lsin.

4-ta'rif. Agar shunday o'zgarmas M soni topilsa va $\forall x\in X$ uchun $f(x)\leq M$ tengsizlik bajarilsa, $f(x)$ fuksiya X to'plamda *yuqoridan chegaralangan* deyiladi.

5-ta'rif. Agar shunday o'zgarmas m soni topilsa va $\forall x\in X$ uchun $f(x)\geq m$ tengsizlik bajarilsa $f(x)$ funksiya X to'plamda *quyidan chegaralangan* deyiladi.

6-ta'rif. Agar $f(x)$ funksiya ham quyidan, ham yuqoridan chegaralangan bo'lsa, y'ani shunday o'zgarmas m va M sonlari topilsa va $\forall x\in X$ uchun $m\leq f(x)\leq M$ tengsizlik bajarilsa, $f(x)$ funksiya X to'plamda *chegaralangan* deyiladi.

Masalan, $y=1-x^4$ funksiya yuqoridan $M=1$ soni bilan chegaralangan, $y=2+x^2$ funksiya quyidan $m=2$ soni bilan chegaralangan, $y=\sin x$ funksiya quyidan $m=-1$ soni bilan va yuqoridan $M=1$ soni bilan chegaralangan.

Funksiyaning davriyligi

$y=f(x)$ funksiya X to'plamda aniqlangan bo'lsin.

7-ta'rif. Agar shunday o'zgarmas $T\,(T\neq 0)$ son topilsa va $\forall x\in X$ uchun $x+T\in X$, $x-T\in X$, $f(x\pm T)=f(x)$ bo'lsa, $f(x)$ funksiyaga *davriy funksiya* deyiladi. Bunda T larning eng kichik musbat qiymati T_0 ga $f(x)$ *funksiyaning davri* deyiladi.

Masalan, $y=\sin x$ funksiyaning davri 2π, tgx funksiyaning davri π.

2-misol. $f(x)=4\sin 3x+3\cos 3x$ funksiyaning eng katta qiymatini va davrini toping.

Yechish. $a\cos x+b\sin x=\sqrt{a^2+b^2}\cos(x-\varphi)\left(\varphi=\arg tg\dfrac{b}{a}\right)$ formulaga ko'ra,

$$f(x)=\sqrt{3^2+4^2}\cos(3x-\varphi)=5\cos(3x-\varphi),\ \varphi=\arg tg\frac{4}{3}.$$

Bu funksiyaning eng katta qiymati $f\left(\dfrac{2k\pi+\varphi}{3}\right)=5$.

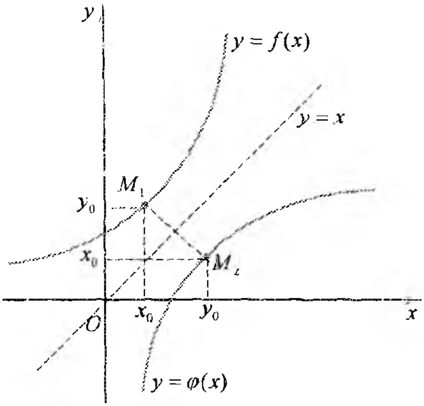
$A\sos(ax\pm\varphi)$ funksiyaning davri $T_0=\dfrac{2\pi}{a}$ bo'ladi. Bundan $T_0=\dfrac{2\pi}{3}$.

4.3.2. Teskari funksiya

Aniqlanish sohasi X va qiymatlar sohasi Y bo'lgan $y=f(x)$ funksiya berilgan bo'lsin. Agar bunda har bir $y\in Y$ qiymat yagona $x\in X$ qiymatga mos qo'yilgan bo'lsa, u holda aniqlanish sohasi Y va qiymatlar sohasi X bo'lgan $x=\varphi(y)$ funksiya aniqlangan bo'ladi. Bu funksiya $y=f(x)$ ga *teskari funksiya* deb ataladi va $x=\varphi(y)=f^{-1}(y)$ kabi belgilanadi.

$y=f(x)$ va $x=\varphi(y)$ funksiyalar *o'zaro teskari funksiyalar* deyiladi. Bunda $y=f(x)$ funksiyaga teskari $x=\varphi(y)$ funksiyani topish uchun $f(x)=y$ tenglamani x ga nisbatan yechish (agar mumkin bo'lsa) yetarli. Masalan, $y=a^x$ funksiyaga teskari funksiya $x=\log_a y$ funksiya bo'ladi. $y=x^2$ funksiyaga $x\in[0;1]$ da $x=\sqrt{y}$ teskari funksiya mavjud, $x\in[-1;1]$ da esa mavjud emas, chunki bunda y ning har bir qiymatiga x ning ikkita qiymati, masalan, $y=1$ ga $x_1=-1$, $x_2=1$ mos keladi.

Teskari funksiya ta'rifiga ko'ra, $y=f(x)$ funksiya X va Y to'plamlar o'rtasida bir qiymatli moslik o'rnatsagina $y=f(x)$ funksiya teskari funksiyaga ega bo'ladi. Bunda *har qanday qat'iy monoton funksiya teskari funksiyaga ega bo'ladi* deyish mumkin bo'ladi. Bunda agar funksiya o'ssa (kamaysa), u holda unga teskari funksiya ham o'sadi (kamayadi).



6-shakl.

$y=f(x)$ va unga teskari $x=\varphi(y)$ funksiyalar bitta egri chiziq bilan ifodalanadi, ya'ni ularning grafigi ustma-ust tushadi. Odatdagidek, argument (erkli o'zgaruvchi) x bilan va funksiya (bog'liq o'zgaruvchi) y bilan belgilansa, $y=f(x)$ funksiyaga teskari funksiya $y=\varphi(x)$ deb yoziladi. Bu $y=f(x)$ egri chiziqning $M_1(x_0;y_0)$ nuqtasi $y=\varphi(x)$ egri chiziqning $M_2(y_0;x_0)$ nuqtasi bo'lishini bildiradi. Bu nuqtalar $y=x$ to'g'ri chiziqqa nisbatan simmetrik bo'ladi (6-shakl). Shu sababli *o'zaro teskari* $y=f(x)$ va $y=\varphi(x)$ *funksiyalarning grafiklari I va III choraklar koordinata burchaklarining bissektrisalariga nisbatan simmetrik* bo'ladi.

4.3.3. Murakkab funksiya

X to'plamda qiymatlar sohasi Z bo'lgan $z=\varphi(x)$ funksiya aniqlangan bo'lsin. Agar Z to'plamda $y=f(z)$ funksiya aniqlangan bo'lsa, u holda X to'plamda $y=f(\varphi(x))$ *murakkab funksiya* (yoki $z=\varphi(x)$ va $y=f(z)$ funksiyalarning superpozitsiyasi) aniqlangan deyiladi.

$z=\varphi(x)$ o'zgaruvchi *murakkab funksiyaning oraliq argumenti* deb ataladi. Murakkab funksiyaning oraliq argumentlari bir nechta bo'lishi ham mumkin.

Masalan, $y=\cos 5x$ murakkab funksiya, chunki u $y=f(z)=\cos z$ va $z=\varphi(x)=5x$ funksiyalarning superpozitsiyasidan iborat.

4.3.4. Elementar funksiyalar sinfi

Quyida keltirilgan funksiyalarga *asosiy elementar funksiyalar* deyiladi.

1. ***O'zgarmas funksiya*** $y=C\,(C\in R)$.

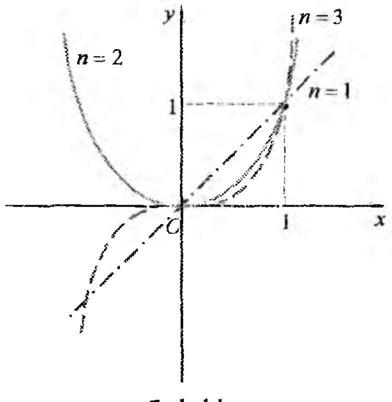
O'zgarmas funksiya: $D(f)=(-\infty;+\infty)$, $E(f)=\{C\}$ chegaralangan, juft, davri ixtiyoriy T.

O'zgarmas funksiyaning grafigi abssissalar o'qiga parallel to'g'ri chiziqdan iborat bo'ladi.

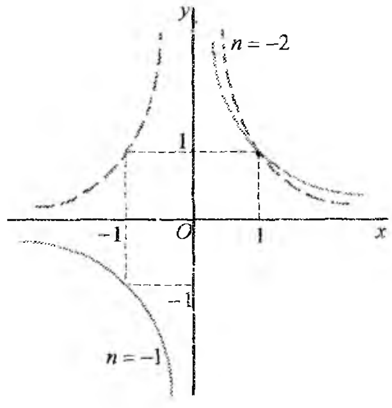
2. ***Darajali funksiya*** $y=x^{\alpha}$, $\alpha\in R,\alpha\neq 0$.

Hamma darajali funksiyaning grafiklari (1;1) nuqtadan o'tadi.

1) $\alpha=n$, $n-$butun musbat son. Bunda funksiyaning grafigi koordinatalar boshida abssissalar o'qiga urunadi ($n\geq 2$ da); n juft son bo'lganda ordinatalar o'qiga nisbatan simmetrik, n toq son bo'lganda esa koordinatalar boshiga nisbatan simmetrik bo'ladi (7-shakl). $n=1$da *I va III* choraklar koordinata burchaklari bissektrisalarining grafigini ifodalaydi (7-shakl).



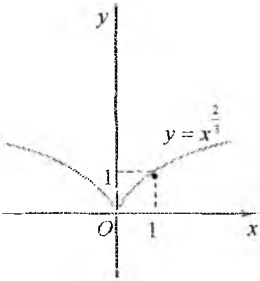
7-shakl.



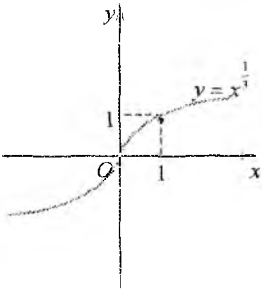
8-shakl.

2) $\alpha=-n$, $n-$butun musbat son. Bunda funksiyaning grafigi n juft son bo'lganda ordinatalar o'qiga nisbatan simmetrik, n toq son bo'lganda esa koordinatalar boshiga nisbatan simmetrik bo'ladi (8-shakl). $n=1$ da teskari proporsional bog'lanish grafigini ifodalaydi (8-shakl).

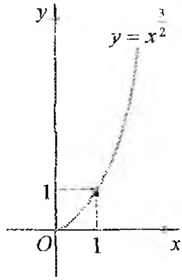
3) $\alpha = r$, $r = \dfrac{m}{n}$, m va n – o‘zaro tub butun sonlar. Bunda n juft son bo‘lganda $D(f) = [0;+\infty)$, n toq son bo‘lganda $D(f) = (-\infty;+\infty)$. Funksiyaning grafigi n toq va m juft son bo‘lganda ordinatalar o‘qiga nisbatan simmetrik (9-shakl), n va m toq sonlar bo‘lgnida koordinatalar boshiga nisbatan simmetrik bo‘ladi (10-shakl). $r < 1$ da grafik koordinatalar boshida ordinatalar o‘qiga urinadi (9, 10-shakl), $r > 1$ da grafik koordinatalar boshida abssissalar o‘qiga urinadi (11-shakl).



9-shakl.

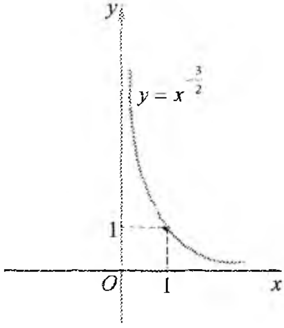


10-shakl.

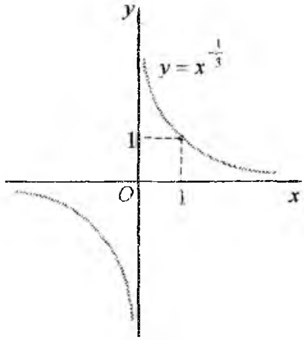


11-shakl.

4) $\alpha = q$, $q = \dfrac{m}{n} < 0$, m va n – o‘zaro tub butun sonlar, $n \neq -1$. Bunda n juft son bo‘lganda $D(f) = (0;+\infty)$ (12-shakl), n toq son bo‘lganda $D(f) = (-\infty;0) \cup (0;+\infty)$. Funksiyaning grafigi n toq va m juft son bo‘lganda ordinatalar o‘qiga nisbatan simmetrik, n va m toq sonlar bo‘lganda esa koordinatalar boshiga nisbatan simmetrik bo‘ladi (13-shakl).



12-shakl.



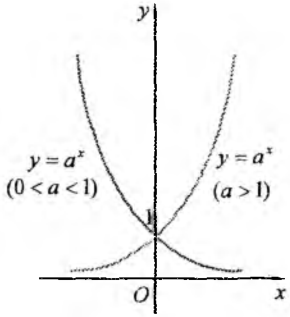
13-shakl.

3. ***Ko'rsatkichli funksiya*** $y=a^x$, $a\in R, a>0, a\neq 1$.

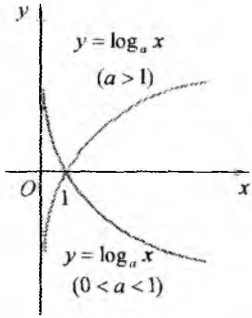
Ko'rsatkichli funksiyada $D(f)=(-\infty;+\infty)$, $E(f)=(0;+\infty)$. Bu funksiya $a>1$ bo'lsa, R da monoton o'suvchi, $0<a<1$ bo'lsa, R da monoton kamayuvchi.

Ko'rsatkichli funksiyaning grafiklari $(0;1)$ nuqtadan o'tadi.

Ko'rsatkichli funksiyalar grafiklari 14-shaklda keltirilgan.



14-shakl.

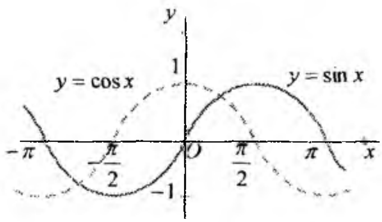


15-shakl.

4. ***Logarifmik funksiya*** $y=\log_a x$, $a\in R$, $a>0$, $a\neq 1$.

Logarifmik funksiyada $D(f)=(0;+\infty)$ va $E(f)=(-\infty;+\infty)$. $a>1$ bo'lsa, $D(f)$da monoton o'suvchi, $0<a<1$ bo'lsa, $D(f)$da monoton kamayuvchi; $y=a^x$ ga teskari funksiya.

Logarifmik funksiyalarning grafigi $(1;0)$ nuqtadan o'tadi.



16-shakl.

Logarifmik funksiyalarning grafigi 15-shaklda keltirilgan.

5. ***Trigonometrik funksiyalar***:

- $y=\sin x$: $D(f)=(-\infty;+\infty)$, $E(f)=[-1;1]$, chegaralangan, toq, davri 2π (16-shakl).

• $y=\cos x$: $D(f)=(-\infty;+\infty)$, $E(f)=[-1;1]$, chegaralangan, juft, davri 2π (16-shakl);

• $y=tgx$:

$D(f)=\left((2n-1)\frac{\pi}{2};(2n+1)\frac{\pi}{2}\right)$, $n\in Z$,

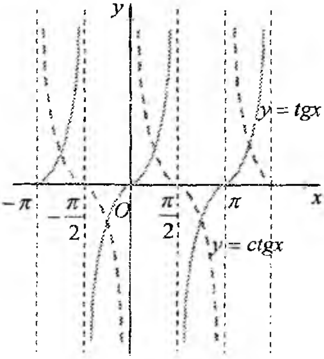
$E(f)=(-\infty;+\infty)$, toq, davri π (17-shakl);

• $y=ctgx$:

$D(f)=(n\pi;(n+1)\pi)$, $n\in Z$,

$E(f)=(-\infty;+\infty)$, toq,

davri π (17-shakl).

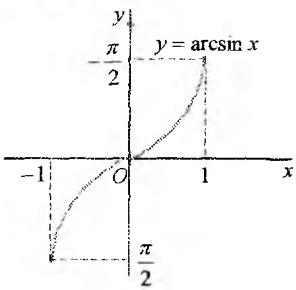


17-shakl.

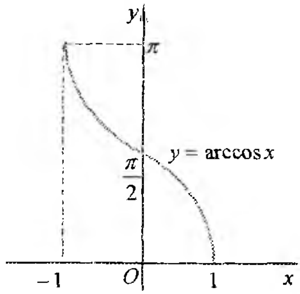
6. ***Teskari trigonometrik funksiyalar***:

• $y=\arcsin x$: $D(f)=[-1;1]$,

$E(f)=\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, chegaralangan, toq, monoton o‘suvchi (18-shakl);

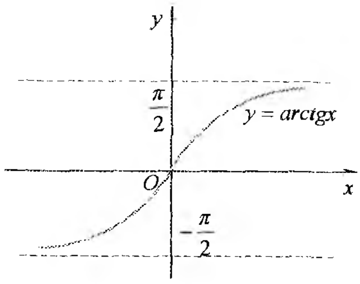


18-shakl.

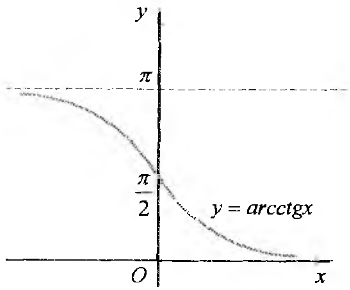


19-shakl.

• $y=\arccos x$: $E(f)=[-1;1]$, $E(f)=[0;\pi]$, chegaralangan, monoton kamayuvchi(19-shakl);



20-shakl.



21-shakl.

• $y = arctgx$: $D(f) = (-\infty;+\infty)$, $E(f) = \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, toq, monoton o'suvchi (20-shakl);

• $y = arcctgx$: $D(f) = (-\infty;+\infty)$, $E(f) = (0;\pi)$, monoton kamayuvchi (21-shakl).

Asosiy elementar funksiyalardan chekli sondagi arifmetik amallar (qo'shish, ayirish, ko'paytirish, bo'lish) va superpozitsiyalash yordamida hosil qilingan bitta formula bilan berilgan funksiyaga *elementar funksiya* deyiladi.

Masalan, $y = P_n(x) = a_0x^m + a_1x^{m-1} + ... + a_{m-1}x + a_m$, $y = \lg^2(\sin 2x) + e^{2x}$,

$y = \arccos\frac{1}{x} + \sqrt[3]{x^2}$ – elementar funksiyalar.

*Elementar bo'lmagan funksiyalar*ga quyidagi funksiyalar misol bo'ladi:

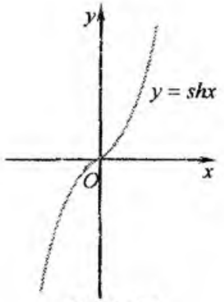
$$y = signx = \begin{cases} 1, & agar\ x > 0, \\ 0, & agar\ x = 0, \\ -1, & agar\ x < 0, \end{cases} \qquad y = \begin{cases} \frac{1}{x^3}, & agar\ x > 0, \\ x^3, & agar\ x \le 0, \end{cases}$$

$$y = 1 - \frac{x^3}{3!\cdot 3} + \frac{x^5}{5!\cdot 5} - \frac{x^7}{7!\cdot 7} + ... + (-1)^n\frac{x^{2n+1}}{(2n+1)!\cdot(2n+1)} + ...$$

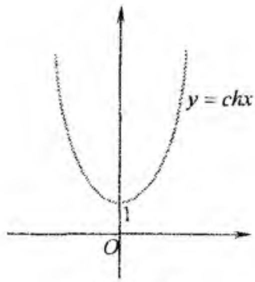
4.3.5. Giperbolik funksiyalar

Ko'rsatkichli funksiyalardan hosil qilinadigan quyidagi elementar funksiyalarga *giperbolik funksiyalar* deyiladi:

• *giperbolik sinus*: $y = shx$, $shx = \frac{e^x - e^{-x}}{2}$ (22-shakl);



22-shakl.

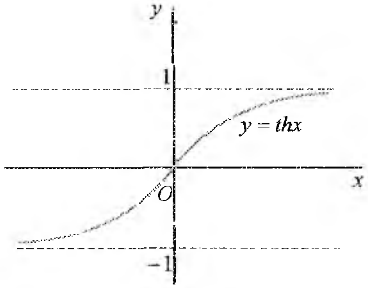


23-shakl.

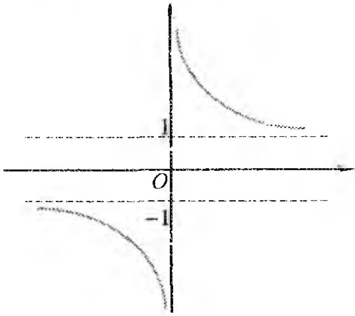
- *giperbolik kosinus*: $y = chx$, $chx = \dfrac{e^x + e^{-x}}{2}$ (23-shakl);

- *giperbolik tangens*: $y = thx$, $thx = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (24-shakl);

- *giperbolik kotangens*: $y = cthx$, $cthx = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ (25-shakl).



24-shakl.



25-shakl.

4.3.6. Oshkormas va parametrik ko'rinishda berilgan funksiyalar

Agar x va y o'zgaruvchilar orasidagi bog'lanish $y = f(x)$ ko'rinishda ifodalansa, bu funksiyaning oshkor ko'rinishdagi berilishi hisoblanadi. Shuningdek, ayrim hollarda funksiyaning oshkormas ko'rinishidan foydalanishga to'g'ri keladi.

Funksiya X to'plamda aniqlangan bo'lsin. Agar har bir $x \in X$ songa mos qo'yilgan yagona y son $F(x,y) = 0$ tenglamani qanoatlantirsa, $y = f(x)$ funksiyaga $F(x,y) = 0$ *tenglama bilan oshkormas ko'rinishda berilgan funksiya* deyiladi.

Agar x va y o'zgaruvchilar orasidagi bog'lanish ikkita $x = x(t)$ va $y = y(t)$, $t \in X$ funksiyalar berilgan bo'lsin. U holda Oxy koordinatalar tekisligining koordinatalari $(x(t);y(t))$ bo'lgan barcha nuqtalari to'plamiga parametrik ko'rinishda berilgan chiziq (egri chiziq yoki to'g'ri chiziq) deyiladi. Bunda t parametr deb ataladi.

Agar parametrik ko'rinishda berilgan chiziq $y = f(x)$ funksiyaning grafigini ifodalasa, bu funksiyaga *parametrik ko'rinishda berilgan funksiya* deyiladi.

4.3.7. Mashqlar

1. Funksiyaning aniqlanish sohasini toping:

1) $f(x)=\frac{1+x^2}{x^3+8}$;
2) $f(x)=\frac{1+x}{x^2+5x+6}$;
3) $f(x)=\sqrt{4-x^2}$;
4) $f(x)=\frac{5}{(x-1)\sqrt{x+2}}$;
5) $f(x)=\sqrt{\frac{10-x}{x^2-11x+18}}$;
6) $f(x)=\frac{\sqrt{4-3x^2-x^4}}{\cos\pi x}$;
7) $f(x)=\sqrt{x-7}+\sqrt{10-x}$;
8) $f(x)=\sqrt{2x+1}-\sqrt{x+1}$;
9) $f(x)=\sqrt{x-2}+\sqrt{2-x}+\sqrt{x^2+4}$;
10) $f(x)=\sqrt{x^3-8}+\frac{3}{\sqrt[3]{2-x}}$;
11) $f(x)=\arcsin x-\arccos(4-x)$;
12) $f(x)=\arcsin(x-2)+3\ln(x-2)$;
13) $f(x)=\log_3 \ln \lg x$;
14) $f(x)=\ln \sin x$;
15) $f(x)=e^{\sqrt{x}}\log_2(2-3x)$;
16) $f(x)=\ln\left(\frac{\sqrt{x-3}+\sqrt{7-x}}{\sqrt[3]{(x-6)^2}}\right)$;
17) $f(x)=\sqrt{3-4x}+\arccos x\frac{3-4x}{6}$;
18) $f(x)=\arccos\frac{x+2}{3}+2^{\frac{1}{x}}$;
19) $f(x)=\frac{3}{\sqrt[3]{x^2-3x+2}}-5\sin 2x$.
20) $f(x)=\frac{x-\ln(x+3)}{\sqrt{8-x^3}}$.

2. Funksiyaning qiymatlar sohasini toping:

1) $f(x)=x^2+4x+2$;
2) $f(x)=\sqrt{7-x}+2$;
3) $f(x)=2\sin x-5$;
4) $f(x)=\sin x+\cos x$;
5) $f(x)=2^{x^2}-1$;
6) $f(x)=2e^{-x^2}+1$;
7) $f(x)=\sqrt{9-x^2}$;
8) $f(x)=\frac{1}{\pi}arctgx$;
9) $f(x)=3|x|-\frac{1}{5}$;
10) $f(x)=\frac{2x-3}{|2x-3|}$;
11) $f(x)=\frac{9}{2x^2+4x+5}$;
12) $f(x)=\frac{2}{\sqrt{2x^2-4x+3}}$;

3. $f(x)=x^3 3^x$ funksiya berilgan. Quyidagilarni toping:

1) $f(1)$; 2) $f(-\sqrt[3]{4})$; 3) $f(-x)$; 4) $f\left(\frac{1}{x}\right)$.

4. Funksiyaning monotonlik oraliqlarini toping:

1) $f(x)=x^2-5x+6$; 2) $f(x)=x^3+\arcsin x$;

3) $f(x)=\frac{1}{x^3}$; 4) $f(x)=arctgx-x$.

5. Funksiyaning juft, toq yoki umumiy ko'rinishda ekanini aniqlang:

1) $f(x)=x^3-3x-x^5$; 2) $f(x)=x^4+5x^2+1$;

3) $f(x)=\frac{tg2x}{x}$; 4) $f(x)=ctg3x+\cos 2x$;

5) $f(x)=\ln\left(\frac{3+x}{3-x}\right)$; 6) $f(x)=\ln(x+\sqrt{x^2+1})$;

7) $f(x)=2|x|-3$; 8) $f(x)=x|x|$;

9) $f(x)=3^{x^2}(x+\sin x)$; 10) $f(x)=\left(\frac{2^x-2^{-x}}{2}\right)x$.

6. Funksiyaning eng katta va eng kichik qiymatlarini toping:

1) $f(x)=(k-n)\cos^2 x+n \ (0<k<n)$; 2) $f(x)=4\sin x^3$;

3) $f(x)=\sin 2x+\cos 2x$; 4) $f(x)=3\sin x+4\cos x$;

5) $f(x)=\sin^4 x+\cos^4 x$; 6) $f(x)=|\cos 4x|$.

7. Funksiyaning monoton, qat'iy monoton yoki chegaralangan ekanini aniqlang:

1) $f(x)=\sin^2 x$; 2) $f(x)=\frac{x+2}{x+7}$;

3) $f(x)=\sqrt{3x-4}$; 4) $f(x)=\begin{cases} x, & agar\ x<0, \\ -3, & agar\ x\geq 0. \end{cases}$

8. Funksiyaning davrini toping:

1) $f(x)=-2\cos\frac{x}{3}$; 2) $f(x)=ctg(2x-3)$;

3) $f(x)=tgx-\cos\frac{x}{2}$; 4) $f(x)=\sin\frac{x}{2}\cos\frac{x}{2}\cos x\cos 2x$;

5) $f(x)=\sin^4 x-\cos^4 x$; 6) $f(x)=\sin 2x+\cos 3x$;

7) $f(x)=|\sin 2x|$; 8) $f(x)=|\cos 3x|$;

9) $f(x)=\sin\frac{3x}{2}+\cos\frac{2x}{3}$;

10) $f(x)=tg\frac{2x}{3}-ctg\frac{3x}{2}+\sin\frac{x}{3}$.

9. Funksiyaga teskari funksiyani toping:

1) $y=3x+5$;

2) $y=\frac{x}{1+x}$;

3) $y=4+\log_3 x$;

4) $y=2\sin 3x$.

10. $f(g(x))$ va $g(f(x)$ murakkab funksiyalarni toping:

1) $f(x)=3x+1,\ g(x)=x^3$;

2) $f(x)=\sin x,\ g(x)=|x|$;

3) $f(x)=\frac{x+1}{x},\ g(x)=\frac{1}{4-x}$;

4) $f(x)=2^{3x},\ g(x)=\log_2 x$.

11. Funksiyaning grafigini chizing:

1) $y=x^2+4x+3$;

2) $y=-2\sin 3x$;

3) $y=\frac{2x-1}{2x+1}$;

4) $y=-x^2|x|$;

5) $y=x\sin x$;

6) $y=x+\sin x$.

7) $y=\arccos|x|$;

8) $y=3^{\frac{1}{x}}$.

12. Ayniyatni isbotlang:

1) $1-th^2x=\frac{1}{ch^2x}$;

2) $cth^2x-1=\frac{1}{sh^2x}$;

3) $ch^2x=\frac{ch2x+1}{2}$;

4) $sh^2x=\frac{ch2x-1}{2}$;

5) $sh(\ln x)=\frac{x^2-1}{2x}$;

6) $ch(\ln x)=\frac{x^2+1}{2x}$.

13. Qaysi nuqta $y+\cos y-x=0$ funksiya grafigiga tegishli ekanini aniqlang: $A(1;0)$; $B(0;0)$; $C\left(\frac{\pi}{2};\frac{\pi}{2}\right)$; $D(\pi-1;\pi)$.

14. Qaysi nuqta $\begin{cases} x=t-1, \\ y=t^2+1 \end{cases}$ parametrik tenglamalar bilan berilgan egri chiziqqa tegishli ekanini aniqlang: $A(1;5)$; $B\left(\frac{1}{2};\frac{13}{4}\right)$; $C(2;8)$; $D(0;1)$.

15. Berilgan funksiyani $y=y(x)$ ko‘rinishga keltiring:

1) $\begin{cases} x=t+2, \\ y=t^2+4t+5; \end{cases}$

2) $\begin{cases} x=3\sin t, \\ y=2\cos t. \end{cases}$

4.4. FUNKSIYANING LIMITI

4.4.1. Funksiyaning limiti

Funksiya limitining ta'riflari

Biror X sonli to'plam berilgan bo'lsin.

1-ta'rif. Agar $x_0 \in X$ nuqtaning ixtiyoriy ε $(\varepsilon > 0)$ atrofida X to'plamning cheksiz ko'p elementlari yotsa, x_0 nuqtaga X toplamning *limit nuqtasi* deyiladi.

Masalan, $X = \left\{\frac{1}{n} : n \in N\right\}$ to'plam uchun $x_0 = 0$ limit nuqta bo'ladi.

$f(x)$ funksiya X toplamda aniqlangan va x_0 nuqta X toplamning limit nuqtasi bo'lsin.

2- ta'rif. (*funksiya limitining «ketma-ketlik tilidagi» yoki Geyne ta'rifi*). Agar X toplamning nuqtalaridan tuzilgan x_0 nuqtaga yaqinlashuvchi har qanday $\{x_n\}$ ketma-ketlik $(x_n \neq x_0)$ olinganda ham, bu ketma-ketlikka mos $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona A limitga intilsa, A soniga $f(x)$ funksiyaning x_0 *nuqtadagi* yoki $x \to x_0$ *dagi limiti* deyiladi va bu $\lim\limits_{x \to x_0} f(x) = A$ deb yoziladi.

3- ta'rif. (*funksiya limitining «$\varepsilon - \delta$ tilidagi» yoki Koshi ta'rifi*). Agar $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsa va x ning $0 < |x - x_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha $x \in X,\ x \neq x_0$ qiymatlarida $|f(x) - A| < \varepsilon$ tengsizlik bajarilsa, A soniga $f(x)$ funksiyaning x_0 *nuqtadagi* yoki $x \to x_0$ *dagi limiti* deyiladi va bu $\lim\limits_{x \to x_0} f(x) = A$ deb yoziladi.

Funksiya limiti uchun berilgan Geyne va Koshi ta'riflari o'zaro ekvivalent ekanligi isbotlangan. Shu sababli funksiyaning nuqtadagi limitini topishda bu ta'riflarning istalgan biridan foydalanish mumkin.

x_0 ga intiluvchi $\{x_n\}$ ketma-ketlikni yetarlicha ko'p usul bilan tanlash mumkin bo'lganligi uchun Geyni ta'rifidan funksiyaning limitini topishdan ko'ra, funksiyaning nuqtada limitga ega bo'lmasligini ko'rsatishda foydalanish qulaylikka ega bo'ladi. Buning uchun x_0 ga nuqtada limitga ega bo'lmagan birorta $\{f(x_n)\}$ ketma-ketlikni topish yetarli yoki har xil limitlarga ega bo'lgan $\{f(x_n')\}$ va $\{f(x_n'')\}$ ketma-ketliklarni ko'rsatish kifoya.

1-misol. $f(x)=\sin\frac{1}{x}$ funksiya $x=0$ nuqtada limitga ega bo'lmasligini ko'rsating.

Yechish. $x=0$ nuqtaga intiluvchi ikkita $\left\{\frac{1}{n\pi}\right\}$ va $\left\{\frac{1}{\frac{\pi}{2}+2n\pi}\right\}$ ketma-ketliklarni qaraymiz. Mos $\{f(x)\}$ ketma-ketliklar har xil limitlarga intiladi: $\{\sin n\pi\}$ ketma-ketlik nolga intiladi, $\left\{\sin\left(\frac{\pi}{2}+2n\pi\right)\right\}$ ketma-ketlik esa birga intiladi.

Demak, $f(x)=\sin\frac{1}{x}$ funksiya $x=0$ nuqtada limitga ega bo'lmaydi.

2- misol. $\lim\limits_{x\to1}(3x-2)=1$ ekanini ko'rsating.

Yechish. $\forall\varepsilon>0$ son olamiz. Shunday $\delta>0$ sonni ko'rsatishimiz kerakki, $|x-1|<\delta$ bo'lganida $|f(x)-1|<\varepsilon$ bo'lsin.

Bunda $f(x)=3x-2$: $|f(x)-1|=|(3x-2)-1|=|3x-3|=|3(x-1)|=3|x-1|$ bo'lgani uchun $\delta=\frac{\varepsilon}{3}$ deb olsak, $|x-1|<\delta$ bo'lganda $|f(x)-1|<\varepsilon$ bo'ladi.

Demak, $\lim\limits_{x\to1}(3x-2)=1$.

Xususan, $\varepsilon=1$ da $\delta=\frac{1}{3}$, $\varepsilon=\frac{1}{2}$ da $\delta=\frac{1}{6}$.

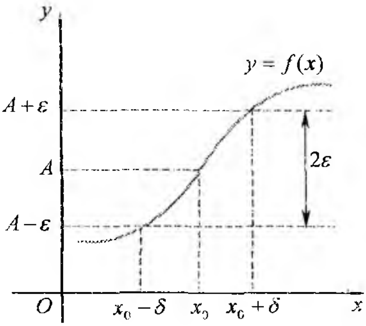
Shunday qilib, δ son ε songa bog'liq bo'ladi. Shu sababli keyingi ta'riflarda $\delta=\delta(\varepsilon)$ deb olamiz.

Izoh. Funksiyaning x_0 nuqtadagi limiti ta'rifida x_0 nuqtaning o'zi qaralmaydi. Shunday qilib, funksiyaning x_0 nuqtadagi qiymati funksiyaning bu nuqtdagi limitiga ta'sir qilmaydi. Bundan tashqari, funksiya x_0 nuqtada aniqlanmagan bo'lishi ham mumkin. Shu sababli x_0 nuqtaning atrofida ($x\neq x_0$ bo'lganda) teng bo'lgan (($x=x_0$ da har xil qiymatga ega bo'lgan yoki ulardan bittasi yoki har ikkalasi aniqlanmagan) ikkita funksiya $x\to x_0$ da bitta limitga ega bo'lishi yoki ularning har ikkalasi limitga ega bo'lmasligi mumkin.

3- misol. $f(x)=\begin{cases}x^2, & x\neq0,\\ 1, & x=0\end{cases}$ bo'lsa, $\lim\limits_{x\to0}f(x)$ limitni toping.

Yechish. $g(x)=x^2$, $-\infty<x<+\infty$ funksiya $x=0$ dan tashqari, barcha nuqtalarda $f(x)$ funksiya bilan ustma-ust tushadi va $\lim\limits_{x\to 0} g(x)=0$ bo'ladi. Shu sababli $\lim\limits_{x\to 0} f(x)=0$.

Funksiyaning nuqtadagi limiti ta'rifini *geometrik nuqtayi nazardan* shunday talqin qilish mumkin: agar A soni $f(x)$ funksiyaning x_0 nuqtadagi limiti bo'lsa, A nuqtaning istalgan ε atrofi uchun x_0 nuqtaning shunday δ atrofi topiladi va δ atrofdagi barcha x $(x \neq x_0)$ nuqtalarda $f(x)$ funksiyaning mos qiymatlari A nuqtaning ε atrofida yotadi. Boshqacha aytganda $f(x)$ funksiyaning δ atrofdagi grafigi $y=A-\varepsilon$ va $y=A+\varepsilon$ to'g'ri chiziqlar bilan chegaralangan, kengligi 2ε ga teng bo'lgan tasmada joylashadi (26-shakl).



26-shakl.

4-ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topilsa va x *ning* $x_0<x<x_0+\delta$ $(x_0-\delta<x<x_0)$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)-A|<\varepsilon$ tengsizlik bajarilsa, A soniga $f(x)$ funksiyaning x_0 nuqtadagi *o'ng* (*chap*) *limiti* deyiladi va $\lim\limits_{x\to x_0+0} f(x)=A$ yoki $f(x+0)=A$ ($\lim\limits_{x\to x_0-0} f(x)=A$ yoki $f(x-0)=A$) kabi belgilanadi.

$f(x)$ funksiyaning x_0 nuqtadagi o'ng va chap limitlari *bir tomonlama limitlar* deb ataladi. Agar $f(x)$ funksiyaning x_0 nuqtadagi o'ng va chap limitlari mavjud va bir-biriga teng, ya'ni $f(x_0+0)=f(x_0-0)=A$ bo'lsa, x_0 nuqtada $f(x)$ funksiyaning limiti mavjud va $\lim\limits_{x\to x_0} f(x)=A$ bo'ladi.

$y=f(x)$ funksiya $(-\infty;+\infty)$ intervalda aniqlangan bo'lsin.

5- ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topilsa va x ning $x>\delta$ $(x<-\delta)$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)-A|<\varepsilon$ tengsizlik bajarilsa, A soniga $f(x)$ funksiyaning $x\to+\infty$ $(x\to-\infty)$ dagi limiti deyiladi va $\lim\limits_{x\to+\infty} f(x)=A$ ($\lim\limits_{x\to-\infty} f(x)=A$) kabi belgilanadi.

Funksiyaning cheksizlikdagi limiti ta'rifini *geometrik nuqtayi* *nazardan* bunday talqin qilish mumkin: agar $\lim_{x\to+\infty} f(x)=A$ ($\lim_{x\to-\infty} f(x)=A$) bo'lsa, $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topiladiki, $x\in(\delta;+\infty)$ $(x\in(-\infty;-\delta))$ larda $f(x)$ funksiyaning qiymatlari A nuqtaning ε atrofida yotadi.

Limitlar haqidagi teoremalar

Funksiya limitining «ketma-ketlik tilidagi» ta'rifi yaqinlashuvchi (limitga ega) ketma-ketliklarning xossalarini funksiyaning limiti uchun o'tkazish imkonini beradi. Bu xossalarni ifodalovchi teoremalar bilan tanishamiz va ularning ayrimlarini isbotlaymiz. Bu teoremalarda qaralayotgan funksiyalar $x\to x_0$ da limitga ega deb hisoblaymiz.

1-teorema. Funksiya $x\to x_0$ da yagona limitga ega bo'ladi.

2-teorema. Ikkita funksiya algebraik yig'indisining limiti bu funksiyalar limitlarining algebraik yig'indisiga teng, ya'ni

$$\lim_{x\to x_0}(f(x)\pm g(x))=\lim_{x\to x_0} f(x)\pm\lim_{x\to x_0} g(x).$$

Isboti. Ixtiyoriy $\{x_n\}$ kema-ketlik olamiz.
Bu ketma-ketlik uchun $x_n\to x_0$, $x_n\neq x_0$, $x_n\in D(f)\cap D(g)$ bo'lsin.

U holda

$$\lim_{x\to x_0}(f(x)\pm g(x))=\lim_{n\to\infty}((f(x_n)\pm g(x_n))=$$

$$=\lim_{n\to\infty} f(x_n)\pm\lim_{n\to\infty} g(x_n)=\lim_{x\to x_0} f(x)\pm\lim_{x\to x_0} g(x).$$

Demak,

$$\lim_{x\to x_0}(f(x)\pm g(x))=\lim_{x\to x_0} f(x)\pm\lim_{x\to x_0} g(x).$$

3-teorema. Ikkita funksiya ko'paytmasining limiti bu funksiyalar limitlarining ko'paytmasiga teng, ya'ni

$$\lim_{x\to x_0}(f(x)\cdot g(x))=\lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0} g(x).$$

1-natija. O'zgarmas funksiyaning limiti uning o'ziga teng , ya'ni

$$\lim_{x\to x_0} C=C.$$

2-natija. O'zgarmas ko'paytuvchini limit belgisidan tashqariga chiqazish mumkin, ya'ni

$$\lim_{x\to x_0}(k\cdot f(x))=k\cdot\lim_{x\to x_0} f(x),\ k\in R.$$

3-natija. Funksiyaning musbat ko'rsatkichli darajasining limiti bu funksiya limitining shu tartibli darajasiga teng, ya'ni

$$\lim_{x \to x_0}(f(x))^p = (\lim_{x \to x_0} f(x))^p, \ p > 0.$$

4-teorema. Ikki funksiya bo'linmasining limiti bu funksiyalar limitlarining nisbatiga teng, ya'ni

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}, \quad \lim_{x \to x_0} g(x) \neq 0.$$

5-teorema. Agar x_0 nuqtaning biror atrofidagi barcha x uchun $f(x) \le g(x)$ tengsizlik bajarilsa, u holda $\lim\limits_{x \to x_0} f(x) \le \lim\limits_{x \to x_0} g(x)$ bo'ladi.

6-teorema. Agar x_0 nuqtaning biror atrofidagi barcha x uchun $f(x) \le \varphi(x) \le g(x)$ tengsizlik bajarilsa va $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = A$ bo'lsa, u holda $\lim\limits_{x \to x_0} \varphi(x) = A$ bo'ladi.

7-teorema. $\lim\limits_{x \to x_0} g(x) = 0$, $\lim\limits_{x \to x_0} f(x) = C \neq 0$ bo'lsin.

U holda:

1) agar $|x - x_0| < \delta$ $(\delta > 0)$ tengsizlikni qanoatlantiruvchi barcha x uchun $\frac{f(x)}{g(x)} > 0$ bo'lsa, $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = +\infty$ bo'ladi;

2) agar $|x - x_0| < \delta$ $(\delta > 0)$ tengsizlikni qanoatlantiruvchi barcha x uchun $\frac{f(x)}{g(x)} < 0$ bo'lsa, $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = -\infty$ bo'ladi.

8-teorema. Agar $\lim\limits_{x \to x_0} f(x) = A \neq \infty$ bo'lsa, u holda x_0 nuqtaning yetarlicha kichik atrofidan olingan x $(x \neq x_0)$ qiymatlarda $f(x)$ funksiya chegaralangan bo'ladi.

Isboti. Shartga ko'ra, $\lim\limits_{x \to x_0} f(x) = A \neq \infty$. Funksiya limitining Koshi ta'rifiga ko'ra, $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ son topiladi va x ning $0 < |x - x_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - A| < \varepsilon$, ya'ni $A - \varepsilon < f(x) < A + \varepsilon$ tengsizlik bajariladi. Demak, x ning $0 < |x - x_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$f(x)$ funksiyaning qiymatlari $(A-\varepsilon;A+\varepsilon)$ oraliqda bo‘ladi. Bu funksiyaning $(x_0-\delta;x_0+\delta)$, $x\neq x_0$ oraliqda chegaralanganligini bildiradi.

9-teorema. Agar: 1) $\lim\limits_{x\to x_0}\varphi(x)=t_0$ va x_0 nuqtaning shunday $(x_0-\delta;x_0+\delta)$, $\delta>0$ atrofi mavjud va bu atrofdan olingan barcha x lar uchun $\varphi(x)\neq t_0$ bo‘lsa, 2) $\lim\limits_{t\to t_0}f(t)=B$ bo‘lsa, u holda $x\to x_0$ da murakkab $f(\varphi(x))$ funksiya limitga ega va $\lim\limits_{x\to x_0}f(\varphi(x))=B$ bo‘ladi.

Yuqorida keltirilgan teoremalar $x\to\pm\infty$ da ham o‘rinli bo‘ladi.

4- misol. $\lim\limits_{x\to 5}\dfrac{x^2-8x+15}{x^2-25}$ limitni toping.

Yechish. Bu limit uchun ikki funksiya bo‘linmasining limiti haqidagi teoremani qo‘llab bo‘lmaydi, chunki $x\to 5$ da kasrning maxraji nolga teng bo‘ladi. Bundan tashqari, suratning limiti ham nolga teng. Bunday hollarda $\dfrac{0}{0}$ ko‘rinishdagi aniqmaslik berilgan deyiladi. Bu aniqmaslikni ochish uchun kasrning surati va maxrajini ko‘paytuvchilarga ajratamiz va kasrni $x-5\neq 0$ $(x\to 5$, lekin $x\neq 5)$ ga bo‘lib, topamiz:

$$\lim_{x\to 5}\frac{(x-5)(x-3)}{(x-5)(x+5)}=\lim_{x\to 5}\frac{x-3}{x+5}=\frac{2}{10}=\frac{1}{5}.$$

5- misol. $\lim\limits_{x\to\infty}\dfrac{2x^3+3x^2+1}{x^3+4x^2-x}$ limitni toping.

Yechish. Bu misolda $x\to\infty$ da $\dfrac{\infty}{\infty}$ ko‘rinishdagi aniqmaslik hosil bo‘ladi. Kasrning surat va maxrajini x ning yuqori darajasiga, ya’ni x^3 ga bo‘lib, topamiz:

$$\lim_{x\to\infty}\frac{2x^3+3x^2+1}{x^3+4x^2-x}=\lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{1}{x^3}}{1+\frac{4}{x}-\frac{1}{x^2}}=\frac{2+0+0}{1+0-0}=2.$$

Ajoyib limitlar

Birinchi ajoyib limit: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

Isboti. $0<x<\dfrac{\pi}{2}$ bo‘lsin.

Radiusi $R=1$ ga teng bo'lgan aylananing radian o'lchovi x ga teng bo'lgan markaziy burchagiga mos yoyini qaraymiz (27-shakl).

Shakldan quyidagilarga ega bo'lamiz:

ΔMOA *yuzi* $<$ MOA *sektor yuzi* $<$ ΔLOA *yuzi*;

ΔMOA *yuzi*: $S_1 = \frac{1}{2} OA \cdot MK = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x;$

MOA *sektor yuzi*: $S_2 = \frac{1}{2} OA \cdot \breve{MA} = \frac{1}{2} \cdot 1 \cdot x = \frac{1}{2} x;$

ΔLOA *yuzi*: $S_1 = \frac{1}{2} OA \cdot LA = \frac{1}{2} \cdot 1 \cdot tgx = \frac{1}{2} tgx.$

Bundan $\sin x < x < tgx$ kelib chiqadi. Tengsizlikni $\sin x > 0$ ga bo'lamiz:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \text{yoki} \quad \cos x < \frac{\sin x}{x} < 1.$$

Endi $x < 0$ bo'lsin.

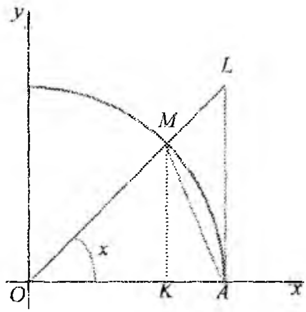
$\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$, $\quad \cos(-x) = \cos x$ ekanidan

$x < 0$ da ham

$$\cos x < \frac{\sin x}{x} < 1.$$

$\lim\limits_{x\to 0} \cos x = 1, \lim\limits_{x\to 0} 1 = 1$ bo'lgani uchun oxirgi tenglikdan 6-teoremaga ko'ra,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$



27-shakl.

6-misol. $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$ limitni toping.

Yechish. $x \to 0$ da $\frac{0}{0}$ ko'rinishdagi aniqmaslik berilgan.

Almashtirishlar bajaramiz:

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} = \frac{1}{2}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2.$$

$x \to 0$ da $\frac{x}{2} \to 0$ va 1– ajoyib limitga ko'ra, $\lim\limits_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 1$.

Demak,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2 = \frac{1}{2}\cdot 1^2 = \frac{1}{2}.$$

Ikkinchi ajoyib limit : $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Isboti. Ma'lumki, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,\ n \in N$.

$x > 1$ bo'lsin. $n = [x]$ deb olamiz. U holda $x = n + \alpha$, bu yerda $0 \le \alpha \le 1$.

$n < x < n+1$ tengsizlikdan topamiz:

$$\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n}$$

yoki

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^{n+1}. \qquad (4.1)$$

$x \to \infty$ da $n \to \infty$.

U holda

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1\cdot e = e;$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)} = \frac{e}{1} = e.$$

Shuning uchun (4.1) tengsizlikdan 6-teoremaga ko'ra,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

kelib chiqadi.

Endi $x<-1$ bo'lsin. $x=-y$ deb olamiz.

U holda

$$\lim_{n\to-\infty}\left(1+\frac{1}{x}\right)^x=\lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y}=\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^y=$$

$$=\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\lim_{y\to\infty}\left(1+\frac{1}{y-1}\right)=e\cdot 1=e.$$

Demak,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

7- *misol.* $\lim\limits_{x\to\infty}\left(\frac{2x}{2x-1}\right)^{3x-1}$ limitni toping.

Yechish. $x\to\infty$ da 1^∞ ko'rinishdagi aniqmaslik berilgan.

Qavs ichidagi kasrning butun qismini ajratib, almashtirishlar bajaramiz:

$$\left(1+\frac{1}{2x-1}\right)^{3x-1}=\left[\left(1+\frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{3x-1}{2x-1}}=\left[\left(1+\frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{3-\frac{1}{x}}{2-\frac{1}{x}}}.$$

$x\to\infty$ da $2x-1\to\infty$ bo'lgani sababli 2-ajoyib limitni qo'llab, topamiz:

$$\lim_{x\to\infty}\left(1+\frac{1}{2x-1}\right)^{2x-1}=e.$$

$$\lim_{x\to\infty}\frac{3-\frac{1}{x}}{2-\frac{1}{x}}=\frac{3}{2} \text{ ekanidan } \lim_{x\to\infty}\left(\frac{2x}{2x-1}\right)^{3x-1}=\sqrt{e^3}.$$

4.4.2. Cheksiz kichik funksiyalar

Ta'riflar va asosiy teoremalar

6-ta'rif. Agar $\lim\limits_{x\to x_0}f(x)=0$ bo'lsa, $f(x)$ funksiyaga $x\to x_0$ *da cheksiz kichik funksiya* deyiladi.

Funksiyaning limiti ta'rifiga ko'ra, $\lim\limits_{x\to x_0}f(x)=0$ tenglik quyidagicha talqin qilinadi: $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topiladi va x ning $0<|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarda $|f(x)|<\varepsilon$ tengsizlik bajariladi.

$x\to x_0+0, x\to x_0-0, x\to+\infty, x\to-\infty$ da cheksiz kichik funksiya shu kabi ta'riflanadi.

Cheksiz kichik funksiyalar ko'pincha cheksiz kichik kattaliklar yoki cheksiz kichik deb ataladi va odatda, grek alifbosining α,β kabi harflari bilan belgilanadi.

Cheksiz kichik funksiyalarga $x\to 0$ da $\alpha=x^3$, $x\to 3$ da $\beta=x-3$, $x\to k\pi, k\in Z$ da $\gamma=\sin x$ funksiyalar misol bo'ladi.

7-ta'rif. Agar $\lim\limits_{x\to x_0} f(x)=\infty$ bo'lsa, $f(x)$ funksiyaga $x\to x_0$ *da cheksiz katta funksiya* deyiladi.

Bunda $f(x)$ funksiya faqat musbat qiymatlar qabul qilsa, $\lim\limits_{x\to x_0} f(x)=+\infty$ deb, faqat manfiy qiymatlar qabul qilsa, $\lim\limits_{x\to x_0} f(x)=-\infty$ deb yoziladi. Masalan, $x\to 1$ da $f(x)=\dfrac{1}{x-1}$ cheksiz katta funksiya bo'ladi.

Cheksiz kichik funksiyalar uchun o'rinli bo'ladigan teoremalar bilan tanishamiz.

10-teorema. $\lim\limits_{x\to x_0} f(x)=A$ bo'lishi uchun $x\to x_0$ da $\alpha(x)=f(x)-A$ funksiya cheksiz kichik bo'lishi zarur va yetarli.

Isboti. Zarurligi. $\lim\limits_{x\to x_0} f(x)=A$ bo'lsin. $f(x)-A=\alpha(x)$ funksiyani olamiz.

U holda

$$\lim_{x\to x_0}\alpha(x)=\lim_{x\to x_0}|f(x)-A|=\lim_{x\to x_0}f(x)-\lim_{x\to x_0}A=A-A=0.$$

Demak, $\alpha(x)=f(x)-A$ funksiya cheksiz kichik.

Yetarliligi. $f(x)-A=\alpha(x)$, bu yerda $\alpha(x)$ cheksiz kichik funksiya bo'lsin. Bundan $f(x)=A+\alpha(x)$.

U holda

$$\lim_{x\to x_0}f(x)=\lim_{x\to x_0}(A+\alpha(x))=\lim_{x\to x_0}A+\lim_{x\to x_0}\alpha(x)=A+0=A.$$

Quyidagi teoremalar $x\to x_0$ da deb qaraladi.

11-teorema. Chekli sondagi cheksiz kichik funksiyalarning algebraik yig'indisi cheksiz kichik funksiya bo'ladi.

12-teorema. Cheksiz kichik funksiyaning chegaralangan funksiyaga ko'paytmasi cheksiz kichik funksiya bo'ladi.

4-natija. Chekli sondagi cheksiz kichik funksiyalarning ko‘paytmasi cheksiz kichik funksiya bo‘ladi.

5-natija. Cheksiz kichik funksiyaning chekli o‘zgarmas songa ko‘paytmasi cheksiz kichik funksiya bo‘ladi.

13-teorema. Cheksiz kichik funksiyaning nolga teng bo‘lmagan limitga ega funksiyaga bo‘linmasi cheksiz kichik funksiya bo‘ladi.

Yuqorida keltirilgan teoremalar $x\to\infty$, $x\to x_0-0$, $x\to x_0+0$ da ham o‘rinli bo‘ladi.

14-teorema. Agar $x\to x_0$ da $\alpha(x)$ funksiya cheksiz kichik bo‘lsa, u holda $x\to x_0$ da $\dfrac{1}{\alpha(x)}$ funksiya cheksiz katta bo‘ladi va aksincha agar $x\to x_0$ da $f(x)$ funksiya cheksiz katta bo‘lsa, u holda $x\to x_0$ da $\dfrac{1}{f(x)}$ funksiya cheksiz kichik bo‘ladi.

8-misol. $\alpha(x)=(x-2)^3\sin^2\dfrac{1}{x-2}$ funksiya $x\to 2$ da cheksiz kichik bo‘lishini ko‘rsating.

Yechish. $\lim\limits_{x\to l}(x-2)^3=0$ ekanidan $\beta(x)=(x-2)^3$ funksiya cheksiz kichik.

$g(x)=\sin^2\dfrac{1}{x-2}$, $x\neq 2$ funksiya chegaralangan, chunki $\left|\sin^2\dfrac{1}{x-2}\right|\le 1$.

$\alpha(x)$ funksiya cheksiz kichik $\beta(x)$ funksiyaning chegaralangan $g(x)$ funksiyaga ko‘paytmasidan iborat. Demak, 12-teoremaga ko‘ra, $\alpha(x)$ funksiya $x\to 2$ da cheksiz kichik.

Cheksiz kichik funksiyalarni taqqoslash

Ma’lumki, cheksiz kichik funksiyalarning yig‘indisi, ayirmasi va ko‘paytmasi cheksiz kichik funksiyalar bo‘ladi. Bu tasdiqni cheksiz kichik funksiyalarning bo‘linmasi uchun ta’kidlab bo‘lmaydi, chunki bitta cheksiz kichik funksiyaning boshqa cheksiz kichik funksiyaga nisbati har xil natijaga olib kelishi, ya’ni chekli son bo‘lishi, cheksiz katta bo‘lishi, cheksiz kichik bo‘lishi yoki limitga ega bo‘lmasligi mumkin.

Cheksiz kichik funksiyalar bir-biri bilan nisbati yordamida taqqoslanadi.

$\alpha(x)$ va $\beta(x)$ funksiyaar $x\to x_0$ da cheksiz kichik funksiyalar bo‘lsin.

1. Agar $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=A\neq 0\,(A-chekli\ son)$ bo‘lsa, $\alpha(x)$ va $\beta(x)$ *bir xil tartibli cheksiz kichik funksiyalar* deyiladi.

2. Agar $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=0$ bo‘lsa, $\alpha(x)$ funksiya $\beta(x)$ funksiyaga nisbatan *yuqori tartibli cheksiz kichik funksiya* deyiladi va bu $\alpha=o(\beta)$ kabi belgilanadi.

3. Agar $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=\infty$ bo‘lsa, $\alpha(x)$ funksiya $\beta(x)$ funksiyaga nisbatan *quyi tartibli cheksiz kichik funksiya* deyiladi.

4. Agar $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}$ mavjud bo‘lmasa, $\alpha(x)$ va $\beta(x)$ funksiyalarga *taqqoslanmaydigan cheksiz kichik funksiyalar* deyiladi.

Misollar keltiramiz:

1) $x\to 0$ da $\sin 3x$ va $\sin x$ funksiyalar bir xil tartibli cheksiz kichik funksiyalar, chunki $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin x}=\lim\limits_{x\to 0}\frac{\frac{3\sin 3x}{3x}}{\frac{\sin x}{x}}=\frac{3\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}=\frac{3}{1}=3$;

2) $x\to 0$ da $2x^3$ funksiya $5x$ funksiyaga nisbatan yuqori tartibli cheksiz kichik funksiya, chunki $\lim\limits_{x\to 0}\frac{2x^3}{5x}=\lim\limits_{x\to 0}\frac{2x^2}{5}=0$;

3) $x\to 0$ da $\sin x$ funksiya x^2 funksiyaga nisbatan quyi tartibli cheksiz kichik funksiyalar, chunki $\lim\limits_{x\to 0}\frac{\sin x}{x^2}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{x}=1\cdot\frac{1}{0}=\infty$;

4) $x\sin\frac{1}{x}$ va x funksiyalar $x\to 0$ da taqqoslanmaydigan cheksiz kichik funksiyalar, chunki $\lim\limits_{x\to 0}\frac{x\sin\frac{1}{x}}{x}=\lim\limits_{x\to 0}\sin\frac{1}{x}$ limit mavjud emas.

Bir xil tartibli cheksiz kichik funksiyalar orasida ekvivalent cheksiz kichik funksiyalar muhim ahamiyatga ega.

$\alpha(x)$ va $\beta(x)$ funksiyalar $x \to x_0$ da cheksiz kichik funksiyalar bo'lsin.

8-ta'rif. Agar $\lim\limits_{x\to x_0}\dfrac{\alpha(x)}{\beta(x)}=1$ bo'lsa, u holda $x \to x_0$ da $\alpha(x)$ va $\beta(x)$ *ekvivalent cheksiz kichik funksiyalar* deyiladi va $\alpha(x) \sim \beta(x)$ kabi belgilanadi.

Masalan, $x \to 0$ da $\sin x$ va x funksiyalar ekvivalent cheksiz kichik funksiyalar, chunki $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

Cheksiz kichik funksiyalar uchun o'rinli bo'ladigan teoremalar bilan tanishamiz.

15-teorema. Agar ikkita cheksiz kichik funksiya nisbatida cheksiz kichik funksiyalarning har ikkalasini yoki ulardan bittasini ekvivalent cheksiz kichik funksiya bilan almashtirilsa, u holda bu nisbatning limiti o'zgarmaydi.

Isboti. $x \to x_0$ da $\alpha \sim \alpha'$ va $\beta \sim \beta'$ bo'lsin.

U holda

$$\lim_{x\to x_0}\frac{\alpha}{\beta}=\lim_{x\to x_0}\frac{\alpha}{\beta}\cdot\frac{\alpha'}{\alpha'}\cdot\frac{\beta'}{\beta'}=\lim_{x\to x_0}\frac{\alpha}{\alpha'}\cdot\lim_{x\to x_0}\frac{\beta'}{\beta}\cdot\lim_{x\to x_0}\frac{\alpha'}{\beta'}=1\cdot 1\cdot\lim_{x\to x_0}\frac{\alpha'}{\beta'}=\lim_{x\to x_0}\frac{\alpha'}{\beta'},$$

ya'ni

$$\lim_{x\to x_0}\frac{\alpha}{\beta}=\lim_{x\to x_0}\frac{\alpha'}{\beta'}.$$

16-teorema. Ikkita ekvivalent cheksiz kichik funksiyaning ayirmasi ularning har biriga nisbatan yuqori tartibli cheksiz kichik funksiya bo'ladi.

17-teorema. Chekli sondagi har xil tartibli cheksiz kichik funksiyalarning yig'indisi eng quyi tartibli qo'shiluvchiga ekvivalent bo'ladi.

Cheksiz kichik funksiyalarning yig'indisiga ekvivalent bo'lgan cheksiz kichik funksiyaga *bu yig'indining bosh qismi* deyiladi. Cheksiz kichik funksiyalarning yig'indisini uning bosh qismi bilan almasatirish *yuqori tartibli cheksiz kichik funksiyalarni tashlab yuborish* deb yuritiladi.

9-misol. $\lim\limits_{x\to 0}\dfrac{3x+7x^2+4x^5}{2\sin x}$ limitni toping.

Yechish. $\lim\limits_{x\to 0}\dfrac{3x+7x^2+4x^5}{2\sin x}=\lim\limits_{x\to 0}\dfrac{3x}{2x}=\lim\limits_{x\to 0}\dfrac{3}{2}=\dfrac{3}{2}$, chunki $x\to 0$ da $\sin x \sim x$ va 17-teoremaga ko'ra, $x\to 0$ da $3x+7x^2+4x^{54}\sim 3x$.

$\dfrac{0}{0}$ ko'rinishdagi aniqmasliklarni ochishda ekvivalent cheksiz kichik funksiyalarni almashtirish prinsipidan va ekvivalent cheksiz kichik funksiyalarning xossalaridan foydalanish mumkin.

Limitlarni hisoblashda quyidagi ekvivalentliklar qo'llaniladi.

1. $x\to 0$ da $\sin x\sim x$;
2. $x\to 0$ da $tgx\sim x$;
3. $x\to 0$ da $\arcsin x\sim x$;
4. $x\to 0$ da $arctgx\sim x$;
5. $x\to 0$ da $1-\cos x\sim\dfrac{x^2}{2}$;
6. $x\to 0$ da $e^x-1\sim x$;
7. $x\to 0$ da $a^x-1\sim x\ln a$;
8. $x\to 0$ da $\ln(1+x)\sim x$;
9. $x\to 0$ da $\log_a(1+x)\sim x\cdot\log_a e$;
10. $x\to 0$ da $(1+x)^m-1\sim mx$.

10- misol. $\lim\limits_{x\to 0}\dfrac{2^{3x}-7^x}{\sin 4x - arctg3x}$ limitni toping.

Yechish. $\lim\limits_{x\to 0}\dfrac{2^{3x}-7^x}{\sin 4x - arctg3x}=\lim\limits_{x\to 0}\dfrac{(2^{3x}-1)-(7^x-1)}{\sin 4x - arctg3x}$.

$x\to 0$ da $2^{3x}-1\sim 3x\ln 2$, $7^x-1\sim x\ln 7$, $\sin 4x\sim 4x$ va $arctg3x\sim 3x$ ekvivalentliklardan foydalanamiz:

$$\lim_{x\to 0}\frac{2^{3x}-7^x}{\sin 4x - arctg3x}=\lim_{x\to 0}\frac{3x\ln 2 - x\ln 7}{4x-3x}=\frac{3\ln 2-\ln 7}{1}=\ln\frac{8}{7}.$$

4.4.3. Mashqlar

1. Funksiya limitining ta'rifi yordamida isbotlang:

1) $\lim\limits_{x\to 2}(2x-3)=1$;
2) $\lim\limits_{x\to -1}(1-3x)=4$;
3) $\lim\limits_{x\to 1}x^2=1$;
4) $\lim\limits_{x\to 3}\left(\dfrac{2}{4-x}\right)=2$.

2. $f(x)$ funksiyaning $x=x_0$ nuqtalardagi chap va o‘ng limitlarini toping:

1) $f(x)=[x],\ x_0=3;$ 2) $f(x)=2^{\frac{1}{x}},\ x_0=0;$

3) $f(x)=\begin{cases} x, & x<2, \\ x^2-4, & x\ge 2, \end{cases}\ x_0=2;$ 4) $f(x)=\dfrac{2(1-x)-|1-x|}{4(1-x)+|1-x|},\ x_0=1.$

3. $f(x)=signx$ funksiyaning $x_0=0$ nuqtada limitga ega emasligini ko‘rsating.

4. $f(x)=x-[x]$ funksiyaning $x_0=2$ nuqtada limitga ega emasligini ko‘rsating.

5. Limitlarni toping:

1) $\lim\limits_{x\to -3}(2x^2+3x-1);$ 2) $\lim\limits_{x\to 2}\dfrac{3^x-9}{3^x+9};$

3) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-2x-3};$ 4) $\lim\limits_{x\to 5}\dfrac{x^2-7x+10}{2x^2-11x+5};$

5) $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2};$ 6) $\lim\limits_{x\to 1}\dfrac{\sqrt{2-x}-1}{\sqrt{5-x}-2};$

7) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{8-x}-2}{x};$ 8) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-1}{x};$

9) $\lim\limits_{x\to -1}\dfrac{x^3+4x^2+6x+3}{2x^2+3x+1};$ 10) $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^3-x^2-x+1};$

11) $\lim\limits_{x\to 2}\left(\dfrac{2x+1}{x-2}-\dfrac{x-7}{x^2-5x+6}\right);$ 12) $\lim\limits_{x\to -1}\left(\dfrac{3}{x^3-1}+\dfrac{1}{1-x}\right);$

13) $\lim\limits_{x\to\infty}\dfrac{4x^4-3x+2}{x^2-3x^4};$ 14) $\lim\limits_{x\to\infty}\dfrac{3x^5-4}{x^4+3x-x^5};$

15) $\lim\limits_{x\to\infty}\dfrac{x^3+2x}{x^4-2x^2+3};$ 16) $\lim\limits_{x\to\infty}\dfrac{x^5-2x^2}{2x^3+x-4};$

17) $\lim\limits_{x\to +\infty}x(\sqrt{4x^2-1}-2x);$ 18) $\lim\limits_{x\to -\infty}(\sqrt{x^2-4}+x);$

19) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{x^2-2}-x\right);$ 20) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{5x^2+1}-\dfrac{x^2}{5x+2}\right);$

21) $\lim\limits_{x\to 0}\dfrac{tg2x}{x};$ 22) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x+\sin x};$

23) $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\pi}{2}-x\right)tgx;$ 24) $\lim\limits_{x\to\pi}\dfrac{\sin 3x}{\sin 2x};$

25) $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x};$ 26) $\lim\limits_{x\to 0}\dfrac{tgx-\sin x}{x^3};$

27) $\lim\limits_{x\to 0}\frac{\sin 3x}{\sqrt{x+2}-\sqrt{2}}$;

28) $\lim\limits_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$;

29) $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-ctgx\right)$;

30) $\lim\limits_{x\to \frac{\pi}{2}}\left(tgx-\frac{1}{\cos x}\right)$;

31) $\lim\limits_{x\to 1}(x-1)ctg\pi x$;

32) $\lim\limits_{x\to \frac{1}{2}}\left(\frac{1}{2}-x\right)tg\pi x$;

33) $\lim\limits_{x\to -1}\frac{\arcsin(x+1)}{x^2+x}$;

34) $\lim\limits_{x\to 2}\frac{arctg(x-2)}{x^2-2x}$;

35) $\lim\limits_{x\to \infty}\left(\frac{2x-1}{2x+1}\right)^{3x-2}$;

36) $\lim\limits_{x\to \infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{4-x}{2}}$;

37) $\lim\limits_{x\to \infty}\left(\frac{3x-2}{x+3}\right)^{x-4}$;

38) $\lim\limits_{x\to -\infty}\left(\frac{2x+3}{x+2}\right)^{4x}$;

39) $\lim\limits_{x\to 2}\frac{e^x-e^2}{x-2}$;

40) $\lim\limits_{x\to e}\frac{\ln x-1}{x-e}$;

41) $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$;

42) $\lim\limits_{x\to 0}(\cos 2x)^{1+ctg^2 x}$;

43) $\lim\limits_{x\to 1}(3-2x)^{\frac{x}{2(1-x)}}$;

44) $\lim\limits_{x\to 2}(3-x)^{\frac{2x-3}{2-x}}$.

45) $\lim\limits_{x\to 0}\frac{e^{2x}-e^{3x}}{tgx-2\sin x}$;

46) $\lim\limits_{x\to 0}\frac{e^{3x}-e^x}{\arcsin x+3x}$;

47) $\lim\limits_{x\to +\infty}(4x+1)(\ln(3x+2)-\ln(3x-1))$;

48) $\lim\limits_{x\to +\infty}x(\ln(x+1)-\ln x)$.

6. Quyidagilarni isbotlang:

1) $x\to 0$ da $\alpha(x)=tg2x$ va $\beta(x)=3x+x^3$ funksiyalar bir xil tartibli;

2) $x\to 1$ da $\alpha(x)=\frac{x-1}{x+1}$ va $\beta(x)=\sqrt{x}-1$ funksiyalar ekvivalent;

3) $x\to +\infty$ da $\alpha(x)=\frac{1}{1+x^2}$ va $\beta(x)=\frac{1}{x\sqrt{x+2}}$ funksiyalar uchun $\alpha=o(\beta)$;

4) $x\to 0$ da $\alpha(x)=\arcsin 2x+x^2$ va $\beta(x)=1-\cos x$ funksiyalar uchun $\beta=o(\alpha)$.

7. Limitlarni ekvivalent cheksiz kichik funksiyalardan foydalanib toping:

1) $\lim\limits_{x\to 0}\frac{tg2x}{\ln(1+3x)}$;

2) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2+2x^3+3x^4}$;

3) $\lim\limits_{x\to 0}\frac{arctg3x}{\sin x-\sin 4x}$;

4) $\lim\limits_{x\to 0}\frac{3^{2x}-1}{\arcsin 2x}$;

5) $\lim\limits_{x\to 2}\dfrac{tg5(x-2)}{x^2+x-6}$; 6) $\lim\limits_{x\to 1}\dfrac{x^2+3x-4}{arctg(x-1)}$;

7) $\lim\limits_{x\to 0}\dfrac{3^{\sin x}-1}{tg2x}$; 8) $\lim\limits_{x\to 0}\dfrac{e^{\sin 2x}-1}{\arcsin x+2x^2}$;

9) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\sin^2 x}-1}{1-\cos x}$; 10) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+xtgx}-1}{x\arcsin 3x}$;

11) $\lim\limits_{x\to 0}\dfrac{\sin\sqrt{x}}{e^{\sqrt{x}}-e^{x\sqrt{x}}}$; 12) $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-e^{3x}}{arctg2x-\arcsin 3x}$;

13) $\lim\limits_{x\to 0}\dfrac{e^{tgx}-1}{\ln(1+\arcsin 2x)}$; 14) $\lim\limits_{x\to 0}\dfrac{3^{2x}-5^{x}}{\arcsin 2x-x^3}$;

15) $\lim\limits_{x\to\pi}\dfrac{\sin 3x}{3tg4x}$; 16) $\lim\limits_{x\to\pi}\dfrac{\ln(2+\cos x)}{\sin x(e^{tgx}-1)}$;

17) $\lim\limits_{x\to 0}\dfrac{tgx-\sin x}{x^3+3x^4}$; 18) $\lim\limits_{x\to 0}\dfrac{x\ln(\cos 3x)}{tgx-\sin x}$;

19) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos 2x}{x\sin x}$; 20) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{e^{\cos x}-1}{x\cos x}$;

21) $\lim\limits_{x\to\infty}x\cdot(e^{1/x^2}-1)$; 22) $\lim\limits_{x\to\infty}x\cdot(2^{1/x}-3^{1/x})$

23) $\lim\limits_{x\to 0}\dfrac{(e^{2x^3}-1)\cdot tg3x}{\ln(1-3x^2)(1-\cos 2x)}$; 24) $\lim\limits_{x\to 0}\dfrac{(\sqrt{1+tgx}-1)\cdot\sin 3x}{x(e^{\arcsin x}-1)}$.

4.5. FUNKSIYANING UZLUKSIZLIGI

4.5.1. Funksiya uzluksizligining ta'riflari

$f(x)$ funksiya x_0 nuqtada va uning biror atrofida aniqlangan bo'lsin.

1-ta'rif. Agar $f(x)$ funksiya x_0 nuqtada chekli limitga ega bo'lib, bu limit funksiyaning shu nuqtadagi qiymatiga teng, ya'ni

$$\lim_{x\to x_0} f(x)=f(x_0) \tag{5.1}$$

bo'lsa, $f(x)$ *funksiya* x_0 *nuqtada uzluksiz* deyiladi.

$\lim\limits_{x\to x_0} f(x)=f(x_0)$ tenglik uchta shartning bajarilishini anglatadi:

1) $f(x)$ funksiya x_0 nuqtada va uning atrofida aniqlangan;

2) $f(x)$ funksiya $x \to x_0$ da limitga ega;

3) funksiyaning x_0 nuqtadagi limiti uning shu nuqtadagi qiymatiga teng.

$x_0 = \lim\limits_{x \to x_0} x$ ekanidan (5.1) tenglikni

$$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) \tag{5.2}$$

ko'rinishda yozish mumkin. Demak, uzluksiz funksiya uchun limitga o'tish va funksiya belgilarining o'rnini almashtirish mumkin.

Funksiya limitining ta'rifi asosida funksiya uzluksizligining ta'rifini «$\varepsilon - \delta$ tilida» quyidagicha ifodalash mumkin.

2- ta'rif. Agar $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, x ning $|x - x_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - f(x_0)| < \varepsilon$ tengsizlik bajarilsa, $f(x)$ *funksiya* x_0 *nuqtada uzluksiz* deyiladi.

(5.1) tenglikni

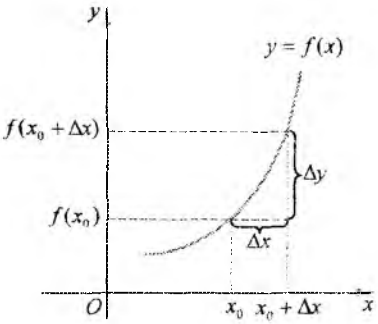
$$\lim_{x - x_0 \to 0} (f(x) - f(x_0)) = 0 \tag{5.3}$$

ko'rinishda yozamiz.

$x - x_0$ ayirmaga x *argumentning* x_0 *nuqtadagi orttirmasi* deyiladi va Δx bilan belgilanadi, $f(x) - f(x_0)$ ayirmaga esa $f(x)$ *funksiyaning* x_0 *nuqtadagi orttirmasi* deyiladi va Δy bilan belgilanadi.

Shunday qilib, $\Delta x = x - x_0$, $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

Demak, $f(x)$ funksiyaning x_0 nuqtadagi orttirmasi x ning fiksirlangan x_0 qiymatida argument orttirmasining funksiyasi bo'ladi (28-shakl).



28-shakl.

(5.3) tenglik yangi belgilashlarda

$$\lim_{\Delta x \to 0} \Delta y = 0 \tag{5.4}$$

ko'rinishni oladi.

(5.4) tenglikni uzluksizlikning argument orttirmasi va funksiya orttirmasi tushunchalariga asoslan-gan ta'rifi sifatida quyidagicha ifodalash mumkin.

3-ta'rif. Agar x argumentning x_0 nuqtadagi cheksiz kichik orttirmasiga $f(x)$ funksiyaning shu nuqtadagi cheksiz kichik orttirmasi mos kelsa, *$f(x)$ funksiya x_0 nuqtada uzluksiz* deyiladi.

Funksiyaning nuqtadagi uzlukizligini tekshirishda keltirilgan ta'riflarning istalgan biridan foydalanish mumkin.

1-misol. $y = \cos x$ funksiyani uzluksizlikka tekshiring.

Yechish. $y = \cos x$ funksiya $x \in R$da aniqlangan. Istalgan $x$nuqtani olamiz va bu nuqtada Δyni topamiz:

$$\Delta y = \cos(x + \Delta x) - \cos x = -2\sin\left(x + \frac{\Delta x}{2}\right) \cdot \sin\frac{\Delta x}{2}.$$

Bundan $\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0}\left(-2\sin\left(x + \frac{\Delta x}{2}\right) \cdot \sin\frac{\Delta x}{2}\right) = 0$ kelib chiqadi, chunki chegaralangan $\sin\left(x + \frac{\Delta x}{2}\right)$ funksiyaning cheksiz kichik $\sin\frac{\Delta x}{2}$ funksiyaga ko'paytmasi cheksiz kichik bo'ladi.

Demak, 3-ta'rifga ko'ra, $y = \cos x$ funksiya $x$nuqtada uzluksiz.

4-ta'rif. Agar $\lim\limits_{x \to x_0 + 0} f(x) = f(x_0)$ $\left(\lim\limits_{x \to x_0 - 0} f(x) = f(x_0)\right)$ bo'lsa, *$f(x)$ funksiya x_0 nuqtada o'ngdan (chapdan) uzluksiz* deyiladi.

1-ta'rif va 4-ta'riflardan quyidagi xulosa kelib chiqadi: $f(x)$ funksiya x_0 nuqtada uzluksiz bo'lishi uchun u shu nuqtada ham chapdan, ham o'ngdan uzluksiz bo'lishi zarur va yetarli.

4.5.2. Uzluksiz funksiyalarning xossalari

Nuqtada uzluksiz funksiyalarning xossalari

1-teorema (*uzluksiz funksiyalar ustida arifmetik amallar*). $f(x)$ va $g(x)$ funksiyalar x_0 nuqtada uzluksiz bo'lsa, u holda $f(x) \pm g(x)$, $f(x) \cdot g(x)$ va $\frac{f(x)}{g(x)}$ $(g(x_0) \neq 0)$ funksiyalar ham x_0 nuqtada uzluksiz bo'ladi.

Isboti. $f(x)$ va $g(x)$ funksiyalar x_0 nuqtada uzluksiz bo'lgani uchun ular bu nuqtada $f(x_0)$ va $g(x_0)$ limitlarga ega. U holda funksiyaning limiti haqidagi teoremalarga ko'ra, $f(x)\pm g(x)$, $f(x)\cdot g(x)$ va $\dfrac{f(x)}{g(x)}(g(x_0)\neq 0)$ funksiyalarning x_0 nuqtadagi limitlari mavjud va ular mos ravishda $f(x_0)\pm g(x_0)$, $f(x_0)\cdot g(x_0)$ va $\dfrac{f(x_0)}{g(x_0)}(g(x_0)\neq 0))$ ga teng bo'ladi. U holda 1-ta'rifga ko'ra, $f(x)\pm g(x)$, $f(x)\cdot g(x)$ va $\dfrac{f(x)}{g(x)}$ $(g(x_0)\neq 0)$ funksiyalar x_0 nuqtada uzluksiz.

Bu teorema chekli sondagi funksiyalarning algebraik yig'indisi va ko'paytmasi uchun ham o'rinli bo'ladi.

2-teorema. (*murakkab funksiyaning uzluksizligi*). $z=\varphi(x)$ funksiya x_0 nuqtada uzluksiz, $y=f(z)$ funksiya esa $z_0=\varphi(x_0)$ nuqtada uzluksiz bo'lsin. U holda $y=f(\varphi(x))$ murakkab funksiya x_0 nuqtada uzluksiz bo'ladi.

Isboti. $z=\varphi(x)$ funksiya x_0 nuqtada uzluksizligidan $z=\varphi(x)$, $\lim\limits_{x\to x_0}\varphi(x)=\varphi(x_0)$, ya'ni $x\to x_0$ da $z\to z_0$ bo'ladi.

Shu sababli $z=\varphi(x)$ funksiyaning uzluksiligidan

$$\lim_{x\to x_0} f(\varphi(x))=\lim_{z\to z_0} f(z)=f(z_0)=f(\varphi(x_0))$$

kelib chiqadi. Bu $f(\varphi(x))$ murakkab funksiyaning x_0 nuqtada uzluksizligini bildiradi.

2-teorema yordamida (5.2) tenglikni quyidagicha umumlashtirish mumkin.

Agar $z=\varphi(x)$ funksiya x_0 nuqtada A limitga ega bo'lib, $y=f(z)$ funksiya $z=A$ nuqtada uzluksiz bo'lsa, u holda $y=f(\varphi(x))$ murakkab funksiya uchun

$$\lim_{x\to x_0} f(\varphi(x))=f(\lim_{x\to x_0}\varphi(x)) \tag{5.5}$$

bo'ladi.

Bu tenglik *uzluksiz funksiya belgisi ostida limitga o'tish qoidasini* ifodalaydi va funksiyaning limitini topishda foydalaniladi.

2- misol. $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}$ $(a>0, a\neq 1)$ limitni toping.

Yechish. $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{1}{x}\cdot\log_a(1+x)=\lim\limits_{x\to 0}\log_a(1+x)^{\frac{1}{x}}.$

$\log_a(1+x)^{\frac{1}{x}}$ funksiya $y=\log_a z$ va $z=(1+x)^{\frac{1}{x}}$ funksiyalarning murakkab funksiyasi. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$ va $y=\log_a z$ funksiya $z=e$ nuqtada uzluksiz. U holda (5.5) tenglikka ko'ra,

$$\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}=\log_a\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)=\log_a e.$$

Xususan, $a=1$ da $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$.

Sonlar o'qida aniqlangan $f(x)=C$ funksiyani qaraymiz. $\forall x_0\in R$ da $\lim\limits_{x\to x_0}f(x)=C=f(x_0)$ bo'ladi. Demak, $f(x)=C$ o'zgarmas funksiya istalgan x_0 nuqtada uzluksiz.

$f(x)=x$ funksiya ham x_0 nuqtada uzluksiz, chunki $\lim\limits_{x\to x_0}x=x_0$.

Bundan 1-teoremaga ko'ra, $f(x)=x$ funksiya ko'paytmalaridan iborat $y=x^n$ $(n\in N)$ darajali funksiya hamda o'zgarmas va darajali funksiyalardan arifmetik amallar orqali hosil qilingan $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ ko'phad (butun-ratsional funksiya) istalgan $x_0\in R$ nuqtada uzluksiz bo'ladi.

Shu kabi yuqorida keltirilgan teoremalar va limitlar haqidagi teoremalar yordamida asosiy elementar funksiyalar o'zining aniqlanish sohasida uzluksiz bo'lishini ko'rsatish va ushbu teoremani isbotlash mumkin.

3-teorema. Elementar funksiyalar o'zining aniqlanish sohasidagi barcha nuqtalarda uzluksiz bo'ladi.

4-teorema. Agar $f(x)$ funksiya x_0 nuqtada uzluksiz va $f(x_0)>A$ $(f(x_0)<A)$ bo'lsa, u holda shunday $\delta>0$ son topiladi va $\forall x\in(x_0-\delta;x_0+\delta)$ uchun $f(x)>A$ $(f(x)<A)$ bo'ladi.

Isboti. $f(x_0)>A$ bo'lsin. Aniqlik uchun $f(x_0)=A+h$ deymiz, bu yerda $h>0$. $\varepsilon=\dfrac{h}{2}$ son olamiz. $f(x)$ funksiyaning x_0 nuqtada uzluksizligidan shunday $\delta>0$ son topiladi va x ning $|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)-f(x_0)|<\dfrac{h}{2}$ tengsizlik bajariladi. Bundan $\forall x\in(x_0-\delta;x_0+\delta)$ uchun

$$f(x) > f(x_0) - \frac{h}{2} = A + h - \frac{h}{2} = A + \frac{h}{2} > A$$

bo‘ladi.

$f(x_0) < A$ bo‘lsin. $-f(x)$ funksiyani qaraymiz. $-f(x_0) > -A$ bo‘lgani sababli yuqoridagi isbotga asosan, x_0 nuqtaning $\delta > 0$ atrofi topiladi va bu atrofda $-f(x) > -A$ yoki $f(x) < A$ bo‘ladi.

5-teorema (*uzluksiz funksiya ishorasining turg‘unligi*). Agar $f(x)$ funksiya x_0 nuqtada uzluksiz va $f(x_0) \neq 0$ bo‘lsa, u holda shunday $\delta > 0$ son topiladi va $(x_0 - \delta; x_0 + \delta)$ intervalda $f(x)$ funksiya ishorasini saqlaydi, ya’ni $f(x_0)$ funksiya bilan bir ishorali bo‘ladi.

Teoremaning isboti 4-teoremadan $A = 0$ bo‘lganda kelib chiqadi.

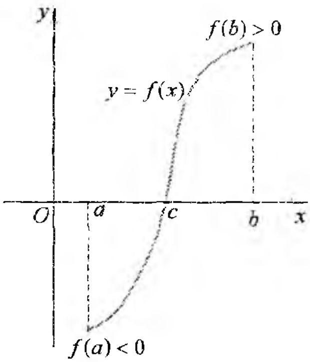
Kesmada uzluksiz funksiyalarning xossalari

Agar $f(x)$ funksiya $(a;b)$ intervalning har bir nuqtasida uzluksiz bo‘lsa, u holda u $(a;b)$ *intervalda uzluksiz* deyiladi.

Agar $f(x)$ funksiya $(a;b)$ intervalda uzluksiz bo‘lib, a nuqtada o‘ngdan uzluksiz va b nuqtada chapdan uzluksiz bo‘lsa, u holda $f(x)$ funksiyaga $[a;b]$ *kesmada uzluksiz* deyiladi.

Kesmada uzluksiz funksiyalar bir qancha muhim xossalarga ega. Bu xossalarni teoremalar orqali ifodalaymiz. Bunda teoremalarning isbotini keltirmasdan, faqat geometrik talqinini ko‘rsatish bilan kifoyalanamiz.

6-teorema (*Bolsano-Koshining birinchi teoremasi*). $f(x)$ funksiya $[a;b]$ kesmada uzluksiz va kesmaning oxirlarida turli ishorali qiymatlar qabul qilinsin. U holda shunday $c \in (a;b)$ nuqta topiladiki, bu nuqtada $f(c) = 0$ bo‘ladi.



29-shakl.

Teoremaning geometrik talqini: uzluksiz funksiyaning grafigi Ox o‘qning bir tomonidan ikkinchi tomoniga o‘tganida Ox o‘qni kesadi (29-shakl).

7-teorema (*Bolsano-Koshining ikkinchi teoremasi*). $f(x)$ funksiya $[a;b]$ kesmada uzluksiz va $f(a) = A$, $f(b) = B$, $C - A$ va B orasidagi ixtiyoriy son bo‘lsin. U holda shunday $c \in [a;b]$ nuqta topiladiki, $f(c) = C$ bo‘ladi.

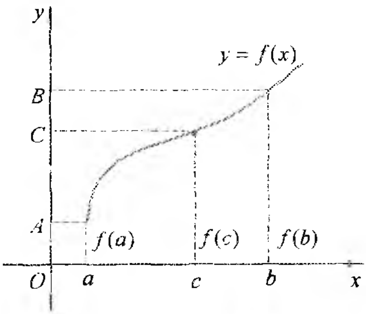
Teoremaning geometrik talqini: uzluksiz funksiya bir qiymatdan ikkinchi qiymatga o'tganida barcha oraliq qiymatlarni qabul qiladi (30-shakl).

8-teorema (*Veyershtrassning birinchi teoremasi*). Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda u bu kesmada chegaralangan bo'ladi.

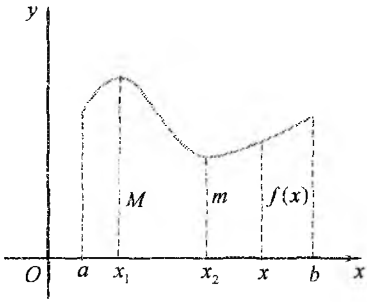
31-shaklda keltirilgan $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz. Bunda $\forall x\in[a;b]$ uchun $m\le f(x)\le M$.

1-izoh. Teorema $[a;b]$ kesma $(a;b)$ interval bilan almashtirilganida o'rinli bo'lmasligi mumkin.

Masalan, $f(x)=\dfrac{1}{x}$ funksiya $(0;1)$ intervalda uzluksiz, lekin chegaralanmagan, chunki $\lim\limits_{x\to+0}\dfrac{1}{x}=+\infty$.



30-shakl.



31-shakl.

9-teorema (*Veyershtrassning ikkinchi teoremasi*). Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda u shu kesmada o'zining eng kichik va eng katta qiymatlariga erishadi.

31-shaklda keltirilgan $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz. Bunda u x_1 nuqtada o'zining eng katta M qiymatini va x_2 nuqtada o'zining eng kichik m qiymatini qabul qiladi.

2-izoh. Bu teorema $(a;b)$ interval uchun o'rinli bo'lmasligi mumkin. Masalan, $f(x)=x$ funksiya $(0;1)$ intervalda uzluksiz, lekin o'zining eng kichik va eng katta qiymatlariga erishmaydi.

10-teorema (*teskari funksiyaning uzluksizligi haqidagi*). Agar $y = f(x)$ funksiya $[a;b]$ kesmada uzluksiz va qat'iy monoton bo'lib, $[c;d]$ uning qiymatlar sohasi bo'lsa, u holda berilgan funksiyaga teskari $y = \varphi(x)$ funksiya $[c;d]$ kesmada uzluksiz va qat'iy monoton bo'ladi.

4.5.3. Funksiyaning uzulish nuqtalari

Agar $f(x)$ funksiya uchun x_0 nuqtada funksiya uzluksizligi 1-ta'rifining hech bo'lmaganda bitta sharti bajarilmasa, *funksiya* x_0 ***nuqtada uzilishga ega*** deyiladi. Bunda x_0 nuqta $f(x)$ *funksiyaning* ***uzilish nuqtasi*** deb ataladi.

32-shakldа gfrafiklar bilan berilgan funksiylarga qaraymiz.

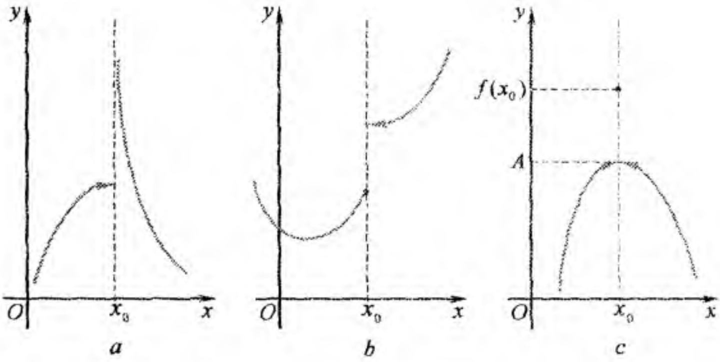
Bu funksiyalarning har biri uchun x_0 – uzilish nuqtasi.

Birinchi holda (32,a-shakl) ta'rifning 1-sharti bajarilmaydi, chunki funksiya x_0 nuqtada aniqlanmagan.

Ikkinchi holda (32,b-shakl) ta'rifning 2-sharti buzilgan, chunki $\lim\limits_{x \to x_0} f(x)$ limit mavjud emas.

Uchinchi holda (32,c-shakl) ta'rifning 3-sharti bajarilmaydi, chunki $\lim\limits_{x \to x_0} f(x) = A \neq f(x_0)$.

Funksiyaning barcha uzilish nuqtalari birinchi va ikkinchi tur uzilish nuqtalariga bo'linadi.



32-shakl.

5-ta'rif. Agar x_0 uzilish nuqtasida $f(x)$ funksiya chekli bir tomonlama limitlarga ega, ya'ni $\lim\limits_{x \to x_0 - 0} f(x) = A_1$ va $\lim\limits_{x \to x_0 + 0} f(x) = A_2$ bo'lsa, x_0 nuqtaga $f(x)$ funksiyaning *birinchi tur uzilish nuqtasi* deyiladi. Bunda:

a) $A_1 = A_2$ bo'lsa, x_0 *bartaraf qilinadigan uzilish nuqtasi* deb ataladi;

b) $A_1 \neq A_2$ bo'lsa, x_0 *sakrash nuqtasi*, $|A_1 - A_2|$ kattalik *funksiyaning sakrashi* deb ataladi.

Masalan: $g(x) = \begin{cases} 2x - 1, & -1 \le x < 1, \\ 4 - 2x, & 1 \le x \le 3 \end{cases}$ funksiya uchun $x_0 = 1$ – sakrash nuqtasi, bunda funksiyaning sakrashi $|1 - 2| = 1$ ga teng;

$\varphi(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 2, & x = 0 \end{cases}$ funksiya uchun $x_0 = 0$ – bartaraf qilinadigan uzilish nuqtasi, bunda $\varphi(x) = 2$ o'rniga $\varphi(x) = 1$ deb olinsa uzilish bartaraf qilinadi, ya'ni $\varphi(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$ uzluksiz funksiya hosil bo'ladi.

6-ta'rif. Agar x_0 uzilish nuqtasida $f(x)$ funksiyaning bir tomonlama limitlaridan kamida bittasi mavjud bo'lmasa yoki cheksizlikka teng bo'lsa, x_0 nuqtaga $f(x)$ funksiyaning *ikkinchi tur uzilishi nuqtasi* deyiladi.

Masalan, $f(x) = \dfrac{1}{x}$ funksiya uchun $x_0 = 0$ – ikkinchi tur uzilish nuqtasi.

3-misol. $f(x) = \dfrac{|2x - 3|}{2x - 3}$ funksiyaning uzilish nuqtalarini toping va har bir uzilish nuqtasining turini aniqlang.

Yechish. Funksiya sonlar o'qining $x = \dfrac{3}{2}$ nuqtasidan boshqa nuqtalarida aniqlangan va uzluksiz.

Bunda

$$f(x)=\begin{cases}-1, & x<\frac{3}{2},\\ 1, & x>\frac{3}{2}.\end{cases}$$

U holda

$$f\left(\frac{3}{2}-0\right)=-1,\quad f\left(\frac{3}{2}+0\right)=1.$$

Demak, $x=\frac{3}{2}$ sakrash nuqtasi va funksiyaning sakrashi $\mu=|1-(-1)|=2$.

4.5.4. Tekis uzluksizlik

$f(x)$ funksiya $(a;b)$ intervalda uzluksiz bo'lsin. U holda istalgan $x_0\in(a;b)$ nuqtada $\forall\varepsilon>0$ son uchun shunday $\delta>0$ son topiladi va $|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha $x_0\in(a;b)$ uchun $|f(x)-f(x_0)|<\varepsilon$ tengsizlik bajariladi. Bunda δ ham ε ga, ham x_0 ga bog'liq bo'ladi: $\delta=\delta(\varepsilon;x_0)$. Bitta $\varepsilon>0$ son uchun har xil $x\in(a;b)$ nuqtalarda δ son turlicha bo'lishi mumkin va bunda barcha $x\in(a;b)$ da yagona δ sonning mavjud bo'lishi kelib chiqmaydi. Bunday $\delta=\delta(\varepsilon)>0$ son mavjud bo'lishining talabi $f(x)$ funksiyaning $(a;b)$ intervalda uzluksiz bo'lishi talabiga nisbatan kuchli talab hisoblanadi.

7-ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topilsa va $(a;b)$ intervalning $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy x' va x'' sonlari uchun $|f(x')-f(x'')|<\varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya $(a;b)$ *intervalda tekis uzluksiz* deyiladi.

Masalan, $f(x)=x$ funksiya butun sonlar o'qida tekis uzluksiz. Bunda $\delta=\varepsilon$ deb olish yetarli.

Agar $f(x)$ funksiya $(a;b)$ intervalda tekis uzluksiz bo'lsa, u holda u har bir $x\in(a;b)$ nuqtada uzluksiz bo'ladi. Teskari tasdiq o'rinli bo'lmaydi. Agar bunda $(a;b)$ interval $[a;b]$ kesma bilan almashtirilsa, teskari tasdiq ham o'rinli bo'ladi.

11-teorema (*Kantor teoremasi*). Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda u $[a;b]$ kesmada tekis uzluksiz bo'ladi.

4.5.5. Mashqlar

1. Funksiyaning uzluksizligi ta'rifidan foydalanib, berilgan funksiyalarning $\forall x_0 \in R$ da uzluksiz ekanini isbotlang:

1) $f(x) = 3x^2 - 7$;   2) $f(x) = x^3 + 7x - 6$.

2. Uzluksiz funksiyalarning xossalaridan foydalanib, berilgan funksiyalarning $(-\infty;+\infty)$ intervalda uzluksiz ekanini isbotlang:

1) $f(x) = \cos 3x - e^{2x-1}$;   2) $f(x) = \sqrt[3]{x-3} + \sin^2 x + \dfrac{3}{x^2+2}$.

3. Berilgan funksiyalarni uzluksizlikka tekshiring va grafigini chizing:

1) $f(x) = \dfrac{x}{|x|}$;   2) $f(x) = x^2 + \dfrac{|x+1|}{x+1}$;

3) $f(x) = \begin{cases} x^2, & x \neq 2, \\ 3, & x = 2; \end{cases}$ 4) $f(x) = \begin{cases} 3x-1, & x < 0, \\ \dfrac{1}{x-1}, & x \geq 0; \end{cases}$

5) $f(x) = 2^{\frac{x}{x^2-1}}$;   6) $f(x) = \dfrac{3}{1+2^{1/x}}$;

7) $f(x) = \begin{cases} 1, & x < -3, \\ \sqrt{9-x^2}, & -3 \leq x \leq 3, \\ x-3, & x > 3; \end{cases}$ 8) $f(x) = \begin{cases} x^2, & x \leq 2, \\ 4, & 2 < x < 5, \\ -x+7, & x \geq 5; \end{cases}$

9) $f(x) = \dfrac{|x-3|}{x^2-2x-3}$;   10) $f(x) = \dfrac{|\sin x|}{(x-1)\sin x}$.

4. a ning qanday qiymatlarida berilgan funksiyalar uzluksiz bo'ladi?

1) $f(x) = \begin{cases} \dfrac{x^2+3x-10}{x-2}, & x < 2, \\ a^2 - x, & x \geq 2; \end{cases}$ 2) $f(x) = \begin{cases} 3^x, & x \geq 0, \\ a\cos x + 2, & x < 0. \end{cases}$

5. $f(x)$ funksiyaning x_0 nuqtadagi uzulish turini aniqlang:

1) $f(x) = \dfrac{3x+4}{x-3}$, $x_0 = 3$;   2) $f(x) = \dfrac{x^2-9}{x+3}$, $x_0 = -3$;

3) $f(x) = arctg\dfrac{5}{2x-1}$, $x_0 = \dfrac{1}{2}$;   4) $f(x) = \dfrac{3}{4^{x-3}-1}$, $x_0 = 3$.

6. Murakkab funksiyani uzluksizlikka tekshiring:

1) $f(z) = \dfrac{2}{z^2+1}$, $z = \begin{cases} x+2, & x < 0, \\ x-2, & x \geq 0; \end{cases}$ 2) $f(z) = 2z^2 - 3$, $z = tgx$.

7. $f(x)=\dfrac{1}{(x+3)(x-4)}$ funksiyani $[a;b]$ kesmada uzluksizlikka tekshiring:

1) $[a;b]=[-4;1]$; 2) $[a;b]=[-2;3]$.

8. $f(x)$ funksiyani $[0;2],[-3;1],[4;5]$ kesmalarda uzluksizlikka tekshiring:

1) $f(x)=\dfrac{1}{x^2+2x-3}$; 2) $f(x)=\ln\dfrac{x-4}{x+5}$.

9. Tenglamalar berilgan kesmada kamida bitta ildizga ega bo'lishini ko'rsating:

1) $x^3-5x^2+3x+2=0$, $[-1;1]$; 2) $\sin x-x+1=0$, $[1;2]$.

10. Limitlarni toping:

1) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$ $(a>0, a\neq 0)$, 2) $\lim\limits_{x\to 0}\dfrac{(1+x)^m-1}{x}$.

5

BIR O'ZGARUVCHI FUNKSIYASINING DIFFERENSIAL HISOBI

- Funksiyaning hosilasi va differensiali
- Differensiallash qoidalari va formulalari
- Differensial hisobning asosiy teoremalari
- Funksiyalarni hosilalar yordamida tekshirish

Gotfrid Vilgelm Leybnis (1646–1716) – nemis matematigi, faylasufi, fizigi, xuquqshunosi va tilshunosi.

Leybnis cheksiz kichiklarga asoslangan differensial va integral hisobni yaratgan, kombinatorikani fan sifatida asoslagan, matematik mantiqqa asos solgan, 0 va 1 sonlari bilan ikkilik sanoq sistemasini tasvirlagan.

Mexanikada energiyaning saqlanish qonunini asoslab bergan.

Differensial hisob – bu matematik analizning hosila va differensial tushunchalari hamda ularning funksiyalarni tekshirishga tatbiqi masalalari o'rganiladigan bo'limidir. Differensial hisobning rivojlanishi integral hisobning rivojlanishi bilan uzviy bog'liq. Ular birgalikda tabiatshunoslik va texnika uchun muhim ahamiyatga ega bo'lgan matematik analizning asosini tashkil qiladi.

Differensial hisobning matematik fan sifatida yuzaga chiqishini, odatda, *I.Nyuton* va *G.Leybnis* (XVII asrning ikkinchi yarmida) nomlari bilan bog'lashadi. Ular differensial hisobning asosiy qoidalarini mustaqil ravishda ishlab chiqishgan.

5.1. FUNKSIYANING HOSILASI VA DIFFERENSIALI

5.1.1. Hosila tushunchasiga olib keluvchi masalalar

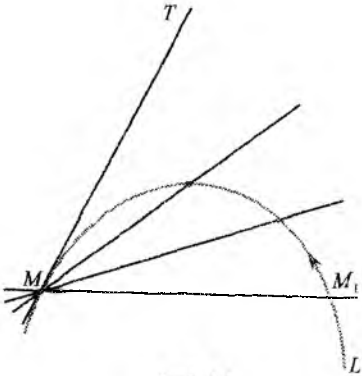
Egri chiziqqa o'tkazilgan urinma

Avval egri chiziqqa o'tkazilgan urinmaning umumiy ta'rifini beramiz. Uzluksiz L egri chiziqda M va M_1 nuqtalarni olamiz (1-shakl).

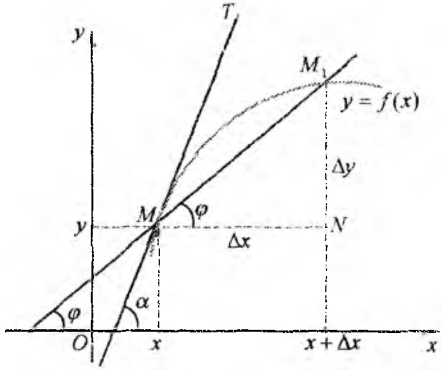
M va M_1 nuqtalar orqali o'tuvchi MM_1 to'g'ri chiziqqa *kesuvchi* deyiladi.

M_1 nuqta L egri chiziq bo'ylab siljib, M nuqtaga yaqinlashsin. U holda MM_1 kesuvchi M nuqta atrofida buriladi va qandaydir MT limit holatiga intiladi.

Berilgan L egri chiziqqa berilgan M nuqtada o'tkazilgan urinma deb, MM_1 kesuvchining M_1 nuqta L egri chiziq bo'ylab siljib, M nuqtaga yaqinlashgandagi MT limit holatiga (agar mavjud bo'lsa) aytiladi.



1-shakl.



2-shakl.

Endi $M(x;y)$ nuqtada vertikal bo'lmagan urinmaga ega bo'lgan $y=f(x)$ uzluksiz funksiya grafigini qaraymiz va uning $k=tg\alpha$ burchak koeffitsiyentini topamiz, bu yerda α – urinmaning Ox o'q bilan tashkil qilgan burchagi. Buning uchun M nuqta va grafikning $x+\Delta x$ abssissali M_1 nuqtasi orqali kesuvchi o'tkazamiz (2-shakl). Kesuvchining Ox o'q bilan tashkil qilgan burchagini φ bilan belgilaymiz.

2-shakldan topamiz:

$$tg\varphi = \frac{M_1N}{MN} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

$\Delta x \to 0$ da funksiyaning uzluksizligiga asosan, Δy ham nolga intiladi. Shu sababli $\Delta x \to 0$ da M_1 nuqta egri chiziq bo'ylab siljib, M nuqtaga yaqinlashadi. Bunda MM_1 kesuvchi M nuqta atrofida buriladi va MT urinmaga yaqinlashib boradi, ya'ni $\varphi \to \alpha$. Bundan $\lim\limits_{\Delta x\to 0}\varphi = \alpha$ yoki $\lim\limits_{\Delta x\to 0} tg\varphi = tg\alpha$.

Shuning uchun urinmaning burchak koeffitsiyenti

$$k = tg\alpha = \lim_{\Delta x \to 0} tg\varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \qquad (1.1)$$

To'g'ri chiziqli harakat tezligi

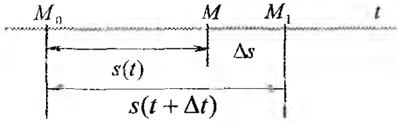
M moddiy nuqta (biror jism) qandaydir to'g'ri chiziq bo'ylab harakat qilayotgan bo'lsin. Vaqtning har bir t qiymatiga boshlang'ich M_0 holatdan M holatgacha bo'lgan muayyan $s = M_0M$ masofa mos keladi. Bu masofa t vaqtga bog'liq, ya'ni s masofa vaqtning funksiyasi bo'ladi: $s = s(t)$.

$s(t)$ funksiyaga *nuqtaning harakat qonuni* deyiladi.

Nuqtaning t vaqtdagi harakat tezligini aniqlash masalasini qaraymiz.

Agar biror t vaqtda nuqta M holatda bo'lsa, u holda $t+\Delta t$ (Δt – vaqtning orttirmasi) vaqtda nuqta M_1 holatga o'tadi, bu yerda $M_0M_1 = s + \Delta s$ (Δs – masofaning orttirmasi). Demak, M nuqtaning Δt vaqt oralig'idagi ko'chishi $\Delta s = s(t+\Delta t) - s(t)$ ga teng bo'ladi (3-shakl).

$\dfrac{\Delta s}{\Delta t}$ nisbat *nuqtaning* Δt *vaqt oralig'idagi o'rtacha tezligini* ifodalaydi: $v_{o'r} = \dfrac{\Delta s}{\Delta t}$. Bunda o'rtacha tezlik Δt qiymatga bog'liq bo'ladi: Δt qancha kichik bo'lsa, o'rtacha tezlik nuqtaning berilgan t vaqtdagi tezligini shuncha aniq ifodalaydi.



3-shakl.

Harakat o'rtacha tezligining Δt vaqt oralig'i nolga intilgandagi limitiga *nuqtaning berilgan vaqtdagi harakat tezligi* (yoki *oniy tezligi*) deyiladi. Bu t ondagi tezlikni $v(t)$ bilan belgilaymiz.

U holda

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \quad \text{yoki} \quad v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}. \qquad (1.2)$$

Shunday qilib, nuqtaning berilgan t vaqtdagi harakat tezligini aniqlash uchun (1.2) limitni hisoblash kerak bo'ladi.

Tabiatning turli sohalariga tegishli ko'plab masalalar (1.1) va (1.2) ko'rinishdagi limitlarni topishga olib keladi.

Bunday masalalardan yana ayrimlarini keltiramiz:

1) agar $Q=Q(t)$ o'tkazgichning ko'ndalang kesimi orqali vaqtning t onida o'tuvchi elektr toki bo'lsa, u holda *elektr tokining t ondagi momenti*

$$I(t)=\lim_{\Delta t\to 0}\frac{\Delta Q}{\Delta t}=\lim_{\Delta t\to 0}\frac{Q(t+\Delta t)-Q(t)}{\Delta t};\qquad(1.3)$$

2) agar $N=N(t)$ vaqtning t onida reaksiyaga kirishuvchi kimyoviy modda miqdori bo'lsa, u holda *kimyoviy moddaning t ondagi reaksiyaga kirishish tezligi*

$$V(t)=\lim_{\Delta t\to 0}\frac{\Delta N}{\Delta t}=\lim_{\Delta t\to 0}\frac{N(t+\Delta t)-N(t)}{\Delta t};\qquad(1.4)$$

3) agar $m=m(x)$ bir jinsli bo'lmagan sterjenning $O(0;0)$ va $M(x;0)$ nuqtalar orasidagi massasi bo'lsa, u holda *sterjenning x nuqtadagi zichligi*

$$\gamma(t)=\lim_{\Delta x\to 0}\frac{\Delta m}{\Delta x}=\lim_{\Delta t\to 0}\frac{m(x+\Delta x)-m(x)}{\Delta x}.\qquad(1.5)$$

Ko'rilgan masalalar fizik mazmunining turliligiga qaramasdan, (1.1)-(1.5) limitlar bir xil ko'rinishga ega: ularda funksiya orttirmasining argument orttirmasiga nisbatining argument orttirmasi nolga intilgandagi limitini topish talab qilinadi.

5.1.2. Hosilaning ta'rifi, geometrik va mexanik ma'nolari

Hosilaning ta'riflari

$y=f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lsin. Ixtiyoriy $x_0\in(a;b)$ nuqtani olamiz va bu nuqtada x argumentga Δx orttirma $(x_0+\Delta x\in(a;b))$ beramiz. Bunda funksiya $\Delta y=f(x_0+\Delta x)-f(x_0)$ orttirma oladi.

1-ta'rif. Agar $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ limit mavjud va chekli bo'lsa, bu limitga $f(x)$ *funksiyaning x_0 nuqtadagi hosilasi* deyiladi va u $f'(x_0)$ (yoki $y'(x_0)$ yoki $y'|_{x=x_0}$) kabi belgilanadi.

Shunday qilib,

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}. \qquad (1.6)$$

Agar x_0 ning biror qiymatida $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=+\infty$ $\left(\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\infty\right)$ bo'lsa, u holda funksiya x_0 nuqtada musbat ishorali (manfiy ishorali) cheksiz hosilaga ega deyiladi. Shu sababli 1-ta'rif bilan aniqlanadigan hosila chekli hosila deb yuritiladi.

1-misol. $f(x)=x^3$ funksiyaning $x=x_0$ nuqtadagi hosilasini toping.

Yechish. x_0 nuqtada x argumentga Δx orttirma beramiz va funksiyaning mos orttirmasini topamiz:

$$\Delta y=f(x_0+\Delta x)-f(x_0)=(x_0+\Delta x)^3-x_0^3=$$

$$=(x_0+\Delta x-x_0)(x_0^2+2x_0\Delta x+\Delta x_0^2+x_0^2+x_0\Delta x+x_0^2)=\Delta x(3x_0^2+3x_0\Delta x+\Delta x^2).$$

Orttirmalar nisbatini tuzamiz:

$$\frac{\Delta y}{\Delta x}=3x_0^2+3x_0\Delta x+\Delta x^2.$$

Bu nisbatning $\Delta x\to 0$ dagi limitini topamiz:

$$y'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(3x_0{}^2+3x_0\Delta x+\Delta x^2)=3x_0{}^2.$$

2-ta'rif. $y=f(x)$ *funksiyaning* x_0 *nuqtadagi o'ng (chap) hosilasi* deb $f'_+(x_0)=\lim\limits_{\Delta x\to 0+}\frac{\Delta y}{\Delta x}$ $\left(f'_-(x_0)=\lim\limits_{\Delta x\to 0-}\frac{\Delta y}{\Delta x}\right)$ limitga aytiladi.

2- misol. $f(x)=|x-3|$ funksiyaning $x_0=3$ nuqtadagi o'ng va chap hosilalarini toping .

Yechish. Berilgan funksiyaning $x_0=3$ nuqtadagi orttirmasini topamiz:

$$\Delta y=f(3+\Delta x)-f(3)=|3+\Delta x-3|-|3-3|=|\Delta x|.$$

U holda

$$\lim_{\Delta x\to 0+}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0+}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0+}\frac{\Delta x}{\Delta x}=1,$$

$$\lim_{\Delta x\to 0-}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0-}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0-}\frac{-\Delta x}{\Delta x}=-1.$$

Bu misolda $f'_{+}(0) \neq f'_{-}(0)$. Shu sababli $f(x)=|x-3|$ funksiya uchun $\Delta x \to 0$ da $\frac{\Delta y}{\Delta x}=\frac{|\Delta x|}{\Delta x}$ nisbatning limiti mavjud emas va $f(x)=|x-3|$ funksiya $x_0=3$ nuqtada hosilaga ega bo'lmaydi.

Funksiya hosilasining yuqorida keltirilgan ta'riflaridan ushbu tasdiqlar kelib chiqadi: agar funksiya x_0 nuqtada hosilaga ega bo'lsa, funksiya shu nuqtada bir-biriga teng bo'lgan o'ng va chap hosilalarga ega bo'lib, $f'_{+}(x_0)=f'_{-}(x_0)=f'(x_0)$ bo'ladi; agar funksiya x_0 nuqtada o'ng va chap hosilalarga ega bo'lib, $f'_{+}(x_0)=f'_{-}(x_0)$ bo'lsa, funksiya shu nuqtada hosilaga ega va $f'_{+}(x_0)=f'_{-}(x_0)=f'(x_0)$ bo'ladi.

Funksiyaning hosilasini topish amaliga *funksiyani differensiallash* deyiladi.

Agar $y=f(x)$ funksiya biror oraliqda aniqlangan va bu oraliqning har bir nuqtasida $f'(x)$ hosila mavjud bo'lsa, u holda

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

formula $f'(x)$ hosilani x ning funksiyasi sifatida aniqlaydi. Bundan keyin, agar $y=f(x)$ funksiyani differensiallashda differensiallash nuqtasi ko'rsatilmagan bo'lsa, hosilani x ning mumkin bo'lgan barcha qiymatlarida topamiz va $y'(x)$ deb yozamiz.

Hosilaning geometrik va mexanik ma'nolari

Egri chiziqqa o'tkazilgan urinma haqidagi masalada urinmaning burchak koeffitsiyenti uchun ushbu

$$k=tg\alpha==\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

tenglik hosil qilingan edi.

Bu tenglikni $k=f'(x)$ ko'inishda yozamiz, ya'ni $f'(x)$ hosila $y=f(x)$ funksiya grafigiga $M(x,f(x))$ nuqtada o'tkazilgan urinmaning burchak koeffitsiyentiga teng. Bu jumla hosilaning *geometrik ma'nosini* ifodalaydi.

To'g'ri chiziqli harakat haqidagi masalada ushbu

$$v(t)=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

limit hosil qilingan edi.

Bu limitni $v(t) = s'(t)$ ko'rinishda yozamiz, ya'ni moddiy nuqta harakat qonunidan t vaqt bo'yicha olingan hosila nuqtaning t ondagi to'g'ri chiziqli harakati tezligiga teng. Bu jumla *hosilaning mexanik ma'nosini* ifodalaydi.

Umumlashtirgan holda, agar $y = f(x)$ funksiya biror fizik jarayonni ifodalasa, u holda y' hosila bu jarayon tezligini ifodalaydi deyish mumkin. Bu jumla *hosilaning fizik ma'nosini* anglatadi.

Funksiya grafigiga o'tkazilgan urinma va normal tenglamalari

$y = f(x)$ funksiya grafigiga $M(x_0; y_0)$(bu yerda $y_0 = f(x_0)$) nuqtada o'tkazilgan urinma tenglamasini hosilaning geometrik ma'nosidan keltirib chiqaramiz.

Urinma $M_0(x_0; y_0)$ nuqtadan o'tadi. Shu sababli uning tenglamasini

$$y - y_0 = k(x - x_0)$$

ko'rinishda izlaymiz.

Hosilaning geometrik ma'nosiga ko'ra,

$$k_{ur} = f'(x_0).$$

Bundan

$$y - y_0 = f'(x_0)(x - x_0) \tag{1.7}$$

urinma tenglamasi kelib chiqadi.

Egri chiziqqa o'tkazilgan normal deb, urinish nuqtasida urinmaga perpendikular bo'lgan to'g'ri chiziqqa aytiladi.

Egri chiziqqa $M_0(x_0; y_0)$ nuqtada o'tkazilgan normal shu nuqtada o'tkazilgan urunmaga perpendikulyar bo'lgani sababli

$$k_{norm} = -\frac{1}{k_{ur}} = -\frac{1}{f'(x_0)}.$$

Bundan, agar $f'(x_0) \neq 0$ bo'lsa

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \tag{1.8}$$

normal tenglamasi kelib chiqadi.

5.1.3. Funksiyaning differensiallanuvchiligi

3-ta'rif. Agar $y=f(x)$ funksiyaning x_0 nuqtadagi Δx orttirmaga mos orttirmasini

$$\Delta y = A\Delta x + \alpha(\Delta x)\Delta x \qquad (1.9)$$

ko'rinishda ifodalash mumkin bo'lsa, $f(x)$ *funksiya* x_0 *nuqtada differensiallanuvchi* deyiladi, bu yerda A – Δx ga bog'liq bo'lmagan son, $\alpha(\Delta x)$ – $\Delta x \to 0$ da cheksiz kichik funksiya, ya'ni $\lim\limits_{\Delta x\to 0}\alpha(\Delta x)=0$.

Funksiyaning nuqtada differensiallanuvchanligi bilan shu nuqtadagi hosilasi orasidagi bog'lanishni aniqlaymiz.

1-teorema. $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lishi uchun u shu nuqtada hosilaga ega bo'lishi zarur va yetarli.

Isboti. Zarurligi. $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lsin.

U holda ta'rifga ko'ra,

$$\Delta y = A\Delta x + \alpha(\Delta x)\Delta x$$

yoki

$$\frac{\Delta y}{\Delta x} = A + \alpha(\Delta x).$$

Bundan

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = A = f'(x_0),$$

ya'ni $y=f(x)$ funksiya x_0 nuqtada hosilaga ega.

Yetarliligi. $y=f(x)$ funksiya x_0 nuqtada hosilaga ega bo'lsin. U holda $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$. $A=f'(x_0)$ belgilash kiritamiz, bunda $\alpha(\Delta x)=\dfrac{\Delta y}{\Delta x}-A$ funksiya $\Delta x\to 0$ da cheksiz kichik bo'ladi.

Bundan

$$\Delta y = A\Delta x + \alpha(\Delta x)\Delta x,$$

ya'ni funksiyaning x_0 nuqtada differensiallanuvchi bo'lishi kelib chiqadi.

2-teorema. Agar $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lsa, u holda u shu nuqtada uzluksiz bo'ladi.

Isboti. $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lsin. Ta'rifga ko'ra, $\Delta y = A\Delta x + \alpha(\Delta x)\Delta x$.

Bundan $\lim\limits_{\Delta x\to 0}\Delta y = \lim\limits_{\Delta x\to 0}(A\Delta x + \alpha(\Delta x)\Delta x) = 0$, ya'ni funksiya x_0 nuqtada uzluksiz.

Teoremaning teskarisi hamma vaqt ham o'rinli bo'lmaydi, ya'ni funksiyaning biror nuqtada uzluksiz bo'lishidan uning shu nuqtada differensiallanuvchi bo'lishi hamma vaqt ham kelib chiqmaydi. Masalan, $y=|x-3|$ funksiya $x=3$ nuqtada uzluksiz bo'lsa ham, u shu nuqtada hosilaga ega emas (2-misol), ya'ni differentsiallanuvchi emas.

Agar $y=f(x)$ funksiya $(a;b)$ intervalning ($[a;b]$ kesmaning) har bir nuqtasida hosilaga ega bo'lsa, u shu *intervalda* (*kesmada*) *differensiallanuvchi* deyiladi.

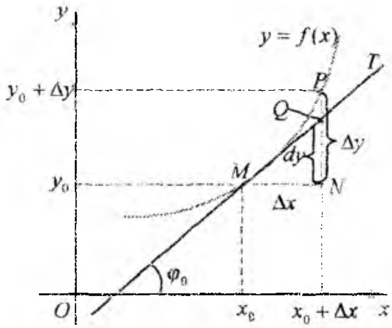
5.1.4. Funksiyaning differensiali

Differensial tushunchasi

$y=f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lib, $x_0\in(a;b)$ nuqtada differensiallanuvchi bo'lsin. U holda $\Delta y = f'(x_0)\Delta x + \alpha(\Delta x)\Delta x$ bo'ladi.

3-ta'rif. $y=f(x)$ funksiya orttirmasining Δx ga nisbatan chiziqli bo'lgan bosh qismi $f'(x_0)\Delta x$ ga $y=f(x)$ *funksiyaning* x_0 *nuqtadagi differensiali* deyiladi va dy (yoki $df(x)$) bilan belgilanadi:

$$dy = f'(x_0)\Delta x. \qquad (1.10)$$



4- shakl.

Erkli o'zgaruvchi x ning, ya'ni $y=x$ funksiyaning differensialini topamiz.

$y'=x'=1$ bo'lgani uchun (1.10) formuladan $dy=dx=\Delta x$ kelib chiqadi, ya'ni erkli o'zgaruvchining differensiali uning orttirmasiga teng: $dx=\Delta x$.

Shu sababli (1.10) tenglikni quyidagicha yozish mumkin:

$$dy = f'(x_0)dx. \qquad (1.11)$$

boshqacha aytganda, *funksiyaning differensiali funksiya hosilasi bilan* *erkli o'zgaruvchi differensialining ko'paytmasiga teng.*

(1.11) tenglikdan $f'(x_0)=\dfrac{dy}{dx}$ bo'ladi, ya'ni funksiyaning x_0 nuqtadagi hosilasi funksiyaning shu nuqtadagi differensialining argument differensiali nisbatiga teng.

Differensialning geomettik ma'nosi

Differensialning geometrik ma'nosini aniqlaymiz. Bunig uchun $y=f(x)$ funksiya grafigiga $M_0(x_0;f(x_0))$ nuqtada MT urinma o'tkazamiz va bu urinmaning $x_0+\Delta x$ nuqtadagi ordinatasini qaraymiz (4-shakl).

MNQ uchburchakdan $NQ=tg\varphi_0\Delta x=dy$ ekanligi kelib chiqadi.

Urinmaning geometrik ma'nosiga ko'ra, $tg\varphi_0=f'(x_0)$.

Bundan $NQ=f'(x_0)\Delta x=dy$.

Demak, $y=f(x)$ funksiyaning x_0 nuqtadagi differensiali funksiya grafigiga $M_0(x_0;f(x_0))$ nuqtada o'tkazilgan urinmaning orttirmasiga teng. Bu jumla *differensialning geometrik ma'nosini* ifodalaydi.

Differensialning taqribiy hisoblashlarga tatbiqi

Ko'pchilik masalalarni yechishda $y=f(x)$ funksiyaning x_0 nuqtadagi orttirmasi funksiyaning shu nuqtadagi differensialiga taqriban almashtiriladi, ya'ni $\Delta y\approx dy$ deb olinadi. Buni $f(x)\approx f(x_0)+f'(x_0)\Delta x$ ko'rinishda yozish mumkin.

Bunday almashtirish yordamida biror A miqdorning taqribiy qiymati quyidagi tartibda hisoblanadi:

1°. A ni x nuqtada biror $f(x)$ funksiya qiymatiga tenglashtiriladi: $A=f(x)$;

2°. x_0 nuqta x ga yaqin va $f(x_0)$ ni hisoblash qulay qilib tanlanadi;

3°. $f(x_0)$ hisoblanadi;

4°. $f'(x_0)$ hisoblanadi;

5°. x_0, $f(x_0)$, $f'(x_0)$ qiymatlar $f(x)\approx f(x_0)+f'(x_0)\Delta x$ formulaga qo'yiladi.

3-misol. $2{,}008^3$ ning taqribiy qiymatini toping .

Yechish.

1.° $A=2{,}008^3$, $f(x)=x^3$ deymiz, u holda $f(x)=A$ va $x=2{,}008$;

2°. $x_0 = 2$ deb olamiz;

3°. $f(x_0) = 2^3 = 8$;

4°. $f'(x) = 3x^2$, $f'(x_0) = 12$;

5°. $f(x) \approx f(x_0) + f'(x_0)\Delta x = 8 + 12 \cdot 0{,}08 = 8{,}096$.

5.1.5. Mashqlar

1. Hosila ta'rifidan foydalanib, funksiyalarning hosilasini toping:

1) $f(x) = \sqrt{3x-1}$; 2) $f(x) = \dfrac{1}{2-5x}$;

3) $f(x) = ctg2x$; 4) $f(x) = ch2x$.

2. $f'(x_0)$ni hosila ta'rifidan foydalanib hisoblang:

1) $f(x) = e^{-3x}$, $x_0 = 0$; 2) $f(x) = \ln(1-4x)$, $x_0 = 0$;

3) $f(x) = tg\left(2x + \dfrac{\pi}{4}\right)$, $x_0 = \pi$; 4) $f(x) = \dfrac{1-x}{1+x}$, $x_0 = 1$.

3. Berilgan funksiyalarning $f'_-(x_0)$ va $f'_+(x_0)$ hosilalarini toping:

1) $f(x) = |3x-2|$, $x_0 = \dfrac{2}{3}$; 2) $f(x) = |x-2| + |x+2|$, $x_0 = 2$;

3) $y = \begin{cases} x \text{ agar } x \le 2 \text{ bo'lsa}, \\ -x^2 + 3x \text{ agar } x > 2 \text{ bo'lsa}, \end{cases} x_0 = 2$; 4) $f(x) = \sqrt{e^{x^2} - 1}$, $x_0 = 0$.

4. Moddiy nuqta Ox o'qi bo'ylab $x = \dfrac{t^3}{3} - 2t^2 + 3t$ qonun bilan harakatlanmoqda. Qaysi nuqtalarda moddiy nuqtaning harakat yo'nalishi o'zgaradi?

5. Moddiy nuqta $s = s(t)$ qonun bilan to'g'ri chiziqli harakat qilmoqda. Qaysi vaqtda material nuqtaning tezlanishi $a(m/c^2)$ ga teng bo'ladi?

1) $s(t) = 2t^3 - \dfrac{5}{2}t^2 + 3t + 1(m)$, $a = 19$; 2) $s(t) = t^3 + \dfrac{3}{2}t^2 - 4t + 3(m)$, $a = 9$.

6. O'tkazgich orqali o'tuvchi tok miqdori $t = 0$ vaqtdan boshlab $q = 3t^2 - 1$ qonun bilan aniqlanadi. Ikkinchi sekund oxiridagi tok kuchini aniqlang.

5.2. DIFFERENSIALLASH QOIDALARI VA FORMULALARI

5.2.1. Yig'indi, ayirma, ko'paytma va bo'linmani differensiallash

Funksiyaning hosilasi ta'rifidan foydalanib, ikki funksiya yig'indisi, ayirmasi, ko'paytmasi va bo'linmasini differensiallash qoidalarini keltirib chiqaramiz.

1-teorema. Agar $u=u(x)$ va $v=v(x)$ funksiyalar x nuqtada differensiallanuvchi bo'lsa, u holda bu funksiyalarning yig'indisi, ayirmasi, ko'paytmasi va bo'linmasi (bo'linmasi $v(x)\neq 0$ shart bajarilganda) ham x nuqtada differensiallanuvchi va quyidagi formulalar o'rinli bo'ladi:

$$\boxed{1.\ (u\pm v)'=u'\pm v';\qquad 2.\ (u\cdot v)'=u'v+v'u;\qquad 3.\ \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2},\ (v\neq 0).}$$

Isboti. 1. Funksiyaning hosilasi va limitlar haqidagi teoremalardan foydalanib topamiz:

$$(u\pm v)'=\lim_{\Delta x\to 0}\frac{(u(x+\Delta x)\pm v(x+\Delta x))-(u(x)\pm v(x))}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x)-u(x)}{\Delta x}\pm\frac{v(x+\Delta x)-v(x)}{\Delta x}\right)=$$

$$=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}\pm\lim_{\Delta x\to 0}\frac{v(x+\Delta x)-v(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\pm\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=u'\pm v'.$$

2. Formulani isbotlashda 5.1.3. banddagi 2-teoremadan foydalanamiz: x nuqtada differensiallanuvchi $u=u(x)$ va $v=v(x)$ funksiyalar shu nuqtada uzluksiz bo'ladi. Shu sababli $\Delta x\to 0$ da $\Delta u\to 0$ va $\Delta v\to 0$.

$$(u\cdot v)'=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)\cdot v(x+\Delta x)-u(x)\cdot v(x)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{(u(x)+\Delta u)\cdot(v(x)+\Delta v)-u(x)\cdot v(x)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{u(x)\cdot v(x)+u(x)\cdot\Delta v+v(x)\cdot\Delta u+\Delta u\cdot\Delta v-u(x)\cdot v(x)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\left(v(x)\cdot\frac{\Delta u}{\Delta x}+u(x)\cdot\frac{\Delta v}{\Delta x}+\Delta v\cdot\frac{\Delta u}{\Delta x}\right)=v(x)\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+u(x)\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}+$$

$$+\lim_{\Delta x\to 0}\Delta v\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=u'\cdot v+u\cdot v'+0\cdot u'=u'v+v'u.$$

3. $$\left(\frac{u}{v}\right)'=\lim_{\Delta x\to 0}\frac{\frac{u(x+\Delta x)}{v(x+\Delta x)}-\frac{u(x)}{v(x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{u(x)+\Delta u}{v(x)+\Delta v}-\frac{u(x)}{v(x)}}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{u(x)\cdot v(x)+v(x)\cdot\Delta u-u(x)\cdot v(x)-u(x)\cdot\Delta v}{\Delta x\cdot\big(v(x)+\Delta v\big)\cdot v(x)}=\lim_{\Delta x\to 0}\frac{v\cdot\Delta u-u\cdot\Delta v}{\Delta x\cdot(v^2+v\cdot\Delta v)}=$$

$$=\lim_{\Delta x\to 0}\frac{v\cdot\frac{\Delta u}{\Delta x}-u\cdot\frac{\Delta v}{\Delta x}}{v^2+v\cdot\Delta v}=\frac{v\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}-u\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta v}{\Delta x}}{v^2+v\cdot\lim\limits_{\Delta x\to 0}\Delta v}=\frac{u'v-v'u}{v^2}.$$

5.2.2. Teskari funksiyani differensiallash

$y=f(x)$ funksiya teskari funksiya mavjudligi haqidagi teoremaning shartlarini qanoatlantirsin va $x=\varphi(y)$ teskari funksiyaga ega bo'lsin.

2-teorema. Agar $y=f(x)$ funksiya x nuqtada $f'(x)\neq 0$ hosilaga ega bo'lsa, u holda $x=\varphi(y)$ funksiya mos $y=f(x)$ nuqtada differensiallanuvchi bo'ladi va

$$\varphi'(y)=\frac{1}{f'(x)},\ \text{ya'ni}\ \ x'_y=\frac{1}{y'_x}.$$

Isboti. $x=\varphi(y)$ funksiyaning argumenti y ga $\Delta y\neq 0$ orttirma beramiz. U holda $y=f(x)$ funksiyaning qat'iy monotonlidan $x=\varphi(y)$ funksiya $\Delta x\neq 0$ orttirma oladi. Shu sababli $\frac{\Delta x}{\Delta y}=\frac{1}{\frac{\Delta y}{\Delta x}}$ kabi yozish mumkin.

$x=\varphi(y)$ teskari funksiya y nuqtada uzluksiz bo'lgani uchun $\Delta y\to 0$ da $\Delta x\to 0$ bo'ladi: $\Delta x\to 0$da oxirgi tenglikning o'ng tomoni $\frac{1}{f'(x)}$ ($f'(x)\neq 0$) ga, chap tomoni $\varphi'(y)$ hosilaga teng bo'ladi.

Demak,

$$\varphi'(y) = \frac{1}{f'(x)}.$$

Shunday qilib, *teskari funksiyaning hosilasi berilgan funksiya hosilasining teskari qiymatiga teng.*

5.2.3. Murakkab funksiyani differensiallash

$y = f(u)$ va $u = \varphi(x)$ bo'lsin. U holda $y = f(\varphi(x))$ funksiya erkli argumenti x dan va oraliq argumenti u dan iborat murakkab funksiya bo'ladi.

3-teorema. Agar $u = \varphi(x)$ funksiya x nuqtada $\varphi'(x)$ hosilaga va $y = f(u)$ funksiya mos $u = \varphi(x)$ nuqtada $f'(u)$ hosilaga ega bo'lsa, u holda $f(\varphi(x))$ murakkab funksiya x nuqtada differensiallanuvchi bo'ladi va

$$y'(x) = f'(u)\varphi'(x).$$

Isboti. $y = f(u)$ funksiya u nuqtada differensiallanuvchi bo'lgani uchun $\Delta y = f'(u)\Delta u + \alpha(\Delta u)\Delta u$ bo'ladi.

Bundan

$$\frac{\Delta y}{\Delta x} = f'(u)\frac{\Delta u}{\Delta x} + \alpha(\Delta u)\frac{\Delta u}{\Delta x}.$$

$u = \varphi(x)$ funksiya x nuqtada hosilaga ega. Shu sababli $u = \varphi(x)$ funksiya x nuqtada uzluksiz va $\Delta x \to 0$ da $\Delta u \to 0$.

U holda

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = f'(u)\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0}\alpha(\Delta u) \cdot \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}.$$

Bundan

$$y'(x) = f'(u) \cdot \varphi'(x) + 0 \cdot \varphi'(x)$$

yoki

$$y'(x) = f'(u) \cdot \varphi'(x).$$

Shunday qilib, *murakkab funksiyaning hosilasi berilgan funksiyaning oraliq argument bo'yicha hosilasi bilan oraliq argumentning erkli argument bo'yicha hosilasining ko'paytmasiga teng.*

Bu qoida oraliq argumentlar bir nechta bo'lganda ham o'z kuchida qoladi. Masalan, $y = f(u), u = g(v), v = h(x)$ bo'lsa, $y'_x = y'_u \cdot u'_v \cdot v'_x$ bo'ladi.

$y=f(u)$ va $u=\varphi(x)$ differensiallanuvchi va $y=f(\varphi(x))$ murakkab funksiyani hosil qiluvchi funksiyalar bo'lsin.

Murakkab funksiyaning hosilasi haqidagi teoremaga ko'ra, $y'_x = y'_u u'_x$ bo'ladi.

Bu tenglikning har ikkala tomonini dx ga ko'paytiramiz:

$$y'_x dx = y'_u u'_x dx.$$

Endi $y'_x dx = dy$ va $u'_x dx = du$ ekanini hisobga olsak,

$$dy = y'_u du.$$

$dy = y'_x dx$ va $dy = y'_u du$ formulalarni solishtirish ko'rsatadiki, $y=f(x)$ funksiyaning differensiali argument erkli o'zgaruvchi yoki biror argumentning funksiyasi bo'lishidan qat'iy nazar bir xil formula bilan topiladi.

Differensialning bu xossasiga *differensialning invariantlik xossasi* deyiladi.

$dy = y'_x dx$ formula tashqi ko'rinishidan $dy = y'_u du$ formulaga o'xshasada, aslini olganda ular orasida farq mavjud: birinchi formulada x erkli o'zgaruvchi va shu sababli $dx = \Delta x$, ikkinchi formulada esa u funksiya x ning funksiyasi va shuning uchun $du \neq \Delta u$.

5.2.4. Asosiy elementar funksiyalarning hosilalari

Asosiy elementar funksiyalarning hosilalarini topishda ekvivalent cheksiz kichik funksiyalardan, teskari va murakkab funksiyalarni differensiallash formulalaridan hamda yig'indi, ayirma, ko'paytma va bo'linmani differensiallash qoidalaridan foydalanamiz.

1. ***O'zgarmas funksiya***: $y = C\,(C \in R)$.

O'zgarmas funksiya $x \in R$ da o'zining qiymatini saqlagani uchun ixtiyoriy nuqtada uning orttirmasi nolga teng bo'ladi.
Shu sababli

$$(C)' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

2. ***Darajali funksiya***: $y = x^\alpha$, bunda $\alpha \in R,\ \alpha \neq 0$.

Bu funksiya uchun $x > 0$ da

$$\Delta y = (x+\Delta x)^\alpha - x^\alpha = x^\alpha\left(\left(1+\frac{\Delta x}{x}\right)^\alpha - 1\right).$$

Bundan

$$\frac{\Delta y}{\Delta x}=x^{\alpha}\frac{\left(1+\frac{\Delta x}{x}\right)^{\alpha}-1}{\Delta x}.$$

Endi $\Delta x\to 0$ da $\left(1+\frac{\Delta x}{x}\right)^{\alpha}-1\sim\alpha\frac{\Delta x}{x}$ ekanligini hisobga olsak,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=x^{\alpha}\lim_{\Delta x\to 0}\frac{\left(1+\frac{\Delta x}{x}\right)^{\alpha}-1}{\Delta x}=x^{\alpha}\lim_{\Delta x\to 0}\frac{\alpha\Delta x}{\Delta x\cdot x}=x^{\alpha}\lim_{\Delta x\to 0}\frac{\alpha}{x}=\alpha x^{\alpha-1}$$

bo‘ladi.

Demak,

$$(x^{\alpha})'=\alpha x^{\alpha-1}.$$

Xususan, $\left(\frac{1}{x}\right)'=-\frac{1}{x^2}$, $(\sqrt{x})'=\frac{1}{2\sqrt{x}}$.

3. ***Ko‘rsatkichli funksiya***: $y=a^x$, bunda $a\in R$, $a>0$, $a\neq 1$.

Bu funksiyaning orttirmasi $\Delta y=a^{x+\Delta x}-a^x=a^x(a^{\Delta x}-1)$ bo‘lgani uchun $\frac{\Delta y}{\Delta x}=a^x\cdot\frac{a^{\Delta x}-1}{\Delta x}$ bo‘ladi. Bundan $\Delta x\to 0$ da $a^{\Delta x}-1\sim\Delta x\ln a$ ekanini hisobga olsak,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}a^x\cdot\frac{a^{\Delta x}-1}{\Delta x}=a^x\lim_{\Delta x\to 0}\frac{\Delta x\ln a}{\Delta x}=a^x\ln a$$

bo‘ladi.

Demak,

$$(a^x)'=a^x\ln a.$$

Xususan, $(e^x)'=e^x$.

4. ***Logarifmik funksiya***: $y=\log_a x$, bunda $a\in R$, $a>0$, $a\neq 1$.

$y=\log_a x$ funksiya $x=a^y$ funksiyaga teskari funksiya. Bunda oldingi banddagi formulaga ko‘ra, $x'(y)=a^y\ln a$.

U holda

$$y'(x) = \frac{1}{x'(y)} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

Demak,

$$(\log_a x)' = \frac{1}{x \ln a}.$$

Xususan, $(\ln x)' = \frac{1}{x}$.

5. *Trigonometrik funksiyalar*.

1) $y = \sin x$ funksiyaning orttirmasi

$$\Delta y = \sin(x + \Delta x) - \sin x = 2\sin\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right)$$

va

$$\frac{\Delta y}{\Delta x} = \frac{2\sin\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x}.$$

Bu tenglikdan $\Delta x \to 0$ da $\sin\frac{\Delta x}{2} \sim \frac{\Delta x}{2}$ ni hisobga olib, topamiz:

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{2\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} = \lim_{\Delta x \to 0}\cos\left(x + \frac{\Delta x}{2}\right) = \cos(x + 0) = \cos x.$$

Demak,

$$(\sin x)' = \cos x.$$

2) $y = \cos x$ funksiyaning hosilasini murakkab funksiyaning hosilasi formulasidan foydalanib topamiz:

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2} - x\right)\right)' = \cos\left(\frac{\pi}{2} - x\right)\cdot\left(\frac{\pi}{2} - x\right)' = \cos\left(\frac{\pi}{2} - x\right)\cdot(-1) = -\sin x.$$

Demak,

$$(\cos x)' = -\sin x.$$

3) $y = tgx$ funksiyaning hosilasini bo'linmaning hosilasi formulasidan foydalanib topamiz:

$$(tgx)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Demak,

$$(tgx)' = \frac{1}{\cos^2 x}.$$

4) $y = ctgx$ funksiyaning hosilasini topishda murakkab funksiyaning hosilasi formulasidan foydalanamiz:

$$(ctgx)' = \left(tg\left(\frac{\pi}{2} - x\right)\right)' = \frac{1}{\cos^2\left(\frac{\pi}{2} - x\right)} \cdot (-1) = -\frac{1}{\sin^2 x}.$$

Demak,

$$(ctgx)' = -\frac{1}{\sin^2 x}.$$

6. *Teskari trigonometrik funksiyalar.*

1) $y = \arcsin x$ funksiya $x = \sin y$ funksiyaga teskari.

Bunda $x'(y) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

U holda

$$y'(x) == \frac{1}{x'(y)} = \frac{1}{\sqrt{1 -- x^2}} .$$

Demak,

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$

2) $y = \arccos x$ funksiyaning hosilasini $\arcsin x + \arccos x = \frac{\pi}{2}$ formuladan foydalanib topamiz:

$$(\arccos x)' = \left(\frac{\pi}{2} - \arcsin x\right)' = -(\arcsin x)' = -\frac{1}{\sqrt{1 - x^2}}.$$

Demak,

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}.$$

3) $y = arctgx$ funksiyaning hosilasini teskari funksiyaning

hosilasi formulasidan foydalanib topamiz:

$$(arctgx)' = \frac{1}{(tgy)'} = \cos^2 y = \frac{1}{1+tg^2 y} = \frac{1}{1+x^2}.$$

Demak,

$$(arctgx)' = \frac{1}{1+x^2}.$$

4) $arctgx$ va $arcctgx$ funksiyalar $arctgx + arcctgx = \frac{\pi}{2}$ bog‘lanishga ega.

Bundan

$$(arcctgx)' = \left(\frac{\pi}{2} - arcctgx\right)' = -(arctgx)' = -\frac{1}{1+x^2}.$$

Demak,

$$(arcctgx)' = -\frac{1}{1+x^2}.$$

1-misol. Giperbolik funksiyalarning hosilalarini toping.

Yechish. Differensiallash qoidalari va $y = e^x$ funksiyaning hosilasidan foydalanib topamiz:

$$(shx)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = chx, \text{ ya'ni } (shx)' = chx;$$

$$(chx)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = shx, \text{ ya'ni } (chx)' = shx;$$

$$(thx)' = \left(\frac{shx}{chx}\right)' = \frac{(shx)'chx - shx\cdot(chx)'}{ch^2x} = \frac{ch^2x - sh^2x}{ch^2x} = \frac{1}{ch^2x}, \text{ ya'ni } (thx)' = \frac{1}{ch^2x};$$

$$(cthx)' = \left(\frac{chx}{shx}\right)' = \frac{sh^2x - ch^2x}{sh^2x} = -\frac{1}{sh^2x}, \text{ ya'ni } (cthx)' = -\frac{1}{sh^2x}.$$

5.2.5. Differensiallash qoidalari va hosilalar jadvali

Keltirib chiqarilgan differensiallash qoidalarini va asosiy elementar funksiyalarning hosilalari formulalarini jadval ko‘rinishida yozamiz.

Amalda ko'pincha murakkab funksiyalarning hosilalarini topishga to'g'ri keladi. Shu sababli quyida keltiriladigan formulalarda «x» argument «u» oraliq argumentga almashtiriladi.

Differensiallash qoidalari.

1. $(u \pm v)' = u' \pm v'$, $u = u(x)$, $v = v(x)$ – differensiallanuvchi funksiyalar;
2. $(u \cdot v)' = u'v + uv'$, xususan $(Cu)' = Cu'$, C – o'zgarmas son;
3. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, xususan $\left(\frac{C}{v}\right)' = -\frac{Cv'}{v^2}$;
4. $y'_x = \frac{1}{x'_y}$, agar $y = f(x)$ va $x = \varphi(y)$;

Asosiy elementar funksiyalarning hosilalar jadvali.

1. $(C)' = 0$;
2. $(u^{\alpha})' = \alpha u^{\alpha-1} \cdot u'$, xususan $\left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$, $(\sqrt{u})' = \frac{1}{2\sqrt{u}} \cdot u'$;
3. $(a^u)' = a^u \ln a \cdot u'$, xususan $(e^u)' = e^u \cdot u'$;
4. $(\log_a u)' = \frac{1}{u \ln a} \cdot u'$, xususan $(\ln u)' = \frac{1}{u} \cdot u'$;
5. $(\sin u)' = \cos u \cdot u'$;
6. $(\cos u)' = -\sin u \cdot u'$;
7. $(tgu)' = \frac{1}{\cos^2 u} \cdot u'$;
8. $(ctgu)' = -\frac{1}{\sin^2 u} \cdot u'$;
9. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$;
10. $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$;
11. $(arctgu)' = \frac{1}{1+u^2} \cdot u'$;
12. $(arcctgu)' = -\frac{1}{1+u^2} \cdot u'$;
13. $(shu)' = chu \cdot u'$;
14. $(chu)' = shu \cdot u'$;
15. $(thu)' = \frac{1}{ch^2 u} \cdot u'$;
16. $(cthu)' = -\frac{1}{sh^2 u} \cdot u'$.

Keltirilgan diferensiallash qoidalari va asosiy elementar funksiyalarning hosilalar jadvali bir o'zgaruvchi funksiyasi differensial hisobining asosini tashkil qiladi. Ularni bilgan holda har qanday elementar funksiyaning hosilasini topish mumkin. Bunda yana elementar funksiya hosil bo'ladi.

4-misol. $f(x)=5^x+\arcsin x+x\ln x$ funksiyaning hosilasini toping.

Yechish. Hosilani topishda differensiallashning 1,2 qoidalari va 3,4,9-formulalaridan foydalanildi.

$$f'(x)=(5^x+\arcsin x+x\ln x)'=(5^x)'+(\arcsin x)'+(x\ln x)'=5^x\ln 5+$$

$$+\frac{1}{\sqrt{1-x^2}}+x'\ln x+x(\ln x)'=5^x\ln 5+\frac{1}{\sqrt{1-x^2}}+\ln x+x\cdot\frac{1}{x}=$$

$$\frac{1}{x}=5^x\ln 5+\frac{1}{\sqrt{1-x^2}}+\ln x+1.$$

5.2.6. Logarifmik differensiallash

Ayrim hollarda funksiyaning hosilasini topish uchun avval berilgan funksiyani logarifmlash, so'ngra differensiallash maqsadga muvofiq bo'ladi. Bu jarayonga *logarifmik differensiallash* deyiladi.

3- misol. $y=\dfrac{(x^3+1)\cdot\sqrt[5]{(x-2)^4}\cdot 2^x}{(x-4)^3}$ funksiyaning hosilasini toping.

Yechish. Bu hosilani differensiallash qoidalari va formulalari orqali topish mumkin. Ammo bu jaroyon katta hajmga ega. Shu sababli logarifmik differensiallashni qo'llaymiz. Buning uchun funksiyani logarifmlaymiz:

$$\ln y=\ln(x^3+1)+\frac{4}{5}\ln(x-2)+x\ln 2-3\ln(x-4).$$

Bu tenglikning har ikki qismini x bo'yicha differensiallaymiz:

$$\frac{1}{y}\cdot y'=\frac{1}{x^3+1}\cdot 3x^2+\frac{4}{5}\cdot\frac{1}{x-2}+\ln 2-3\cdot\frac{1}{x-4}.$$

Endi bu tenglikdan y' ni topamiz:

$$y'=y\cdot\left(\frac{3x^2}{x^3+1}+\frac{4}{5(x-2)}+\ln 2-\frac{3}{x-4}\right),$$

yoki

$$y' = \frac{(x^3+1)\cdot\sqrt[5]{(x-2)^4}\cdot 2^x}{(x-4)^3}\cdot\left(\frac{3x^2}{x^3+1}+\frac{4}{5(x-2)}+\ln 2-\frac{3}{x-4}\right).$$

Shunday funksiyalar borki, ularning hosilalari faqat logarifmik differensiallash orqali topiladi. Bunday funksiyalarning qatoriga *darajali-ko'rsatkichli funksiya* deb ataluvchi $y=u^v$ funksiya kiradi, bu yerda $u=u(x)$, $v=v(x)-$ x ning differensiallanuvchi funksiyalari.

Bu funksiyalaning hosilasini logarifmik differensiallash yordamida topamiz:

$$\ln y = v\cdot\ln u,\quad \frac{1}{y}y' = v'\cdot\ln u + v\cdot\frac{1}{u}\cdot u',\quad y' = y\left(v'\ln u+\frac{vu'}{u}\right).$$

Bundan

$$(u^v)' = u^v\cdot\ln u\cdot v' + v\cdot u^{v-1}u'. \tag{2.1}$$

(2.1) formulani eslab qolish qoidasini ifodalaymiz: darajali-ko'rsatkichli funksiyaning hosilasi $u=const$ shartidagi ko'rsatkichli funksiya hosilasi bilan $v=const$ shartidagi darajali funksiya hosilasining yig'indisiga teng.

4- misol. $y=x^{\cos 3x}$ funksiyaning hosilasini toping.

Yechish. (2.1) formula bilan topamiz:

$$y' = x^{\cos 3x}\cdot\ln x\cdot(-\sin 3x)\cdot 3+\cos 3x\cdot x^{\cos 3x-1}\cdot 1$$

yoki

$$y' = x^{\cos 3x-1}\cdot\cos 3x - 3x^{\cos 3x}\cdot\ln x\cdot\sin 3x.$$

5.2.7. Parametrik va oshkormas ko'rinishda berilgan funksiyalarni differensiyallash

t o'zgaruvchining $x=\varphi(t)$ va $y=\psi(t)$ funksiyalari biror $T=(\alpha;\beta)$ intervalda aniqlangan bo'lib, bu intervalda $\varphi'(t)$, $\psi'(t)$ hosilalar va $x=\varphi(t)$ funksiyaga teskari $t=\phi(x)$ funksiya mavjud bo'lsin. Agar $x=\varphi(t)$ funksiya qat'iy monoton bo'lsa, $t=\phi(x)$ teskari funksiya bir qiymatli, uzluksiz va qat'iy monoton bo'ladi. Shu sababli $y=\psi(\phi(x))$ murakkab funksiya mavjud bo'ladi. Bunda $y=f(x)$ funksiya $x=\varphi(t)$ va

$y=\psi(t)$ tenglamalar bilan *parametrik ko'rinishda* (t parametrli) berilgan deyiladi.

$y=f(x)$ funksiya

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \ t\in T \end{cases}$$

parametrik tenglamalar bilan berilgan bo'lsin. U holda $t=\phi(x)$ teskari funksiya mavjud va uning hosilasi $\phi'(x)=\frac{1}{\varphi'(t)}$ bo'ladi. Shuningdek, $y=\psi(\phi(x))$ murakkab funksiya hosilasi $y'_x=\psi'(\phi(x))\phi'(x)$ bo'ladi.

Bundan

$$y'(x)=\frac{\psi'(t)}{\varphi'(t)}$$

yoki

$$y'_x=\frac{y'_t}{x'_t}. \tag{2.2}$$

5- misol. $\begin{cases} x=3\cos t, \\ y=4\sin t \end{cases}$ bo'lsa, y'_x ni toping.

Yechish. $y'_x=\frac{(3\cos t)'_t}{(4\sin t)'_t}=-\frac{3\sin t}{4\cos t}=-\frac{3}{4}ctgt.$

Agar funksiya y ga nisbatan yechilmagan, ya'ni x va y o'zgaruvchilar orasidagi bog'lanish $F(x,y)=0$ ko'rinishda berilgan bo'lsa, funksiya *oshkormas ko'rinishda* berilgan deyiladi.

Oshkor berilgan har qanday $y=f(x)$ funksiyani oshkormas ko'rinishda $f(x)-y=0$ kabi yozish mumkin, ammo teskarisini hamma vaqt bajarib bo'lmaydi, $F(x,y)=0$ tenglamani y ga nisbatan yechish hamma vaqt ham oson emas, ayrim hollarda esa umuman mumkin emas.

Funksiya oshkormas ko'rinishda berilgan bo'lsa, $F(x,y)$ funksiya x ning murakkab funksiyasi, ya'ni bunda $y=y(x)$ deb qaraladi va $F(x,y)=0$ tenglikning chap va o'ng tomoni x bo'yicha differensialanadi, so'ngra hosil bo'lgan tenglamadan y' topiladi.

6- misol. $y-\arg tgy-x^3=0$ bo‘lsa, y' ni toping.

Yechish. Tenglikning har ikkala tomonini x bo‘yicha differensiallaymiz:

$$y'-\frac{1}{1+y^2}y'-3x^2=0.$$

Bundan

$$y'\left(1-\frac{1}{1+y^2}\right)=3x^2,\quad y'\frac{y^2}{1+y^2}=3x^2$$

yoki

$$y'=3x^2\left(1+\frac{1}{y^2}\right).$$

5.2.8. Yuqori tartibli hosilalar va differensiallar

Yuqori tartibli hosilalar

$f(x)$ funksiya biror $(a;b)$ intervalda aniqlangan bo‘lib, shu intervalda differensiyallanuvchi bo‘lsin. U holda $f'(x)$ hosila $x\in(a;b)$ ning funksiyasi bo‘ladi. Shu sababli bu funksiya uchun hosilaning mavjudligi va uni hisoblash masalalarini qarash mumkin.

$f'(x)$ ga ***birinchi tartibli hosila*** deyiladi. $f(x)$ funksiyaning $f'(x)$ hosilasidan olingan hosilaga *ikkinchi tartibli hosila* deyiladi. Ikkinchi tartibli hosila mavjud bo‘lsa, bu hosiladan olingan hosila *uchinchi tartibli hosila* deyiladi va hokazo. Bunday hosilalar ikkinchi tartiblidan boshlab *yuqori tartibli hosila* deyiladi.

Yuqori tartibli hosilalar $y'',y''',y^{(4)},\ldots,y^{(n)},\ldots$

(yoki $f''(x),f'''(x),f^{IV}(x),\ldots,f^{(n)}(x),\ldots$ yoki $\left(\frac{d^2y}{dx^2},\frac{d^3y}{dx^3},\ldots,\frac{d^ny}{dx^n},\ldots\right)$) kabi

belgilanadi.

7- misol. $y=x^3\ln x$ bo‘lsa, $y^{(4)}(3)$ ni toping.

Yechish. $y'=(x^3)'\ln x+x^3(\ln x)'=3x^2\ln x+x^3\frac{1}{x}=x^2(3\ln x+1);$

$$y''=(x^2(3\ln x+1))'=(x^2)'(3\ln x+1)+x^2(3\ln x+1)'=$$

$$= 2x(3\ln x + 1) + x^2 \cdot \frac{3}{x} = x(6\ln x + 5);$$

$$y''' = (x(6\ln x + 5))' = x'(6\ln x + 5) + x(6\ln x + 5)' = 6\ln x + 5 + x \cdot \frac{6}{x} = 6\ln x + 11;$$

$$y^{(4)} = (6\ln x + 11)' = \frac{6}{x}.$$

Bundan

$$y^{(4)}(3) = \frac{6}{3} = 2.$$

Funksiyaning yuqori tartibli hosilasini topish uchun uning barcha oldingi tartibli hosilalarini topish kerak bo‘ladi. Biroq, ayrim funksiyalarning n-tartibli hosilalarini topish imkonini beruvchi formulalar mavjud. Masalan, quyida keltirilgan formulalar bunday formulalar qatoriga kiradi:

1. $(a^x)^{(n)} = a^x \ln^n a \ (a > 0),\ (e^x)^{(n)} = e^x;$	2. $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right);$
3. $(x^\alpha)^{(n)} = \alpha(\alpha - 1)...(\alpha - n + 1)x^{\alpha - n}, \alpha \in R;$	4. $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right);$
5. $(\ln x)^{(n)} = \frac{(-1)^n (n-1)!}{x^n};$	6. $(u \pm v)^{(n)} = u^{(n)} \pm v^{(n)};$
7. $(Cu)^{(n)} = Cu^{(n)};$	8. $(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} \cdot v^{(n-k)}.$

Formulalardan ayrimlarining isbotini keltiramiz.

3. $(x^\alpha)^{(n)} = \alpha(\alpha - 1)...(\alpha - n + 1)x^{\alpha - n}, \alpha \in R$ ning *isboti.*

$$y = x^\alpha, \qquad y' = \alpha x^{\alpha - 1}, \qquad y'' = \alpha(\alpha - 1)x^{\alpha - 2},$$

$$y''' = \alpha(\alpha - 1)(\alpha - 2)x^{\alpha - 3},\ ..., \qquad y^{(n)} = \alpha(\alpha - 1)...(\alpha - n + 1)x^{\alpha - n}.$$

Shunday qilib,

$$(x^\alpha)^{(n)} = \alpha(\alpha - 1)...(\alpha - n + 1)x^{\alpha - n}, \alpha \in R.$$

4. $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$ ning *isboti.*

$$y=\cos x,\ \ y'=-\sin x=\cos\left(x+\frac{\pi}{2}\right),\quad y''=-\cos x=\cos\left(x+2\cdot\frac{\pi}{2}\right),$$

$$y'''=\sin x=\cos\left(x+3\cdot\frac{\pi}{2}\right),\ldots,\quad y^{(n)}=\cos\left(x+(n-1)\cdot\frac{\pi}{2}\right)'=\cos\left(x+n\cdot\frac{\pi}{2}\right).$$

Demak,

$$(\cos x)^{(n)}=\cos\left(x+\frac{n\pi}{2}\right).$$

5. $(\ln x)^{(n)}=\dfrac{(-1)^n(n-1)!}{x^n}$ ning *isboti.*

$$y=\ln x,\quad y'=\frac{1}{x}=x^{-1},\quad y''=-1\cdot x^{-2},\ y'''=(-1)(-2)x^{-3}=(-1)^2\cdot 1\cdot 2\cdot x^{-3},\ldots,$$

$$y^{(n)}=(-1)^{n-1}\cdot 1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)x^{-n}=\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}.$$

Shunday qilib,

$$(\ln x)^{(n)}=\frac{(-1)^n(n-1)!}{x^n}.$$

Ikkinchi tartibli hosilaning mexanik ma'nosi

M moddiy nuqta $s=s(t)$ qonun bilan to'g'ri chiziqli harakat qilsin. U holda $s_t'(t)$ moddiy nuqtaning t vaqtdagi tezligini ifodalaydi:

$$s_t'(t)=v(t).$$

Nuqtaning t vaqtdagi tezligi $v(t)$, $t+\Delta t$ vaqtdagi tezligi $v(t)+\Delta v$ bo'lsin, ya'ni Δt vaqt oralig'ida nuqtaning tezligi Δv ga o'zgarsin.

$\dfrac{\Delta v}{\Delta t}$ nisbat *to'g'ri chiziqli harakatda nuqtaning* Δt *vaqt oralig'idagi o'rtacha tezlanishini* ifodalaydi. Bu nisbatning $\Delta t\to 0$ dagi limiti M nuqtaning berilgan t vaqtdagi tezlanishi deyiladi va u $a(t)$ bilan belgilanadi: $\lim\limits_{\Delta t\to 0}\dfrac{\Delta v}{\Delta t}=a(t)$, ya'ni $v'(t)=a(t)$.

Shu sababli

$$a(t)=s_t''(t).$$

Demak, moddiy nuqta harakat qonunidan t vaqt bo'yicha olingan ikkinchi tartibli hosila to'g'ri chiziqli harakatda nuqtaning t vaqtdagi tezlanishiga teng. Bu tasdiq *ikkinchi tartibli hosilaning mexanik ma'nosini* ifodalaydi.

Parametrik va oshkormas ko'rinishda berilgan funksiyalarning yuqori tartibli hosilalari

$y=f(x)$ funksiya

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \ t\in T \end{cases}$$

parametrik tenglamalar bilan berilgan bo'lsin.

U holda

$$y'_x=\frac{y'_t}{x'_t}.$$

Bundan murakkab va teskari funksiyalarni differensiallash qoidalariga ko'ra,

$$y''_{x^2}=(y'_x)'_t\cdot t'_x=\frac{(y'_x)'_t}{x'_t},$$

$$y''_{x^2}=\frac{(y'_x)'_t}{x'_t}. \qquad (2.3)$$

Shu kabi

$$y'''_{x^3}=\frac{(y''_{xx})'_t}{x'_t}, \qquad y^{(4)}_{x^4}=\frac{(y'''_{xxx})'_t}{x'_t}$$

va boshqa hosilalar topiladi.

8- misol. $\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t) \end{cases}$ bo'lsa, ikkinchi tartibli hosilani toping.

Yechish. (2.2) formula bilan topamiz:

$$y'_x=\frac{(a(1-\cos t))'_t}{(a(t-\sin t))'_t}=\frac{a\sin t}{a(1-\cos t)}=\frac{\sin t}{1-\cos t}=ctg\frac{t}{2}.$$

Bundan (2.3) formulaga ko'ra,

$$y''_{x^2}=\frac{\left(ctg\frac{t}{2}\right)'_t}{(a(t-\sin t))'_t}=\frac{-\frac{1}{2\sin^2\frac{t}{2}}}{a(1-\cos t)}=-\frac{1}{1-\cos t}\cdot\frac{1}{a(1-\cos t)}=-\frac{1}{a(1-\cos t)^2}.$$

y''_{x^2} hosilani (2.3) formula ustida almashtirishlar bajarib, quyidagicha yozish mumkin

$$y''_{x^2}=\frac{y''_t x'_t - x''_t y'_t}{(x'_t)^3}. \qquad (2.4)$$

$y=f(x)$ funksiya $F(x,y)=0$ tenglama bilan oshkormas ko'rinishda berilgan bo'lsin.

Bu tenglama x bo'yicha differensiallansa va y' ga nisbatan yechilsa, birinchi tartibli hosila topiladi. Topilgan birinchi tartibli hosila x bo'yicha differensiallansa, ikkinchi tartibli hosila kelib chiqadi. Bu hosilada x, y, y' lar qatnashadi. Ikkinchi tartibli hosilaga topilgan y' ni qo'yib, y'' ni x va y orqali ifodasi aniqlanadi.

9- misol. $b^2x^2+a^2y^2=a^2b^2$ bo'lsa, ikkinchi tartibli hosilani toping.

Yechish. Oshkormas ko'rinishdagi funksiyani differensiallash qoidasidan foydalanib topamiz:

$$(b^2x^2+a^2y^2)'=(a^2b^2)', \quad b^2\cdot 2x+a^2\cdot 2yy'=0.$$

Bundan

$$y'=-\frac{b^2}{a^2}\frac{x}{y}.$$

U holda

$$y''=-\frac{b^2}{a^2}\left(\frac{x}{y}\right)'=-\frac{b^2}{a^2}\cdot\frac{x'y-y'x}{y^2}=\frac{b^2}{a^2}\cdot\frac{xy'-y}{y^2}=\frac{b^2}{a^2}\cdot\frac{-x\cdot\frac{b^2}{a^2}\frac{x}{y}-y}{y^2}=$$

$$=-\frac{b^2}{a^2}\cdot\frac{x^2b^2+a^2y^2}{a^2y^3}=-\frac{b^2a^2b^2}{a^4y^3}=-\frac{b^4}{a^2y^3}.$$

Yuqori tartibli differensiallar

Biror $(a;b)$ intervalda differensiyallanuvchi $y=f(x)$ funksiyaning $dy=y'(x)dx$ differensiyali *birinchi tartibli differensial* deyiladi.

U holda

$$d(dy)=d(f'(x)dx)=(f'(x)dx)'dx=f''(x)dx\cdot dx$$

differensialga *ikkinchi tartibli differensial* deyiladi va

$$d^2y=f''(x)dx^2 \qquad (2.5)$$

kabi yoziladi, bu yerda dx^2 bilan $(dx)^2$ belgilanadi.

Ikkinchi tartibli differensialdan olingan differensial *uchinchi tartibli differensial* deyiladi va hokazo. n-*tartibli differensial* deb $(n-1)$-tartibli
differensialdan olingan differensialga aytiladi va $d^n y = f^{(n)}(x)dx^n$ kabi yoziladi.

Bundan $y^{(n)}(x) = \frac{d^n y}{dx^n}$, ya'ni $y = f(x)$ funksiyaning n-tartibli hosilasi funksiya n-tartibli differensialning argument differensialining n-darajasiga nisbatiga teng bo'lishi kelib chiqadi.

10- misol. $y = x^4 + 3x - 1$ bo'lsa, $d^3 y$ ni toping.

Yechish. $y' = 4x^3 + 3$, $y'' = 12x^2$, $y''' = 24x$.

Bundan

$$d^3 y = y'''(x)dx^3 = 24xdx^3.$$

Yuqorida keltirilgan formulalar x erkli o'zgaruvchi bo'lganda o'rinli bo'ladi. Agar $y = f(x)$ funksiyada x *boshqa bir erkli o'zgaruvchining funksiyasi bo'lsa,* yuqori tartibli differesiallar invariantlik xossasiga bo'ysunmaydi.

Buni ikkinchi tartibli differensial uchun ko'rsatamiz.

Ko'paytmaning differensiali formulasiga ko'ra,

$$d^2 y = d(f'(x)dx) = d(f'(x))dx + f'(x)d(dx) = f''(x)dx \cdot dx + f'(x)d^2 x,$$

ya'ni

$$d^2 y = f''(x)dx^2 + f'(x)d^2 x. \tag{2.6}$$

(2.5) va (2.6) formulalarni solishtiramiz: murakkab funksiya uchun ikkinchi tartibli differensial o'zgaradi, ya'ni bunda ikkinchi qo'shiluvchi $f'(x)d^2 x$ hosil bo'ladi.

Agar bunda x erkli o'zgaruvchi bo'lsa, u holda

$$d^2 x = d(dx) = d(1 \cdot dx) = dx \cdot d1 = dx \cdot 0 = 0$$

va (2.5) ifoda (2.6) formulaga o'tadi.

11- misol. $y = x^2$ va $x = \sin t$ bo'lsa, $d^2 y$ ni toping.

Yechish. $y' = 2x$, $y'' = 2$, $dx = \cos tdt$, $d^2 x = -\sin tdt^2$

(2.6) formulaga formula bilan topamiz:

$$d^2 y = 2dx^2 - 2x\sin tdt^2 = 2(\cos tdt)^2 - 2\sin t \cdot \sin tdt^2 =$$
$$= 2\cos^2 tdt^2 - 2\sin^2 tdt^2 = 2\cos 2tdt^2.$$

Boshqa yechim: $y=x^2$ va $x=\sin t$.

Bundan

$$y=\sin^2 t,\ \ y'=2\sin t\cos t=\sin 2t,\ \ y''=2\cos 2t.$$

U holda

$$d^2y=y''dt^2=2\cos 2t dt^2.$$

5.2.9. Mashqlar

1. Differensiallash qoidalari va formulalaridan foydalanib, berilgan funksiyalarning hosilasini toping:

1) $y=3x^4-\frac{1}{3}x^3+\ln 2;$ 2) $y=\frac{1}{6}x^6+3x^4-2x;$

3) $y=\frac{2}{\sqrt{x}}+3x^2\sqrt[3]{x}-\frac{6}{\sqrt[3]{x^2}};$ 4) $y=\sqrt{x}-\frac{3}{x}+\frac{1}{3x^3};$

5) $y=\frac{xe^x-e^{-x}}{x^2};$ 6) $y=\frac{2^x+3^x}{2^x-3^x};$

7) $y=\frac{x\ln x}{\ln x-1};$ 8) $y=\frac{\ln x+e^x}{\ln x-e^x};$

9) $y=\frac{1+\cos x}{1-\cos x};$ 10) $y=\frac{1+tgx}{1-tgx};$

11) $y=tgx-ctgx;$ 12) $y=\frac{x\sin x-\cos x}{x\cos x+\sin x};$

13) $y=\frac{xchx-shx}{xshx-chx};$ 14) $y=thx+cthx;$

15) $y=\log_x e;$ 16) $y=4\sin^2 x-3\lg x+4\cos^2 x;$

17) $y=\sqrt{4-3x^2};$ 18) $y=\arcsin\sqrt{x};$

19) $y=\cos^4 x-\sin^4 x;$ 20) $y=\frac{1}{6}\ln\frac{x-3}{x+3};$

21) $y=\sqrt{1-x^2}+x\arcsin x;$ 22) $y=\ln(e^{2x}+1)-2arctge^x;$

23) $y=\frac{1}{2}\ln\frac{1+3^x}{1-3^x};$ 24) $y=\log_{x^3}x^x;$

25) $y=\frac{tg3x+\ln\cos^2 3x}{3};$ 26) $y=e^{-3x}(\sin 3x+\cos 3x);$

27) $y=\sqrt{e^x-1}-arctg\sqrt{e^x-1}$;   28) $y=\ln ctg\left(\frac{\pi}{4}+\frac{x}{2}\right)$;

29) $y=3\arccos\frac{x-3}{\sqrt{5}}+\sqrt{6x-4-x^2}$;   30) $y=\frac{2-x}{4(x^2+2)}-\frac{1}{4\sqrt{2}}arctg\frac{x}{\sqrt{2}}+\ln\sqrt{x^2+2}$.

2. Berilgan $x=\varphi(y)$ funksiyalar uchun y' hosilani toping:

1) $x=\frac{1-y}{1+y}$;   2) $x=e^{-y}$;   3) $x=2\sin y$;   4) $x=3ctg\,y$.

3. Quyidagi sonlarni differensial yordamida taqriban hisoblang:

1) $\sqrt[5]{33}$;   2) $\lg 10{,}21$;   3) $ctg\,45^0 10'$;   4) $3{,}013^3$.

4. Quyidagi funksiyalarning berilgan nuqtadagi taqribiy qiymatini differensial yordamida hisoblang:

1) $y=\sqrt{x^2-7x+10}$, $x=0{,}98$;   2) $y=\sqrt[5]{\frac{2-x}{2+x}}$, $x=0{,}15$;

3) $y=\sqrt{\frac{x^2-3}{x^2+5}}$, $x=2{,}037$;   4) $y=\sqrt[3]{2x-\sin\frac{\pi x}{2}}$, $x=1{,}02$.

5. Berilgan funksiyalarning birinchi tartibli differensialini toping:

1) $y=x(\ln x-1)$;   2) $y=\frac{\ln x}{x}$;   3) $y=\cos^2 2x$;

4) $y=a\sin^3 x$.   5) $y=3^{\cos x}$;   6) $y=\ln^3\cos x$.

6. Berilgan hosilalar uchun y''' ni toping:

1) $y=(x^2-1)^3$;   2) $y=e^{2x}\cos x$;   3) $y=(1+x^2)arctgx$;   4) $y=x^2(\ln x-1)$.

7. Berilgan funksiyalar uchun $y^{(n)}(0)$ ni toping:

1) $y=\sin 5x\cos 2x$;   2) $y=x\cos x$;   3) $y=x^2\sin x$;   4) $y=x^2e^x$.

8. Oshkormas funksiyalarning hosilasini toping:

1) $b^2x^2+a^2y^2=a^2b^2$;   2) $y^3=x^3+3xy$;   3) $e^{x+y}=xy$;

4) $\cos(xy)=x^2$;   5) $e^y+xy=e$;   6) $x\sin y+y\sin x=0$.

9. Berilgan funksiyalar uchun $\frac{d^2y}{dx^2}$ ni toping:

1) $\begin{cases}x=t^2+1,\\ y=t^3-1;\end{cases}$ 2) $\begin{cases}x=a\cos t,\\ y=a\sin t;\end{cases}$

3) $\begin{cases}x=\ln(1+t^2),\\ y=t-arctg\,t;\end{cases}$ 4) $\begin{cases}x=\arcsin t,\\ y=\sqrt{1-t^2}.\end{cases}$

10. Berilgan egri chiziqqa $M_0(x_0,y_0)$ nuqtada o'tkazilgan urinma va normal tenglamalarini tuzing:

1) $y=\frac{x^3}{3}$, $M_0\left(-1,-\frac{1}{3}\right)$; 2) $y=\sin x$, $M_0(\pi,0)$;

3) $y=x^3+x^2-1$ egri chiziqqa $y=x^2$ parabola bilan kesishish nuqtasida;

4) $\frac{x^2}{9}+\frac{y^2}{25}=1$, $M_0\left(\frac{9}{5};4\right)$; 5) $\begin{cases} x=\frac{1+t}{t^3}, \\ y=\frac{3}{t^2}-\frac{1}{t}, \end{cases}$ $M_0(2,2)$; 6) $\begin{cases} x=\sin t, \\ y=\cos 2t, \end{cases}$ $M_0\left(\frac{1}{2};\frac{1}{2}\right)$.

5.3. DIFFERENSIAL HISOBNING ASOSIY TEOREMALARI

5.3.1. Ferma teoremasi

Differensiallanuvchi funksiyalarning nazariy va amaliy ahamiyatga ega bo'lgan teoremalari bilan tanishamiz.

1-teorema (*Ferma teoremasi*). $f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lib, bu intervalning biror c nuqtasida o'zining eng katta (eng kichik) qiymatiga erishsin. Agar funksiya c nuqtada chekli hosilaga ega bo'lsa, u holda

$$f'(c)=0$$

bo'ladi.

Isboti. Aytaylik, $y=f(x)$ funksiya $c\in(a;b)$ nuqtada o'zining eng katta qiymatiga ega bo'lsin. U holda $\forall x\in(a;b)$ uchun $f(x)\le f(c)$, ya'ni $f(x)-f(c)\le 0$ bo'ladi.

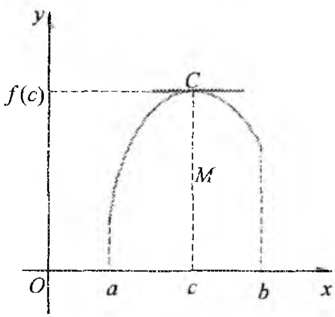
$y=f(x)$ funksiya c nuqtada hosilaga ega. Shu sababli bu nuqtada funksiyaning o'ng va chap hosilalari mavjud va

$$f'_+(c)=\lim_{x\to c+0}\frac{f(x)-f(c)}{x-c}\le 0 \ (x>c),$$

$$f'_-(c)=\lim_{x\to c-0}\frac{f(x)-f(c)}{x-c}\ge 0 \ (x<c),$$

$$f'_+(c)=f'_-(c)=f'(c)$$

bo'ladi.



5-shakl.

Bu munosabatlardan $f'(c)=0$ bo'lishi kelib chiqadi.

Funksiya c nuqtada eng kichik qiymatga ega bo'lgan hol uchun teorema shu kabi isbotlanadi.

Ferma teoremasining geometrik talqini quyidagicha bo'ladi: $y=f(x)$ funksiya c nuqtada eng katta (eng kichik) qiymatga erishsa va $f'(c)$ hosila mavjud bo'lsa, $f(x)$ funksiya grafigiga $M(c;f(c))$ nuqtada o'tkazilgan urinma Ox o'qqa parallel bo'ladi (5-shakl).

$[a;b]$ kesma uchun Ferma teoremasi hamma vaqt ham o'rinli bo'lmaydi. Masalan, $f(x)=x$ funksiya $[0;1]$ kesmada o'zining eng katta ($x=1$ da) va eng kichik ($x=0$ da) qiymatiga erishadi. Bu nuqtalarda hosila $f'(x)=1\neq 0$.

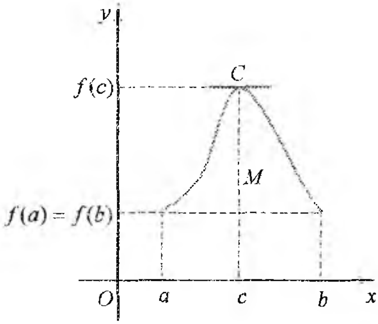
5.3.2. Roll teoremasi

2-teorema (*Roll teoremasi*). $f(x)$ funksiya $[a;b]$ kesmada aniqlangan, uzluksiz va $f(a)=f(b)$ bo'lsin. Agar funksiya $(a;b)$ intervalda differensiallanuvchi bo'lsa, u holda shunday $c\in(a;b)$ nuqta topiladiki,

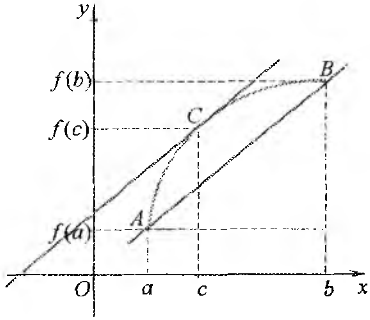
$$f'(c)=0$$

bo'ladi.

Isboti. Shartga ko'ra, $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz. U holda Veershtrassning 2-teoremasiga ko'ra, funksiya shu kesmada o'zining eng katta qiymati M ga va eng kichik qiymati m ga ega bo'ladi. Bunda $M=m$ bo'lsa, $f(x)$ funksiya $[a;b]$ kesmada o'zgarmas va shu sababli $\forall x\in(a;b)$ uchun $f'(x)=0$ bo'ladi.



6-shakl.



7-shakl.

Agar $M \neq m$ bo'lsa, $f(x)$ funksiya M va m qiymatlardan biriga biror $c \in (a;b)$ nuqtada ega bo'ladi. Bunda Ferma teoremasiga asosan, $f'(c)=0$ bo'ladi.

Roll teoremasi ushbu geometrik talqinga ega: $[a;b]$ kesmada uzluksiz, $(a;b)$ intervalda differensiallanuvchi va kesmaning chetki nuqtalarida teng qiymatlar qabul qiluvchi funksiya grafigida shunday $(c;f(c))$ nuqta topiladi va bu nuqtada funksiya grafigiga o'tkazilgan urinma Ox o'qiga parallel bo'ladi (6-shakl).

1-misol. Roll teoremasi o'rinli bo'lishini tekshiring:
1) $f(x)=x^2-3x-4$ funksiya uchun $[0;3]$ kesmada; 2) $f(x)=\sqrt[3]{x^2}-1$ funksiya uchun $[-1;1]$ kesmada.

Yechish. 1) $f(x)=x^2-3x-4$ funksiya $[0;3]$ kesmada uzluksiz, differensiallanuvchi va uning chetki nuqtalarida bir xil qiymatga ega: $f(0)=f(3)=-4$. Shu sababli, bu funksiya uchun Roll teoremasi o'rinli bo'ladi. x ning $f'(x)=0$ bo'lgan qiymatini topamiz: $f'(x)=4x-3=0$. Bundan $x=\frac{3}{4}$.

2) $f(x)=\sqrt[3]{x^2}-1$ funksiya $[-1;1]$ kesmada uzluksiz, $f(-1)=f(1)=0$, $f'(x)=\frac{2}{3}\frac{1}{\sqrt[3]{x}}$. Bu hosila $x=0\in(-1;1)$ nuqtada mavjud emas. Demak, bu funksiya uchun Roll teoremasi o'rinli bo'lmaydi.

5.3.3. Lagranj teoremasi

3-teorema (*Lagranj teoremasi*). $f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsin. Agar funksiya $(a;b)$ intervalda chekli hosilaga ega bo'lsa, u holda shunday $c \in (a;b)$ nuqta topiladiki,

$$f'(c)=\frac{f(b)-f(a)}{b-a} \tag{3.1}$$

bo'ladi.

Isboti. Teoremani isbotlash uchun yordamchi

$$F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}\cdot(x-a)$$

funksiyadan foydalanamiz. Shartga ko‘ra, $f(x)$ funksiya $[a;b]$ kesmada aniqlangan, uzluksiz va $(a;b)$ intervalda hosilaga ega bo‘lgani uchun $F(x)$ funksiya ham $[a;b]$ kesmada aniqlangan, uzluksiz va $(a;b)$ intervalda hosilaga ega bo‘ladi.

Bunda

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \quad F(a) = F(b) = 0. \tag{3.2}$$

Demak, $F(x)$ funksiya Roll teoremasining barcha shartlarini qanoatlantiradi. U holda biror $c \in (a;b)$ nuqta uchun $F'(c) = 0$, $f'(c) - \frac{f(b) - f(a)}{b - a} = 0$ bo‘ladi.

Bundan

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Teoremaning geometrik talqinini beramiz.

$\frac{f(b) - f(a)}{b - a}$ qiymat funksiya grafigining $A(a; f(a))$ va $B(b; f(b))$ nuqtalari orqali o‘tivchi kesuvchining burchak koeffitsiyentini aniqlaydi. Teoremaga ko‘ra, shunday $c \in (a;b)$ topiladiki, $C(c; f(c))$ nuqtada funksiya grafigiga o‘tkazilgan urinma AB kesuvchiga parallel bo‘ladi (7-shakl).

(3.1) tenglikdan

$$f(b) - f(a) = f'(c)(b - a) \tag{3.3}$$

kelib chiqadi. Bu formulaga *Lagranj formulasi* yoki *chekli ayirmalar formulasi* deyiladi.

Agar $a = x$, $b = x + \Delta x$ desak, bu formulani

$$f(x + \Delta x) - f(x) = f'(c)\Delta x \tag{3.4}$$

ko‘rinishda yozish mumkin.

$c \in (a;b)$ bo‘lgani uchun $c = a + \theta(b - a)$, $0 < \theta < 1$, deyish mumkin.

U holda (3.4) tenglik

$$f(x + \Delta x) - f(x) = f'(x + \theta\Delta x)\Delta x$$

ko‘rinishni oladi.

Langranj teoremasi yordamida $\Delta y \approx dy$ taqribiy tenglikning aniqligini baholash mumkin. Buning uchun $f(x)$ funksiya ikkinchi tartibli uzluksiz $f''(x)$ hosilaga ega bo'lsin deb, topamiz:

$$\Delta y - dy = (f(x+\Delta x) - f(x)) - f'(x)dx = f'(c)\Delta x - f'(x)\Delta x =$$

$$= (f'(c) - f'(x))\Delta x = f''(c_1)(c-x)\Delta x, \text{ bu yerda } c_1 \in (c;x).$$

Demak, $\Delta y - dy = f''(c_1)(c-x)\Delta x$. $M = \max\limits_{[x,x+\Delta x]}|f''(x)|$ bo'lsin.

$|c-x| < \Delta x$ va $f''(c_1) \le M$ tengsizliklarni hisobga olib, topamiz:

$$|\Delta y - dy| \le M|\Delta x|^2.$$

Lagranj teoremasidan quyidagi natijalar kelib chiqadi.

1-natija. Agar biror intervalda funksiyaning hosilasi nolga teng bo'lsa, funksiya shu intervalda o'zgarmas bo'ladi.

2-natija. Agar biror intervalda ikkita funksiya teng hosilalarga ega bo'lsa, funksiyalar bir-biridan o'zgarmas qo'shiluvchiga farq qiladi.

2- misol. $\arcsin x + \arccos x = \dfrac{\pi}{2}$, $x \in [-1;1]$ ekanini isbotlang.

Yechish. $f(x) = \arcsin x + \arccos x$ deb olsak, $(-1;1)$ oraliqda

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$$

U holda 1-natijaga ko'ra, $f(x) = C$, ya'ni $\arcsin x + \arccos x = C$. C ning qiymatini topish uchun x ga $(-1;1)$ intervaldagi qiymatlardan birini, masalan, $x = 0$ ni qo'yamiz: $\arcsin 0 + \arccos 0 = C$ yoki $\dfrac{\pi}{2} = C$. Bundan

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

5.3.4. Koshi teoremasi

4-teorema (*Koshi teoremasi*). $f(x)$ va $g(x)$ funksiyalar $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsin. Agar funksiyalar $(a;b)$ intervalda differensiallanuvchi va $\forall x \in (a;b)$ uchun $g'(x) \ne 0$ bo'lsa, u holda shunday $c \in (a;b)$ nuqta topiladi va

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \tag{3.5}$$

bo'ladi.

Isboti. Teoremaning $g'(x)\neq 0$ shartiga ko'ra, (3.5) tenglik ma'noga ega bo'lishi uchun $g(b)\neq g(a)$ bo'lishi kerak. $f(x)$ va $g(x)$ funksiyalardan ushbu

$$F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}\cdot(g(x)-g(a))$$

funksiyani tuzamiz. Bu funksiya $[a;b]$ kesmada aniqlangan, uzluksiz va $(a;b)$ intervalda hosilaga ega.

Bundan tashqari,

$$F'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}\cdot g'(x),\quad F(a)=F(b)=0.$$

Demak, $F(x)$ funksiya Roll teoremasining barcha shartlarini qanoatlantiradi va biror $c\in(a;b)$ nuqta uchun $F'(c)=0$, ya'ni

$$f'(c)-\frac{f(b)-f(a)}{g(b)-g(a)}\cdot g'(c)=0$$

bo'ladi.

Bundan

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

5.3.5. Lopital teoremasi

5-teorema $\left(\frac{0}{0}\right.$ *ko'rinishdagi aniqmasliklarni ochishning Lopital qoidasi*$\left.\right)$. x_0 nuqtaning biror atrofida $f(x)$ va $g(x)$ funksiyalar uzluksiz, differensiallanuvchi va $g'(x)\neq 0$ bo'lsin. Agar $\lim\limits_{x\to x_0} f(x)=0$, $\lim\limits_{x\to x_0} g(x)=0$ va $\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}=k$ (chekli yoki cheksiz) limit mavjud bo'lsa, u holda

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)} \tag{3.6}$$

bo'ladi.

Isboti. $f(x)$ va $g(x)$ funksiyalar uchun x_0 nuqtaning biror atrofida yotuvchi $[x_0;x]$ kesmada Koshi teoremasini qo'llaymiz.

Bunda

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(c)}{g'(c)}, \quad f(x_0)=g(x_0)=0$$

hisobga olinsa,

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)} \tag{3.7}$$

hosil bo'ladi, bu yerda c nuqta x va x_0 nuqtalar orasida yotuvchi biror son.

$x \to x_0$ da c ham x_0 ga intiladi.

(3.7) tenglikda limitga o'tamiz:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{c\to x_0}\frac{f'(c)}{g'(c)}.$$

$\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}=k$ ekanidan $\lim\limits_{c\to x_0}\frac{f'(c)}{g'(c)}=k$.

Shu sababli

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

Izohlar: 1. 1-teorema $f(x)$ va $g(x)$ funksiyallar $x=x_0$ da aniqlanmagan, ammo $\lim\limits_{x\to x_0} f(x)=0$ va $\lim\limits_{x\to x_0} g(x)=0$ bo'lganda ham o'rinli bo'ladi. Bunda $f(x_0)=\lim\limits_{x\to x_0} f(x)=0$ va $g(x_0)=\lim\limits_{x\to x_0} g(x)=0$ deb olish yetarli.

2. 1-teorema $x\to\infty$ da ham o'rinli bo'ladi. Haqiqatan ham, $x=\frac{1}{z}$ deb, topamiz:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{z\to 0}\frac{f\left(\frac{1}{z}\right)}{g\left(\frac{1}{z}\right)}=\lim_{z\to 0}\frac{\left(f\left(\frac{1}{z}\right)\right)'}{\left(g\left(\frac{1}{z}\right)\right)'}=\lim_{z\to 0}\frac{f'\left(\frac{1}{z}\right)\cdot\left(-\frac{1}{z^2}\right)}{g'\left(\frac{1}{z}\right)\cdot\left(-\frac{1}{z^2}\right)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

3. $f'(x)$ va $g'(x)$ funksiyalar 1-teoremaning shartlarini qanoatlantirsa, teorema takror qo'llanishi mumkin:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\lim_{x\to x_0}\frac{f''(x)}{g''(x)}.$$

3- misol. $\lim\limits_{x\to 1}\dfrac{x^2-1+\ln x}{e^x-e}$ limitni toping.

Yechish. $f(x)=x^2-1+\ln x$, $g(x)=e^x-e$ funksiyalar $x=1$ nuqta atrofida aniqlangan. $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1} g(x)=0$, ya'ni $\dfrac{0}{0}$ ko'rinishdagi aniqmaslik berilgan.

U holda 1-teoremaga ko'ra,

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{f'(x)}{g'(x)}=\lim_{x\to 1}\frac{2x+\frac{1}{x}}{e^x}=\frac{3}{e}.$$

Demak,

$$\lim_{x\to 1}\frac{x^2-1+\ln x}{e^x-e}=\frac{3}{e}.$$

$\dfrac{\infty}{\infty}$ ko'rinishdagi aniqmasliklarni ochish haqidagi teoremani isbotsiz keltiramiz.

6-teorema $\left(\dfrac{\infty}{\infty}\right.$ *ko'rinishdagi aniqmasliklarni ochishning Lopital qoidasi*$\left.\right)$.

x_0 nuqtaning biror atrofida $f(x)$ va $g(x)$ funksiyalar uzluksiz, differensiallanuvchi va $g'(x)\neq 0$ bo'lsin. Agar $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=\infty$ bo'lib, $\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}$ limit mavjud bo'lsa, u holda

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

bo'ladi.

4- misol. $\lim\limits_{x\to a}\dfrac{\ln(x-a)}{\ln(e^x-e^a)}$ limitni toping.

Yechish. $\lim\limits_{x\to a}\dfrac{\ln(x-a)}{\ln(e^x-e^a)}=\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to a}\dfrac{\frac{1}{x-a}}{\frac{e^x}{e^x-e^a}}=\lim\limits_{x\to a}\dfrac{e^x-e^a}{e^x(x-a)}=\left(\dfrac{0}{0}\right)=$

$$=\lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x}=\lim_{x\to a}\frac{1}{1+(x-a)}=\frac{1}{1+(a-a)}=\frac{1}{1+0}=1.$$

$\frac{0}{0}$ va $\frac{\infty}{\infty}$ ko'rinishdagi aniqmasliklarga asosiy aniqmasliklar deyiladi.

$0\cdot\infty$ yoki $\infty-\infty$ ko'rinishdagi aniqmasliklar algebraik almashtirishlar yordamida asosiy aniqmasliklarga keltiriladi.

0^0, ∞^0 yoki 1^∞ ko'rinishdagi aniqmasliklar

$$f(x)^{g(x)} = e^{g(x)\ln f(x)}$$

formula yordamida $0\cdot\infty$ aniqmaslikka keltiriladi.

5- misol. $\lim\limits_{x\to 0} x^3 \ln x$ limitni toping.

Yechish. $\lim\limits_{x\to 0} x^3 \ln x = (0\cdot\infty) = \lim\limits_{x\to 0}\frac{\ln x}{\frac{1}{x^3}} = \left(\frac{\infty}{\infty}\right) = \lim\limits_{x\to 0}\frac{\frac{1}{x}}{-3\frac{1}{x^4}} = -\frac{1}{3}\lim\limits_{x\to 0} x^3 = 0$.

6- misol. $\lim\limits_{x\to 0}\left(\frac{1}{x} - ctgx\right)$ limitni toping.

Yechish. $\lim\limits_{x\to 0}\left(\frac{1}{x} - ctgx\right) = (\infty-\infty) = \lim\limits_{x\to 0}\left(\frac{\sin x - x\cos x}{x\sin x}\right) = \left(\frac{0}{0}\right) =$

$$= \lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{x\sin x}{\sin x + x\cos x} = \left(\frac{0}{0}\right) =$$

$$= \lim_{x\to 0}\frac{\sin x + x\cos x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0.$$

7- misol. $\lim\limits_{x\to 0} x^{\sin x}$ limitni toping.

Yechish. $\lim\limits_{x\to 0} x^{\sin x} = (0^0) = \lim\limits_{x\to 0} e^{\ln x^{\sin x}} = e^{\lim\limits_{x\to 0}\sin x\ln x} = e^{\lim\limits_{x\to 0}\frac{\ln x}{\frac{1}{\sin x}}} =$

$$= e^{-\lim\limits_{x\to 0}\frac{\frac{1}{x}}{\frac{\cos x}{\sin^2 x}}} = e^{-\lim\limits_{x\to 0}\frac{\sin^2 x}{x\cos x}} = e^{-\lim\limits_{x\to 0}\left(tgx\cdot\frac{\sin x}{x}\right)} = e^0 = 1.$$

5.3.6. Teylor teoremasi

7-teorema (*Teylor teoremasi*). $f(x)$ funksiya x_0 nuqtaning biror atrofida aniqlangan bo'lib, bu atrofda $(n+1)$-tartibligacha hosilalarga ega va $f^{(n+1)}(x)$ hosila x_0 nuqtada uzluksiz bo'lsin.

U holda

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \qquad (3.8)$$

bo‘ladi, bunda $c=x_0+\theta(x-x_0),\ 0<\theta<1.$

(3.8) tenglikka *Teylor formulasi* deyiladi.

Isboti. Avval

$$\varphi(x,x_0)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

$$R_{n+1}(x)=f(x)-\varphi(x,x_0)$$

belgilashlar kiritamiz.

Bunda biror c son uchun $R_{n+1}(x)=\frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1}$ bo‘lishi ko‘rsatilsa, teorema isbot bo‘ladi.

Endi x_0 nuqtaning atrofida x ($x>x_0$ bo‘lsin) nuqtani tanlaymiz va $[x_0;x]$ kesmada

$$F(t)=f(x)-\varphi(x,t)-\frac{(x-t)^{n+1}R_{n+1}(x)}{(x-x_0)^{n+1}},\ t\in[x_0;x]$$

yordamchi funksiyani tanlaymiz.

$F(t)$ funksiya $[x_0;x]$ kesmada usluksiz va differensiallanuvchi va

$$F'(t)=-f'(t)+\frac{f'(t)}{1!}-\frac{f''(t)}{1!}(x-t)+\frac{f''(t)}{2!}2(x-t)-\frac{f'''(t)}{3!}(x-t)^2+\ldots+$$

$$+\frac{f^{(n)}(t)}{n!}n(x-t)^{n-1}-\frac{f^{(n+1)}(t)}{n!}(x-t)^n+\frac{(n+1)(x-t)^nR_{n+1}(x)}{(x-x_0)^{n+1}}=$$

$$=-\frac{f^{(n+1)}(t)}{n!}(x-t)^n+\frac{(n+1)(x-t)^nR_{n+1}(x)}{(x-x_0)^{n+1}}. \qquad (3.9)$$

$t=x_0$ da

$$F(x_0)=f(x)-\varphi(x,x_0)-R_{n+1}(x)=R_{n+1}(x)-R_{n+1}(x)=0;$$

$t=x$ da

$$F(x)=f(x)-f(x)-\frac{f'(x)}{1!}(x-x)-\ldots-\frac{f^{(n)}(x)}{n!}(x-x)^n-\frac{(x-x)^{n+1}R_{n+1}(x)}{(x-x_0)^{n+1}}=0.$$

Demak, $F(t)$ funksiya $[x_0;x]$ kesmada Roli teoremasining barcha shartlarini qanoatlantiradi. U holda shunday c $(x_0 < c < x)$ nuqta topiladi va $F'(c) = 0$ bo'ladi.

(3.9) tenglikka ko'ra,

$$-\frac{f^{(n+1)}(t)}{n!}(x-c)^n + \frac{(n+1)(x-c)^n R_{n+1}(x)}{(x-x_0)^{n+1}} = 0.$$

Bundan

$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

$$\varphi(x,x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

ko'phadga n-tartibli *Teylor ko'phadi*,

$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1}$$

funksiyaga Teylor formulasining *Lagranj ko'rinishdagi qoldiq hadi* deyiladi.

$n = 0$ da Teylor formulasidan $f(x) = f(x_0) + f'(c)(x-x_0)$ yoki $f(x) - f(x_0) = f'(c)(x-x_0)$ tenglik, ya'ni Lagranj formulasi kelib chiqadi. Demak, Lagranj formulasi Teylor formulasining xususiy holi bo'ladi.

$x_0 = 0$ da Teylor formulasining xususiy hollaridan yana biri

$$f(x) = f(0) + \frac{f^1(0)}{1!}x + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \cdots + = \frac{f^{(n+1)}(cx)}{(n+1)!}x^{n+1},\ 0 < c < 1$$

hosil bo'ladi.

Bu formulaga *Makloren formulasi* deyiladi.

Ayrim funksiyalarning Makloren formulasiga yoyilmasini keltiramiz:

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1},\ x \in R$;

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \sin cx \frac{x^{2n+2}}{(2n+2)!},\ x \in R;$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1}\cos cx \frac{x^{2n+1}}{(2n+1)!},\ x \in R;$

4. $(1+x)^m = 1+\frac{m}{1!}x+\frac{m(m-1)}{2!}x^2+...+\frac{m(m-1)...(m-n+1)}{n!}x^2+$

$$+\frac{m(m-1)...(m-n)}{(n+1)!}(1+cx)^{m-n+1}x^{n+1},\ x\in(-1;1);$$

Xususan, $n=m$ da

$$(1+x)^n = 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+...+\frac{n(n-1)...(n-(n-2))}{(n-1)!}x^{n-1}+x^n.$$

Formulalardan ba'zilarining isbotini keltiramiz.

1. $f(x)=e^x$ bo'lsin.

U holda

$$f(x)=f'(x)=f''(x)=...=f^{(n+1)}(x)=e^x,$$

$$f(0)=f'(0)=f''(0)=...=f^{(n+1)}(0)=1.$$

Makloren formulasi quyidagi ko'rinishga keladi:

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}. \tag{3.10}$$

2. $f(x)=\sin x$ bo'lsin.

U holda

$$f^{(n)}(x)=\sin\left(x+n\cdot\frac{\pi}{2}\right),\quad f^{(n)}(0)=\sin\left(n\cdot\frac{\pi}{2}\right)=\begin{cases}0, & n\ \textit{juft bo'lsa},\\ (-1)^{\frac{n-1}{2}}, & n\ \textit{toq bo'lsa}.\end{cases}$$

Bundan

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+(-1)^n\sin cx\frac{x^{2n+2}}{(2n+2)!}.$$

4. $f(x)=(1+x)^m$ bo'lsin.

U holda

$$f^{(n)}(x)=m(m-1)...(m-n+1)(1+x)^{m-n+1},$$

$$f^{(n)}(0)=m(m-1)...(m-n+1).$$

Bundan

$$(1+x)^m = 1+\frac{m}{1!}x+\frac{m(m-1)}{2!}x^2+...+\frac{m(m-1)...(m-n+1)}{n!}x^2+$$

$$+\frac{m(m-1)...(m-n)}{(n+1)!}(1+cx)^{m-n+1}x^{n+1},\ x\in(-1;1).$$

Teylor formulasi funksiyalar qiymatlari va limitlarini berilgan aniqlikda hisoblash imkonini beradi. Masalan, $f(x)$ funksiyaning $x=a$ nuqtadagi qiymatini xatoligi ε dan katta bo'lmagan aniqlikda hisoblash uchun Teylor ko'phadini shunday k darajasigacha olinadiki, bunda k son $|R_n(a)|<\varepsilon$ tengsizlikni qanoatilaniradigan n larning eng kichigi qilib tanlanadi.

n	e_n
0	1.0000000000000
1	2.0000000000000
2	2.5000000000000
3	2.6666666666667
4	2.7083333333333
5	2.7166666666667
6	2.7180555555556
7	2.7182539682540
8	2.7182787698413
9	2.7182815255732
10	2.7182818011464

8- misol. e sonini 0,001 aniqlikda hisoblang.

Yechish. $f(x)=e^x$ funksiyani qaraymiz. Shartga ko'ra, $x=a=1$, $\varepsilon=0{,}001$.

(3.10) formulaga binoan n ning $R_n(1)=\frac{e^c}{(n+1)!}<\varepsilon=0{,}001$ shartni qanoatlantiruvchi eng kichik qiymati $n=6$, bunda $0<c<1$.

Demak,

$$e\approx 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{6!}=$$

$$=2+0{,}5+0{,}16667+0{,}04167+0{,}00833+0{,}00139=2{,}718.$$

Shu kabi e sonining istalgan aniqlikdagi qiymatlari jadvalini topish mumkin.

5.3.7. Mashqlar

1. Funksiya uchun berilgan kesmada Roll teoremasi o'rinli bo'lishini tekshiring. Agar o'rinli bo'lsa, teoremadagi c ning tegishli qiymatini toping:

1) $f(x)=4x-x^2+5,\ [0;2]$; 2) $f(x)=\sin 2x,\ \left[\frac{\pi}{2};\pi\right]$;

3) $f(x)=2-\sqrt[5]{x^2},\ [-1;1]$; 4) $f(x)=3-|\ x\ |,\ [-2;2]$.

2. Funksiya uchun berilgan kesmada Lagranj formulasidagi c ning tegishli qiymatini toping:

1) $f(x)=\frac{1}{3}x^3-x+1,\ [0;1]$; 2) $f(x)=e^x,\ [0;1]$;

3) $f(x)=\ln x,\ [1;e]$; 4) $f(x)=x^2-6x+1,\ [0;1]$.

3. Berilgan funksiya grafigining urinmasi AB vatarga parallel bo'lgan nuqtasini toping:

1) $f(x)=x^2+3x,\ A(-2;-2),\ B(1;4);$ 2) $f(x)=\sqrt{x+1},\ A(0;1),\ B(3;2).$

4. Funksiyalar uchun berilgan kesmada Koshi formulasini yozing va bu formuladagi c ning tegishli qiymatini toping:

1) $f(x)=\sin 2x,\ g(x)=\cos 2x,\ \left[0;\frac{\pi}{4}\right];$ 2) $f(x)=x^4-3,\ g(x)=x^3+2,\ [0;2].$

5. Funksiyaning o'zgarmas bo'lishlik alomatidan foydalanib, quyidagilarni isbotlang:

1) $\arccos\frac{1-x^2}{1+x^2}=2arctgx,\ 0\le x<+\infty;$ 2) $\arcsin\frac{2x}{1+x^2}=\begin{cases}-\pi-2arctgx, & x\le-1,\\ 2arctgx, & -1\le x<1,\\ -\pi-2arctgx, & x\ge 1.\end{cases}$

6. Loopital qoidasidan foydalanib, limitlarni toping:

1) $\lim\limits_{x\to1}\frac{\sin\pi x}{\ln x};$ 2) $\lim\limits_{x\to0}\frac{x-arctgx}{x^3};$

3) $\lim\limits_{x\to0}\frac{\ln tg2x}{\ln\sin x};$ 4) $\lim\limits_{x\to+0}\frac{\ln x}{ctgx};$

5) $\lim\limits_{x\to+\infty}\frac{\log_3 x}{3^x},$ 6) $\lim\limits_{x\to+\infty}\frac{\pi-2arctgx}{\ln\left(1+\frac{1}{x}\right)};$

7) $\lim\limits_{x\to0}\frac{e^{x^2}-x^2-1}{\sin^4 x};$ 8) $\lim\limits_{x\to0}\frac{\ln\cos(3x^2-x)}{\sin 2x^2};$

9) $\lim\limits_{x\to\infty}xtg\frac{3}{x};$ 10) $\lim\limits_{x\to0}(1-e^{3x})ctgx;$

11) $\lim\limits_{x\to\frac{\pi}{2}}(\sec x-tgx);$ 12) $\lim\limits_{x\to0}\left(\frac{1}{x}-\frac{1}{arctgx}\right);$

13) $\lim\limits_{x\to\frac{\pi}{2}-0}(\pi-2x)^{\cos x};$ 14) $\lim\limits_{x\to0}x^{\frac{1}{\ln(e^x-1)}};$

15) $\lim\limits_{x\to3}\left(2-\frac{x}{3}\right)^{tg\frac{\pi x}{6}};$ 16) $\lim\limits_{x\to0}(\cos 3x)^{\frac{2}{x^2}};$

17) $\lim\limits_{x\to\frac{\pi}{2}}(tgx)^{1-\sin x};$ 18) $\lim\limits_{x\to+\infty}(x+3x)^{\frac{1}{x}}.$

7. Ko‘phadni $(x-x_0)$ ning darajasi bo‘yicha yoying:

1) $P(x)=x^3+5x^2-3x+1,\ x_0=-2;$ 2) $P(x)=x^4-2x^3+5x-6,\ x_0=2.$

8. Funksiyaning berilgan nuqtada uchinchi tartibli Teylor formulasini yozing:

1) $f(x)=\sqrt{1+x},\ x_0=3;$ 2) $f(x)=\frac{1}{x},\ x_0=-2.$

9. Funksiyalarni Makloren formulasi yordamida x ning darajalari bo‘yicha yoying:

1) $f(x)=xe^x;$ 2) $f(x)=chx.$

10. Berilganlarni 0,001 aniqlikda hisoblang:

1) $\sin 36^0;$ 2) $\cos 32^0;$

3) $\sqrt[3]{e};$ 4) $\lg 10{,}09.$

11. Limitlarni Makloren formulasi yordamida toping:

1) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{e^x-1-x-\dfrac{x^2}{2}};$ 2) $\lim\limits_{x\to 0}\dfrac{\cos x-1+\dfrac{x^2}{2}}{x^4};$

3) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}};$ 4) $\lim\limits_{x\to 0}\dfrac{1+x\cos x-\sqrt{1+2x}}{\ln(1+x)-x}$

5.4. FUNKSIYAILARNI HOSILALAR YORDAMIDA TEKSHIRISH

Differensial hisobning tatbiqlaridan biri funksiyalarni tekshirish va grafigini chizishga hosilaning qo‘llanilishi hisoblanadi. Quyida bu tatbiqlarni ifodalovchi teoremalarni keltiramiz.

5.4.1. Funksiyaning monotonlik shartlari

1-teorema (*funksiya monoton bo‘lishining zaruriy sharti*). Agar $(a;b)$ intervalda differensiallanuvchi $f(x)$ funksiya shu intervalda o‘suvchi (kamayuvchi) bo‘lsa, u holda $\forall x\in(a;b)$ da

$$f'(x)\geq 0\ \ (f'(x)\leq 0)$$

bo‘ladi.

Isboti. $f(x)$ funksiya $(a;b)$ intervalda o'suvchi bo'lsin. $(a;b)$ intervalning ixtiyoriy x va $x+\Delta x$ nuqtalarini olamiz.

U holda $\Delta x > 0$ bo'lsa, $f(x+\Delta x) > f(x)$ va $\Delta x < 0$ bo'lsa, $f(x+\Delta x) < f(x)$ bo'ladi.

Ikkala holda ham

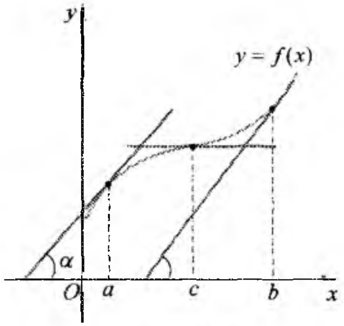
$$\frac{f(x+\Delta x)-f(x)}{\Delta x} > 0.$$

Teoremaning shartiga ko'ra $f(x)$ funksiya $(a;b)$ intervalda differensiallanuvchi.

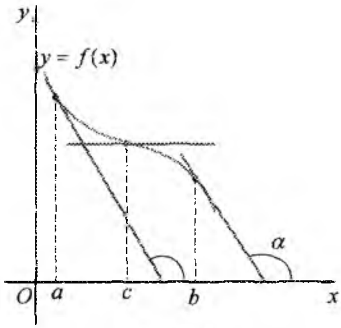
Shu sababli

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geq 0.$$

$f(x)$ funksiya $(a;b)$ intervalda kamayuvchi bo'lganda teorema shu kabi isbotlanadi.



8-shakl.



9-shakl.

Bu teorema ushbu geometrik talqinga ega: biror intervalda differensiallanuvchi bo'lgan o'suvchi (kamayuvchi) funksiya grafigiga o'tkazilgan urinmalar Ox o'qning musbat yo'nalishi bilan o'tkir (o'tmas) burchak tashkil qiladi yoki ayrim nuqtalarda Ox o'qiga parallel bo'ladi (8-shakl) ((9-shakl)).

2-teorema (*funksiya monoton bo'lishining yetarli sharti*). Agar $(a;b)$ intervalda differensiallanuvchi $f(x)$ funksiya uchun $\forall x \in (a;b)$ da $f'(x) > 0$ $(f'(x) < 0)$ bo'lsa, $f(x)$ funksiya $(a;b)$ intervalda o'sadi (kamayadi).

Isboti. $(a;b)$ intervalda $f'(x)>0$ bo'lsin. $\forall x_1,x_2\in(a;b)$, $x_1<x_2$ nuqtalarni olamiz.

$f(x)$ funksiya uchun $[x_1,x_2]$ kesmada Lagranj teoremasining shratlari bajariladi. Shu sababli Lagranj formulasiga binoan, biror $c\in(x_1,x_2)$ da $f(x_2)-f(x_1)=f'(c)(x_2-x_1)$ bo'ladi.

Teoremaning shartiga ko'ra, $\forall x\in(a;b)$ da $f'(x)>0$, shu jumladan, $c\in(x_1,x_2)$ da $f'(c)>0$. $x_2-x_1>0$ va shuning uchun $f'(c)(x_2-x_1)>0$. Bundan $f(x_2)-f(x_1)>0$ yoki $f(x_2)>f(x_1)$. Shunday qilib, $f(x)$ funksiya $\forall x\in(a;b)$ da o'sadi.

$f'(x)<0$ bo'lganda teorema shu kabi isbotlanadi.

Eslatmalar. 1. Funksiya o'suvchi va kamayuvchi bo'lgan intervallarga funksiyaning monotonlik intervallari deyiladi.

2. $(a;b)$ intervalda o'suvchi (kamayuvchi) va $[a;b]$ kesmada uzluksiz bo'gan $f(x)$ funksiya $[a;b]$ kesmada o'suvchi (kamayuvchi) bo'ladi.

1- misol. $f(x)=x^3-12x+5$ funksiyaning monotonlik intervallarini toping.

Yechish. $D(f)=R$, $f'(x)=3x^2-12=3(x^2-4)$. U holda: $f'(x)>0$ dan $x^2-4>0$ yoki $|x|>2$; $f'(x)<0$ dan $x^2-4<0$ yoki $|x|<2$.

Demak, $f(x)$ funksiya $(-\infty;-2)\cup(2;+\infty)$ intervalda o'sadi, $(-2;2)$ intervalda kamayadi.

5.4.2. Funksiyaning ekstremumlari

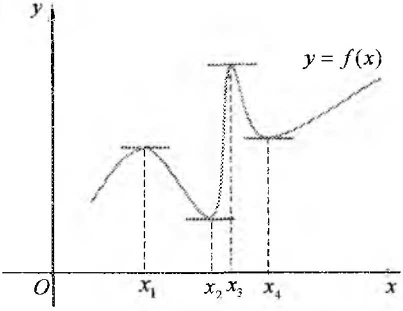
1-ta'rif. Agar x_0 nuqtaning shunday δ atrofi topilsa va bu atrofning barcha $x\neq x_0$ nuqtalarida $f(x)<f(x_0)$ $(f(x)>f(x_0))$ tengsizlik bajarilsa, x_0 nuqtaga $f(x)$ funksiyaning *qat'iy lokal maksimum* (*qat'iy lokal minimum*) nuqtasi deyiladi.

Funksiyaning maksimum va minimum nuqtalariga *ekstremum* nuqtalar deyiladi. Funksiyaning ekstremum nuqtadagi qiymati *funksiyaning ekstremumi* deb ataladi.

Ekstremum tushunchasi funksiya aniqlanish sohasining biror atrofi bilan bog'liq. Shu sababli funksiya ekstremumga aniqlanish sohasining faqat ichki nuqtalarida erishadi. Shu bilan birga, funksiya o'zining aniqlanish sohasida bir nechta minimumga yoki maksimumga erishishi va bunda maksimumlardan ayrimlari qandaydir minimumdan kichik bo'lishi mumkin (10-shakl).

3-teorema (*ekstremum mavjud bo'lishining zaruriy sharti*). Agar x_0 nuqtada differensiallanuvchi $f(x)$ funksiya shu nuqtada ekstremumga ega bo'lsa, u holda $f'(x_0)=0$ bo'ladi.

Isboti. x_0 nuqta $f(x)$ funksiyaning ekstremum nuqtasi bo'lsin. U holda shunday $(x_0-\delta, x_0+\delta)$ interval topiladi va bu intervalda $f(x)$ funksiya o'zining eng katta yoki eng kichik qiymatiga ega bo'ladi.

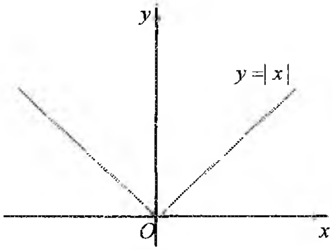


10-shakl.

U holda Ferma teoremasiga ko'ra, $f'(x_0)=0$ bo'ladi.

Bu teorema quyidagicha geometrik talqinga ega. Agar x nuqta $f(x)$ funksiyaning ekstremum nuqtasi bo'lsa (masalan, 10-shaklda x_1 nuqta), funksiya grafigiga shu nuqtada urinma o'tkazish mumkin va bu urinma Ox o'qiga parallel bo'ladi.

Izohlar. 1. $f(x)$ funksiya x_0 nuqtada uzluksiz bo'lib, differensiallanuvchi bo'lmaganida ham, ekstremumga ega bo'lishi mumkin. Masalan, uzluksiz $y=|x|$ funksiya $x=0$ nuqtada hosilaga ega emas, ammo $x=0$ minimum nuqta (11-shakl).



11-shakl.

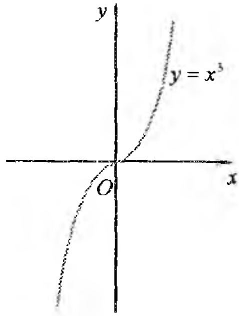
Shunday qilib, $f(x)$ funksiya x_0 nuqtada ekstremumga ega bo'lsa, bu nuqtada $f'(x)$ hosila nolga teng yoki mavjud bo'lmaydi.

$f'(x)$ hosilasi nolga teng bo'lgan yoki mavjud bo'lmagan nuqtaga *birinchi tur kritik nuqta* deyiladi.

2. Hamma birinchi tur kritik nuqta ham ekstremum nuqta bo'lavermaydi. Masalan, $f(x)=x^3$ funksiya uchun $x=0$ da $f'(x)=3x^2=0$. Demak, $x=0$ birinchi tur kritik nuqta, ammo u ekstremum nuqta emas (12-shakl).

4-teorema (*ekstremum mavjud bo'lishining yetarli sharti*). Agar $f(x)$ funksiya x_0 birinchi tur kritik nuqtaning biror δ atrofida differensiallanuvchi bo'lib, x nuqta x_0 nuqtadan chapdan o'ngga o'tganida $f'(x)$ hosila: ishorasini musbatdan manfiyga o'zgartirsa x_0 nuqta maksimum nuqta bo'ladi; manfiydan musbatga o'zgartirsa x_0 nuqta minimum nuqta bo'ladi.

Isboti. x_0 – birinchi tur kritik nuqta, $x \in (x_0 - \delta; x_0)$ da $f'(x) > 0$ va $x \in (x_0, x_0 + \delta)$ da $f'(x) < 0$ bo'lsin. U holda 1-teoremaga ko'ra, funksiya $(x_0 - \delta; x_0)$ intervalda o'sadi va $(x_0; x_0 + \delta)$ intervalda kamayadi. Demak, $f(x)$ funksiyaning x_0 nuqtadagi qiymati uning $\forall x \in (x_0 - \delta; x_0 + \delta)$ nuqtadagi qiymatidan katta bo'ladi, ya'ni $f(x)$ funksiya x_0 nuqtada maksimumga erishadi.



12-shakl.

$x \in (x_0 - \delta; x_0)$ da $f'(x) < 0$ va $x \in (x_0, x_0 + \delta)$ da $f'(x) > 0$ bo'lgan hol uchun teorema shu kabi isbotlanadi.

Izoh. Agar x nuqta x_0 nuqtadan chapdan o'ngga o'tganida ishorasini o'zgartirmasa x_0 nuqtada ekstremum mavjud bo'lmaydi.

Funksiyani ekstremumga tekshirish – bu funksiyaning barcha ekstremumlarini topish demakdir. Ekstremum mavjud bo'lishining zaruriy va yetarli shartlaridan funksiyani ekstremumga tekshirishning quyidagi qoidasi kelib chiqadi:

1°. $y = f(x)$ funksiyaning birinchi tur kritik nuqtalari topiladi;

2°. Bu nuqtalardan funksiyaning aniqlanish sohasiga tegishli bo'lganlari tanlanadi;

3°. tanlangan nuqtadan chapdan o'ngga o'tishda $f'(x)$ hosilaning ishorasi tekshiriladi;

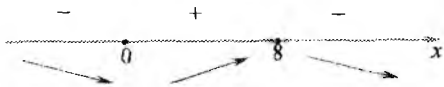
4°. 4- teoremaga asosan funksiyalarning ekstremum nuqtalari (agar ular bor bo'lsa) aniqlanadi va funksiyaning bu nuqtalardagi qiymatlari hisoblanadi.

2- misol. $f(x) = \sqrt[3]{x^2} - \frac{x}{3}$ funksiyaning ekstremumlarini toping.

Yechish. Ravshanki,

$$D(f) = R, \quad f'(x) = \frac{2}{3 \cdot \sqrt[3]{x}} - \frac{1}{3} = \frac{1}{3} \cdot \frac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}.$$

Hosila $x_1=0$ nuqtada mavjud emas va $x_2=8$ nuqtada nolga teng. Bu kritik nuqtalar berilgan funksiyaning aniqlanish sohasini uchta $(-\infty;0)$, $(0;8)$, $(8;+\infty)$ intervallarga ajratadi. Hosilaning har bir birinchi tur kritik nuqtadan chapdan o‘ngga o‘tishdagi ishoralarini chizmada belgilaymiz (13-shakl). Bunda strelkalar funksiyaning tegishli intervalda o‘suvchi yoki kamayuvchi ekanligini bildiradi.



13-shakl.

Demak, $x_1=0$ – minimum nuqta, $y_{min}=f(0)=0$ va $x_2=8$– maksimum nuqta, $y_{max}=f(8)=\frac{4}{3}$.

Funksiyaning eng katta va eng kichik qiymatini topish matematika, fizika, kimyo, iqtisodiyot va boshqa fanlarning ko‘plab masalalarini yechishda keng qo‘llaniladi. Masalan: minimal xarajat sarflab yukni tashish haqidagi transport masalasi, maksimal daromad olish maqsadida ishlab chiqarishni tashkil etish masalasi, eng katta va eng kichik qiymatlarni izlash usullarini takomillashtirish va rivojlantirishga olib keluvchi optimal yechimlarni izlash haqidagi boshqa masalalar. Bunday masalalarni yechish bilan matematikaning maxsus bo‘limi – chiziqli programmalashtirish shug‘ullanadi.

Biz soddaroq masalalardan birini ko‘rib chiqamiz.

3-misol. Tomoni 12 uzunlik birligiga teng kvadrat tunukaning burchaklaridan bir xil o‘lchamli kvadratlar kesib olingan va usti ochiq quti yasalgan. Qutining sig‘imi eng katta bo‘lishi uchun tomoni necha uzunlik birligiga teng kvadratlar kesilishi kerak?

Yechish. Kesib olinadigan kvadratlarning tomoni x bo‘lsin.

U holda qutining asosi $12-2x$ va balandligi x ga teng bo‘ladi.

Qutining hajmini topamiz:

$$V(x)=x(12-2x)^2=144x-48x^2+4x^3,\ x\in[0;6].$$

Bu funksiyaning maksimumini aniqlaymiz.

$$V'(x)=144-96x+12x^2=12(2-x)(6-x)$$

hosila $x=2$ da ishorasini musbatdan manfiyga almashtiradi, ya’ni $x=2$ maksimum nuqta bo‘ladi.

Demak, kesib olinadigan kvadratlar tomoni $x=2$ $(uz.b)$, bunda $V(2)=128$ $(kub.b)$.

5.4.3. Kesmada uzluksiz funksiyaning eng katta va eng kichik qiymatlari

$y = f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsin. U holda Veershtrassning ikkinchi teoremasiga ko'ra, funksiya bu kesmada o'zining eng katta va eng kichik qiymatlariga ega bo'ladi. Bu qiymatlarga funksiya yoki ekstremum nuqtalarda yoki $[a;b]$ kesmaning chetki nuqtalarida erishadi.

Bundan $[a;b]$ kesmada uzluksiz $y = f(x)$ funksiyaning eng katta va eng kichik qiymatlarini topishning quyidagi qoidasi kelib chiqadi:

1°. $y = f(x)$ funksiyaning $(a;b)$ intervaldagi birinchi tur kritik nuqtalari topiladi;

2°. funksiyaning topilgan kritik nuqtalardagi va $[a;b]$ kesmaning chetki nuqtalaridagi qiymatlari hisoblanadi;

3°. hisoblangan qiymatlar orasidan eng kattasi va eng kichigi tanlanadi.

Izohlar. 1. Agar $y = f(x)$ funksiya $[a;b]$ kesmada faqat bitta kritik nuqtaga ega bo'lib, u maksimum (minimum) nuqta bo'lsa, bu nuqtada funksiya o'zining eng katta (eng kichik) qiymatiga ega bo'ladi, ya'ni $\max\limits_{x\in[a;b]} f(x) = f(x_0)$ $(\min\limits_{x\in[a;b]} f(x) = f(x_0))$ bo'ladi.

2. Agar $y = f(x)$ funksiya $[a;b]$ kesmada kritik nuqtaga ega bo'lmasa, bu funksiyaning kesmada monoton o'sishini yoki monoton kamayishini bildiradi. Bunda $y = f(x)$ funksiya $[a;b]$ kesmadagi eng katta va eng kichik qiymatlariga kesmaning chetki nuqtalarida erishadi.

4- misol. $y = x^3 - 3x$ funksiyaning $[0,2]$ kesmada eng katta va eng kichik qiymatlarini toping.

Yechish.

1°. $f'(x) = 3x^2 - 3 = 0$ dan $x_1 = -1$, $x_2 = 1$, $x_2 \in [0,2]$;

2°. $f(1) = -2$, $f(0) = 0$, $f(2) = 2$;

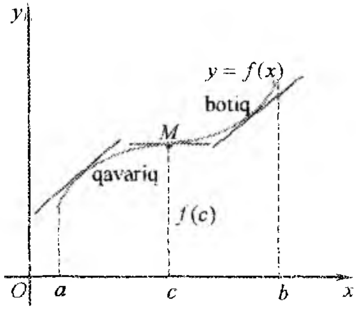
3°. $\max\limits_{x\in[0;2]} f(x) = f(2) = 2$; $\min\limits_{x\in[0;2]} f(x) = f(1) = -2$.

5.4.4. Funksiya grafigining botiqligi, qavariqligi va egilish nuqtalari

$y = f(x)$ funksiya $(a;b)$ intervalda differensiallanuvchi bo'lsin. U holda $y = f(x)$ funksiya grafigining $\forall M(x; f(x))$, $x \in (a,b)$ nuqtada urinmasi mavjud bo'ladi.

2-ta'rif. Agar $(a;b)$ intervalda $y = f(x)$ funksiyaning grafigi unga intervalning ixtiyoriy nuqtasida o'tkazilgan urinmadan yuqorida (pastda) yotsa, *funksiya grafigi* $(a;b)$ *intervalda botiq* (*qavariq*) deyiladi(14-shakl).

Funksiya grafigining botiq qismini qavariq qismidan ajratuvchi $M(c;f(c))$ nuqta *funksiya grafigining egilish nuqtasi* deb ataladi.



14-shakl.

5-teorema. Agar $y = f(x)$ funksiya $(a;b)$ intervalda ikkinchi tartibli hosilaga ega va $\forall x \in (a;b)$ da $f''(x) < 0$ ($f''(x) > 0$) bo'lsa, u holda $y = f(x)$ funksiya grafigi $(a;b)$ intervalda qavariq (botiq) bo'ladi.

Isboti. $\forall x \in (a;b)$ da $f''(x) < 0$ bo'lsin. Funksiya grafigida $x_0 \in (a;b)$ abssissali ixtiyoriy M nuqta olamiz (15-shakl). Funksiyaning grafigi bu urunmadan pastda yotishini ko'rsatamiz. Buning uchun $x \in (a;b)$ nuqtada $y = f(x)$ egri chiziqning y ordinatasi bilan urunmaning y_{ur} ordinatasini solishtiramiz.

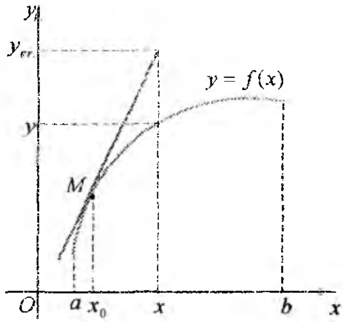
Urunma tenglamasi

$$y_{ur} = f(x_0) + f'(x_0)(x - x_0).$$

bo'lgani uchun

$$y - y_{ur} = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

Lagranj teoremasiga ko'ra $f(x) - f(x_0) = f'(c)(x - x_0)$, bu yerda c nuqta x_0 bilan x ning orasida yotadi.



15-shakl.

Shu sababli

$$y - y_{ur} = f'(c)(x - x_0) - f'(x_0)(x - x_0)$$

yoki

$$y - y_{ur} = (f'(c) - f'(x_0))(x - x_0).$$

$f'(c)-f'(x_0)$ ayirmaga Lagranj teoremasini takror qo'llaymiz: $f'(c)-f'(x_0)=f''(c_1)(x-x_0)$, bu yerda c_1 nuqta c bilan x_0 ning orasida yotadi. Demak, $y-y_{ur}=f''(c_1)(c-x_0)(x-x_0)$.

Bu tengsizlikni tekshiramiz:

1) agar $x>x_0$ bo'lsa, u holda $x-x_0>0$, $c-x_0>0$ bo'ladi va $f''(c_1)<0$;

2) agar $x<x_0$ bo'lsa, u holda $x-x_0<0$, $c-x_0<0$ bo'ladi va $f''(c_1)<0$.

Har ikkala holda ham $y-y_{ur}=f''(c_1)(c-x_0)(x-x_0)<0$, ya'ni $y<y_{ur}$.

Demak, $\forall x\in(a;b)$ da urunmaning ordinatasi funksiya grafigining ordinatasidan katta bo'ladi va $(a;b)$ intervalda funksiya grafigi qavariq.

$f''(x)>0$ da funksiya grafigi botiq bo'lishi shu kabi isbotlanadi.

Funksiya grafigining egilish nuqtasini topish quyidagi teoremalarga asoslanadi.

6-teorema (*egilish nuqta mavjud bo'lishining zaruriy sharti*). Agar $y=f(x)$ funksiya $(a;b)$ intervalda uzluksiz ikkinchi tartibli hosilaga ega va $M(x_0;f(x_0))$ nuqta funksiya grafigining egilish nuqtasi bo'lsa, u holda $f''(x_0)=0$ bo'ladi.

Isboti. Teskarisini faraz qilamiz: $f''(x_0)\neq 0$, aniqlik uchun $f''(x_0)>0$; x_0 nuqtaning biror atrofida $f''(x)>0$ bo'lsin. U holda 5-teoremaga ko'ra, funksiya grafigi bu atrofda botiq bo'ladi. Bu x_0 nuqta egilish nuqtaning abssissasi bo'ladi mulohazasiga zid. Demak, qilingan faraz noto'g'ri va $f''(x)=0$.

$f''(x)=0$ bo'ladigan nuqtalarning hammasi ham funksiya grafigining egilish nuqtasi bo'lavermaydi. Masalan, $f(x)=x^4$ funksiya grafigining $M(0;0)$ nuqtasi egilish nuqta emas, ammo $x=0$da $f''(x)=12x^2=0$.

Demak, $f''(x_0)=0$ shart egilish nuqta mavjud bo'lishining zaruriy sharti bo'ladi.

7-teorema (*egilish nuqta mavjud bo'lishining yetarli sharti*) $y=f(x)$ funksiya x_0 nuqtaning biror δ atrofida ikkinchi tartibli hosilaga ega bo'lsin. Agar δ atrofning x_0 nuqtadan chap va o'ng qismlarida $f''(x)$ hosila har xil ishoraga ega bo'lsa, u holda $M(x_0;f(x_0))$ nuqta funksiya grafigining egilish nuqtasi bo'ladi.

Isboti. $x\in(x_0-\delta;x_0)$da $f''(x)>0$ va $x\in(x_0,x_0+\delta)$ da $f''(x)<0$ bo'lsin. U holda 5-teoremaga ko'ra, x_0 nuqtadan chapda funksiya grafigi botiq va o‘ngda qavariq bo‘ladi. Demak, $M(x_0; f(x_0))$ nuqta funksiya grafigining egilish nuqtasi bo‘ladi.

$x \in (x_0 - \delta; x_0)$ da $f''(x) < 0$ va $x \in (x_0, x_0 + \delta)$ da $f''(x) > 0$ bo‘lgan hol uchun teorema shu kabi isbotlanadi.

Izoh. $y = f(x)$ funksiya x_0 nuqtaning biror δ atrofida ikkinchi tartibli hosilaga ega bo‘lib, uning $f''(x_0)$ hosilasi mavjud bo‘lmaganida ham egilish nuqtaga ega bo‘lishi mumkin. Shu sababli egilish nuqtalarni ikkinchi tartibli hosila nolga teng bo‘lgan yoki uzilishga ega bo‘lgan nuqtalardan izlash kerak bo‘ladi.

$f''(x)$ hosilasi nolga teng bo‘lgan yoki mavjud bo‘lmagan nuqtaga *ikkinchi tur kritik nuqta* deyiladi.

5- misol. $y = \dfrac{x}{1-x^2}$ funksiya grafigini botiq va qavariqlikka tekshiring.

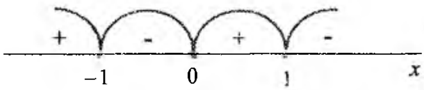
Yechish. $D(f) = (-\infty; -1) \cup (-1; 1) \cup (1; \infty)$.

$$y' = \left(\frac{x}{1-x^2}\right)' = \frac{x^2+1}{(1-x^2)^2}, \qquad y'' = \left(\frac{x^2+1}{(1-x^2)^2}\right)' = \frac{2x(x^2+3)}{(1-x^2)^3}.$$

Ikkinchi tartibli hosila $x_2 = 0$ nuqtada nolga teng va $x_1 = -1$, $x_3 = 1$ nuqtalarda mavjud emas.

$f''(x)$ hosilaning ishorasini oraliqlar usuli bilan tekshiramiz:

Demak, funksiyaning grafigi $(-1;0)$ va $(1;\infty)$ intervallarda qavariq, $(-\infty;-1)$ va $(0;1)$ intervallarda botiq bo‘ladi. $O(0;0)$ nuqta funksiya grafigining egilish nuqtasi bo‘ladi.



5.4.5. Funksiya grafigining asimptotalari

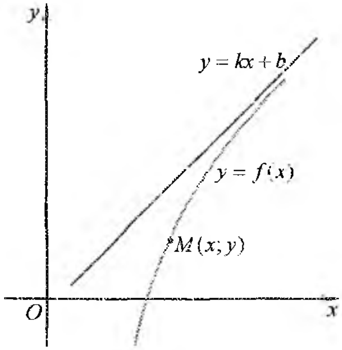
Egri chiziqning asimptotasi deb shunday to‘g‘ri chiziqqa aytiladiki, egri chiziqda yotuvchi M nuqta egri chiziq bo‘ylab harakat qilib koordinata boshidan cheksiz uzoqlashgani sari M nuqtadan bu to‘g‘ri chiziqqacha bo‘lgan masofa nolga intiladi (16-shakl).

Uch turdagi, ya’ni vertikal, gorizontal va og‘ma asimptotalar mavjud.

Agar $\lim\limits_{x\to x_0+0} f(x)$ yoki $\lim\limits_{x\to x_0-0} f(x)$ limitlardan hech bo'lmaganda bittasi cheksiz ($+\infty$ yoki $-\infty$) bo'lsa, $x = x_0$ to'g'ri chiziq $y = f(x)$ funksiya grafigining asimptotasi bo'ladi. Bunday asimptota *vertikal asimptota* deb ataladi.

Masalan, $f(x) = \dfrac{1}{x}$ funksiya grafigi uchun $x = 0$ to'g'ri chiziq vertikal asimptota, chunki $\lim\limits_{x\to x_0+0} f(x) = +\infty$ va $\lim\limits_{x\to x_0-0} f(x) = -\infty$.

Shunday qilib, vertikal asimptotalarni izlash uchun x ning unga yaqin qiymatlarida $f(x)$ funksiya modul bo'yicha cheksiz o'sadigan x_0 qiymatini topish kerak. Odatda, bu x_0 ikkinchi tur uzilish nuqtasi bo'ladi.



16-shakl.

Agar shunday k va b sonlari mavjud bo'lib, $x\to\infty$ $(x\to-\infty)$ da $f(x)$ funksiya

$$f(x) = kx + b + \alpha(x),\ \lim_{x\to\pm\infty}\alpha(x) = 0$$

ko'rinishda ifodalansa, $y = kx + b$ to'g'ri chiziq $y = f(x)$ funksiya grafigining asimptotasi bo'ladi. Bunday asimptota *og'ma asimptota* deb ataladi.

8-teorema. $y = f(x)$ funksiya grafigi $y = kx + b$ og'ma asimptotaga ega bo'lishi uchun

$$\lim_{x\to\pm\infty}\frac{f(x)}{x} = k, \qquad \lim_{x\to\pm\infty}(f(x) - kx) = b$$

bo'lishi zarur va yetarli.

Isboti. Zarurligi. $y = f(x)$ funksiya grafigi $y = kx + b$ og'ma asimptotaga ega bo'lsin.
U holda og'ma asimptotaning ta'rifiga ko'ra, $y = kx + b + \alpha(x)$, $\lim\limits_{x\to+\infty}\alpha(x) = 0$ bo'ladi.

Bundan

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\left(k + \frac{b}{x} + \frac{\alpha(x)}{x}\right) = k, \quad \lim_{x\to+\infty}(f(x) - kx) = \lim_{x\to+\infty}(b - \alpha(x)) = b$$

kelib chiqadi.

Yetarliligi. $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=k$, $\lim\limits_{x\to+\infty}(f(x)-kx)=b$ bo'lsin.

U holda $\lim\limits_{x\to+\infty}(f(x)-kx)=b$ dan $f(x)-kx=b+\alpha(x)$, $\lim\limits_{x\to+\infty}\alpha(x)=0$ kelib chiqadi. Demak, $f(x)=kx+b+\alpha(x)$ bo'ladi. Bu esa $y=kx+b$ to'g'ri chiziq $f(x)$ funksiya grafigining asimptotasi ekanini bildiradi.

Agar $\lim\limits_{x\to-\infty}\frac{f(x)}{x}$, $\lim\limits_{x\to+\infty}(f(x)-kx)$ limitlardan hech bo'lmaganda bittasi mavjud bo'lmasa yoki cheksiz bo'lsa, $f(x)$ funksiya grafigi og'ma asimptotaga ega bo'lmaydi.

Agar $k=0$ bo'lsa, $b=\lim\limits_{x\to+\infty}f(x)$ bo'ladi. Bunda $y=b$ to'g'ri chiziqqa $f(x)$ funksiya *grafigining gorizontal asimptotasi* deyiladi.

Izoh. $y=f(x)$ funksiya grafigining asimptotalari $x\to+\infty$ da va $x\to-\infty$ da har xil bo'lishi mumkin. Shu sababli $\lim\limits_{x\to\pm\infty}\frac{f(x)}{x}$, $\lim\limits_{x\to\pm\infty}(f(x)-kx)$ limitlarni aniqlashda $x\to+\infty$ va $x\to-\infty$ hollarini alohida qarash lozim.

5- misol. $y=\frac{x^2-3}{x}$ funksiya grafigining asimptotalarini toping.

Yechish. $\lim\limits_{x\to0+}\frac{x^2-3}{x}=-\infty$, $\lim\limits_{x\to0-}\frac{x^2-3}{x}=+\infty$.

Demak, $x=0$ to'g'ri chiziq vertikal asimptota.

$$k=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x^2-3}{x^2}=1,$$

$$b=\lim_{x\to+\infty}[f(x)-kx]=\lim_{x\to+\infty}\left(\frac{x^2-3}{x}-x\right)=\lim_{x\to+\infty}\frac{-3}{x}=0,$$

$$k=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{x^2-3}{x^2}=1,$$

$$b=\lim_{x\to-\infty}[f(x)-kx]=\lim_{x\to-\infty}\left(\frac{x^2-3}{x}-x\right)=\lim_{x\to-\infty}\frac{-3}{x}=0$$

Bundan

$$y=kx+b=x.$$

Demak, $y=x$ to'g'ri chiziq og'ma asimptota.

5.4.6. Funksiyani tekshirish va grafigini chizishning umumiy sxemasi

Funksiyani tekshirish va grafigini chizish ma'lum tartibda (sxema asosida) bajariladi. Shunday sxemalardan birini keltiramiz.

1°. Funksiyaning aniqlanish sohasini topish.

2°.Funksiya grafigining koordinata o'qlari bilan kesishadigan nuqtalarini (agar ular mavjud bo'lsa) aniqlash.

3°. Funksiyaning ishorasi o'zgarmaydigan oraliqlarni ($f(x)>0$ yoki $f(x)<0$ bo'ladigan oraliqlarni) aniqlash.

4°. Funksiyaning juft-toqligini tekshirish.

5°. Funksiya grafigining asimptotalarini topish.

6°. Funksiyaning monotonlik oraliqlarini aniqlash va ekstremumlarini topish.

7°. Funksiyaning qavariqlik va botiqlik oraliqlarini hamda egilish nuqtalarini aniqlash.

8°. $1^\circ-7^\circ$ bandlardagi tekshirishlar asosida funksiyaning grafigini chizish.

Funksiya grafigini chizish uchun keltirilgan sxemaning hamma bandlari, albatta, bajarilishi shart emas. Soddaroq hollarda keltirilgan bandlardan ayrimlarini, masalan $1^\circ,2^\circ,6^\circ$ni bajarish yetarli bo'ladi. Agar funksiya grafigi juda tushunarli bo'lmasa $1^\circ-7^\circ$ bandlardan keyin funksiyaning davriyligini tekshirish, funksiyaning bir nechta qo'shimcha nuqtalarini topish va funksiyaning boshqa xususiyatlarini aniqlash bo'yicha qo'shimcha tekshirishlar o'tkazish mumkin.

7- misol. $y=\dfrac{x^2+1}{x^2-1}$ funksiyani tekshiring va grafigini chizing.

Yechish. 1°.Funksiyaning aniqlanish sohasi:

$$D(f)=(-\infty;-1)\cup(-1;1)\cup(1;+\infty).$$

2°. $x=0$ da $y=-1$ bo'ladi. Funksiya Oy o'qini $(0;-1)$ nuqtada kesadi. $y\neq 0$ bo'lgani uchun funksiya Ox o'qini kesmaydi.

3°. Funksiya $(-\infty;-1)$ va $(1;+\infty)$ intervallarda musbat ishorali va $(-1;1)$ intervalda manfiy ishorali.

4°. Funksiya uchun $x\in D(f)$ da $f(-x)=f(x)$ bo'ladi. Demak, u juft.

5°. $\lim\limits_{x\to -1-0}\dfrac{x^2+1}{x^2-1}=+\infty,\quad \lim\limits_{x\to -1+0}\dfrac{x^2+1}{x^2-1}=-\infty,$

$$\lim_{x\to 1-0}\frac{x^2+1}{x^2-1}=-\infty,\quad \lim_{x\to -1+0}\frac{x^2+1}{x^2-1}=+\infty.$$

Demak, $x=-1$ va $x=1$ to‘g‘ri chiziqlar vertikal asimptotalar bo‘ladi.

$$k=\lim_{x\to+\infty}\frac{x^2+1}{x(x^2-1)}=0\,(x\to+\infty \text{ da ham } x\to-\infty \text{ da ham } k=0),$$

$$b=\lim_{x\to+\infty}\left(\frac{x^2+1}{x^2-1}-0\cdot x\right)=1.$$

Demak, $y=1$ to‘g‘ri chiziq $x\to+\infty$ da ham $x\to-\infty$ da ham gorizontal asimptota bo‘ladi.

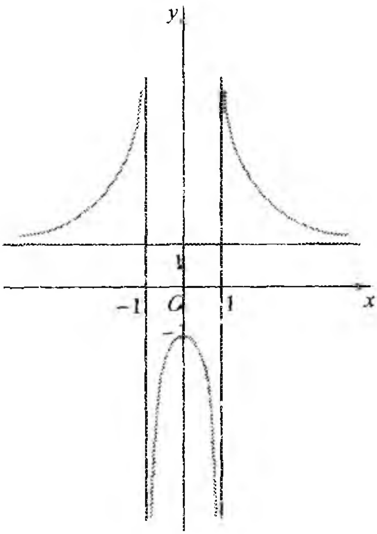
6°. Funksiyaning monotonlik oraliqlarini aniqlaymiz va ekstremumlarini topamiz.

$$y'=\frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}=-\frac{4x}{(x^2-1)^2}.$$

Birinchi tartibli hosila $x=-1$ va $x=1$ da mavjud emas va $x=0$ da nolga teng. Bu nuqtalar berilgan funksiyaning aniqlanish sohasini to‘rtta

$(-\infty;-1)$, $(-1;0)$, $(0;1)$, $(1;+\infty)$

intervallarga ajratadi. Hosilaning bu intervallardagi va har bir birinchi tur kritik nuqtadan chapdan o‘ngga o‘tishdagi ishoralarini chizmada belgilaymiz:



17-shakl.

Demak, funksiya $(-\infty;-1)$ va $(-1;0)$ intervallarda o‘sadi, $(0;1)$ va $(1;+\infty)$ intervallarda kamayadi. $x=0$ maksimum nuqta, $y_{max}=f(0)=-1$.

7°. Funksiyaning qavariqlik va botiqlik oraliqlarini hamda egilish nuqtalarini aniqlaymiz.

$$y'' = \left(-\frac{4x}{(x^2-1)^2}\right)' = -4\frac{(x^2-1)^2 - x\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} = \frac{4(1+3x^2)}{(x^2-1)^3}.$$

Ikkinchi tartibli hosila $x_1=-1$ va $x_3=1$ nuqtalarda mavjud emas. y'' hosilaning $(-\infty;-1)$, $(-1;1)$, $(1;+\infty)$ intervallardagi ishoralarini tekshiramiz:

Demak, funksiyaning grafigi $(-1;1)$ intervalda qavariq, $(-\infty;-1)$ va $(1;+\infty)$ intervallarda botiq bo‘ladi. Funksiya grafigining egilish nuqtasi yo‘q.

8°. $1^{\circ}-7^{\circ}$ bandlar asosida funksiya grafigini chizamiz (17-shakl).

8- misol. $y=\dfrac{x^3}{(x+1)^2}$ funksiyani tekshiring va grafigini chzing.

Yechish. Misolni *Maple* paketida bajaramiz.

> $with(plots):$

> $restart:$

> $readlib(extrema):$

> $f := \dfrac{x^3}{(x+1)^2}:$

> $\lim\limits_{x\to -1^+} f;$

$$-\infty$$

> $\lim\limits_{x\to -1^-} f;$

$$-\infty$$

> $\lim\limits_{x\to \infty} f;$

$$\infty$$

> $\lim\limits_{x\to -\infty} f;$

$$-\infty$$

> $\lim\limits_{x\to \infty} \dfrac{f}{x};$

$$1$$

> $\lim_{x\to\infty}(f-x);$

$$-2$$

> $g := x-2;$

$$g := x-2$$

> $solve(\{f=0\},x);$

$$\{x=0\},\{x=0\},\{x=0\}$$

> $solve(\{f>0\},x);$

$$\{0<x\}$$

> $solve(\{f<0\},x);$

$$\{x<-1\},\{-1<x,x<0\}$$

> $\frac{d}{dx}f;$

$$\frac{3x^2}{(x+1)^2}-\frac{2x^3}{(x+1)^3}$$

$$\{x=-3\},\{x=0\},\{x=0\}$$

> $extrema(f,\{\ \},\{x\},'s');\ s;$

$$\left\{0,-\frac{27}{4}\right\}$$

$$\{\{x=-3\},\{x=0\}\}$$

> $solve\left(\left\{\frac{d}{dx}f>0\right\},x\right);$

$$\{x<-3\},\{-1<x,x<0\},\{0<x\}$$

> $solve\left(\left\{\frac{d}{dx}f<0\right\},x\right);$

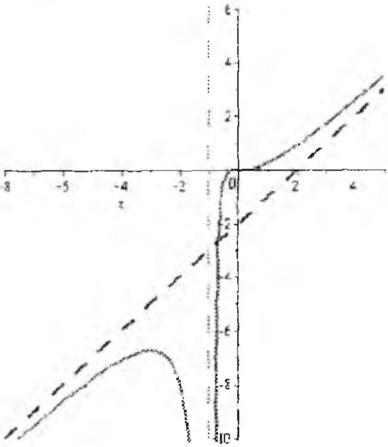
$$\{-3<x,x<-1\}$$

> $\frac{d^2}{dx^2}f;$

$$\frac{6x}{(x+1)^2}-\frac{12x^2}{(x+1)^3}+\frac{6x^3}{(x+1)^4}$$

> simplify( );

$$\frac{6x}{(x+1)^4}$$



18-shakl.

> $solve\left(\left\{\frac{d^2}{dx^2}\,f=0\right\},x\right);$

$$\{x=0\}$$

> $solve\left(\left\{\frac{d^2}{dx^2}\,f>0\right\},x\right);$

$$\{0<x\}$$

> $solve\left(\left\{\frac{d^2}{dx^2}\,f<0\right\},x\right);$

$$\{x<-1\},\{-1<x,x<0\}$$

>

plot([*f*(*x*), *g*(*x*), [-1, *t*, *t* =-10 ..10]], *x* =-8 ..5, -10 ..6, *color* = [*red*, *blue*, *green*], *linestyle* = [1, 6, 6], *thickness* = 2, *discont* = *true*, *grid* = [50, 50]);

5.4.7. Mashqlar

1. Funksiyalarning monotonlik intervallarini va ekstremumlarini toping:

1) $f(x)=x^3-9x^2+15x;$ 2) $f(x)=\frac{x^3}{3}-\frac{x^2}{2}-2x;$

3) $f(x)=\frac{x^2}{4-x^2};$ 4) $f(x)=\frac{4x}{x^2+4};$

5) $f(x)=x\sqrt{1-x^2};$ 6) $f(x)=3\sqrt[3]{x^2}-x^2;$

7) $f(x)=xe^{-x};$ 8) $f(x)=ch^2x;$

9) $f(x)=\ln(x^2+1);$ 10) $f(x)=\frac{x}{\ln x};$

11) $f(x)=x-2\sin x,\ 0\le x\le 2\pi;$ 12) $f(x)=x+2\cos^2 x,\ 0\le x\le \pi.$

2. Funksiyalarning berilgan kesmadagi eng katta va eng kichik qiymatlarini toping:

1) $f(x)=x^3-3x,\ [0;2];$ 2) $f(x)=x^3+3x^2-9x-10,\ [-4;0];$

3) $f(x)=x+\cos 2x,\ \left[0;\frac{\pi}{3}\right];$ 4) $f(x)=x^3\ln x,\ [1;e].$

3. Jism $S=21t+3t^2-t^3$ qonun bilan harakatlanmoqda. Jismning eng katta tezligini toping.

4. Ko'ndalang kesimi to'g'ri to'rtburchakdan iborat to'sinning bukilishga qarshiligi ko'ndalang kesimning eni bilan bo'yi kvadratining ko'paytmasiga proporsional. D diametrli xodadan kesilgan to'sinning bukilishga qarshiligi eng katta bo'lishi uchun to'sinning o'lchamlari qanday bo'lishi kerak?

5. Uzunligi l ga teng simdan to‘g‘ri to‘rtburchak yasalgan. To‘g‘ri to‘rtburchakning yuzasi eng katta bo‘lishi uchun uning o‘lchamlari qanday bo‘lishi kerak?

6. $\frac{x^2}{16}+\frac{y^2}{9}=1$ ellipsga to‘g‘ri to‘rtburchak ichki chizilgan. To‘g‘ri to‘rtburchakning mumkin bo‘lgan eng katta yuzasini toping.

7. R radiusli sharga yon sirti eng katta bo‘lgan silindr ichki chizilgan. Silindrning balandligi qanday bo‘lishi kerak?

8. Silindrning hajmi V ga teng. Eng kichik to‘la sirtga ega bo‘lgan silindrning balandligini toping?

9. Kemada yoqilg‘i sarfi uning tezligining kubiga proporsional bo‘lib, u 20 km/soat tezlikda soatiga 40 so‘m sarflaydi. Kemaning boshqa xarajatlari soatiga 270 so‘mni tashkil qiladi. Kemaning 1 km. yo‘l uchun umumiy xarajati eng kam bo‘lishi uchun uning tezligi qanday bo‘lishi kerak?

10. A zavoddan B shahar orqali o‘tuvchi temir yo‘lgacha bo‘lgan qisqa masofa a km. Agar yuk tashish narxi avtomobil yo‘lida temir yo‘ldagiga qaraganida ikki baravar kam bo‘lsa, yukni A zavoddan B shaharga eng arzon yetkazish uchun A zavoddan avtomobil yo‘li temir yo‘lga qanday burchak o‘tkazilishi kerak?

11. Funksiyalar grafigining botiqlik-qavariqlik intervallarini va egilish nuqtalarini toping:

1) $f(x)=x^4-4x^3+6x;$ 2) $f(x)=(x-5)^5+4x-13;$

3) $f(x)=2x-3\sqrt[3]{x^2};$ 4) $f(x)=1+\sqrt[3]{(x-3)^5};$

5) $f(x)=x-\ln(1+x);$ 6) $f(x)=\ln(1+x^2);$

7) $f(x)=\frac{1}{1+x^2};$ 8) $f(x)=x^3-\frac{3}{x}.$

12. Funksiyalar grafigining asimptotalarini toping:

1) $f(x)=\frac{x}{x^2-1};$ 2) $f(x)=\frac{\sqrt{1+x^2}}{x};$

3) $f(x)=\sqrt[3]{x^3-3x};$ 4) $f(x)=-x\,\mathrm{arcctg}x;$

5) $f(x)=\frac{e^x}{x+2};$ 6) $f(x)=\frac{\ln^2 x}{x};$

7) $f(x)=3x-\frac{\sin x}{x};$ 8) $f(x)=\sqrt{\frac{x^3}{x-1}}.$

13. Funksiyalarni tekshiring va grafigini chizing:

1) $f(x)=x-x^3$; 2) $f(x)=x^3-3x+2$;

3) $f(x)=\frac{x-2}{x^2}$; 4) $f(x)=\frac{x^2}{1-x^2}$;

5) $f(x)=x+\frac{1}{x}$; 6) $f(x)=x-\frac{1}{x}$;

7) $f(x)=x-\frac{1}{x^2}$; 8) $f(x)=\frac{x}{3}-\frac{9}{2x^2}$.

5.5. HOSILANING GEOMETRIK TATBIQLARI

5.5.1. Yassi egri chiziq yoyining differensiali

$[a;b]$ kesmada differensiallanuvchi $y=f(x)$ funksiya berilgan va uning grafigi $\breve{AB}$ yoydan iborat bo‘lsin (19-shakl). $[a;b]$ kesmani $x_1,x_2,...,x_{n-1}$ nuqtalar bilan n ta bo‘lakka bo‘lamiz. Bu nuqtalarga $\breve{AB}$ yoyning $M_1,M_2,...,M_{n-1}$ nuqtalari mos keladi va ularni siniq chiziq bilan tutashtiramiz. Bu siniq chiziqqa $\breve{AB}$ *yoyga ichki chizilgan siniq chiziq* deyiladi. Siniq chiziqning perimetrini s_n bilan belgilaymiz, ya’ni

$$s_n=\sum_{i=1}^{n}|M_{i-1}M_i|\ (M_0=A,\ M_n=B).$$

$M_{i-1}M_i$ bo‘g‘inlardan eng kattasining uzunligini λ_i bilan belgilaymiz, bunda $M_{i-1}M_i$ bo‘g‘inlar soni cheksiz ortganida, ya’ni $n\to\infty$ da $\lambda_i\to 0$.

$\breve{AB}$ yoyga ichki chizilgan siniq chiziq perimetrining $\lambda_i\to 0$ dagi chekli limiti $M_1,M_2,...,M_{n-1}$ nuqtalarni tanlash usuliga bog‘liq bo‘lmagan holda mavjud bo‘lsa, bu limitga *chiziq yoyining uzunligi* deyiladi.

Yoy uzunligini s bilan belgilab, topamiz:

$$s=\lim_{n\to\infty}s_n=\lim_{\lambda\to 0}\sum_{i=1}^{n}|M_{i-1}M_i|.$$

Yoy uzunligini uning biror nuqtasidan, masalan, A nuqtadan hisoblaymiz.

$M(x;y)$ nuqtada $\breve{AM}$ yoy uzunligi s ga, $M'(x+\Delta x;y+\Delta y)$ nuqtada $\breve{AM'}$ yoy uzunligi $s+\Delta s$ ga teng bo‘lsin, bu yerda

$\Delta s - \breve{MM'}$ yoy uzunligi (20-shakl).

MNM' uchburchakdan MM' uzunlikni topamiz:

$$MM' = \sqrt{\Delta x^2 + \Delta y^2}.$$

Geometrik mulohazalardan

$$\lim_{M'\to M} \frac{\breve{MM'}}{MM'} = 1, \qquad \lim_{\Delta x\to 0} \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} = 1$$

kelib chiqadi, ya'ni chiziqning cheksiz kichik yoyi va unga tortilgan to'g'ri chiziq ekvivalent bo'ladi. Ushbu ekvivalentlikni hisobga olib, MM' uzunlikni ifodalovchi formulada almashtirish bajaramiz:

$$\frac{MM'}{\Delta s} \cdot \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

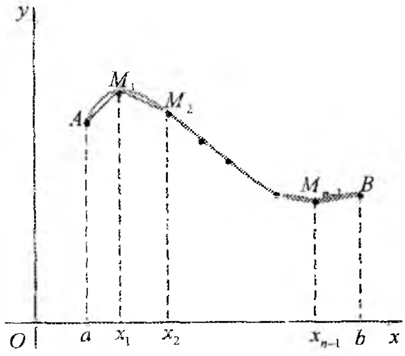
Oxirgi tenglikda limitga o'tib, $s = s(x)$ funksiyaning hosilasi uchun formula topamiz:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{5.1}$$

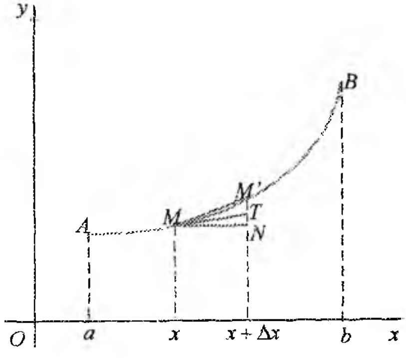
Bundan

$$ds = \sqrt{dx^2 + dy^2}. \tag{5.2}$$

(5.1) formula yassi egri chiziq yoyi differensialini ifodalaydi va ushbu *geometrik ma'no*ga ega: yoy differensialini chiziq urinmasining tegishli kesmasi bilan ifodalash mumkin (20-shaklda MT kesma).



19-shakl.



20-shakl.

Agar egri chiziq $x = x(t), y = y(t), t \in T$ parametrik tenglamalar bilan berilgan bo'lsa, $dx = x'(t)dt,\ dy = y'(t)dt$ bo'ladi va (5.2) ifoda ushbu ko'rinishni oladi:

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt. \tag{5.3}$$

Agar egri chiziq qutb koordinatalarida $r = r(\varphi)$ tenglama bilan berilgan bo'lsa, $x = r\cos\varphi,\ y = r\sin\varphi$ ifodalarni φ parametrli tenglamalar deb, topamiz:

$$\frac{dx}{d\varphi} = \frac{dr}{d\varphi}\cos\varphi - r\sin\varphi, \quad \frac{dy}{d\varphi} = \frac{dr}{d\varphi}\sin\varphi + r\cos\varphi,$$

$$\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = r^2 + \left(\frac{dr}{d\varphi}\right)^2;$$

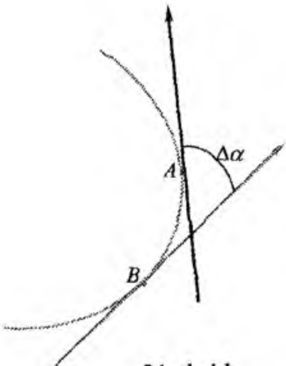
$$ds = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\, d\varphi. \tag{5.4}$$

5.5.2. Yassi egri chiziqning egriligi

Egrilik

Egri chiziq shaklini xarakterlovchi elementlardan biri uning egilganlik darajasidir.

O'z-o'zini kesib o'tmaydigan va har bir nuqtasida tayin urinmaga ega bo'lgan yassi egri chiziqni qaraymiz. Egri chiziqda ikkita A va B nuqtalarni olamiz. Bu nuqtalarda egri chiziqqa o'tkazilgan urinmalar orasidagi burchakni $\Delta\alpha$ bilan belgilaymiz (21-shakl).



21-shakl.

$\Delta\alpha$ burchakka $\breve{AB}$ *yoyning qo'shnilik burchagi* deyiladi.

Bir xil uzunlikka ega ikkita yoydan qo'shnilik burchagi katta bo'lgani ko'proq egilgan bo'ladi. Har xil uzunlikka ega yoylarning egilganlik darajasini qo'shnilik burchagi orqali baholab bo'lmaydi. Bunda $\breve{AB}$ yoy uzunligida $\Delta\alpha$ burchakning o'rtacha qancha ulushi to'g'ri kelishi muhim hisoblanadi.

$\breve{AB}$ *yoyning o'rtacha egriligi* deb, tegishli qo'shnilik burchagining $\breve{AB}$ yoy uzunligiga nisbatining absolut qiymatiga aytiladi:

$$K_{o'r} = \left|\frac{\Delta\alpha}{\Delta s}\right|. \qquad (5.5)$$

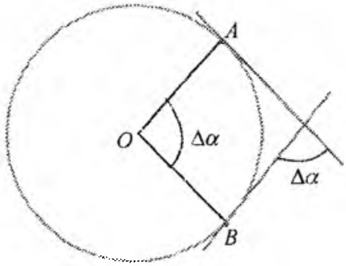
Bitta egri chiziqning turli qismlarida o'rtacha egrilik har xil bo'lishi mumkin.

Egri chiziqning bevosita A nuqtadagi egilganlik darajasini baholash uchun egri chiziqning berilgan nuqtadagi egriligi tushunchasini kiritamiz.

Berilgan egri chiziqning A *nuqtadagi egriligi* deb $\breve{AB}$ yoy o'rtacha egriligining $B \to A$ dagi ($s \to 0$ dagi) limitiga aytiladi:

$$K = \lim_{B\to A}\left|\frac{\Delta\alpha}{\Delta s}\right| \quad \left(K = \lim_{s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right|\right). \qquad (5.6)$$

$\Delta\alpha$ burchak urunish nuqtasi egri chiziq yoyi bo'ylab Δs masofaga ko'chishida urinmaning burilish burchagini ifodalaydi. Shu sababli egri chiziqning egriligiga quyidagicha ta'rif bersa ham bo'ladi: *egri chiziqning egriligi* bu – urinish nuqtasi egri chiziq bo'ylab harakatlanganida urinma aylanishining burchak tezligidir.



22-shakl.

Misol uchun to'g'ri chiziq va aylananing o'rtacha egriligi va egriligini topaylik.

1. To'g'ri chiziqning istalgan nuqtasiga o'tkazilgan urinma to'g'ri chiziqning o'zi bilan ustma-ust tushadi. Shu sababli $\Delta\alpha = 0$, uning ixtiyoriy nuqtasida o'rtacha egrilik va egrilik nolga teng bo'ladi.

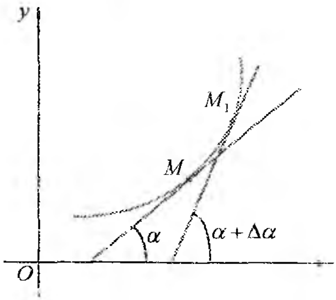
2. Aylana uchun $\Delta\alpha$ qo'shnilik burchagi uning OA va OB radiuslari orasidagi burchakka, $\breve{AB}$ yoy uzunligi $\Delta s = R\Delta\alpha$ ga teng bo'ladi, bu yerda R – aylana radiusi (22-shakl).

U holda

$$K_{o'r} = \frac{\Delta\alpha}{R\Delta\alpha} = \frac{1}{R}, \quad K = \frac{1}{R},$$

ya'ni aylananing egriligi o'zgarmaydi va radiusining teskarisiga teng bo'ladi.

Uzluksiz ikkinchi tartibli hosilaga ega $y=f(x)$ funksiya bilan aniqlanuvchi yassi egri chiziqning $M(x;y)$ nuqtadagi egriligini hisoblaymiz. Egri chiziqqa $M(x;y)$ va $M_1(x+\Delta x; y+\Delta y)$ nuqtalarda urinma o'tkazamiz, urinmalarning og'ish burchaklarini α va $\alpha+\Delta\alpha$ bilan belgilaymiz (23-shakl).



23-shakl.

$\breve{MM_1}$ yoyga mos keluvchi qo'shnilik burchagi $\Delta\alpha$ ga teng (qavariq yoy uchun $\Delta\alpha<0$, botiq yoy uchun $\Delta\alpha>0$).

$\alpha=\alpha(x)$ va $s=s(x)$ bo'lgani uchun

$$K=\lim_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right|=\left|\frac{d\alpha}{ds}\right|=\left|\frac{\frac{d\alpha}{dx}}{\frac{ds}{dx}}\right|. \tag{5.7}$$

$tg\alpha=y'$, va demak, $\alpha=arctgy'$ bo'ladi. Bu tenglikni differensiallab, topamiz:

$$\frac{d\alpha}{dx}=\frac{y''}{1+y'^2}. \tag{5.8}$$

(5.8) va (5.1) ifodalarni (5.7) tenglamaga qo'yib, $y=f(x)$ egri chiziqning $M(x;y)$ nuqtadagi egriligini topish formulasini hosil qilamiz:

$$K=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}. \tag{5.9}$$

Agar egri chiziq $x=x(t), y=y(t), t\in T$ parametrik tenglamalar bilan berilgan bo'lsa,

$$y'_x=\frac{y'_t}{x'_t},\quad y''=\frac{x'_t y''_t-x''_t y'_t}{(x'^2_t+y'^2_t)^{\frac{3}{2}}}$$

bo'ladi va (5.9) formula ushbu ko'rinishni oladi:

$$K=\frac{|x'_t y''_t-x''_t y'_t|}{(x'^2_t+y'^2_t)^{\frac{3}{2}}}. \tag{5.10}$$

Agar egri chiziq qutb koordinatalarida $r=r(\varphi)$ tenglama bilan berilgan bo'lsa, $x=r\cos\varphi$, $y=r\sin\varphi$ ifodalarni φ parametrli tenglamalar deb, topamiz:

$$K=\frac{|r^2+2r'^2-rr''|}{(r'^2+r^2)^{\frac{3}{2}}}. \tag{5.11}$$

1-misol. $y=-x^3$ egri chiziqning $x=\frac{1}{2}$ nuqtasidagi egriligini toping.

Yechish. Hosilalarni topamiz: $y'=-3x^2$, $y''=-6x$. Hosilalarning $x=\frac{1}{2}$ nuqtadagi qiymatlarini hisoblaymiz:

$$y'=-\frac{3}{4},\ y''=-3.$$

U holda

$$K=\frac{|-3|}{\left(1+\left(-\frac{3}{4}\right)^2\right)^{\frac{3}{2}}}=\frac{3}{\frac{125}{64}}=\frac{192}{125}.$$

2-misol. $x=t^2, y=2t^3$ egri chiziqning $t=1$ parametrga mos nuqtasidagi egriligini toping.

Yechish. $x'(t)=2t$, $x''(t)=2$, $y'(t)=6t^2$, $y''(t)=12t$ hosilalarning qiymatlarini $t=1$ da topamiz: $x'=2$, $x''=2$, $y'=6$, $y''=12$.

Bundan

$$K=\frac{|2\cdot12-2\cdot6|}{(2^2+6^2)^{3/2}}=\frac{3}{20\sqrt{10}}.$$

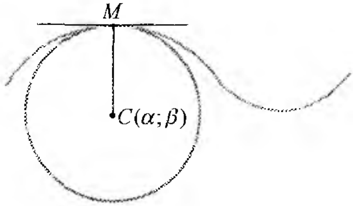
Egrilik radiusi, markazi va aylanasi, evolyuta, evolventa

Chiziqning berilgan M nuqtadagi egriligi K ga teskari R miqdorga shu chiziqning qaralayotgan nuqtadagi *egrilik radiusi* deyiladi:

$$R=\frac{1}{K} \tag{5.12}$$

yoki

$$R=\frac{(1+y'^2)^{\frac{3}{2}}}{|y''|}. \tag{5.13}$$



24-shakl.

Egri chiziqning M nuqtadagi normaliga egri chiziqning botiqligi yo'nalishida $MC = R$ kesmani qo'yamiz (24-shakl).

C nuqtaga berilgan egri chiziqning M nuqtadagi *egrilik markazi* deyiladi.

Markazi C nuqtada bo'lgan R radiusli aylana berilgan egri chiziqning M nuqtadagi *egrilik aylanasi* deb ataladi (24-shakl).

Egrilik aylanasining bu ta'rifidan berilgan M nuqtada egri chiziqning egriligi va egrilik aylanasining egriligi bir-biriga teng bo'lishi kelib chiqadi.

Egrilik markazining koordinatalarini α va β bilan belgilab, ularni hisoblash uchun formulalar chiqaramiz.

$y = f(x)$ egri chiziqqa $M(x;y)$ nuqtada o'tkazilgan normal tenglamasini tuzamiz:

$$Y - y = -\frac{1}{y'}(X - x), \tag{5.14}$$

bu yerda X, Y – normalning o'zgaruvchi koordinatalari.

$C(\alpha;\beta)$ nuqta normalga tegishli bo'lgani uchun uning koordinatalari (5.14) tenglamani qanoatlantiradi:

$$\beta - y = -\frac{1}{y'}(\alpha - x). \tag{5.15}$$

$C(\alpha;\beta)$ nuqta $M(x;y)$ nuqtadan egrilik radiusi R ga teng masofada yotadi, shu sababli

$$(\alpha - x)^2 + (\beta - y)^2 = R^2. \tag{5.16}$$

(5.15) va (5.16) tenglamalarni birgalikda yechib, α va β larni topamiz:

$$\alpha = x \pm \frac{y'}{\sqrt{1 + y'^2}}R, \qquad \beta = y \mp \frac{1}{\sqrt{1 + y'^2}}R. \tag{5.17}$$

Bu formulalarga (5.13) tenglikdan R ni qo'yamiz:

$$\alpha = x \pm \frac{y'(1 + y'^2)}{|y''|}, \qquad \beta = y \mp \frac{1 + y'^2}{|y''|}. \tag{5.18}$$

Bu formulalarda qaysi ishoralar olinishini aniqlashtirish uchun $y'' > 0$ va $y'' < 0$ bo'lgan hollar alohida qaraladi. Bunda $y'' > 0$ bo'lsa, mos nuqtada egri chiziq botiq va $\beta > y$ bo'ladi. Shu sababli bunda quyi ishoralar olinadi, ya'ni egrilik markazining koordinatalari

$$\alpha = x - \frac{y'(1+y'^2)}{|y''|}, \qquad \beta = y + \frac{1+y'^2}{|y''|} \tag{5.19}$$

formulalar bilan aniqlanadi. Bu formulalar $y''<0$ bo'lganida ham o'rinli bo'ladi.

Agar egri chiziq $x = x(t), y = y(t), t \in T$ parametrik tenglamalar bilan berilgan bo'lsa, (5.19) ifoda ushbu ko'rinishni oladi:

$$\alpha = x - \frac{y'(x'^2+y'^2)}{x'y''-x''y'}, \qquad \beta = y + \frac{x'(x'^2+y'^2)}{x'y''-x''y'}. \tag{5.20}$$

3-misol. $r = 3(1+\cos\varphi)$ kardioidaning istalgan nuqtasida egrilik radiusini toping.

Yechish. Hosilalarni topamiz: $r' = -3\sin\varphi,\ r'' = -3\cos\varphi.$

U holda

$$R = \frac{1}{K} = \frac{(r^2+r'^2)^{\frac{3}{2}}}{|r^2+2r'^2-rr''|} =$$

$$= \frac{(9(1+2\cos\varphi+\cos^2\varphi)+9\sin^2\varphi)^{\frac{3}{2}}}{|9(1+2\cos\varphi+\cos^2\varphi)+18\sin^2\varphi+9(\cos\varphi+\cos^2\varphi)|} =$$

$$= \frac{27(1+2\cos\varphi+\cos^2\varphi+\sin^2\varphi)^{\frac{3}{2}}}{9|1+2\cos\varphi+\cos^2\varphi+2\sin^2\varphi+\cos\varphi+\cos^2\varphi|} =$$

$$= \frac{3(2(1+\cos\varphi))^{\frac{3}{2}}}{3(1+\cos\varphi)} = 2(2(1+\cos\varphi)^{\frac{1}{2}} = 4\cos\frac{\varphi}{2}.$$

Egri chiziqning barcha egrilik markazlari to'plamiga *evolyuta* deyiladi.

Berilgan chiziq o'z evolyutasiga nisbatan *evolventa* (yoki *yoyilma*) deb yuritiladi.

Agar egri chiziq $y = f(x)$ tenglama bilan berilgan bo'lsa, (5.19) tenglamani uning evolyutasi uchun parametrik (x parametrli) tenglama deb qarash mumkin.

Egri chiziq parametrik tenglamalar bilan berilgan holda (5.20) tenglama evolyutaning parametrik tenglamalarini (bu tenglamalarning o'ng tomoniga kiruvchi kattaliklar t parametrga bog'liq bo'ladi) ifodalaydi.

4-misol. $y = x^2$ parabola evolyutasi tenglamasini toping.

Yechish. Hosilalarni topamiz: $y' = 2x$, $y'' = 2$.

Egrilik markazining koordinatalarini aniqlaymiz:

$$\alpha = x - \frac{2x(1+4x^2)}{2} = -4x^3, \quad \beta = x^2 + \frac{1+4x^2}{2} = \frac{6x^2+1}{2}.$$

Bu tenglamalarni $y = x^2$ parabola evolyutasining parametrik tenglamalari deb qarash mumkin. Tenglamadan x parametrni chiqarib, topamiz:

$$\alpha^2 = \frac{16}{27}\left(\beta - \frac{1}{2}\right)^3.$$

Bu tenglama yarim kubik parabola tenglamasi bo'ladi.

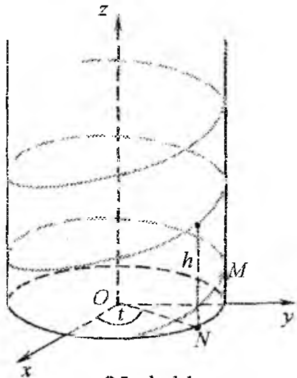
5.5.3. Skalyar argumentning vektor funksiyasi

Fazodagi egri chiziqning parametrik tenglamalari

Egri chiziq tekislikdagi kabi fazoda ham parametrik tenglamalar bilan berilishi mumkin.

t argumentning bir xil aniqlanish sohasiga ega bo'lgan uchta funksiyasini qaraymiz:

$$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \ t \in T. \end{cases} \qquad (5.21)$$



25-shakl.

Bunda $t \in T$ ning har bir qiymatiga tayin x, y, z qiymatlar va fazoviy $M(x; y; z)$ nuqta mos keladi. t o'zgarishi bilan M nuqta fazoda biror γ egri chiziqni chizadi. Bunda egri chiziq (5.21) parametrik tenglamalar bilan berilgan va t ga parametr deymiz.

Biz, fazodagi to'g'ri chiziqning quyidagi parametrik tenglamalari bilan avvaldan tanishmiz:

$$\begin{cases} x = x_0 + pt, \\ y = y_0 + qt, \\ z = z_0 + rt, \ t \in R. \end{cases}$$

Yana bitta misol keltiramiz. Ushbu parametrik tenglamalar bilan berilgan egri chiziqni qaraymiz:

$$\begin{cases} x = a\cos t, \\ y = a\sin t, \\ z = bt. \end{cases}$$

Bu egri chiziqqa *vint chizig'i* deyiladi (25-shakl).

t parametrning istalgan qiymatida:

$$x^2 + y^2 = a^2(\cos^2 t + \sin^2 t) = a^2.$$

Bu tenglik vint chizig'ining $x^2 + y^2 = a^2$ silindrda joylashishini bildiradi. Bundan, M nuqta vint chizig'i bo'ylab harakatlanganida uning Oxy tekisligidagi proeksiyasi radiusi a ga teng aylanada harakatlanishi kelib chiqadi, bunda t parametr N nuqtaning qutb burchagi bo'ladi. t parametr 2π ga o'sganida N nuqta aylanani to'liq aylanib o'tadi, vint chizig'i M nuqtasining z applikatasi esa $h = 2\pi b$ kattalikka o'zgaradi. Bu kattalikka vint chizig'ining *qadami* deyiladi.

Skalyar argumentning vektor funksiyasi

Fazodagi egri chiziq (5.21) parametrik tenglamalari bilan berilgan bo'lsin. Bunda $x(t)$, $y(t)$, $z(t)$ funksiyalarning aniqlanish sohasi T ga tegishli har bir t parametrga biror $M(x, y, z)$ nuqta, har bir M nuqtaga esa boshi koordinata boshida va oxiri M nuqtada bo'lgan $\vec{r} = \overline{OM}$ radius vektor mos keladi (26-shakl). Bu vektorning koordinata o'qlaridagi proeksiyalari M nuqtaning koordinatalari bo'ladi va shu sababli vektor (5.21) formulalar bilan aniqlanadi.

Shunday qilib, $t \in T$ parametrning har bir qiymatiga biror

$$\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \tag{5.22}$$

vektor mos keladi. Bu vektorni t *skalyar argumentning vektor funksiyasi* (yoki *vektor funksiya*) deb ataymiz va $\vec{r}(t)$ bilan belgilaymiz.

$\vec{r}(t)$ radius vektorning oxiri chizadigan L chiziqqa $\vec{r}(t)$ *vektorning godografi* deyiladi.

$\vec{r}(t)$ vektor funksiyaning berilishi uchta skalyar funksiyalarning berilishiga teng kuchli, uning koordina o'qlaridagi proeksiyalari $x(t)$, $y(t)$, $z(t)$ bo'ladi.

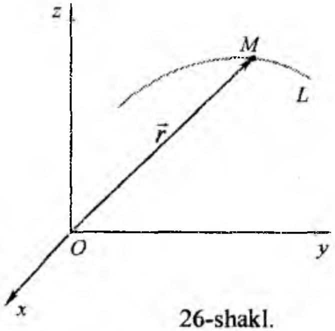
Vektor funksiya uchun limit, uzluksizlik va hosila tushunchalarini kiritamiz.

1-ta'rif. Agar $\lim\limits_{t\to t_0} x(t)=x_0$, $\lim\limits_{t\to t_0} y(t)=y_0$, $\lim\limits_{t\to t_0} z(t)=z_0$ bo'lsa, $\vec{r}_0=x_0\vec{i}+y_0\vec{j}+z_0\vec{k}$ vektorga $\vec{r}=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$ vektor funksiyaning $t\to t_0$ dagi limiti deyiladi va $\lim\limits_{t\to t_0}\vec{r}(t)=\vec{r}_0$ deb yoziladi.

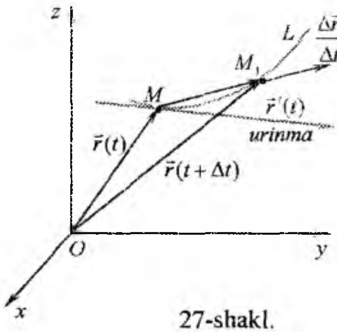
$\vec{r}(t)$ vektor funksiya $t=t_0$ da va t_0 nuqtani o'z ichiga olgan biror intervalda aniqlangan bo'lsin.

2-ta'rif. Agar t_0 nuqtada $\lim\limits_{t\to t_0}\vec{r}(t)=\vec{r}(t_0)$ bo'lsa, $\vec{r}(t)$ vektor funksiya t_0 nuqtada uzluksiz deyiladi.

$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$ vektor funksiya $M(x,y,z)$ nuqtaning radius vektori, ya'ni $\vec{r}=\overrightarrow{OM}$ bo'lsin (27-shakl). t parametrning o'zgarishi bilan M nuqta L godografni chizadi. Parametrning tanlangan tayin t qiymatiga M nuqta va $\vec{r}(t)$ vektor mos kelsin.



26-shakl.



27-shakl.

Perimetrning boshqa $t+\Delta t$ qiymatini olamiz. Unga M_1 nuqta va $\vec{r}(t+\Delta t)$ vektor mos keladi. $\vec{r}(t+\Delta t)=\overrightarrow{OM_1}$ va $\vec{r}(t)=\overrightarrow{OM}$ vektorlarning ayirmasi $\Delta\vec{r}=\overline{MM_1}$ vektorni qaraymiz:

$$\Delta\vec{r}=\vec{r}(t+\Delta t)-\vec{r}(t).$$

Uni $\vec{r}(t)$ vektorning t nuqtadagi orttirmasi deymiz.

$\frac{\Delta r}{\Delta t}$ nisbat $\Delta\vec{r}$ vektorga kollinear, chunki $\frac{1}{\Delta r}$ skalyar ko'paytuvchi.

3-ta'rif. $\Delta\vec{r}$ vektor funksiya orttirmasining argumentning mos Δt orttirmasiga nisbatining $\Delta t\to 0$ dagi limitiga $\vec{r}(t)$ *vektor funksiyaning t nuqtadagi t skalyar argument bo'yicha hosilasi* deyiladi va $\vec{r}'(t)$ yoki $\frac{d\vec{r}(t)}{dt}$ kabi belgilanadi.

Shunday qilib,

$$r'(t)=\frac{d\vec r(t)}{dt}=\lim_{\Delta t\to 0}\frac{d\vec r}{dt}. \tag{5.23}$$

Limit ta'rifiga ko'ra, $r'(t)$ – vektor.

$\vec r(t)$ funksiya hosilasini uning koordinatalar o'qlaridagi proeksiyalari bilan bog'laymiz.

$$\vec r(t)=x(t)\vec i+y(t)\vec j+z(t)\vec k$$

va

$$\vec r(t+\Delta t)=x(t+\Delta t)\vec i+y(t+\Delta t)\vec j+z(t+\Delta t)\vec k$$

bo'lgani uchun

$$\Delta\vec r=\vec r(t+\Delta t)-\vec r(t)=(x(t+\Delta t)-x(t))\vec i+(y(t+\Delta t)-y(t))\vec j+(z(t+\Delta t)-z(t))\vec k$$

yoki

$$\Delta\vec r=\Delta x\vec i+\Delta y\vec j+\Delta z\vec k.$$

Shunday qilib,

$$\frac{\Delta\vec r}{\Delta t}=\frac{\Delta x}{\Delta t}\vec i+\frac{\Delta y}{\Delta t}\vec j+\frac{\Delta z}{\Delta t}\vec k.$$

Bundan

$$\vec r'(t)=\lim_{\Delta t\to 0}\frac{\Delta\vec r}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\vec i+\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\vec j+\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}\vec k=$$

$$=x'(t)\vec i+y'(t)\vec j+z'(t)\vec k.$$

Demak,

$$\vec r'(t)=x'(t)\vec i+y'(t)\vec j+z'(t)\vec k. \tag{5.24}$$

(5.24) ifodadan vektor funksiyani differensiallash qoidalari hamda vektor funksiya hosilasining geometrik va mexanik ma'nolari differensial hisobning asosiy differensiallash qoidalariga hamda hosilaning geometrik va mexanik ma'nolariga o'xshash bo'lishligi kelib chiqadi.

2. $\vec r'(t)$ vektor $\vec r(t)$ radius vektor godografiga t parametrning o'sish yo'nalishida o'tkazilgan urinma bo'ylab yo'nalgan bo'ladi (27-shakl) (*vektor funksiya hosilasining geometrik ma'nosi*).

3. Vektor funksiyaning t vaqt bo'yicha $\vec{r}'(t)$ hosilasi moddiy nuqtaning t ondagi harakati tezligi $\vec{v}(t)$ ga teng (*vektor funksiya hosilasining mexanik ma'nosi*).

I. Skalyar argumentning vektor funksiyasini differensiallashning asosiy qoidalari.

1°. $\frac{d}{dt}(\vec{r}_1+\vec{r}_2-\vec{r}_3)=\frac{d\vec{r}_1}{dt}+\frac{d\vec{r}_2}{dt}-\frac{d\vec{r}_3}{dt}$;

2°. $\frac{d\vec{c}}{dt}=0$, $\vec{c}$ –o'zgarmas vektor;

3°. $\frac{d}{dt}(\lambda\vec{r}_1)=\lambda\frac{d\vec{r}_1}{dt}+\vec{r}\frac{d\lambda}{dt}$, $\lambda=\lambda(t)-t$ argumentning skalyar funksiyasi;

4°. $\frac{d}{dt}(\vec{r}_1\cdot\vec{r}_2)=\frac{d\vec{r}_1}{dt}\cdot\vec{r}_2+\vec{r}_1\cdot\frac{d\vec{r}_2}{dt}$;

5°. $\frac{d}{dt}(\vec{r}_1\times\vec{r}_2)=\frac{d\vec{r}_1}{dt}\times\vec{r}_2+\vec{r}_1\times\frac{d\vec{r}_2}{dt}$.

(5.24) ifoda va keltirilgan o'xshashliklar asosida fazodagi egri chiziqning urinmasi, normal tekisligi, yoyining differensiali va egriligi uchun quyidagi tenglamalar keltirib chiqariladi.

1. (5.21) parametrik tenglamalar bilan berilgan fazodagi egri chiziqqa parametrning $t=t_0$ qiymatiga mos $M_0(x_0;y_0;z_0)$ nuqtasida o'tkazilgan urinma (to'g'ri chiziq)

$$\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)} \tag{5.25}$$

tenglama bilan aniqlanadi.

2. Fazodagi egri chiziq urinish nuqtasida urinmaga perpendikular o'tkazilgan tekislikka *egri chiziqqa o'tkazilgan normal tekislik* deyiladi.

Urinish nuqtasi $M_0(x_0;y_0;z_0)$ bo'lgan normal tekislik

$$x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0 \tag{5.26}$$

tenglama bilan ifodalanadi.

3. Fazoviy egri chiziq yoyining differensiali

$$ds=\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt \tag{5.27}$$

formula bilan topiladi.

4. Fazodagi egri chiziqning egriligi

$$K=\frac{\sqrt{(y'z''-y''z')^2+(x'z''-x''z')^2+(x'y''-x''y')^2}}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} \tag{5.28}$$

formula bilan aniqlanadi.

5-misol. $x=t^3$, $y=t^2$, $z=t$ egri chiziqqa $t=-1$ parametrga mos nuqtada o‘tkazilgan urinma va normal tekislik tenglamalarini toping:

Yechish. Urinish nuqtasining koordinatalarini topamiz:

$$x=-1,\ y=1,\ z=-1.$$

Hosilalarni topamiz va ularning urinish nuqtasidagi qiymatlarini hisoblaymiz:

$$x'(t)=3t^2,\ y'(t)=2t,\ z'(t)=1;$$
$$x'(-1)=3,\ y'(-1)=-2,\ z'(-1)=1.$$

Urinish nuqtasining koordinatalari va hosilalarning hisoblangan qiymatlari asosida izlanayotgan tenglamalarni topamiz:

a) urinma tenglamasi

$$\frac{x+1}{3}=\frac{y-1}{-2}=\frac{z+1}{1};$$

b) normal tekislik tenglamasi

$$3(x+1)-2(y-1)+z+1=0,$$
$$3x-2y+z+6=0.$$

7- misol. Asosining radiusi $R=4$ ga teng silindrda joylashgan, qadami $h=6\pi$ ga teng vint chizig‘i tenglamasini tuzing. Uning yoyi differensialini va egriligini toping.

Yechish. $t=2\pi$ da $z=h=3t$.

Demak, vint chizig‘ining tenglamasi

$$x=4\cos t,\ y=4\sin t,\ z=3t.$$

Bundan

$$x'(t)=-4\sin t,\ y'(t)=4\cos t,\ z'(t)=3,\ x''(t)=-4\cos t,\ y''(t)=-4\sin t,\ z''(t)=0.$$

Hosilalarning hisoblangan qiymatlari asosida izlanayotgan tenglamalarni topamiz:

a) vint chizig‘i yoyining differensiali

$$ds=\sqrt{x'^2+y'^2+z'^2}\,dt=\sqrt{16\sin^2 t+16\cos^2 t+9}\,dt=$$

$$= \sqrt{16(\sin^2 t + \cos^2 t) + 9}dt = 5dt.$$

b) vint chizig‘ining egriligi

$$y'z'' - y''z' = 4\cos t \cdot 0 - (-4\sin t)\cdot 3 = 12\sin t;$$

$$x'z'' - x''z' = (-4\sin t)\cdot 0 - (-4\cos t)\cdot 3 = 12\cos t;$$

$$x'y'' - x''y' = (-4\sin t)\cdot(-4\sin t) - (-4\cos t)\cdot 4\cos t = 16;$$

$$x'^2 + y'^2 + z'^2 = 16\sin^2 t + 16\cos^2 t + 9 = 25;$$

$$K = \frac{\sqrt{144\sin^2 t + 144\cos^2 t + 256}}{25^{\frac{3}{2}}} = \frac{20}{125} = \frac{4}{25}.$$

5.5.4. Mashqlar

1. Egri chiziqning egriligini toping:

1) $y = \cos 2x$, $x = \frac{\pi}{2}$ nuqtada; 2) $y = e^{-x^2}$, Oy o‘q bilan kesishish nuqtasida.

3) $x = 9\cos t$, $y = 3\sin t$ ellipsning istalgan nuqtasida;

4) $x = 4(t - \sin t)$, $y = 4(1 - \cos t)$ sikloidaning istalgan nuqtasida;

5) $r = 2(1 - \cos\varphi)$, $\varphi = \pi$ nuqtada; 6) $r^2 = a^2\sin 2\varphi$, $\varphi = \frac{\pi}{4}$ nuqtada.

2. Egri chiziqning egrilik radiusini toping:

1) $\frac{x^2}{25} + \frac{y^2}{9} = 1$, $x = 0$ nuqtada; 2) $y = \sqrt[3]{x}$, $x = 8$ nuqtada

3) $x = t^2$, $y = t - \frac{1}{3}t^3$, $t = 1$ nuqtada; 4) $r = a\varphi$, istalgan nuqtada.

3. Egri chiziq egrilik markazining koordinatalarini toping:

1) $y = \frac{1}{x}$, $x = 1$ nuqtada; 2) $x = t - \sin t, y = 1 - \cos t$, $t = \frac{\pi}{2}$ nuqtada.

4. Egri chiziq evolyutasi tenglamasini toping:

1) $y^2 = 2(x+1)$; 2) $y^2 - 2x = 0$;

3) $x = 2t$, $y = t^2 - 2$; 4) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$.

5. Vektor funksiyaning godografini toping:

1) $\vec{r}(t) = (2t-1)\vec{i} + (-3t+2)\vec{j} + 4t\vec{k}$, $t \in R$; 2) $\vec{r}(t) = 4ch t\vec{i} - \vec{j} + 3sh t\vec{k}$, $t \in R$;

3) $\vec{r}(t) = 2\cos^3 t\vec{i} + 2\sin^3 t\vec{j} + \vec{k}$, $t \in [0;2\pi]$; 4) $\vec{r}(t) = 6\cos t\vec{i} + 5\sin t\vec{j} + 3\vec{k}$, $t \in [0;2\pi]$.

6. Egri chiziqqa o'tkazilgan urinma va normal tekislik tenglamalarini toping:

1) $\vec{r}(t)=t^2\vec{i}+t^3\vec{j}+t^6\vec{k}$, $\quad t=1$ nuqtada;

2) $\vec{r}(t)=\frac{1}{2}t^2\vec{i}+\frac{1}{3}t^3\vec{j}+\frac{1}{4}t^4\vec{k}$, $t=2$ nuqtada;

3) $\vec{r}(t)=\sin^2 t\vec{i}+2\sin t\cos t\vec{j}+3\cos^2 t\vec{k}$, $\quad t=\frac{\pi}{4}$ nuqtada;

4) $\vec{r}(t)=e^t(\cos t+\sin t)\vec{i}+e^t(\sin t-\cos t)\vec{j}+e^t\vec{k}$, $\quad t=0$ nuqtada.

7. Egri chiziq yoyi differensialini toping:

1) $\vec{r}(t)=3\cos^2 t\vec{i}+5\sin t\cos t\vec{j}+4\sin^2 t\vec{k}$; $\qquad$ 2) $\vec{r}(t)=5\cos t\vec{i}+5\sin t\vec{j}+12t\vec{k}$.

5.6. TENGLAMALARNI TAQRIBIY YECHISH

5.6.1. Asosiy tushunchalar

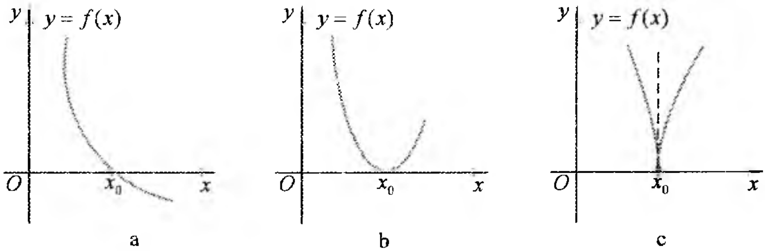
Bir o'zgaruvchining har qanday tenglamasini

$$f(x)=0 \tag{6.1}$$

ko'rinishda yozish mumkin, bu yerda $f(x)-x$ o'zgaruvchining biror funksiyasi.

(6.1) tenglamaning *ildizi* deb shunday $x=x_0$ qiymatga aytiladiki, bunda tenglamaning chap tomoni nolga aylanadi, ya'ni $f(x_0)=0$ bo'ladi.

(6.1) tenglamani *yechish* deb uning ildizlarini topishga aytiladi.



28-shakl.

(6.1) tenglamaning ildizi geometrik jihatdan yo $y=f(x)$ funksiya grafigining Ox o'q bilan kesishish nuqtasining abssissasini (28-a shakl), yo bu grafikning shu o‘q bilan urinish nuqtasini (28-b shakl) yoki grafik va o‘qning umumiy nuqtasini (28-c shakl) ifodalaydi.

Ildizning bu interpritatsiyalaridan bir o‘zgaruvchi tenglamasini yechishning geometrik usuli kelib chiqadi: (6.1) tenglamani yechish uchun uning grafigini chizish va grafikning Ox $(y=0)$ o‘q bilan kesishish nuqtalarini topish kerak.

$f(x)=0$ tenglamani taqribiy yechish, odatda, ikki bosqichda amalga oshiriladi.

Birinchi bosqichda tenglamaning ildizlari ajratiladi, ya’ni tenglama yagona haqiqiy ildiziga ega bo‘lgan chekli oraliq aniqlanadi.

Ikkinchi bosqichda ildizlar aniqlashtiriladi, ya’ni ular berilgan aniqlikda topiladi.

(6.1) tenglamaning ildizlarini ajratish uchun ushbu kriteriya qo‘llaniladi: agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz, monoton va kesmaning chetki nuqtalarida har xil ishorali qiymatlarga ega bo‘lsa, u holda (6.1) tenglama shu kesmada yagona haqiqiy ildizga ega bo‘ladi.

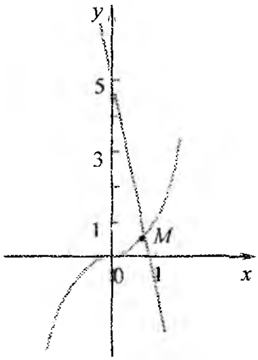
$f(x)$ funksiyaning kesmada monoton bo‘lishining yetarli sharti uning birinchi tartibli hosilasining bu kesmada ishorasini saqlashi hisoblanadi (agar $f(x)>0$ bo‘lsa, funksiya kesmada o‘sadi, agar $f(x)<0$ bo‘lsa, kamayadi).

Agar $y=f(x)$ funksiyaning Ox o‘q bilan kesishish nuqtalari joylashgan oraliqlarni aniq ko‘rsatuvchi grafigini chizish mumkin bo‘lsa, (6.1) tenglamaning ildizlarini grafik usulda ajratsa bo‘ladi.

$f_1(x)$ va $f_2(x)$ funksiya grafiklarini chizish oson bo‘lgan hollarda (6.1) tenglamani

$$f_1(x)=f_2(x) \tag{6.2}$$

ekvivalent ko‘rinishda ifodalash orqali ildizlarni ajratsa bo‘ladi. Bunda (6.2) tenglamaning ildizi $y=f_1(x)$ va $y=f_2(x)$ grafiklarning kesishish nuqtasi bo‘ladi.



29-shakl.

1-misol. $x^3+6x-5=0$ tenglama ildizlarini ajrating.

Yechish. Misolning shartiga ko‘ra,

$$f(x)=x^3+6x-5.$$

$x \in R$ da $f'(x)=3x^2+6>0$, $x \in R$ bo'lgani uchun funksiya ixtiyoriy oraliqda o'sadi. x ning manfiy qiymatlarida $f(x)<0$ va x ning yetarlicha katta qiymatlarida, masalan, $x \geq 1$ da $f(x)>0$.

Demak, funksiya $[0;1]$ kesmada bitta haqiqiy ildizga ega.

Berilgan tenglamaning ildizini grafik usulda ham ajratish mumkin. Tenglamani $x^3=-6x+5$, ya'ni (6.2) ko'rinishga keltiramiz. $y=x^3$ va $y=-6x+5$ funksiya grafiklarini chizamiz (29-shakl). Chizmadan ko'rinadiki, keltirilgan grafiklar abssissasi $(0;1)$ intervalda yotuvchi M nuqtada kesishishadi.

5.6.2. Ildizlarni aniqlashtirishning vatarlar va urinmalar usullari

Vatarlar usuli

Biror (analitik yoki grafik) usul bilan (6.1) tenglamaning ildizi ajratilgan, ya'ni tenglama yagona X ildizga ega bo'lgan $[a;b]$ kesma aniqlangan bo'lsin. Bu kesma shunday tanlanadiki, bunda quyidagi uch shart bajaradi:

1) $f(a)$ va $f(b)$ sonlari har xil ishorali: $f(a)\cdot f(b)<0$;

2) $[a;b]$ kesmada $f'(x)$ hosila nolga teng emas va ishorasini saqlaydi;

3) $[a;b]$ kesmada $f''(x)$ hosila nolga teng emas va ishorasini saqlaydi.

(6.1) tenglama yechimining birinchi yaqinlashishi sifatida $y=f(x)$ funksiya grafigi yoyining chetki $A(a;f(a))$ va $B(b;f(b))$ nuqtalarini tutashtiruvchi vatar bilan Ox o'q kesishish nuqtasining abssissasini olamiz (30-shakl).

Ikki nuqtadan o'tuvchi to'g'ri chiziq formulasi asosida AB to'g'ri chiziq tenglamasini tuzamiz:

$$\frac{x-a}{b-a}=\frac{y-f(a)}{f(b)-f(a)}. \tag{6.3}$$

(6.3) tenglamaga $y=0$ ni qo'yib, ildizning birinchi yaqinlashishini topamiz:

$$x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}. \tag{6.4}$$

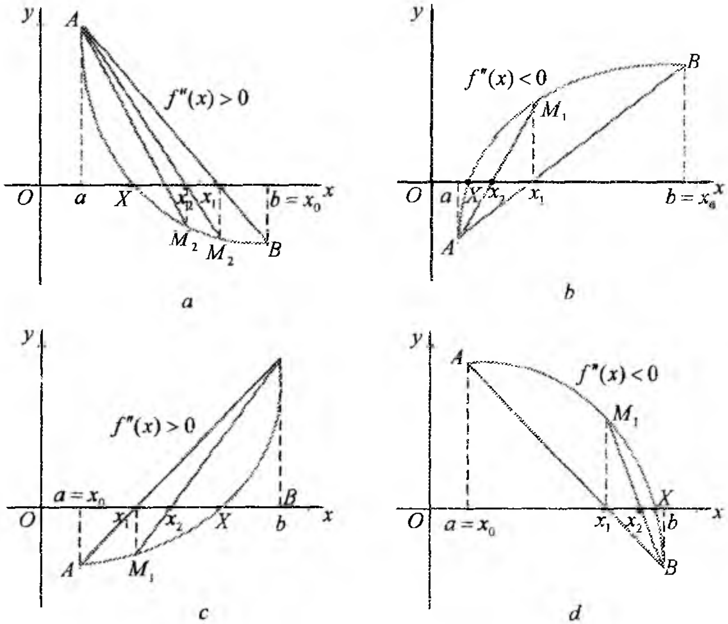
(6.1) tenglamaning izlanayotgan X ildizining ikkinchi x_2 yaqinlashishini topish uchun (6.4) ga monand formula $f(x)$ funksiya chetki nuqtalarida har xil ishoralarga ega bo‘lgan $[a;x_1]$ va $[x_1;b]$ kesmalardan biriga qo‘llaniladi. Bunda, agar $f(a)f''(x)>0$ shart bajarilsa (1-hol) $[a;x_1]$ kesmada

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)} \tag{6.4}$$

kabi topiladi (30,a,b-shakl), agar $f(a)f''(x)<0$ shart bajarilsa (2-hol) $[x_1;b]$ kesmada

$$x_2 = \frac{bf(x_1) - x_1 f(b)}{f(x_1) - f(b)} \tag{6.5}$$

kabi aniqlanadi (30,c,d-shakl).



30-shakl.

Bu jarayonni davom ettirib, ildizning keyingi $x_3, x_4, \dots$ yaqinlashishlari topiladi. Agar $(n-1)$- yaqinlashish mavjud bo‘lsa, n-yaqinlashish 1-hol uchun

$$x_n = \frac{af(x_{n-1}) - x_{n-1}f(a)}{f(x_{n-1}) - f(a)}, \quad x_0 = b, \quad n = 1,2,3,... \tag{6.6}$$

formula bilan, 2-hol uchun

$$x_n = \frac{bf(x_{n-1}) - x_{n-1}f(b)}{f(x_{n-1}) - f(b)}, \quad x_0 = a, \quad n = 1,2,3,... \tag{6.7}$$

formula bilan topiladi.

Geometrik mulohazalarga ko‘ra, x_n $(n = 1,2,3,...)$ sonlarning ketma-ketligi ildizga yaqinlashadi, ya’ni

$$\lim_{n\to\infty} x_n = X. \tag{6.8}$$

(6.6) va (6.7) formulalar bilan hisoblashlar oxirgi kesma uchun talab qilingan aniqlik olinganga qadar davom ettiriladi.

Vatarlar usulining absolut xatoligi

$$|X - x_n| \le \frac{|f(x_n)|}{\mu} \tag{6.9}$$

tengsizlik bilan baholanadi, bu yerda $\mu = \min\limits_{a\le x\le b} |f'(x)|$, $f'(x) \ne 0$.

Urinmalar usuli

$(a;b)$ oraliqda (6.1) tenglama yagona X ildizga ega bo‘lsin. $y = f(x)$ funksiya grafigiga $A(a; f(a))$ nuqtada urinma o‘tkazamiz.

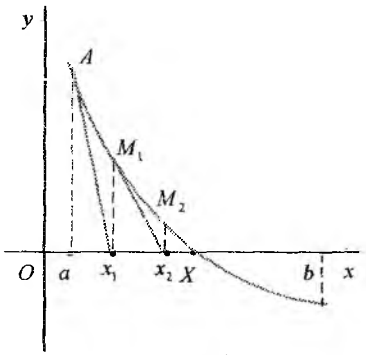
Bu urinmaning tenglamasini tuzamiz:

$$y - f(a) = f'(a)(x - a). \tag{6.10}$$

Bu tenglamada $y = 0$ deb urinmaning Ox o‘q bilan kesishish nuqtasining abssissasini topamiz:

$$x_1 = a - \frac{f(a)}{f'(a)}, \quad f'(a) \ne 0. \tag{6.11}$$

(6.11) formula ildizning birinchi yaqinlashishini beradi. Ikkinchi yaqinlashishni (6.11) ga monand



31-shakl.

formulani $(x_1;b)$ oraliqqa qo‘llab, topamiz:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad (6.12)$$

Ildizning keyingi yaqinlashishlari shu kabi topiladi.

Agar $(n-1)$-yaqinlashish mavjud bo‘lsa, n-yaqinlashish

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n = 1,2,3 \qquad (6.13)$$

formula bilan topiladi.

Bu tenglamada nolinchi yaqinlashish x_0 sifatida $[a;b]$ kesmaning

$$f(x_0)f''(x) > 0 \qquad (6.14)$$

shartni qanoatlantiruvchi nuqtasining abssissasi olinadi (31-shakl). Demak, bu usulni A nuqtadan boshlash shart emas.

Urinmalar usulida ham (6.8) va (6.9) munosabatlar o‘rinli bo‘ladi.

Vatarlar va urinmalar usullarining birgalikda qo‘llanilishi

Vatarlar usuli bilan urinmalar usuli ildizga turli tomondan yaqinlashishni beradi. Shu sababli, odatda, ularni birgalikda qo‘llash qulay bo‘ladi. Bunda ildizni aniqlashtirish tez bajariladi va hisoblashni nazorat qilish mumkin bo‘ladi.

Vatarlar va urinmalar usullarini birgalikda qo‘llanilishi masalasini qaraymiz. Bunda ham vatarlar usulidagi uchta shart bajarilsin deymiz. Bu shartlar bajarilganida to‘rtta hol bo‘lishi mumkin (30-shakl):

a) $f'(x) < 0,\ f''(x) > 0$; b) $f'(x) > 0,\ f''(x) < 0$;

c) $f'(x) > 0,\ f''(x) > 0$; d) $f'(x) < 0,\ f''(x) < 0$.

Aniqlik uchun $f'(x) > 0,\ f''(x) > 0$ bo‘lgan holni qaraymiz (32-shakl).

$A(a; f(a))$ va $B(b; f(b))$ nuqtalar orqali vatar o‘tkazamiz.

Bu vatarning Ox o‘q bilan kesishish nuqtasini topamiz:

$$a_1 = a - \frac{f(a)}{f(b) - f(a)}(b - a).$$

Endi $f(x)$ funksiya grafigiga $B(b; f(b))$ nuqtaga urinma o‘tkazamiz.

Bu urinmaning Ox o'q bilan kesishish nuqtasini topamiz:

$$b_1 = b - \frac{f(b)}{f'(b)}.$$

Shunday qilib, $a < a_1 < X < b_1 < b.$

Bu jarayonni davom ettirib, taqribiy qiymatlarning ikkita ketma-ketligini topamiz:

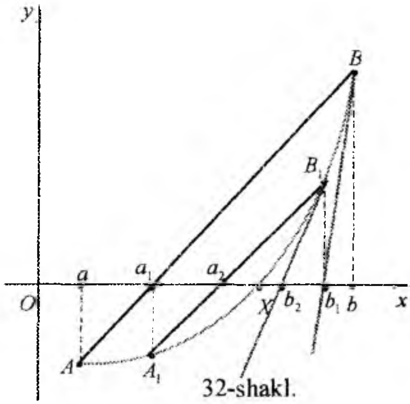
$$a < a_1 < a_2 < ... < X \quad \text{va} \quad b > b_1 > b_2 > ... > X,$$

bu yerda

$$a_n = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}, n = 1,2,..., \; a_0 = a; \tag{6.15}$$

$$b_n = b_{n-1} - \frac{f(b_{n-1})}{f'(b_{n-1})}, n = 1,2,..., \; b_0 = b. \tag{6.16}$$

$\{a_n\}$ va $\{b_n\}$ ketma-ketliklar monoton va chegaralangan. Demak, ular limitga ega. $\lim\limits_{n\to\infty} a_n = \alpha$, $\lim\limits_{n\to\infty} b_n = \beta$ bo'lsin. Agar 1) va 3) shartlar bajarilsa, $\alpha = \beta = X$ $f(x) = 0$ teng-lamaning yagona yechimi bo'ladi.



32-shakl.

Agar ildizning aniqligi ε oldindan berilgan bo'lsa, u holda a_n va b_n lar $|a_n - b_n| < \varepsilon$ yoki $\varepsilon < |a_n - b_n| < 2\varepsilon$ shartlardan biri bajarilishiga qadar davom ettiriladi. Bunda birinchi shart bajarilsa $X \approx a_n$ (yoki $X \approx b_n$) bo'ladi, ikkinchi shart bajarilsa $X \approx \frac{a_n + b_n}{2}$ bo'ladi.

1-misol. $2 - x - \lg x = 0$ tenglmaning haqiqiy ildizini vatarlar va urinmalar usullarini birgalikda qo'llab, $\varepsilon = 0{,}000001$ aniqlikda hisoblang.

Yechish. Izlanayotgan ildizni ajratish uchun berilgan tenglamani $\lg x = 2 - x$ ko'rinishga keltiramiz va $y = \lg x$, $y = 2 - x$ funksiyalarning grafiklarini chizamiz (33-shakl). Chizmadan izlanayotgan ildiz $[1,6;1,8]$ kesmaning ichki nuqtasi bo‘lishi kelib chiqadi.

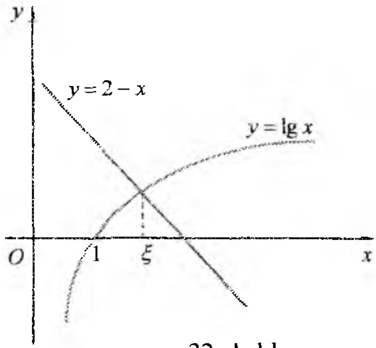
1) va – 3) shartlarning bajarilishini tekshiramiz.

$f(x)=2-x-\lg x$ funksiya uchun:

$f(1,6)=2-1,6-0,2041=0,1959>0;$

$f(1,8)=2-1,8-0,2553=-0,0553<0;$

$f'(x)=-1-\dfrac{1}{x}\lg e;\quad f''(x)=\dfrac{1}{x^2}\lg e;$



33-shakl.

$[1,6;1,8]$ kesmada $f'(x)<0,\ f''(x)>0.$

Demak, $[1,6;1,8]$ kesmada 1) – 3) shartlar bajariladi. Shu sababli $a=1,6$ va $b=1,8$ deb olamiz va aniqlashtirish formulalarini qo‘llaymiz:

$$a_1=1,6-\frac{(1,8-1,6)f(1,6)}{f(1,8)-f(1,6)}=1,6+0,1559=1,7559;$$

$$b_1=1,6-\frac{f(1,6)}{f'(1,6)}=1,6+0,1540=1,7540;$$

$$a_2=1,75558;\quad b_2=1,75557;\quad a_3=1,7555816;\quad b_2=1,7555807.$$

$$|a_3-b_3|=|1,7555816-1,7555807|=0,0000009<\varepsilon.$$

Demak, izlanayotgan yechim $\xi\approx a_3=1,7555816.$

5.6.3.Mashqlar

1. Tenglamalarning haqiqiy ildizini vatarlar va urinmalar usullarini birgalikda qo‘llab, $\varepsilon=0,0001$ aniqlikda toping:

1) $x^3-2x+7=0;$ 2) $x^3+x-1=0;$

3) $x\lg x-1=0;$ 4) $2x+1-\sin x=0.$

6

OLIY ALGEBRA ELEMENTLARI

- Kompleks sonlar
- Ko'phadlar

Leonard Eyler (1707–1783) – shveysar, nemis va rus matematigi hamda mexanigi.

Eyler matematik analiz, differensial geometriya, sonlar nazariyasi, taqribiy hisoblashlar, osmon mexanikasi, matematik fizika, ballistika, kemalar qurilishi, musiqa nazariyasi va fanning boshqa sohalariga oid 850 dan ortiq ishlar muallifi.

Eyler matematikaga qatorlar nazariyasini, kompleks sonlar nazariyasida Eyler formulasini, e sonini, mavhum birlik uchun i belgini va boshqa ko'plab maxsus funksiyalarni kiritgan.

6.1. KOMPLEKS SONLAR

Kvadrat ildiz chiqarish amali barcha haqiqiy sonlar uchun aniqlanmaganligi sababli har qanday kvadrat tenglama ham haqiqiy sonlar to'plamida yechimga ega bo'lmaydi. Bu masala hamda uchinchi va to'rtinchi darajali tenglamalarni yechishda yuzaga kelgan ko'plab masalalar haqiqiy sonlar to'plamini kompleks sonlar to'plamigacha kengaytirishni taqozo etdi.

Keyinchalik kompleks sonlar matematika va amaliy matematikaning ko'pchilik muhim masalalarini yechishga tatbiq qilindi va hozirgi matematikani kompleks sonlar tushunchasisiz tasavvur qilib bo'lmaydi.

6.1.1.Kompleks son tushunchasi va tasviri

Kompleks son tushunchasi

1-ta'rif. z *kompleks son* deb ma'lum tartibda berilgan x va y haqiqiy sonlar juftiga aytiladi va $z=(x,y)$ deb yoziladi.

x va y sonlarga mos ravishda z kompleks sonning *haqiqiy* va *mavhum qismlari* deyilib, $x=\operatorname{Re} z$, $y=\operatorname{Im} z$ kabi belgilanadi.

$z_1=(x_1,y_1)$ va $z_2=(x_2,y_2)$ kompleks sonlarida $x_1=x_2$ va $y_1=y_2$ bo'lganida *ular teng*, ya'ni $z_1=z_2$ deyiladi.

$(0,1)$ son *mavhum birlik* deb ataladi va i bilan belgilanadi:

$$i=(0,1). \tag{1.1}$$

2-ta'rif. $z_1=(x_1,y_1)$ va $z_2=(x_2,y_2)$ *kompleks sonlarining yig'indisi* deb

$$z_1+z_2=(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2) \tag{1.2}$$

songa aytiladi.

3-ta'rif. $z_1=(x_1,y_1)$ va $z_2=(x_2,y_2)$ *kompleks sonlarining ko'paytmasi* deb

$$z_1\cdot z_2=(x_1,y_1)\cdot(x_2,y_2)=(x_1x_2-y_1y_2,x_1y_2+y_1x_2) \tag{1.3}$$

songa aytiladi.

Keltirilgan ta'riflardan haqiqiy sonlardagi kabi

$$(x_1,0)+(x_2,0)=(x_1+x_2,0)$$

va

$$(x_1,0)\cdot(x_2,0)=(x_1x_2,0)$$

kelib chiqadi.

Shunday qilib, kompleks sonlar haqiqiy sonlarni «*to'ldiradi*».

(1.1) va (1.3) formulalar asosida topamiz:

$$x=(x,0),\quad iy=(0,1)y=(0,1)\cdot(y,0)=(0\cdot y-1\cdot 0,0\cdot 0+1\cdot y)=(0,y).$$

Bu ifodalardan $z=(x,y)$ kompleks son yozilishining boshqa bir ko'rinishi kelib chiqadi:

$$z=(x,y)=(x,0)+(0,y)=x+iy.$$

Kompleks sonlarni ko'paytirish ta'rifidan topamiz:

$$i^2=ii=(0,1)\cdot(0,1)=(-1,0)=-1.$$

Demak, $i=(0,1)-$ kvadrati minus birga teng bo'lgan son.

Shunday qilib, $z=x+iy$ ifodaga *kompleks son* deyiladi, bu yerda $x,y-$ haqiqiy sonlar, $i-$ mavhum birlik.

Agar $z=x+iy$ ifodada $y=0$ bo'lsa, $z=x$ *haqiqiy son*, agar $x=0$ bo'lsa, $z=iy$ *sof mavhum son* hosil bo'ladi.

Faqat $x=y=0$ bo'lganida $z=x+iy$ kompleks son nolga teng bo'ladi. Kompleks sonlar uchun «katta» va «kichik» tushunchalari kiritilmaydi.

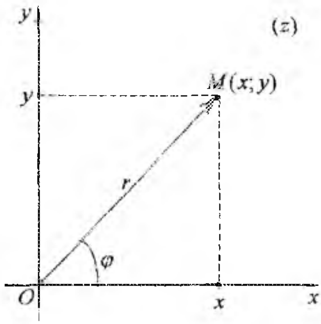
Mavhum qismlarining ishorasi bilan farq qiluvchi $z = x + iy$ va $\bar{z} = x - iy$ sonlariga *qo'shma kompleks sonlar* deyiladi.

Haqiqiy va mavhum qismlarining ishorasi bilan farq qiluvchi $z_1 = x + iy$ va $z_2 = -x - iy$ sonlariga *qarama-qarshi kompleks sonlar* deyiladi.

Kompleks sonlarning geometrik tasviri

Har bir $z = x + iy$ kompleks sonni Oxy koordinatalar tekisligining $M(x;y)$ nuqtasi bilan ifodalash mumkin (bu yerda $x = \operatorname{Re} z$, $y = \operatorname{Im} z$) va aksincha, koordinatalar tekisligining har bir $M(x;y)$ nuqtasini $z = x + iy$ kompleks sonning geometrik tasviri deb qarash mumkin (1-shakl).

Oxy tekislikka *kompleks tekislik* deyiladi va (z) kabi belgilanadi. Bunda, $z = x$ haqiqiy sonlar *haqiqiy o'q* deb ataluvchi Ox o'qning nuqtalari bilan aniqlanadi; $z = iy$ mavhum sonlar *mavhum o'q* deb ataluvchi Oy o'qning nuqtalari bilan aniqlanadi.



1-shakl.

Shuningdek, $z = x + iy$ kompleks sonni $M(x;y)$ nuqtaning radius vektori $\vec{r} = \overrightarrow{OM}$ orqali ifodalash mumkin (1-shakl). Bunda: $\vec{r}$ vektorning uzunligiga *kompleks sonning moduli* deyiladi va $|z|$ yoki r bilan belgilanadi; $\vec{r}$ vektorning Ox o'qning musbat yo'nalishi bilan hosil qilgan burchagiga *kompleks sonning argumenti* deyiladi va $Argz$ bilan belgilanadi.

$z = 0$ kompleks sonning argumenti aniqlanmagan. $z \neq 0$ kompleks sonning argumenti ko'p qiymatli bo'lib, $2\pi k$ $(k = 0,\ -1,\ 1,\ -2,\ 2,...)$ qo'shiluvchigacha aniqlikda topiladi: $Argz = \arg z + 2\pi k, k \in z$, bu yerda $\arg z$ *-argumentning* $(-\pi;\pi]$ oraliqda yotuvchi *bosh qiymati*, ya'ni $-\pi < \arg z \leq \pi$ (ayrim hollarda argumentning bosh qiymati sifatida $[0;2\pi)$ oraliqqa tegishli qiymat olinadi).

6.1.2. Kompleks sonlarning yozilish shakllari

Ushbu

$$z = x + iy$$

ifodaga *kompleks sonning algebraik shakli* deyiladi.

Kompleks sonning r moduli va $\varphi = Argz$ argumentini $z = x + iy$ kompleks sonni ifodalovchi $\bar{r} = \overrightarrow{OM}$ vektorning qutb koordinatalari deb qarash mumkin. Bunda $x = r\cos\varphi$, $y = r\sin\varphi$ bo'ladi (1-shakl).

U holda $z = x + iy$ kompleks sonni $z = r\cos\varphi + ir\sin\varphi$ yoki

$$z = r(\cos\varphi + i\sin\varphi)$$

ko'rinishda yozish mumkin. Bu ifodaga *kompleks sonning trigonometrik shakli* deyiladi.

Bunda $r = |z|$ modul quyidagi formula bilan aniqlanadi:

$$r = |z| = \sqrt{x^2 + y^2}.$$

φ argument esa

$$\cos\varphi = \frac{x}{r}, \quad \sin\varphi = \frac{y}{r}, \quad tg\varphi = \frac{y}{x}$$

formulalardan topiladi.

$\varphi = Argz = \arg z + 2\pi k, k \in z$ ekanidan

$$\cos\varphi = \cos(\arg z + 2\pi k) = \cos(\arg z), \quad \sin\varphi = \sin(\arg z)$$

kelib chiqadi. Shu sababli kompleks sonning algebraik shaklidan trigonometrik shakliga o'tishda kompleks son argumentining bosh qiymatini aniqlash yetarli bo'ladi.

$-\pi < \arg z \le \pi$ bo'lgani ucun $tg\varphi = \frac{y}{x}$ tenglikdan topamiz:

$$\arg z = \begin{cases} arctg\frac{y}{x}, & I, IV \text{ choraklarning ichki nuqtalarida,} \\ arctg\frac{y}{x} + \pi, & II \text{ chorakning ichki nuqtalarida,} \\ arctg\frac{y}{x} - \pi, & III \text{ chorakning ichki nuqtalarida.} \end{cases}$$

Agar nuqta haqiqiy yoki mavhum o'qlarda yotsa, $\arg z$ ni bevosita topish mumkin.

Masalan, $z_1 = 2$ uchun $\arg z_1 = 0$; $z_2 = -3$ uchun $\arg z_2 = \pi$; $z_3 = i$ uchun $\arg z_3 = \frac{\pi}{2}$; $z_4 = -4i$ uchun $\arg z_4 = -\frac{\pi}{2}$ (2-shakl).

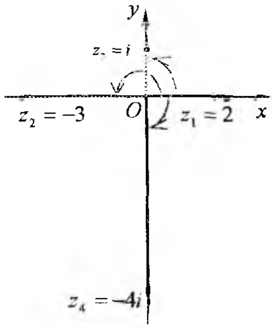
Eyler formulasi deb ataluvchi

$$\cos\varphi + i\sin\varphi = e^{i\varphi}$$

ifoda yordamida $z = r(\cos\varphi + i\sin\varphi)$ tenglikdan $z = re^{i\varphi}$ ifoda keltirib chiqariladi. Ushbu

$$z = re^{i\varphi}$$

ifodaga $z = r(\cos\varphi + i\sin\varphi)$ *kompleks sonning ko'rsatkichli (yoki eksponensial) shakli* deyiladi, bu yerda $r = |z|$ – kompleks sonning moduli; $\varphi = Argz$ – kompleks sonning argumenti.



2-shakl.

Eyler formulasiga ko'ra $e^{i\varphi}$ funksiya 2π davrli davriy funksiya bo'ladi. Shu sababli z kompleks sonni ko'rsatkichli shaklda yozish uchun kompleks son argumentining bosh qiymatini, ya'ni $\varphi = \arg z$ ni aniqlash yetarli bo'ladi.

1-misol. $z = -\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}$ kompleks sonni turli (algebraik, trigonometrik va ko'rsatkichli) shakllarda yozing.

Yechish. $z = -\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}$ kompleks son trigonometrik shaklda berilgan emas. Shu sababli shunday φ burchakni topamizki, $\cos\varphi = -\sin\frac{\pi}{8}$ va $\sin\varphi = \cos\frac{\pi}{8}$ bo'lsin.

Bunday burchak $\varphi = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}$ bo'ladi.

$\cos\frac{5\pi}{8} = -\frac{\sqrt{2-\sqrt{2}}}{2}$ va $\sin\frac{5\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$ ni hisobga olib, kompleks sonining turli shakllarini yozamiz:

$$z = -\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2+\sqrt{2}}}{4} = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8} = e^{\frac{5\pi}{8}i}.$$

6.1.3. Kompleks sonlar ustida amallar

Kompleks sonlarni qo'shish

Agar $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ bo'lsa, yuqorida keltirilgan kompleks sonlarni qo'shish ta'rifiga ko'ra,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \qquad (1.4)$$

Kompleks sonlarni qo'shish kommutativlik va assosiativlik xossalariga ega:

$$z_1 + z_2 = z_2 + z_1, \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(1.4) tenglikdan kompleks sonlar geometrik jihatdan vektorlar kabi qo'shilishi kelib chiqadi (3-shakl).

3-shakldan bevosita ko'rinadiki, $|z_1 + z_2| \leq |z_1| + |z_2|$. Bu tengsizlikka *uchburchak tengsizligi* deyiladi.

Kompleks sonlarni ayirish

Kompleks sonlarni ayirish amali qo'shishga teskari amal sifatida aniqlanadi.

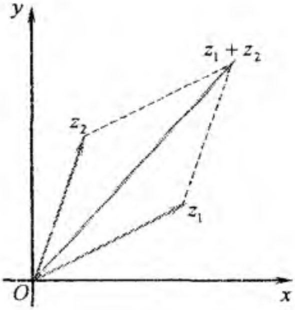
z_1 va z_2 *kompleks sonlarning ayirmasi* deb, z_2 ga qo'shilganida z_1 ni hosil qiluvchi z kompleks soniga aytiladi va $z = z_1 - z_2$ tarzda yoziladi.

Agar $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ bo'lsa, ta'rifga ko'ra,

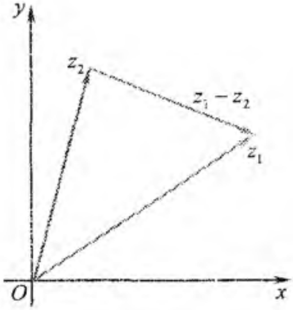
$$z = z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \qquad (1.5)$$

(1.5) tenglikdan kompleks sonlar geometrik jihatdan vektorlar kabi ayrilishi kelib chiqadi. (4-shakl).

4-shakldan ko'rinadiki, $|z_1 - z_2| \geq |z_1| - |z_2|$.



3-shakl.



4-shakl.

Kompleks sonlar uchun

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

bo'ladi, ya'ni ikkita kompleks sonlar ayirmasining moduli tekislikda bu sonlarni ifodalovchi nuqtalar orasidagi masofaga teng.

Shu sababli, masalan $|z - 2i| = 1$ tenglik kompleks tekisligida $z_0 = 2i$ nuqtadan birga teng masofada yotuvchi z nuqtalar to'plamini, ya'ni markazi $z_0 = 2i$ nuqtada joylashgan va radiusi birga teng aylanani aniqlaydi.

Kompleks sonlarni ko'paytirish

Agar $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ bo'lsa, yuqorida keltirilgan kompleks sonlarni ko'paytirish ta'rifiga ko'ra,

$$z = z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \qquad (1.6)$$

Kompleks sonlarni ko'paytirish kommutativlik, assosiativlik va qo'shishga nisbatan distributivlik xossalariga ega:

$$z_1 z_2 = z_2 z_1;$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3);$$
$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

2-misol. $z_1 = 1 + 3i$, $z_2 = -3 + i$ bo'lsa, $z_1 + z_2$, $2z_1 - z_2$, $z_1 \cdot z_2$ larni hisoblang.

Yechish. $z_1 + z_2 = (1 + 3i) + (-3 + i) = -2 + 4i;$

$$2z_1 - z_2 = (2 + 6i) - (-3 + i) = 5 + 5i;$$

$$z_1 \cdot z_2 = (1 + 3i) \cdot (-3 + i) = (-3 - 3) + i(1 - 9) = -6 - 8i.$$

Trigonometrik shaklda berilgan

$$z_1 = r_1 (\cos\varphi_1 + i\sin\varphi_1), \; z_2 = r_2 (\cos\varphi_2 + i\sin\varphi_2)$$

kompleks sonlarni ko'paytiramiz:

$$z_1 z_2 = r_1 (\cos\varphi_1 + i\sin\varphi_1) r_2 (\cos\varphi_2 + i\sin\varphi_2) =$$

$$= r_1 r_2 (\cos\varphi_1 \cos\varphi_2 + i\sin\varphi_1 \cos\varphi_2 + i\cos\varphi_1 \sin\varphi_2 - \sin\varphi_1 \sin\varphi_2) =$$

$$= r_1 r_2 (\cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2) + i(\sin\varphi_1 \cos\varphi_2 + i\cos\varphi_1 \sin\varphi_2) =$$

$$= r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)),$$

ya'ni

$$z_1 z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)). \qquad (1.7)$$

Demak, kompleks sonlar ko'paytirilganda ularning modullari ko'paytiriladi va argumentlari qo'shiladi.

Bu qoida istalgan sondagi ko'paytuvchilar uchun ham o'rinli bo'ladi. Xususan, n ta bir xil ko'paytuvchilar uchun

$$z^n = (r(\cos\varphi + i\sin\varphi))^n = r^n(\cos n\varphi + i\sin n\varphi) \qquad (1.8)$$

bo'ladi.

(1.8) formulaga *Muavr formulasi* deyiladi.

3- misol. $z = \sqrt{3} + i$ bo'lsa, z^6 ni hisoblang.

Yechish. Avval kompleks sonni trigonometrik shaklga keltiramiz.

$x = \sqrt{3}$, $y = 1$ bo'lgani uchun $r = \sqrt{\sqrt{3}^2 + 1^2} = 2$, $\arg z = arctg\frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

Bundan

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

Muavr formulasiga ko'ra:

$$z^6 = 2^6\left(\cos\frac{\pi}{6}\cdot 6 + i\sin\frac{\pi}{6}\cdot 6\right) = 64(\cos\pi + i\sin\pi) = -64.$$

Kompleks sonlarni bo'lish

Kompleks sonlarni bo'lish amali ko'paytirishga teskari amal sifatida aniqlanadi.

z_1 va $z_2 \neq 0$ *kompleks sonlarning bo'linmasi* deb, z_2 ga ko'paytirilganida z_1 ni hosil qiluvchi z kompleks soniga aytiladi va $z = \frac{z_1}{z_2}$ kabi yoziladi.

$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \neq 0$ va $z = x + iy$ bo'lsin.

U holda $(x_2 + iy_2)(x + iy) = x_1 + iy_1$ tenglikdan

$$\begin{cases} xx_2 - yy_2 = x_1, \\ xy_2 + yx_2 = y_1 \end{cases}$$

tenglamalar sistemasi kelib chiqadi.

Sistemadan x va y ni topamiz:

$$x=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2},\quad y=\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}.$$

Shunday qilib,

$$z=\frac{z_1}{z_2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}. \tag{1.9}$$

Amalda ikki kompleks sonning bo'linmasi uning surat va maxrajini maxrajning qo'shmasiga ko'paytirish orqali topiladi (maxraj mavhumlikdan qutqariladi).

4- misol. $z_1=1+2i,\ z_2=3+i$ bo'lsa, $\frac{z_1}{z_2}$ ni toping.

Yechish. $\frac{z_1}{z_2}$ kasrning surat va maxrajini $\bar{z}_2$ ga ko'paytirib, topamiz:

$$\frac{z_1}{z_2}=\frac{1+2i}{3+i}=\frac{(1+2i)(3-i)}{(3+i)(3-i)}=\frac{3+6i-i+2}{9+1}=\frac{5+5i}{10}=\frac{1}{2}+\frac{1}{2}i.$$

Trigonometrik shaklda berilgan $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$ kompleks

sonini $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$ kompleks soniga bo'lamiz:

$$\frac{z_1}{z_2}=\frac{r_1(\cos\varphi_1+i\sin\varphi_1)}{r_2(\cos\varphi_2+i\sin\varphi_2)}=\frac{r_1(\cos\varphi_1+i\sin\varphi_1)\cdot(\cos\varphi_2-i\sin\varphi_2)}{r_2(\cos\varphi_2+i\sin\varphi_2)\cdot(\cos\varphi_2-i\sin\varphi_2)}=$$

$$=\frac{r_1}{r_2}(\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)).$$

Demak,

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}(\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)). \tag{1.10}$$

Shunday qilib, bir kompleks sonni ikkinchisiga bo'lganda ularning modullari bo'linadi va argumentlari ayriladi.

Kompleks sonlardan ildiz chiqarish

Kompleks sondan n-darajali ildiz chiqarish amali n-natural darajaga oshirish amaliga teskari amal sifatida aniqlanadi.

z kompleks sonining n-darajali ildizi deb, $w^n=z$ tenglikni qanoatlantiruvchi w kompleks soniga aytiladi.

$z = r(\cos\varphi + i\sin\varphi)$ va $w = \rho(\cos\theta + i\sin\theta)$ bo‘lsin.

Ildizning ta’rifi va Muavr formulasidan foydalanib topamiz:

$$z = w^n = \rho^n(\cos n\theta + i\sin n\theta) = r(\cos\varphi + i\sin\varphi).$$

Bundan

$$\rho^n = r,\ n\theta = \varphi + 2\pi k,\ k = 0,\ -1,\ 1,\ -2,\ 2,...$$

yoki

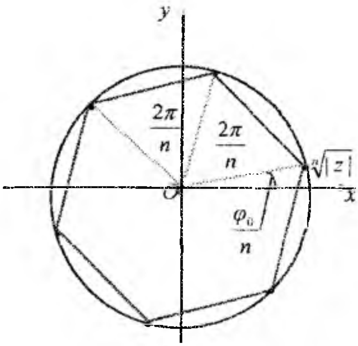
$$\theta = \frac{\varphi + 2\pi k}{n},\ \rho = \sqrt[n]{r}$$

kelib chiqadi.

U holda $w = \sqrt[n]{z}$ tenglik quyidagi ko‘rinishga keladi:

$$w_k = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right),\ k = 0,\ 1,\ ...,\ n-1. \quad (1.11)$$

Sinus va kosinus funksiyala-rning davriyligi sababli z kompleks sonining n- darajali ildizlari soni n ga teng bo‘ladi va ular k ning n ta $k = 0,\ 1,\ ...,\ n-1$ qiymatlarida aniqlanadi. Ildizlarning moduli r haqiqiy sonining n-darajali algebraik ildizidan iborat bo‘ladi, argumentlari esa bir-biridan $\frac{2\pi}{n}$ ga karrali songa farq qiladi. Bunda barcha ildizlar kompleks tekisligida markazi $z = 0$ nuqtada bo‘lgan va radiusi $\sqrt[n]{|z|}$ ga teng aylanaga ichki chizilgan n burchakli muntazam ko‘pburchakning uchlarini tasvirlaydi (5-shakl).



5-shakl.

5-misol. $\sqrt[4]{-1}$ ning barcha ildizlarini toping.

Yechish. Ildiz ostidagi ifodani trigonometrik shaklda yozamiz:

$$-1 = \cos\pi + i\sin\pi.$$

Bundan

$$\sqrt[4]{-1} = \cos\frac{\pi + 2\pi k}{4} + i\sin\frac{\pi + 2\pi k}{4},\ k = 0,\ 1,\ 2,\ 3.$$

k ga 0, 1, 3 va 4 qiymatlar berib, topamiz:

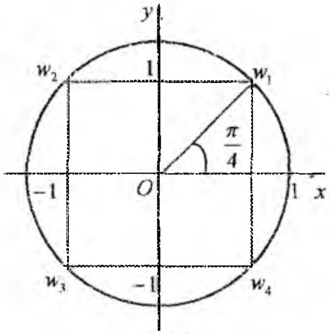
$$w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i),$$

$$w_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}(-1+i),$$

$$w_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}(-1+i),$$

$$w_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{\sqrt{2}}{2}(-1-i),$$

$$w_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2}(1-i).$$



6-shakl.

Bu ildizlar (z) kompleks tekisligida birlik aylanaga ichki chizilgan muntazam to‘rtburchakning (kvadratning) uchlarida yotadi (6-shakl).

6.1.4. Mashqlar

1. x, y ning qanday haqiqiy qiymatlarida $z_1 = x - 3yi - y - \frac{2x}{i}$ va $z_2 = 2x + i^2 - 3xi - yi^3$ kompleks sonlar qo‘shma bo‘ladi?

2. x, y ning qanday haqiqiy qiymatlarida $z_1 = y + 2i^3 + 3 - 2xi$ va $z_2 = 3x + 8i + \frac{2y}{i} + 2i^2$ kompleks sonlar teng bo‘ladi?

3. x, y ning qanday haqiqiy qiymatlarida $z_1 = 3x - 2yi + 5i^3 - 1$ va $z_2 = 3y - i^3 + \frac{8x}{i} + 2i^2$ kompleks sonlar qarama-qarshi bo‘ladi?

4. x, y ning qanday haqiqiy qiymatlarida $z_1 = 5x + \frac{3y}{i^2} + 3yi + i^3$ va $z_2 = 3y(1+i) + \frac{5x}{i} - i^4$ kompleks sonlar nolga teng bo‘ladi?

5. (z) tekislikda berilgan tenglamalarni yeching:

1) $z^2 + 6z + 25 = 0$; 2) $2z^2 + iz + 1 = 0$;

3) $iz^2 - 2z + 3i = 0$; 4) $z^2 - 6iz - 5 = 0$.

6. (z) tekislikda berilgan shartlar bilan qanday nuqtalar to'plami aniqlanadi?

1) $\operatorname{Re} z = a$; 2) $\operatorname{Im} z = b$;

3) $r < |z| < R$; 4) $\varphi < \arg z < \psi$;

5) $r < |z| < R,\ \varphi < \arg z < \psi$, bu yerda $a, b, r, R, \varphi, \psi$ - haqiqiy sonlar.

7. Berilgan kompleks sonlarni turli (algebraik, trigonometrik va ko'rsatkichli) shakllarda yozing:

1) $z = -2 + 2\sqrt{3}i$; 2) $z = \sqrt{3} - i$;

3) $z = \sqrt{3}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$; 4) $z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$;

5) $z = \sqrt{2}e^{-\frac{\pi}{4}i}$; 6) $z = \sqrt{13}e^{i\left(arctg\frac{2}{3} - \pi\right)}$;

7) $z = 2\cos 60^o - 2i\sin 60^o$; 8) $z = -2\cos 45^o - 2i\sin 45^o$.

8. Berilgan kompleks sonlarning yig'indisi, ayirmasi, ko'paytmasi va bo'linmasini toping:

1) $z_1 = -5 + 3i$ va $z_2 = 2 - 4i$; 2) $z_1 = -3 - 4i$ va $z_2 = 2 + 3i$.

9. (z) tekislikda berilgan nuqtalar orasidagi masofani toping:

1) $1 - 3i$ va $4i$; 2) $1 - 5i$ va -4;

3) $1 + 3i$ va $3 + 2i$; 4) $8 - 3i$ va $2 + 5i$.

10. Hisoblang:

1) $\frac{2 - 3i}{1 + 2i} + (1 - i)^2(1 + i)$; 2) $\frac{1 + 3i}{-2 + i}\cdot(-2i) + 1$;

3) $(2 + 3i)^3 - (2 - 3i)^3$; 4) $(-1 + 2i)^4 - (1 + 2i)^4$.

11. Kompleks sonlarning haqiqiy va mavhum qismlarini toping:

1) $\frac{3\sqrt{3} - i^7}{2(\sqrt{3} + 2i^3)}$; 2) $\frac{-1 + i^5}{2 + i} + \frac{8 + 19i^3}{40}$;

3) $\frac{3 + i^5}{(1 - i^3)(1 + 2i^7)}$; 4) $(i^5 - 5)\left(2i + \frac{3}{2 - i^3}\right)$.

12. Berilgan kompleks sonlarning ko'paytmasi va bo'linmasini toping:

1) $z_1 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ va $z_2 = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$;

2) $z_1 = 6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ va $z_2 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$;

3) $z_1=8(\cos135^\circ+i\sin135^\circ)$ va $z_2=2(\cos45^\circ+i\sin45^\circ)$;

4) $z_1=8(\cos90^\circ+i\sin90^\circ)$ va $z_2=\cos30^\circ+i\sin30^\circ$.

13. Darajalarni hisoblang:

1) $\left(\frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{36}+i\sin\frac{7\pi}{36}\right)\right)^{-9}$; 2) $\left(\frac{1+i}{1-i}\right)^{10}$;

3) $(2-2i)^9$; 4) $(\sqrt{2}(\cos20^\circ+i\sin20^\circ))^{12}$.

14. Berilgan sonlarning barcha ildizlarini toping:

1) $\sqrt{2-2\sqrt{3}i}$; 2) $\sqrt[3]{-i}$;

3) $\sqrt[4]{-8+8\sqrt{3}i}$; 4) $\sqrt[5]{1+i}$.

6.2. KO'PHADLAR

6.2.1. Ko'phadlar ustida amallar

Ushbu

$$P_n(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n \tag{2.1}$$

funksiyaga n- *darajali ko'phad* (yoki *butun ratsional funksiya*) deyiladi. Bunda $a_0\neq0$, $a_1,\ldots,a_n$ sonlari ko'phadning koeffitsiyentlari deb ataladi, manfiy bo'lmagan butun n soni ko'phadning daraja ko'rsatkichini bildiradi.

Xususan, $n=0$ da $P_0(x)=a_0$ $(a_0\neq0)$ nolinchi darajali ko'phad hosil bo'ladi.

Agar $P_n(x)$ va $Q_n(x)$ ko'phadlar x ning barcha qiymatlarida bir xil qiymatlar qabul qilsa, u holda $P_n(x)$ va $Q_n(x)$ ko'phadlarda x ning bir xil darajalari oldida turgan koeffitsiyentlar teng bo'ladi va aksincha. Bu ko'phadlarga teng ko'phadlar deyiladi va $P_n(x)=Q_n(x)$ deb yoziladi.

Ko'phadlar ustida qo'shish, ayirish va ko'paytirish amallarini bajarish mumkin.

Ikki ko'phadning yig'indisi, ayirmasi va ko'paytmasi yana ko'phad bo'ladi.

Ko'phadlarni qo'shish, ayirish va ko'paytirish amallari algebraik ifodalardagi kabi bajarilgani uchun arifmetik amallarning asosiy xossalariga ega bo‘ladi.

Ko‘phadlarni bo‘lish qoldiqli va qoldiqsiz bo‘lishi mumkin.

$P_n(x)$ va $Q_m(x)$ ko‘phadlar berilgan bo‘lsin, bunda $n \geq m > 0$.

U holda

$$P_n(x) = Q_m(x) \cdot R_{n-m}(x) + r_k(x), \; 0 \leq k < m \tag{2.2}$$

tenglikni qanoatlantiruvchi $R_{n-m}(x)$ va $r_k(x)$ ko‘phadlar mavjud va yagona bo‘ladi. Bunda $P_n(x)$ bo‘linuvchi, $Q_m(x)$ bo‘luvchi, $R_{n-m}(x)$ bo‘linma, $r_k(x)$ qoldiq deb ataladi.

(2.2) tenglikda $r_k(x) \equiv 0$ bo‘lishi ham mumkin. U holda $P_n(x)$ ko‘phad $R_{n-m}(x)$ ko‘phadga qoldiqsiz bo‘linadi deyiladi.

Ko‘phadlarni bo‘lishdan hosil bo‘ladigan bo‘linma va qoldiqni topishning har xil usullari mavjud. Ko‘p hollarda «burchakli usulida bo‘lish» qoidasidan foydalaniladi.

6.2.2. Ko‘phadlarning ildizi

$P_n(x)$ *ko‘phadning ildizi* deb x o‘zgaruvchining bu ko‘phadning qiymatini nolga aylantiradigan x_0 (haqiqiy yoki kompleks) qiymatiga aytiladi, ya’ni bunda $P_n(x_0) = 0$ bo‘ladi.

$P_n(x)$ ko‘phadni $x - \alpha$ ga bo‘lishdan hosil bo‘ladigan qoldiqni bo‘lish jarayonini bajarmasdan topish imkonini beradigan teoremani isbotlaymiz.

1-teorema. $P_n(x)$ ko‘phadni $x - \alpha$ ikkihadga bo‘lishdan hosil bo‘ladigan qoldiq $P_n(\alpha)$ ga teng bo‘ladi.

Isboti. $P_n(x)$ ko‘phadni $x - \alpha$ ikkihadga bo‘lish natijasi

$$P_n(x) = (x - \alpha) \cdot R_{n-1}(x) + r,$$

bo‘lsin, bu yerda r biror o‘zgarmas son (0- darajali ko‘phad) bo‘ladi.

Bu tenglikda x o‘zgaruvchiga α qiymat berib, topamiz:

$$r = P_n(\alpha).$$

Teorema isbotlandi.

Bu teoremadan quyidagi teorema kelib chiqadi.

2-teorema (*Bezu teoremasi*). α son $P_n(x)$ ko‘phadning ildizi bo‘lishi uchun $P_n(x)$ ko‘phad $x - \alpha$ ikkihadga qoldiqsiz bo‘linishi zarur va yetarli.

6.2.3. Ko'phadni ko'paytuvchilarga ajratish

Ko'phadni nolinchi darajali bo'lmagan ikkita yoki bir nechta ko'phadning ko'paytmasi ko'rinishida ifodalashga *ko'phadni ko'paytuvchilarga ajratish* deyiladi.

Algebraning asosiy teoremasi deb ataluvchi quyidagi teoremani isbotsiz keltiramiz.

3-teorema. n-darajali $(n>0)$ har qanday ko'phad hech bo'lmaganda bitta haqiqiy yoki kompleks ildizga ega bo'ladi.

Bu teoremaning natijasi sifatida quyidagi teoremani isbotlaymiz.

4-teorema. $n-$ darajali har qanday ko'phadni

$$P_n(x)=a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n) \tag{2.3}$$

ko'rinishda ko'paytuvchilarga ajratish mumkin, bu yerda a_0- ko'phadning bosh koeffitsiyenti, α_1, α_2 ,..., α_n- ko'phadning ildizlari.

Isboti. (2.1) ko'phadni qaraymiz. 3-teoremaga ko'ra, u ildizga ega. Bu ildizni α_1 bilan belgilaymiz. U holda Bezu teoremasiga ko'ra, $P_n(x)=(x-\alpha_1)\cdot P_{n-1}(x)$ bo'ladi, bu yerda $P_{n-1}(x)-$ $(n-1)$- darajali ko'phad. $P_{n-1}(x)$ ko'phad bo'lgani uchun u ham ildizga ega. Bu ildizni α_2 bilan belgilaymiz. U holda $P_{n-1}(x)=(x-\alpha_2)\cdot P_{n-2}(x)$ bo'ladi, bu yerda $P_{n-2}(x)-$ $(n-2)$- darajali ko'phad. Demak, $P_n(x)=(x-\alpha_1)(x-\alpha_2)\cdot P_{n-2}(x)$.

Bu jarayonni davom ettirish natijasida

$$P_n(x)=a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$$

yoyilmani hosil qilamiz.

(2.3) tenglikdagi $(x-\alpha_i)$ ko'paytuvchilarga chiziqli ko'paytuvchilar deyiladi.

1-misol. $P_3(x)=x^3-2x^2-x+2$ ko'phadni chiziqli ko'paytuvchilarga ajrating.

Yechish. Berilgan ko'phad $x=-1$, $x=1$, $x=2$ da nolga teng bo'ladi, $a_0=1$.

Demak,

$$x^3-2x^2-x+2=(x+1)(x-1)(x-2).$$

(2.3) tenglikdan shunday *xulosa* kelib chiqadi: n- darajali har qanday ko'phad n ta ildizga (haqiqiy yoki kompleks) ega. Ular orasida tenglari bo'lishi mumkin. Ko'phadning (2.3) yoyilmasida qandaydir bir ildiz k marta uchrashi mumkin. U holda bu ildizga k *karrali ildiz* deyiladi. $k=1$ bo'lganida ildiz *oddiy ildiz* deb ataladi.

Agar (2.1) ko'phad k_1 karrali α_1 ildizga, k_2 karrali α_2 ildizga va umuman k_r karrali x_r ildizga ega bo'lsa, (2.3) formula

$$P_n(x)=a_0(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}...(x-\alpha_r)^{k_r} \tag{2.4}$$

ko'rinishni oladi, bu yerda $k_1+k_2+...+k_r=n$.

(2.3) tenglikda α_1, α_2 ,..., α_n ildizlar orasida komplekslari bo'lishi ham mumkin. Kompleks ildizlar uchun o'rinli bo'ladigan teoremani isbotsiz keltiramiz.

5-teorema. Agar haqiqiy koeffitsiyentli $P_n(x)$ ko'phad $a+ib$ kompleks ildizga ega bo'lsa, u holda u $a-ib$ qo'shma ildizga ham ega bo'ladi.

Bu teoremaga ko'ra, ko'phadning (2.3) yoyilmasida kompleks ildizlar o'z qo'shma juftlari bilan qatnashadi. Bu juftga mos chiziqli ko'paytuvchilarni ko'paytiramiz:

$$(x-(a+ib))(x-(a-ib))=((x-a)-ib)((x-a)+ib)=(x-a)^2+b^2=$$

$$=x^2-2ax+a^2+b^2=x^2+px+q,$$

bu yerda $p=-2a,\ q=a^2+b^2$.

Demak, qo'shma ildizlarga mos chiziqli ko'paytuvchilar ko'paytmasini haqiqiy koeffitsiyentli, diskriminanti manfiy bo'lgan kvadrat uchhad bilan almashtirish mumkin.

Shu kabi

$$(x-(a+ib))^k(x-(a-ib))^k=((x-(a+ib))(x-(a-ib)))^k=(x^2+px+q)^k$$

almashtirish bajarish mumkin.

Shunday qilib, yuqorida keltirilganlar asosida quyidagi tasdiqni hosil qilamiz.

6-teorema. Har qanday haqiqiy koeffitsiyentli ko'phad haqiqiy koeffitsientli chiziqli va kvadrat uchhadlardan iborat ko'paytuvchilarga ajratiladi, ya'ni $P_n(x)$ ko'phadni

$$P_n(x)=a_0(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}...(x-\alpha_r)^{k_r}\times$$

$$\times(x^2+p_1x+q_1)^{s_1}(x^2+p_2x+q_2)^{s_2}...(x^2+p_lx+q_l)^{s_l} \tag{2.5}$$

ko'rinishda ifodalash mumkin. Bunda a_0 – ko'phadning bosh koeffitsiyenti, $\alpha_1, \alpha_2, ..., \alpha_r$ – ko'phadning mos ravishda $k_1, k_2, ..., k_r$ karrali ildizlari, $x^2 + p_i x + q_i$ $(i = \overline{1,l})$ kvadrat uchhadlar uchun diskreminant $D = p^2 - 4q < 0$, $k_1 + k_2 + ... + k_r + 2s_1 + 2s_2 + ... + 2s_l = n$; $r, l, k_1, k_2, ..., k_r, s_1, s_2, ..., s_l$ – natural sonlar.

2-misol. $P_4(x) = x^4 + 3x^3 - x - 3$ ko'phadni ko'paytuvchilarga ajrating.

Yechish. $P_4(x) = x^4 + 3x^3 - x - 3 = x^3(x+3) - (x+3) =$

$$= (x+3)(x^3 - 1) = (x+3)(x-1)(x^2 + x + 1).$$

6.2.4. Ratsional kasrlarni sodda kasrlarga yoyish

Ikkita $Q_m(x)$ va $P_n(x)$ ko'phadning nisbatiga

$$R(x) = \frac{Q_m(x)}{P_n(x)} = \frac{b_0 x^m + b_1 x^{m-1} + ... + b_{m-1} x + b_m}{a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n}$$

ratsional funksiya (*ratsional kasr*) deyiladi.

$m < n$ bo'lganda ratsional kasr *to'g'ri kasr*, $m \geq n$ bo'lganda *noto'g'ri kasr* deyiladi.

Noto'g'ri kasrda uning $Q_m(x)$ suratini $P_n(x)$ maxrajiga odatdagidek bo'lish yo'li bilan kasrdan butun qismi $q(x)$ ajratiladi, ya'ni

$$R(x) = \frac{Q_m(x)}{P_n(x)} = q(x) + \frac{r(x)}{P_n(x)}$$

tenglik hosil qilinadi, bu yerda $q(x)$ – butun qism deb ataluvchi ko'phad, $\frac{r(x)}{P_n(x)}$ – to'g'ri kasr, chunki $r(x)$ qoldiqning darajasi $P_n(x)$ning darajasidan kichik.

3- misol. $R(x) = \frac{3x^4 - 2x^3 + 1}{x^2 + 2x + 2}$ ratsional kasrdan butun qismini ajrating.

Yechish. Ko'phadlarni bo'lish qoidasi bo'yicha kasrning suratini maxrajiga bo'lamiz:

$$\begin{array}{r|l}
-\;3x^4 - 2x^3 + 1 & x^2 + 2x + 2 \\
\cline{2-2}
3x^4 + 6x^3 + 6x^2 & 3x^2 - 8x + 10 \\
\cline{1-1}
-\;-8x^3 - 6x^2 + 1 & \\
-8x^3 - 16x^2 - 16x & \\
\cline{1-1}
-\;10x^2 + 16x + 1 & \\
10x^2 + 20x + 20 & \\
\cline{1-1}
-4x - 19 &
\end{array}$$

Demak,

$$R(x)=3x^2-8x+10+\frac{-4x-19}{x^2+2x+2}.$$

Quyidagi to‘g‘ri kasrlarga *sodda* (*elementar*) *kasrlar* deyiladi:

I. $\frac{A}{x-\alpha}$;

II. $\frac{A}{(x-\alpha)^k}$, $(k\geq 2,\ k\in N)$;

III. $\frac{Mx+N}{x^2+px+q}$, $(p^2-4q<0)$;

IV. $\frac{Mx+N}{(x^2+px+q)^s}$, $(s\geq 2,\ s\in N,\ p^2-4q<0)$,

bu yerda $A,\ M,\ N,\ \alpha,\ p,\ q$ - haqiqiy sonlar.

7-teorema. Maxraji (2.5) ko‘rinishda ko‘paytuvchilarga ajratilgan har qanday $\frac{Q_m(x)}{P_n(x)}$ to‘g‘ri kasrni sodda kasrlar yig‘indisiga yagona tarzda yoyish mumkin.

Bunda:

1) (2.5) ifodaning $(x-\alpha)$ ko‘rinishdagi ko‘paytuvchisiga I turdagi $\frac{A}{x-\alpha}$ kasr mos keladi;

2) (2.5) ifodaning $(x-\alpha)^k$ ko‘rinshidagi ko‘paytuvchisiga I va II turdagi k ta kasrlar yig‘indisi $\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+...+\frac{A_k}{(x-\alpha)^k}$ mos keladi;

3) (2.5) ifodaning x^2+px+q ko‘rinishdagi ko‘paytuvchisiga III turdagi $\frac{Mx+N}{x^2+px+q}$ kasr mos keladi;

4) (2.5.) ifodaning $(x^2+px+q)^s$ ko‘rinishdagi ko‘paytuvchisiga III va IV turdagi s ta kasrlar yig‘indisi

$$\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+...+\frac{M_sx+N_s}{(x^2+px+q)^s}$$ mos keladi.

Shunday qilib, teoremaga ko‘ra,

$$\frac{Q_m(x)}{P_n(x)}=\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\ldots+\frac{A_k}{(x-\alpha)^k}+\ldots+$$

$$+\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+\ldots+\frac{M_sx+N_s}{(x^2+px+q)^s}, \qquad (2.6)$$

bu yerda $A_1,\ A_2,\ldots,A_k,\ M_1,\ N_1,\ M_2,\ N_2,\ldots,M_s,\ N_s$ – koeffitsiyentlar.

(2.6) tenglikdagi noma’lum koeffitsiyentlarini topishning turli usullari mavjud. *Masalan*, noma’lum koeffitsiyentlarni topishda *koeffitsiyentlarni tenglashtirish* usulini qo‘llash mumkin.

Bu usul quyidagi tartibda amalga oshiriladi:

1. (2.6) yoyilmaning o‘ng tomoni umumiy maxraj $P_n(x)$ ga keltiriladi. Natijada $\frac{Q_m(x)}{P_n(x)}=\frac{S_m(x)}{P_n(x)}$ ayniyat hosil bo‘ladi, bu yerda $S_m(x)$ – koeffitsiyentlari noma’lum bo‘lgan ko‘phad.

2. Hosil bo‘lgan ayniyatda maxrajlar teng bo‘lgani uchun, suratlar ham aynan teng bo‘ladi, ya’ni $Q_m(x)=S_m(x)$.

3. $Q_m(x)=S_m(x)$ tenglikda x ning bir xil darajalari oldidagi koeffitsiyentlar tenglashtiriladi (ko‘phadlarning aynan tengligi haqidagi teoremaga ko‘ra);

Natijada tenglamalar sistemasi hosil bo‘ladi va bu sistemadan izlanayotgan $A_1,\ A_2,\ldots,A_k,\ M_1,\ N_1,\ M_2,\ N_2,\ldots,M_s,\ N_s$ koeffitsiyentlar topiladi.

4- misol. $R(x)=\frac{x^4-2x+1}{x^2(x^3+1)}$ to‘g‘ri kasrni oddiy kasrlar yig‘indisiga yoying.

Yechish. $R(x)$ ning maxrajini ko‘paytuvchilarga ajratamiz:

$$x^2(x^3+1)=x^2(x+1)(x^2-x+1)$$

$R(x)$ni 1-teoremaga asosan, sodda kasrlar yig‘indisiga yoyamiz:

$$R(x)=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A}{x+1}+\frac{Mx+N}{x^2-x+1}.$$

Noma’lum koeffitsiyentlarini koeffitsiyentlarni tenglashtirish usuli bilan topamiz. Buning uchun tenglikning o‘ng qismini umumiy maxrajga keltiramiz, hosil bo'lgan tenglikning har ikkala tomonidagi maxrajlarni tashlab yuboramiz va quyidagi tenglikni hosil qilamiz:

$$x^4-2x+1=A_1x(x^3+1)+A_2(x^3+1)+Ax^2(x^2-x+1)+Mx^3(x+1)+Nx^2(x+1).$$

x ning bir xil darajalari oldidagi koeffitsiyentlarni tenglashtiramiz:

$$\begin{cases} x^4: & A+A_1+M=1, \\ x^3: & -A+A_2+M+N=0, \\ x^2: & A+N=0, \\ x^1: & A_1=-2, \\ x^0: & A_2=1. \end{cases}$$

Bu tenglamalar sistemasini yechamiz:

$$A=\frac{4}{3},\ A_1=-2,\ A_2=1,\ M=\frac{5}{3},\ N=-\frac{4}{3}.$$

Demak,

$$R(x)=-\frac{2}{x}+\frac{1}{x^2}+\frac{4}{3(x+1)}+\frac{5x-4}{3(x^2-x+1)}.$$

6.2.5. Mashqlar

1. Berilgan $P(x)$ va $Q(x)$ ko'phadlarning yig'indisini, ayirmasini va ko'paytmasini toping:

1) $P(x)=2x^3-x^2+4,\quad Q(x)=x^5+3x^2-x;$

2) $P(x)=x^4+x^2-5,\quad Q(x)=3x^4+x^3-x^2.$

2. $P(x)$ ko'phadni $Q(x)$ ko'phadga bo'lishda hosil bo'ladigan bo'linma va qoldiqni toping:

1) $P(x)=x^3+2x^2-x+1,\ Q(x)=x^2+x-1;$

2) $P(x)=x^4+5x^3-6x+5,\ Q(x)=x^3+2x^2-1;$

3) $P(x)=3x^5+4x^3+2x-1,\ Q(x)=x^3+3x+7;$

4) $P(x)=2x^4-13x^3+32x^2-24x+1,\quad Q(x)=x^2-5x+6.$

3. $P(x)$ ko'phadni $x-\alpha$ ikkihadga bo'lganda hosil bo'ladigan qoldiqni toping:

1) $P(x)=x^7-3x^4-x^3+1,\ x-2;$

2) $P(x)=x^{11}-6x^7+x^5-8,\ x+1;$

3) $P(x)=3x^6+x^5-64,\ x+2;$

4) $P(x)=x^5-6x^3+x,\ x-3.$

4. $P(x) = ax^3 + bx^2 + x - 1$ va $Q(x) = 3x^3 - x^2 + x - c$ ko‘phadlar bir-biriga aynan teng. a, b, c larni toping.

5. $ax^4 + x^3 - 3x + b \equiv 2x^4 + cx^3 - 3x + 1$. a, b, c larni toping.

6. Berilgan ko‘phadlarni ko‘paytuvchilarga ajrating:

1) $x^4 - 16$; 2) $x^3 - 81x$;

3) $5x^4 - 40x^3 + 115x^2 - 140x + 60$; 4) $x^5 - 6x^4 + 9x^3 - x^2 + 6x - 9$.

7. Berilgan to‘g‘ri kasrlarni sodda kasrlar yig‘indisiga yoying:

1) $\dfrac{x^2 + 4x + 1}{x^3 + x^2}$; 2) $\dfrac{3x^3 - 5x^2 + 8x - 4}{x^4 + 4x^2}$;

3) $\dfrac{3x - 2}{x^3 + x^2 - 2x}$; 4) $\dfrac{x^2 + 5x + 1}{x^4 + x^2 + 1}$.

8. Berilgan to‘g‘ri kasrlarni sodda kasrlar yig‘indisiga yoying va koeffitsiyentlarni noma’lum koeffitsiyentlarni tenglashtirish usuli bilan toping:

1) $\dfrac{x^2 + 2x + 3}{x^4 + x^3}$; 2) $\dfrac{2x^2 - 11x - 6}{x^3 + x^2 - 6x}$;

3) $\dfrac{3x^3 - 2x^2 - 2x + 7}{x^4 - x^2}$; 4) $\dfrac{2x - 1}{x^4 + x}$.

7

- Aniqmas integral
- Ratsional funksiyalarni integrallash
- Trigonometrik funksiyalarni integrallash
- Irratsional ifodalarni integrallash
- Aniq integral
- Xosmas integrallar
- Aniq integralning tatbiqlari

Isaak Nyuton (1642–1727) – ingliz matematigi, fizigi, mexanigi va astronomi

Nyuton "Natural falsafaning matematik asoslari" asarida butun olam tortishish qonunini va mexanikaning uch qonunini bayon qilgan.

U matematik analiz asoslarini, Nyuton binomi formulasini, tenglamalarni yechishning Nyuton usulini yaratgan, matematik hisoblashlarda qatorlardan foydalanishni ko'rsatgan.

BIR O'ZGARUVCHI FUNKSIYASINING INTEGRAL HISOBI

Integral hisob – bu matematikaning integral tushunchasi, uning xossalari va hisoblash usullari o'rganiladigan bo'limidir. Integral hisob differensial hisob bilan uzviy bog'liq va ular birgalikda matematik analizning asosini tashkil qiladi.

Differensial va integral hisobning asosiy tushunchalari, jumladan, differensiallash va integrallash amallari o'rtasidagi bog'lanish, ularmning amaliy masalalarni yechishga tatbiqi XVII asrning ikkinchi yarmida *I.Nyuton* va *G.Leybnis* tomonidan ishlab chiqilgan. Ularning izlanishlari matematik analizning gurkirab rivojlanish davrini boshlab bergan.

Integral hisob yordamida ko'plab yangi masalalar bilan bir qatordam ilgari yechilmagan bir qancha nazariy va amaliy masalalarni yechish imkoni yaratilgan.

Integral hisobning asosiy tushunchalari bir-biri bilan uzviy bog'liq aniqmas va aniq integrallardir.

7.1. ANIQMAS INTEGRAL

7.1.1. Boshlang'ich funksiya va aniqmas integral

Berilgan funksiyaning hosilasini topish differensial hisobning asosiy masalalaridan biri hisoblanadi. Matematik analiz masalalarining turliligi, uning geometriya, mexanika, fizika va texnikadagi keng miqyosdagi tatbiqi berilgan $f(x)$ funksiya uchun hosilasi shu funksiyaga teng bo'lgan $F(x)$ funksiyani topishga olib keladi.

Funksiyaning berilgan hosilasiga ko'ra, uning o'zini topish masalasi integral hisobning asosiy masalalaridan biri hisoblanadi.

$y=f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lsin.

1-ta'rif. Agar $(a;b)$ intervalda differensiallanuvchi $F(x)$ funksiyaning hosilasi berilgan $f(x)$ funksiyaga teng, ya'ni

$$F'(x)=f(x) \quad (\text{ yoki } dF(x)=f(x)dx), \; x\in(a;b)$$

bo'lsa, $F(x)$ funksiyaga $(a;b)$ intervalda $f(x)$ funksiyaning *boshlang'ich funksiyasi* deyiladi.

Masalan: 1) $F(x)=x^3$ funksiya sonlar o'qida $f(x)=3x^2$ funksiyaning boshlang'ich funksiyasi bo'ladi, chunki $x\in R$ da $(x^3)'=3x^2$;

2) $F(x)=\sqrt{1-x^2}$ funksiya $(-1;1)$ intervalda $f(x)=-\dfrac{x}{\sqrt{1-x^2}}$ funksiyaning boshlang'ich funksiyasi bo'ladi, chunki $x\in(-1;1)$ da $(\sqrt{1-x^2})'=-\dfrac{x}{\sqrt{1-x^2}}$.

Lemma. Agar $F(x)$ va $\Phi(x)$ funksiyalar $(a;b)$ intervalda $f(x)$ funksiyaning boshlang'ich funksiyalari bo'lsa, u holda $F(x)$ va $\Phi(x)$ bir-biridan o'zgarmas songa farq qiladi.

Isboti. $F(x)$ va $\Phi(x)$ funksiyalar $(a;b)$ intervalda $f(x)$ funksiyaning boshlang'ich funksiyalari bo'lsin: $F'(x)=f(x)$, $\Phi'(x)=f(x)$.

U holda istalgan $x\in(a;b)$da

$$(\Phi(x)-F(x))'=\Phi'(x)-F'(x)=f(x)-f(x)=0$$

bo'ladi.

Bundan $\Phi(x)-F(x)=C$ yoki $\Phi(x)=F(x)+C$ kelib chiqadi, bu yerda C – ixtiyoriy o'zgarmas son.

Shunday qilib, $f(x)$ funksiya $(a;b)$ intervalda biror $F(x)$ boshlang'ich funksiyaga ega bo'lsa, uning qolgan barcha boshlang'ich funksiyalari $\{F(x)+C \mid C\in R\}$ to'plamni tashkil qiladi.

2-ta'rif. $f(x)$ funksiyaning $(a;b)$ intervaldagi boshlang'ich funksiyalari to'plamiga $f(x)$ funksiyaning *aniqmas integrali* deyiladi va $\int f(x)dx$ kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\int f(x)dx = F(x) + C, \qquad (1.1)$$

bu yerda $f(x)$ – *integral ostidagi funksiya*, $f(x)dx$ – *integral ostidagi ifoda*; x – *integrallash o'zgaruvchisi*, $\int$ – *integrallash belgisi* deyiladi.

Aniqmas integralni topish, ya'ni berilgan funksiyaning boshlang'ich funksiyalari to'plamini aniqlash masalasi *funksiyani integrallash* deb yuritiladi.

Demak, funksiyani integrallash amali funksiyani differensiallashga teskari amal bo'ladi.

Berilgan $f(x)$ funksiya qachon boshlang'ich funksiyaga ega bo'ladi degan savolga quyidagi teorema javob beradi (teoremani isbotsiz keltiramiz).

1-teorema. Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa u holda u bu kesmada uzluksiz bo'lgan boshlang'ich funksiyaga ega bo'ladi.

Ko'p hollarda $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'ladigan $(a;b)$ interval ko'rsatilmaydi. Bunday holda $(a;b)$ interval sifatida $f(x)$ funksiyaning aniqlanish sohasi tushuniladi. Shu sababli bundan keyin integral ostidagi funksiyalar uzluksiz va (1.1) formula ma'noga ega deb hisoblaymiz.

Masalan, $f(x) = \frac{1}{x}$ funksiya $(-\infty;0)$ va $(0;\infty)$ intervalda uzluksiz.

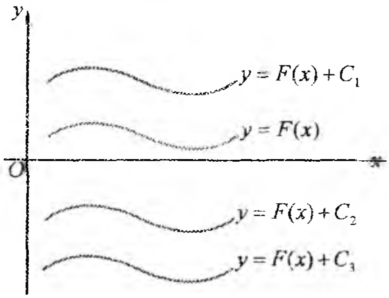
Shu sababli uning aniqmas integrali deb

$$\int \frac{dx}{x} = \begin{cases} \ln x + C, & x > 0, \\ \ln(-x) + C, & x < 0 \end{cases} = \ln|x| + C \quad (x \neq 0)$$

funksiya tushuniladi.

Boshlang'ich funksiyaning grafigi *integral egri chiziq* deb ataladi.

Aniqmas integral *geometrik jihatdan* ixtiyoriy C o'zgarmasga bog'liq bo'lgan barcha integral egri chiziqlar to'plamini ifodalaydi. Agar $F(x)$ funksiyaning grafigi integral egri chiziq bo'lsa, boshqa



1-shakl.

integral egri chiziqlar uni Oy o'qi bo'yicha parallel ko'chirish yordamida hosil qilinadi (1-shakl).

7.1.2. Aniqmas integralning xossalari

Aniqmas integral quyidagi xossalarga ega.

1°. Aniqmas integralning hosilasi (differensiali) integral ostidagi funksiyaga (ifodaga) teng:

$$(\int f(x)dx)' = f(x). \quad (d\int f(x)dx = f(x)dx).$$

Isboti. $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi, ya'ni $F'(x) = f(x)$ bo'lsin.

U holda

$$(\int f(x)dx)' = (F(x)+C)' = F'(x)+0 = f(x).$$

$$(d\int f(x)dx = d(F(x)+C) = dF(x)+dC = F'(x)dx = f(x)dx).$$

Bu xossa *integral amali to'g'ri bajarilganligini differensiallash orqali tekshirish* imkonini beradi.

Masalan, $\int(3x^2+5)dx = x^3+5x+C$ to'g'ri, chunki $(x^3+5x+C)' = 3x^2+5$.

2°. Funksiya differentsialining aniqmas integrali shu funksiya bilan o'zgarmas sonning yig'indisiga teng:

$$\int dF(x) = F(x)+C.$$

Isboti. $F'(x) = f(x)$ bo'lsin.

U holda

$$\int dF(x) = \int F'(x)dx = \int f(x)dx = F(x)+C.$$

3°. Ozgarmas ko'paytuvchini aniqmas integral belgisidan tashqariga chiqarish mumkin:

$$\int kf(x)dx = k\int f(x)dx, \quad k = const, k \neq 0.$$

Isboti. $F'(x) = f(x)$ bo'lsin.

Bundan

$$\int kf(x)dx = \int kF'(x)dx = \int (kF(x))'dx = k(F(x)+C) = kF(x)+C_1 = k\int f(x)dx$$

($C_1 = kC$ deb olindi).

4°. Chekli sondagi funksiyalar algebraik yig'indisining aniqmas integrali shu funksiyalar aniqmas integrallarining algebraik yig'indisiga teng:

$$\int(f(x)\pm g(x))dx=\int f(x)dx\pm\int g(x)dx.$$

Isboti. $F'(x)=f(x),\ G'(x)=g(x)$ bo'lsin.

U holda

$$\int(f(x)\pm g(x))dx=\int(F'(x)\pm G'(x))dx=$$

$$=\int(F(x)\pm G(x))'dx=\int d(F(x)\pm G(x))=$$

$$=F(x)\pm G(x)+C=(F(x)+C_1)\pm(G(x)+C_2)=$$

$$=\int f(x)dx\pm\int g(x)dx,\ C_1\pm C_2=C.$$

5°. Agar $\int f(x)dx=F(x)+C$ bo'lsa, u holda x ning istalgan differensiallanuvchi funksiyasi $u=u(x)$ uchun $\int f(u)du=F(u)+C$ bo'ladi.

Isboti. x erkli o'zgaruvchi, $f(x)$ uzluksiz funksiya, $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsin.

U holda $\int f(x)dx=F(x)+C$ bo'ladi.

$u=\varphi(x)$ bo'lsin, bu yerda $\varphi(x)$ – uzluksiz hosilaga ega bo'lgan funksiya.

Birinchi differensialning invariantlik xossasiga ko'ra,

$$dF(u)=F'(u)du=f(u)du$$

bo'ladi.

Bundan

$$\int f(u)du=\int d(F(u))=F(u)+C.$$

Bu xossa integrallash formulasining invariantligi xossasi deyiladi. Demak, aniqmas integral integrallash o'zgaruvchisi erkli o'zgaruvchi yoki erkli o'zgaruvchining uzluksiz hosilaga ega bo'lgan ixtiyoriy funksiyasi bo'lishidan qat'iy nazar bir xil formula bilan topiladi.

7.1.3. Asosiy integrallar jadvali

Integrallash amali differensiallash amaliga teskari amal bo'lgani uchun asosiy integrallar jadvalini differensial hisobning mos formulalarini (differensiallar jadvalini) qo'llash va aniqmas integralning xossalaridan foydalanish orqali hosil qilish mumkin.

Masalan, $d(\sin u)=\cos u du$ ekanidan $\int \cos u du=\int d(\sin u)=\sin u+C$.

Quyida keltiriladigan integrallar *asosiy integrallar jadvali* deyiladi.

Asosiy integrallar jadvali.

1. $\int u^{\alpha} du=\dfrac{u^{\alpha+1}}{\alpha+1}+C,\ (\alpha \neq -1)$;	2. $\int \dfrac{du}{u}=\ln\lvert u\rvert = C$;
3. $\int a^{u} du=\dfrac{a^{u}}{\ln a}+C,\ (0<a\neq 1)$;	4. $\int e^{u} du=e^{u}+C$;
5. $\int \sin u du=-\cos u+C$;	6. $\int \cos u du=\sin u+C$;
7. $\int tgu du=-\ln\lvert \cos u\rvert = C$;	8. $\int ctgu du=\ln\lvert \sin u\rvert = C$;
9. $\int \dfrac{du}{\cos^{2} u}=tgu+C$;	10. $\int \dfrac{du}{\sin^{2} u}=-ctgu+C$;
11. $\int \dfrac{du}{\sin u}=\ln\left\lvert tg\dfrac{u}{2}\right\rvert+C$;	12. $\int \dfrac{du}{\cos u}=\ln\left\lvert tg\left(\dfrac{u}{2}+\dfrac{\pi}{4}\right)\right\rvert+C$;
13. $\int \dfrac{du}{\sqrt{a^{2}-u^{2}}}=\arcsin\dfrac{u}{a}+C$;	14. $\int \dfrac{du}{\sqrt{u^{2}\pm a^{2}}}=\ln\left\lvert u+\sqrt{u^{2}\pm a^{2}}\right\rvert+C$.
15. $\int \dfrac{du}{a^{2}+u^{2}}=\dfrac{1}{a} arctg\dfrac{u}{a}+C$;	16. $\int \dfrac{du}{u^{2}-a^{2}}=\dfrac{1}{2a}\ln\left\lvert\dfrac{u-a}{u+a}\right\rvert+C$;
17. $\int shudu=chu+C$;	18. $\int chudu=shu+C$;
19. $\int \dfrac{du}{ch^{2}u}=thu+C$;	20. $\int \dfrac{du}{sh^{2}u}=-cthu+C$.

Asosiy integrallar jadvalida integrallash o‘zgaruvchisi u erkli o‘zgaruvchi yoki erkli o‘zgaruvchining funksiyasi (5- xossaga ko‘ra) bo‘lishi mumkin.

Jadvalda keltirilgan formulalarning to‘g‘riligiga uning o‘ng tomonini differensiallash va bu differensialning formula chap tomonidagi integral ostidagi ifodaga teng bo‘lishini tekshirish orqali ishonch hosil qilish mumkin.

Bu integrallardan birining, masalan 13-formulaning to‘g‘riligini ko‘rsatamiz:

$$d\left(\arcsin\frac{u}{a}+C\right)=\frac{1}{\sqrt{1-\left(\frac{u}{a}\right)^{2}}}\cdot\frac{1}{a}du=\frac{du}{\sqrt{a^{2}-u^{2}}}.$$

7.1.4. Integrallash usullari

Bevosita integrallash usuli

Integral ostidagi funksiyada (yoki ifodada) almashtirishlar bajarish va aniqmas integralning xossalarini qo'llash orqali berilgan integralni bir yoki bir nechta jadval integraliga keltirib integrallash usuliga *bevosita integrallash usuli* deyiladi.

Misollar:

$$1)\ \int\left(5\sin x-\frac{2}{x^2+1}+x^3\right)dx=5\int\sin xdx-2\int\frac{dx}{x^2+1}+\int x^3dx=$$

$$=-5\cos x-2arctgx+\frac{x^4}{4}+C;$$

$$2)\ \int\frac{\cos 2x}{\cos^2 x\sin^2 x}dx=\int\frac{\cos^2 x-\sin^2 x}{\cos^2 x\sin^2 x}dx=\int\left(\frac{1}{\sin^2 x}-\frac{1}{\cos^2 x}\right)dx=$$

$$=\int\frac{dx}{\sin^2 x}-\int\frac{dx}{\cos^2 x}=-ctgx-tgx+C=-\frac{2}{\sin 2x}+C;$$

$$3)\ \int\frac{x^4}{1+x^2}dx=-\int\frac{1-x^4-1}{1+x^2}dx=-\int(1-x^2)dx+\int\frac{dx}{1+x^2}=$$

$$=-\int dx+\int x^2dx+\int\frac{dx}{1+x^2}=-x+\frac{x^3}{3}+arctgx+C.$$

Berilgan integralni jadval integrallariga keltirishda differensialning quyidagi almashtirishlari («differensial amali ostiga kiritish» jarayoni) qo'llaniladi:

$$du=d(u+a),\ a-\text{son};\quad du=\frac{1}{a}d(au);\quad udu=\frac{1}{2}d(u^2);\quad \cos udu=d(\sin u);$$

$$\sin udu=-d(\cos u);\quad \frac{1}{u}du=d(\ln u);\quad \frac{1}{\cos^2 u}du=d(tgu).$$

Umuman olganda, $f'(u)du=d(f(u))$.

Bu formuladan integrallarni topishda ko'p foydalaniladi.

Misollar:

$$1)\ \int\frac{dx}{16+9x^2}=\frac{1}{3}\int\frac{d(3x)}{16+(3x)^2}==\frac{1}{3}\cdot\frac{1}{4}arctg\frac{3x}{4}+C=\frac{1}{12}arctg\frac{3x}{4}+C;$$

$$2)\ \int\frac{\cos x+\sin x}{\sin x-\cos x}dx=\int\frac{d(\sin x-\cos x)}{\sin x-\cos x}=\ln|\sin x-\cos x|+C.$$

O'rniga qo'yish (o'zgaruvchini almashtirish) usuli

Ko'p hollarda integraldagi o'zgaruvchini almashtirish uni bevosita integrallashga olib keladi. Integrallashning bu usuli *o'rniga qo'yish* (*o'zgaruvchini almashtirish*) *usuli* deb yuritiladi. Bu usul quyidagi teoremaga asoslanadi.

2-teorema. Biror T oraliqda aniqlangan va differensiallanuvchi $x=\varphi(t)$ funksiyaning qiymatlar sohasi X oraliqdan iborat bo'lib, X da $f(x)$ funksiya aniqlangan va uzluksiz, ya'ni T oraliqda $f(\varphi(t))$ murakkab funksiya aniqlangan va uzluksiz bo'lsin. U holda

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt \qquad (1.2)$$

bo'ladi.

Isboti. X oraliqda $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ichi bo'lsin. U holda $F(\varphi(t))$ murakkab funksiya T oraliqda aniqlangan, differensiallanuvchi hamda

$$F_t'(\varphi(t))=F_x'(\varphi(t))\varphi'(t)=f(\varphi(t))\varphi'(t)$$

bo'ladi.

Bundan

$$\int f(\varphi(t))\varphi'(t)dt=\int F_t'(\varphi(t))dt=F(\varphi(t))+C=$$

$$=F(x)+C\Big|_{x=\varphi(t)}=\int f(x)dx\Big|_{x=\varphi(t)}$$

hisobga olinsa,

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt.$$

(1.2) formula *aniqmas integralda o'zgaruvchini almashtirish* formulasi deb yuritiladi.

Ayrim hollarda $t=\varphi(x)$ o'rniga qo'yish bajarishga to'g'ri keladi. U holda $\int f(\varphi(x))\varphi'(x)dx=\int f(t)dt$ bo'ladi. Demak, (1.2) formula o'ngdan chapga qo'llanishi ham mumkin.

1-misol. $\int x\sqrt{x-3}dx$ integralni toping.

Yechish. $\sqrt{x-3}=t$ almashtirish bajaramiz.
U holda $x=t^2+3,\ dx=2tdt.$

Shu sababli

$$\int x\sqrt{x-3}dx=\int(t^2+3)\,t\cdot 2tdt=2\int(t^4+3t^2)dt=$$

$$=2\int t^4dt+6\int t^2dt=2\cdot\frac{t^5}{5}+6\cdot\frac{t^3}{3}+C=\frac{2}{5}\sqrt{(x-3)^5}+2\sqrt{(x-3)^3}+C.$$

2- misol. $\int\frac{\sqrt{1+\ln x}}{x\ln x}dx$ integralni toping.

Yechish. $1+\ln x=t^2$ bo‘lsin.

Bundan

$$\ln x=t^2-1,\quad \frac{dx}{x}=2tdt.$$

U holda (1.2) formulaga ko‘ra,

$$\int\frac{\sqrt{1+\ln x}}{x\ln x}dx=\int\frac{t\cdot 2tdt}{t^2-1}=2\int\frac{t^2dt}{t^2-1}=$$

$$=2\int\left(1+\frac{1}{t^2-1}\right)dt=2\left(t+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right)+C=$$

$$=2t+\ln\left|\frac{(t-1)^2}{t^2-1}\right|+C=2\sqrt{1+\ln x}+\ln\left|\frac{(\sqrt{1+\ln x})^2}{1+\ln x-1}\right|+C=$$

$$=2\sqrt{1+\ln x}+2\ln\left|\sqrt{1+\ln x}-1\right|-\ln\left|\ln x\right|+C.$$

3- misol. $\int\sqrt{1+\cos^2 x}\sin 2xdx$ integralni toping.

Yechish. $1+\cos^2 x=t^2$ deymiz.

Bundan

$$-2\cos x\sin xdx=2tdt \text{ yoki } \sin 2xdx=-2tdt.$$

U holda

$$\int\sqrt{1+\cos^2 x}\sin 2xdx=\int t(-2t)dt=$$

$$=-2\cdot\frac{t^3}{3}+C=-\frac{2}{3}\sqrt{(1+\cos^2 x)^3}+C.$$

Izoh. Ayrim hollarda integrallashning o‘zgaruvchini almashtirish usuli takroran qo‘llaniladi, ya’ni bunda bajarilgan o‘rniga qo‘yishdan so‘ng shunday integral hosil bo‘ladiki, bu integralni boshqa o‘rniga qo‘yish orqali soddalashtirish yoki jadval integraliga keltirish mumkin bo‘ladi.

Bo'laklab integrallash usuli ikki funktsiya ko'paytmasining differensiali formulasiga asoslanadi.

3-teorema. $u(x)$ va $v(x)$ funksiyalar qandaydir X oraliqda aniqlangan va differentsiallanuvchi bo'lib, $u'(x)v(x)$ funksiya bu oraliqda boshlang'ich funksiyaga ega, ya'ni $\int u'(x)v(x)dx$ integral mavjud bo'lsin. U holda X oraliqda $u(x)v'(x)$ funksiya boshlang'ich funksiyaga ega va

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \tag{1.3}$$

bo'ladi.

Isboti. $(u(x)v(x))' = u'(x)v(x) + v'(x)u(x)$ tenglikdan

$$u(x)v'(x) = (u(x)v(x))' - v(x)u'(x).$$

$(u(x)v(x))'$ va $u'(x)v(x)$ funksiyalar X intervalda boshlang'ich funksiyaga ega bo'lgani uchun $v'(x)u(x)$ ham X intervalda boshlang'ich funksiyaga ega bo'ladi. Oxirgi tenglikning chap va o'ng tomonlarini integrallasak, (1.3) formula kelib chiqadi.

(1.3) formulaga *aniqmas integralni bo'laklab integrallash* formulasi deyiladi.

Ma'lumki,

$$v'(x)dx = dv, \quad u'(x)dx = du.$$

Demak, (1.3) formulani

$$\int udv = uv - \int vdu \tag{1.4}$$

ko'rinishda yozish mumkin.

Bo'laklab integrallash usulining mohiyati berilgan integralda integral ostidagi $f(x)dx$ ifodani udv ko'paytma shaklida tasvirlash va (1.4) formulani qo'llagan holda berilgan $\int udv$ integralni oson integrallanadigan $\int vdu$ integral bilan almashtirib topishdan iborat.

Bo'laklab integrallash orqali topiladigan integrallarning, asosan, uchta guruhini ajratish mumkin:

1) $\int P(x)arctgxdx$, $\int P(x)arcctgxdx$, $\int P(x)\ln xdx$, $\int P(x)\arcsin xdx$, $\int P(x)\arccos xdx$ (bu yerda $P(x)$ – ko'phad) ko'rinishdagi 1-guruh

integrallar. Bunda $dv = P(x)dx$ deb, qolgan ko‘paytuvchilar esa u bilan belgilanadi;

2) $\int P(x)e^{kx}dx$, $\int P(x)\sin kxdx$, $\int P(x)\cos kxdx$ ko‘rinishdagi 2-guruh integrallar. Bunda $u = P(x)$ deb, qolgan ko‘paytuvchilar dv deb olinadi;

3) $\int e^{kx}\sin kxdx$, $\int e^{kx}\cos kxdx$ ko‘rinishdagi 3-guruh integrallar bo‘laklab integrallash ormulasini takroran qo‘llash orqali topiladi.

4- misol. $\int arctgxdx$ integralni toping.

Yechish. $\int arctgxdx = \left| \begin{array}{c} u = arctgx, \ du = \dfrac{dx}{1+x^2}, \\ dv = dx, \ v = x \end{array} \right| = xarctgx - \int \dfrac{x}{1+x^2}dx =$

$$= xarctgx - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}dx = xarctgx - \frac{1}{2}\ln|1+x^2| + C.$$

5- misol. $\int xe^x dx$ integralni toping.

Yechish.

$$\int xe^x dx = \left| \begin{array}{c} u = x, \ du = dx \\ dv = e^x dx, \ v = e^x \end{array} \right| = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C.$$

6- misol. $I = \int \sin xe^{2x}dx$ integralni toping.

Yechish.

$$I = \int \sin xe^{2x}dx = \left| \begin{array}{c} u = e^{2x}, \ du = 2e^{2x}dx \\ dv = \sin xdx, \ v = -\cos x \end{array} \right| = -e^{2x}\cos x + 2\int e^{2x}\cos xdx =$$

$$= \left| \begin{array}{c} u = e^{2x}, \ du = 2e^{2x}dx \\ dv = \cos xdx, \ v = \sin x \end{array} \right| = -e^{2x}\cos x + 2(^{2x}\sin x - 2\int e^{2x}\sin xdx) =$$

$$= e^{2x}(2\sin x - \cos x) - 4I.$$

Bundan

$$I = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C.$$

Ko'rsatilgan uchta guruh bo'laklab integrallanadigan barcha integrallarni o'z ichiga olmaydi. Masalan, $\int\frac{xdx}{\cos^2 x}$ integral yuqorida keltirilgan integrallar guruhlariga kirmaydi, lekin uni bo'laklab integrallash usuli bilan topish mumkin:

$$\int\frac{xdx}{\cos^2 x}=\left|\begin{array}{c} u=x,\ du=dx \\ dv=\dfrac{dx}{\cos^2 x},\ v=tgx \end{array}\right|=xtgx-\int tgxdx=xtgx-\ln|\cos x|+C.$$

7.1.5. Mashqlar

1. Berilgan integrallarni aniqmas integralning xossalari va integrallar jadvalini qo'llab toping:

1) $\int\left(5tgx-\frac{2}{x^2+1}+x^4\right)\cdot dx$;
2) $\int\frac{x^2-7}{x+3}dx$;

3) $\int\frac{\sqrt[3]{x}-x^2e^x-x}{x^2}dx$;
4) $\int\left(\frac{3}{1+x^2}-\frac{2}{\sqrt{1-x^2}}\right)\cdot dx$;

5) $\int\frac{2\cdot 3^x-3\cdot 2^x}{3^x}dx$;
6) $\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx$;

7) $\int e^x\left(1+\frac{e^{-x}}{\cos^2 x}\right)dx$;
8) $\int\frac{1-\sin^3 x}{\sin^2 x}dx$;

9) $\int ctg^2 xdx$;
10) $\int\frac{dx}{\cos^2 x-\cos 2x}$;

11) $\int\frac{dx}{25+4x^2}$;
12) $\int\frac{dx}{\sqrt{3+4x-2x^2}}$.

2. Berilgan integrallarni differensial ostiga kiritish usuli bilan toping:

1) $\int\frac{tgx}{\cos^2 x}dx$;
2) $\int\cos^2 x\sin xdx$;

3) $\int\frac{\sqrt[3]{arctg^5 2x}}{1+4x^2}dx$;
4) $\int\frac{\sqrt[7]{\ln^3(x+5)}}{x+5}dx$;

5) $\int e^{\sin x}\cos xdx$;
6) $\int e^{-x^3}x^2dx$;

7) $\int\frac{\cos x}{\sin^5 x}dx$;
8) $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx$;

9) $\int\frac{e^x dx}{\sqrt{4-e^{2x}}}$;
10) $\int\frac{dx}{\sin^2 4x\sqrt[3]{ctg^2 4x}}$.

3. Berilgan integrallarni o‘rniga qo‘yish usuli bilan toping:

1) $\int \frac{e^x-1}{e^x+1}dx;$ 2) $\int \frac{x^5dx}{x^6+2}$

3) $\int \sqrt{16-x^2}dx;$ 4) $\int \frac{x^3dx}{\sqrt[3]{x^4+4}};$

5) $\int x^2\sqrt{x^3+3}dx;$ 6) $\int \frac{\cos 2x dx}{1+\sin x\cos x};$

7) $\int \frac{dx}{(\arcsin x)^3\sqrt{1-x^2}};$ 8) $\int \frac{4x-5}{x^2+5}dx;$

9) $\int \frac{dx}{\sqrt{5-4x-x^2}};$ 10) $\int \frac{dx}{\sqrt{3x^2-2x-1}};$

11) $\int x(2x+7)^{10}dx;$ 12) $\int \frac{dx}{\sqrt{x(1-x)}};$

13) $\int \frac{e^{2x}dx}{e^{4x}-9};$ 14) $\int \frac{\ln 2x}{\ln 4x}\cdot\frac{dx}{x}.$

4. Integrallarni bo‘laklab integrallash usuli bilan toping:

1) $\int x\,arctgx\,dx$ 2) $\int \arcsin x dx;$

3) $\int x\ln x dx;$ 4) $\int x^2e^x dx.$

5) $\int x3^x dx;$ 6) $\int x\sin 2x dx;$

7) $\int x\ln(x+1)dx;$ 8) $\int \frac{x\sin x dx}{\cos^3 x};$

9) $\int \sin\ln x dx;$ 10) $\int e^{4x}\sin 4x dx.$

11) $\int \frac{\ln tgx dx}{\cos^2 x};$ 12) $\int \frac{\ln arctgx dx}{1+x^2}.$

5. Integrallarni toping:

1) $\int x^3\sqrt[3]{1+x^2}dx;$ 2) $\int \sin 3x\sin 5x dx;$

3) $\int e^x\cos^2(e^x)dx;$ 4) $\int \frac{xdx}{e^{3x}};$

5) $\int \frac{1-tgx}{1+tgx}dx;$ 6) $\int \frac{\ln x dx}{x(1-\ln^2 x)};$

7) $\int \frac{dx}{(x+1)(2x-3)};$ 8) $\int \frac{dx}{x(4+\ln^2 x)};$

9) $\int \frac{x dx}{\cos^2 x}$; 10) $\int \frac{dx}{x\sqrt{2x-9}}$;

11) $\int \frac{e^{arctgx} dx}{1+x^2}$; 12) $\int \frac{e^{2x} dx}{\sqrt{3+e^{2x}}}$;

13) $\int \sin^2 \frac{3x}{2} dx$; 14) $\int x tg^2 x^2 dx$;

15) $\int x^2 \ln^2 x dx$; 16) $\int \frac{1-2\cos x}{\sin^2 x} dx$

7.2. RATSIONAL FUNKSIYALARNI INTEGRALLASH

7.2.1. Sodda kasrlarni integrallash

6.2 banddan bilamizki, quyidagi ratsional kasrlarga *sodda kasrlar* deyiladi:

$I.\ \frac{A}{x-\alpha}$;

$II.\ \frac{A}{(x-\alpha)^k},\ (k \geq 2,\ k \in N)$;

$III.\ \frac{Mx+N}{x^2+px+q},\ (p^2-4q<0)$;

$IV.\ \frac{Mx+N}{(x^2+px+q)^s},\ (s \geq 2,\ s \in N,\ p^2-4q<0)$,

bu yerda $A,\ M,\ N,\ \alpha,\ p,\ q$ – haqiqiy sonlar.

I va II turdagi sodda kasrlar jadval integrallari orqali topiladi:

$$\int \frac{A dx}{x-\alpha} = A\int \frac{d(x-\alpha)}{x-\alpha} = A\ln|x-\alpha| + C; \qquad (2.1)$$

$$\int \frac{A dx}{(x-\alpha)^k} = A\int (x-\alpha)^{-k} d(x-\alpha) = A\frac{(x-\alpha)^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-\alpha)^{k-1}} + C. \quad (2.2)$$

III turdagi sodda kasrni qaraymiz.

$\int \frac{Mx+N}{x^2+px+q} dx$ integralining suratida kasrning maxrajidan olingan hosila $(x^2+px+q)' = 2x+p$ ni ajratamiz va natijani integrallaymiz:

$$\int\frac{Mx+N}{x^2+px+q}dx=\int\frac{\frac{M}{2}(2x+p)+N-\frac{Mp}{2}}{x^2+px+q}dx=\frac{M}{2}\int\frac{2x+p}{x^2+px+q}dx+$$

$$+\left(N-\frac{Mp}{2}\right)\int\frac{dx}{x^2+px+q}=\frac{M}{2}J_1+\left(N-\frac{Mp}{2}\right)J_2$$

yoki

$$\int\frac{Mx+N}{x^2+px+q}dx=\frac{M}{2}J_1+\left(N-\frac{Mp}{2}\right)J_2.$$

Bu tenglikning o'ng tomonidagi integrallardan birinchisi

$$J_1=\ln|x^2+px+q|.$$

Ikkinchi integral maxrajida to'liq kvadrat ajratamiz va integralni quyidagicha hisoblaymiz:

$$J_2=\int\frac{dx}{x^2+px+q}=\int\frac{d\left(x+\frac{p}{2}\right)}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}=\frac{2}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}},$$

bunda $4q-p^2>0$, chunki $D<0$.

Natijada quyidagiga ega bo'lamiz:

$$\int\frac{Mx+N}{x^2+px+q}dx=\frac{M}{2}\ln|x^2+px+q|+\frac{2N-Mp}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C. \qquad (2.3)$$

1- misol. $I=\int\frac{5x+11}{x^2+6x+13}dx$ integralni toping.

Yechish. $I=\int\frac{\frac{5}{2}(2x+6)+11-\frac{5}{2}\cdot 6}{x^2+6x+13}dx=\frac{5}{2}\int\frac{(2x+6)dx}{x^2+6x+13}-4\int\frac{dx}{x^2+6x+13}=$

$$=\frac{5}{2}\ln|x^2+6x+13|-4J.$$

Bu yerda

$$J=\int\frac{dx}{(x+3)^2+4}=\int\frac{d(x+3)}{(x+3)^2+2^2}=\frac{1}{2}arctg\frac{x+3}{2}.$$

Bundan

$$\int \frac{5x+11}{x^2+6x+13}dx = \frac{5}{2}\ln|x^2+6x+13| - 2arctg\frac{x+3}{2} + C.$$

IV turdagi sodda kasrning integralini topamiz:

$$\int \frac{Mx+N}{(x^2+px+q)^s}dx = \frac{M}{2}\int \frac{(2x+p)dx}{(x^2+px+q)^s} +$$

$$+\left(N - \frac{Mp}{2}\right)\int \frac{d\left(x+\frac{p}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right)^s}. \qquad (2.4)$$

Bu tenglikning o‘ng tomonidagi birinchi integral jadvaldagi integralga keltirib, piladi:

$$I = \int \frac{(2x+p)dx}{(x^2+px+q)^s} =$$

$$= \int (x^2+px+q)^{-s} d(x^2+px+q) = \frac{1}{(1-s)(x^2+px+q)^{s-1}}.$$

Ikkinchi integralga (uni I_s bilan belgilaymiz) $\left(x+\frac{p}{2}\right) = t$ almashtirish bajaramiz va $0 < q - \frac{p^2}{4} = a^2$ belgilash kiritamiz.

U holda

$$I_s = \int \frac{d\left(x+\frac{p}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right)^s} = \int \frac{dt}{(t^2+a^2)^s} = \frac{1}{a^2}\int \frac{(t^2+a^2)-t^2}{(t^2+a^2)^s}dt =$$

$$= \frac{1}{a^2}\int \frac{dt}{(t^2+a^2)^{s-1}} - \frac{1}{a^2}\int \frac{t^2 dt}{(t^2+a^2)^s}.$$

Bu tenglikning o‘ng qismidagi birinchi integral I_s ga o‘xshash bo‘lib, unda maxrajning darajasi s dan bir birlikka kichik. Shu sababli, belgilashga ko‘ra, I_{s-1} bo‘ladi. Ikkinchi integralni bo‘laklab integrallaymiz:

$$\int\frac{t^2dt}{(t^2+a^2)^s}=\frac{1}{2}\int\frac{t\cdot 2tdt}{(t^2+a^2)^s}=$$

$$=\frac{1}{2}\left(\frac{-t}{(s-1)(t^2+a^2)^{s-1}}+\frac{1}{s-1}\int\frac{dt}{(t^2+a^2)^{s-1}}\right)=$$

$$=-\frac{t}{2(s-1)(t^2+a^2)^{s-1}}+\frac{1}{2(s-1)}I_{s-1}.$$

Demak, I_s integralni hisoblash uchun s darajani pasaytirish formulasini hosil qilamiz:

$$I_s=\frac{1}{a^2}I_{s-1}+\frac{t}{2a^2(s-1)(t^2+a^2)^{s-1}}-\frac{1}{2a^2(s-1)}=$$

$$=\frac{t}{2a^2(s-1)(t^2+a^2)^{s-1}}+\frac{2s-3}{2a^2(s-1)}I_{s-1}. \tag{2.5}$$

Shunday qilib, (2.5) formula bo'yicha I_s integralni topamiz, keyin I_s dagi barcha t ni $x+\frac{p}{2}$ bilan almashtirib va I, I_s integrallarni (2.4) tenglikka qo'yib, IV turdagi sodda kasr integralini topish uchun ifoda hosil qilamiz.

(2.5) formula bo'yicha I_s integralni topish indeksi bittaga kichik bolgan I_{s-1} integralni topishga, I_{s-1} integralni topish esa o'z navbatida I_{s-2} integralni topishga keltiriladi va bu jarayon quyidagi jadval integralni topishgacha davom ettiriladi:

$$I_1=\int\frac{dt}{t^2+a^2}=\frac{1}{a}arctg\frac{t}{a}+C.$$

Demak, (2.5) formula orqali I_s dan I_{s-1} ga , so'ngra I_{s-2} o'tiladi va hokazo. Shu sababli bunday formulalar *keltirish* yoki *rekurrent* (*qaytuvchan*) formulalar deyiladi.

2- misol. $\int\frac{2x+5}{(x^2+4x+8)^2}dx$ integralni toping.

Yechish. $\int\frac{2x+4+1}{(x^2+4x+8)^2}dx=\int\frac{2x+4}{(x^2+4x+8)^2}dx+\int\frac{dx}{(x^2+4x+8)^2}=$

$$=-\frac{1}{x^2+4x+8}+\int\frac{d(x+2)}{[(x+2)^2+4]^2}=-\frac{1}{x^2+4x+8}+\int\frac{dt}{t^2+a^2},$$

bu yerda $t=x+2,\ a=2.$

(2.5) integraldan foydalanib ayrim hisoblashlardan so'ng

$$I_2=\frac{t}{2a^2(t^2+a^2)}+\frac{1}{2a^3}arctg\frac{t}{a}=\frac{x+2}{8(x^2+4x+8)}+\frac{1}{16}arctg\frac{x+2}{2}$$

ekanligini topamiz.

Demak,

$$\int\frac{2x+5}{(x^2+4x+8)^2}dx=$$

$$=-\frac{1}{x^2+4x+8}+\frac{x+2}{8(x^2+4x+8)}+\frac{1}{16}arctg\frac{x+2}{2}+C=$$

$$=\frac{x-6}{8(x^2+4x+8)}+\frac{1}{16}arctg\frac{x+2}{2}+C.$$

7.2.2. Ratsional kasrlarni integrallash

6.2 banddan va yuqorida aytilganlardan kelib chiqadiki, $R(x)=\frac{Q_m(x)}{P_n(x)}$ ratsional kasr funksiyani integrallash quyidagi tartibda amalga oshiriladi:

1) berilgan ratsional kasrning to'g'ri yoki noto'g'ri kasr ekanini tekshirish; agar kasr noto'g'ri bo'lsa, kasrdan butun qismini ajratish;
2) to'g'ri kasrning maxrajini ko'paytuvchilarga ajratish;
3) to'g'ri kasrni sodda kasrlar yig'indisiga yoyish;
4) hosil bo'lgan ko'phad va sodda kasrlar yig'indisini integrallash.

3- misol. $I=\int\frac{x^4+6}{x^3-2x^2+2x}dx$ integralni toping.

Yechish. $R(x)=\frac{x^4+6}{x^3-2x^2+2x}$ noto'g'ri kasr, chunki

$m=4,\ n=3\ (m>n).$

Bu kasrning suratni maxrajga bo'lish orqali kasrdan butun qismini ajratamiz:

$$\begin{array}{r|l} -\begin{array}{l} x^4+6 \\ x^4-2x^3+2x^2 \end{array} & \dfrac{x^3-2x^2+2x}{x+2} \\ \hline -\begin{array}{l} 2x^3-2x^2+6 \\ 2x^3-4x^2+4x \end{array} & \\ \hline 2x^2-4x+6 & \end{array}$$

Bundan

$$R(x)=x+2+\frac{2x^2-4x+6}{x^3-2x^2+2x}.$$

To'g'ri kasrning maxrajini ko'paytuvchilarga ajratamiz:

$$x^3-2x^2+2x=x(x^2-2x+2).$$

To'g'ri kasrni sodda kasrlarga yoyilmasi ko'rinishida yozamiz:

$$\frac{2x^2-4x+6}{x(x^2-2x+2)}=\frac{A}{x}+\frac{Mx+N}{x^2-2x+2}.$$

Yoyilmaning noma'lum koeffitsiyentlarini topamiz:

$$2x^2-4x+6=A(x^2-2x+2)+Mx^2+Nx,$$

$$\begin{cases} x^2: \; A+M=2, \\ x^1: \; -2A+N=-4, \\ x^0: \; 2A=6. \end{cases}$$

Bundan $A=3,\; M=-1,\; N=2.$

Shunday qilib,

$$R(x)=x+2+\frac{3}{x}+\frac{-x+2}{x^2-2x+2}.$$

Ko'phad va sodda kasrlar yig'indisini integrallaymiz:

$$I=\int(x+2)dx+\int\frac{3dx}{x}+\int\frac{-x+2}{x^2-2x+2}dx=\frac{x^2}{2}+2x+3\ln|x|-$$

$$-\int\frac{\frac{1}{2}(2x-2)+1-2}{x^2-2x+2}dx=\frac{x^2}{2}+2x+3\ln|x|-\frac{1}{2}\int\frac{2x-2}{x^2-2x+2}dx+$$

$$\frac{x^2}{2}+2x+3\ln|x|-\frac{1}{2}\ln|x^2-2x+2|+arctg(x-1)+C.$$

7.2.3. Mashqlar

1. Integrallarni toping:

1) $\int \frac{2x+3}{(x-2)(x+5)}dx;$ 2) $\int \frac{xdx}{(x+1)(2x+1)};$

3) $\int \frac{xdx}{(x+1)(x+2)(x+3)};$ 4) $\int \frac{8xdx}{(x+1)(x^2+6x+5)};$

5) $\int \frac{3x^2+2x-3}{x(x-1)(x+1)}dx;$ 6) $\int \frac{x^3-1}{4x^3-x}dx;$

7) $\int \frac{2x^3+2x^2+4x+3}{x^3+x^2}dx;$ 8) $\int \frac{2+5x^3}{x(x^2-5x+4)}dx;$

9) $\int \frac{x^3-3}{x^3-2x^2-x+2}dx;$ 10) $\int \frac{dx}{x^2(x^2+1)};$

11) $\int \frac{dx}{x(1+x^2)};$ 12) $\int \frac{dx}{1+x^3};$

13) $\int \frac{x^4+3x^3+2x^2+x+1}{x^2+x+1}dx;$ 14) $\int \frac{x^9dx}{x^4-1};$

15) $\int \frac{dx}{x^4-1};$ 16) $\int \frac{dx}{(x^2+9)^3};$

17) $\int \frac{3x+5}{(x^2+2x+2)^2}dx;$ 18) $\int \frac{x^4+2x^2+x}{(x-1)(x^2+4)^2}dx;$

19) $\int \frac{dx}{(x^2+4x+5)(x^2+4x+13)};$ 20) $\int \frac{dx}{(x+1)^2(x^2+1)};$

21) $\int \frac{dx}{(x^2+1)^4};$ 22) $\int \frac{2x-1}{(x^2-2x+5)^2}dx;$

23) $\int \frac{2x+3}{(x^2-3x+3)^2}dx;$ 24) $\int \frac{3x^2-10x+12}{x^4+13x^2+36}dx.$

7.3. TRIGONOMETRIK FUNKSIYALARNI INTEGRALLASH

Trigonometrik funksiyalarni integrallash usullaridan ayrimlari bilan tanishamiz. Faqat trigonometrik funksiyalar ustida ratsional amallar (qo‘shish, ayirish, ko‘paytirish va bo‘lish) bajarilgan ifoda berilgan bo‘lsin. Bunday ifodani barcha trigonometrik funksiyalarni $\sin x$ va $\cos x$

funksiyalar orqali ratsional ravishda ifodalash va $R(\sin x, \cos x)$ ko‘rinishga keltirish mumkin.

7.3.1. $\int R(\sin x, \cos x)dx$ ko‘rinishidagi integrallar

$\int R(\sin x, \cos x)dx$ ko‘rinishidagi integralni $tg\frac{x}{2}=t$ almashtirish orqali hamma vaqt t o‘zgaruvchili ratsional funksiyaning integraliga almashtirish, ya’ni *ratsionallashtirish* mumkin. Shu sababli bu almashtirish *universal trigonometrik almashtirish* deyiladi.

Haqiqatan ham, $\int R(\sin x, \cos x)dx$ ifodadan

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}, \quad x = arctgt, \quad dx = \frac{2dt}{1+t^2}$$

tarzdagi o‘rniga qo‘yishlar yordamida t o‘zgaruvchili

$$\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2dt}{1+t^2} = \int R_1(t)dt$$

ratsional funksiya kelib chiqadi.

1- misol. $I = \int\frac{dx}{3\sin x + 2\cos x + 3}$ integralni toping.

Yechish. $tg\frac{x}{2}=t$ deymiz. U holda

$$I = \int\frac{\frac{2dt}{1+t^2}}{3\cdot\frac{2t}{1+t^2}+2\cdot\frac{1-t^2}{1+t^2}+3} = 2\int\frac{dt}{t^2+6t+5} = 2\int\frac{dt}{(t+1)(t+5)} =$$

$$= \int\left(\frac{A}{t+1}+\frac{B}{t+5}\right)dt = A\ln|t+1| + B\ln|t+5| + C.$$

Noma’lum koeffitsiyentlarni aniqlaymiz: $A=\frac{1}{2}, \ B=-\frac{1}{2}$.

Demak,

$$I = \frac{1}{2}(\ln|t+1| - \ln|t+5|) + C = \frac{1}{2}\ln\left|\frac{t+1}{t+5}\right| = \frac{1}{2}\ln\left|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}+5}\right| + C.$$

Universal trigonometrik o'rniga qo'yish natijasida amalda ko'pincha ancha murakkab ratsional funksiyalar hosil bo'lishi mumkin. Bunday hollarda yuqorida keltirilgan integralni topishda quyidagi sodda almashtirishlardan foydalangan ma'qul:

a) agar $R(\sin x, \cos x)$ ifoda $\sin x$ ga nisbatan toq, ya'ni

$$R(-\sin x, \cos x) = -R(\sin x, \cos x)$$

bo'lsa, u holda $\cos x = t$ o'rniga qo'yish bu funksiyani ratsionallashtiradi;

b) agar $R(\sin x, \cos x)$ ifoda $\cos x$ ga nisbatan toq, ya'ni

$$R(\sin x, -\cos x) = -R(\sin x, \cos x)$$

bo'lsa, u holda $\sin x = t$ o'rniga qo'yish orqali bu funksiya ratsionallashtiriladi;

c) agar $R(\sin x, \cos x)$ ifoda $\sin x$ va $\cos x$ larga nisbatan juft, ya'ni

$$R(-\sin x, -\cos x) = R(\sin x, \cos x)$$

bo'lsa, u holda $tgx = t$ o'rniga qo'yish bu funksiyani ratsionallashtiradi. Bunda quyidagi almashtirishlardan foydalaniladi:

$$\sin^2 x = \frac{tg^2 x}{1 + tg^2 x} = \frac{t^2}{1 + t^2}, \quad \cos^2 x = \frac{1}{1 + tg^2 x} = \frac{1}{1 + t^2},$$

$$x = arctgt, \quad dx = \frac{dt}{1 + t^2}.$$

2-misol. $I = \int \frac{\cos x dx}{\sin^2 x - 4\sin x + 5}$ integralni toping.

Yechish. Integral ostidagi funksiya $\cos x$ ga nisbatan toq funksiya. Shu sababli $\sin x = t$ deb olamiz.
U holda

$$I = \int \frac{dt}{t^2 - 4t + 5} = \int \frac{d(t-2)}{(t-2)^2 + 1} = arctg(t-2) + C = arctg(\sin x - 2) + C.$$

3-misol. $I = \int \frac{dx}{1 - 2\sin^2 x}$ integralni toping.

Yechish. Integral ostidagi funksiya $\sin x$ ga nisbatan juft funksiya. Shu sababli $tgx = t$ o'rniga qo'yishdan foydalanamiz.
U holda

$$I = \int \frac{\frac{dt}{1+t^2}}{1 - \frac{2t^2}{1+t^2}} = \int \frac{dt}{1-t^2} = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| = \frac{1}{2}\ln\left|\frac{tgx+1}{tgx-1}\right| + C.$$

7.3.2. $\int \sin^n x\cos^m x dx$ ko‘rinishidagi integrallar

$\int \sin^n x\cos^m x dx$ ko‘rinishidagi integrallar m va n butun sonlarga bog‘liq holda quyidagicha topiladi:

a) $n>0$ va toq bo‘lganida $\cos x=t$ o‘rniga qo‘yish integralni ratsionallashtiradi;

a) $m>0$ va toq bo‘lganida $\sin x=t$ o‘rniga qo‘yish orqali integral ratsionallashtiriladi;

c) m va n sonlar juft va nomanfiy bo‘lganida

$$\sin^2 x=\frac{1-\cos 2x}{2},\quad \cos^2 x=\frac{1+\cos 2x}{2}$$

formulalaridan foydalanib, darajalar pasaytiriladi;

d) $m+n<0$ hamda m va n juft bo‘lganida $tgx=t$ yoki $ctgx=t$ o‘rniga qo‘yishdan foydalaniladi. Bunda $m<0$ va $n<0$ bo‘lsa, suratda $1=(\sin^2 x+\cos^2 x)^k$, bu yerda $k=\frac{|m+n|}{2}-1$, almashtirishdan iborat usul qo‘llab, ratsional funksiyalarni integrallashga keltiriladi;

e) $m,n\le 0$ va ulardan biri toq bo‘lganida $\sin x$ va $\cos x$ lardan qaysi birining darajasi toqligiga qarab, surat va maxrajni shu funksiyaga qo‘shimcha ko‘paytirishdan foydalaniladi.

4-misol. $\int \sin^5 x\cos^2 x dx$ integralni toping.

Yechish. $\int \sin^5 x\cos^2 x dx$ ($n>0$ *va toq*, $\cos x=t$) $=\int \sin^4 x\cos^2 x\sin x dx=$

$$=-\int (1-t^2)^2 t^2 dt=-\int t^2 dt+2\int t^4 dt-\int t^6 dt=-\frac{t^3}{3}+\frac{2t^5}{5}-\frac{t^7}{7}+C=$$

$$=-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+C.$$

5-misol. $\int \sin^4 x\cos^2 x dx$ integralni toping.

Yechish. $\int \sin^4 x\cos^2 x dx$ ($n,m\ge 0$ *va* n,m – *juft*) $=\int (\sin x\cos x)^2\sin^2 x dx=$

$$=\int\left(\frac{\sin^2 2x}{4}\right)\cdot\left(\frac{1-\cos 2x}{2}\right)dx=\frac{1}{8}\int (\sin^2 2x-\sin^2 2x\cos 2x)dx=$$

$$=\frac{1}{8}\int\frac{1-\cos 4x}{2}dx-\frac{1}{16}\int \sin^2 2x d(\sin 2x)=$$

$$=\frac{1}{16}\left(x-\frac{\sin 4x}{4}\right)-\frac{\sin^3 2x}{48}+C=\frac{1}{16}\left(x-\frac{\sin 4x}{4}-\frac{\sin^3 2x}{3}\right)+C.$$

6-misol. $I=\int\frac{dx}{\sin^4 x\cos^2 x}$ integralni toping.

Yechish. Bunda $n=-4,\ m=-2,\ n+m=-6<0,\ k=\frac{|m+n|}{2}-1=2$.

Demak,

$$I=\int\frac{(\sin^2 x+\cos^2 x)^2}{\sin^4 x\cos^2 x}dx=\int\frac{\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x}{\sin^4 x\cos^2 x}dx=$$

$$=\int\frac{dx}{\cos^2 x}+2\int\frac{dx}{\sin^2 x}+\int\frac{\cos^2 x}{\sin^4 x}dx=tgx-2ctgx-\int ctg^2 x d(ctgx)=$$

$$=tgx-2ctgx-\frac{1}{3}ctg^3 x+C.$$

7.3.3. $\int tg^n xdx$ va $\int ctg^n xdx$ ko'rinishidagi integrallar

$\int tg^n xdx$ va $\int ctg^n xdx$ (bu yerda $n>0$ butun son) ko'rinishidagi integrallar mos rasvishda $tgx=t$ va $ctgx=t$ o'rniga qo'yish orqali topiladi.

Bunday integrallarni o'rniga qo'yishlardan foydalanmasdan, bevosita

$$tg^2 x=\frac{1}{\cos^2 x}-1,\ ctg^2 x=\frac{1}{\sin^2 x}-1$$

formulalar yordamida hisoblash ham mumkin.

7-misol. $\int tg^5 xdx$ integralni hisoblang.

Yechish. 1-usul. $\int tg^5 xdx=\left|tgx=t,\ dx=\frac{dt}{1+t^2}\right|=\int\frac{t^5 dt}{1+t^2}=\int t^3 dt-$

$$-\int tdt+\int\frac{tdt}{1+t^2}=\frac{t^4}{4}-\frac{t^2}{2}+\frac{1}{2}\int\frac{d(1+t^2)}{1+t^2}=\frac{t^4}{4}-\frac{t^2}{2}+\frac{1}{2}\ln|1+t^2|+C=$$

$$=\frac{1}{4}tg^4 x-\frac{1}{2}tg^2 x-\frac{1}{2}\ln|\cos^2 x|+C=\frac{1}{4}tg^4 x-\frac{1}{2}tg^2 x-\ln|\cos x|+C.$$

2-usul. $\int tg^5 xdx = \int tg^3 x \cdot tg^2 xdx = \int tg^3 x \cdot \left(\frac{1}{\cos^2 x} - 1\right)dx =$

$$= \int tg^3 x \cdot \frac{dx}{\cos^2 x} - \int tg^3 xdx = \int tg^3 xd(tgx) - \int tgx \cdot \left(\frac{1}{\cos^2 x} - 1\right)dx =$$

$$= \frac{1}{4}tg^4 x - \int tgxd(tgx) - \int tgxdx = \frac{1}{4}tg^4 x - \frac{1}{2}tg^2 x - \ln|\cos x| + C.$$

7.3.4. $\int \sin mx \cos nxdx$, $\int \sin mx \sin nxdx$, $\int \cos mx \cos nxdx$ ko'rinishidagi integrallar

Bu ko'rinishdagi integrallarni hisoblashda

$$\sin mx \cos nx = \frac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right),$$

$$\sin mx \sin nx = \frac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right),$$

$$\cos mx \cos nx = \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$$

trigonometrik formulalardan foydalaniladi.

8-misol. $\int \cos 3x \cdot \cos 5xdx$ integralni toping.

Yechish. $\int \cos 3x \cdot \cos 5xdx = \frac{1}{2}\int (\cos 8x + \cos 2x)dx =$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x\right) + C = \frac{1}{16}(\sin 8x + 4\sin 2x) + C.$$

7.3.5. Mashqlar

1. Berilgan integrallarni toping:

1) $\int \frac{dx}{5+4\sin x}$; 2) $\int \frac{dx}{2\sin x + \sin 2x}$;

3) $\int \frac{dx}{3+5\sin x+3\cos x}$; 4) $\int \frac{dx}{4+2\sin x+3\cos x}$;

5) $\int \frac{\sin xdx}{\sqrt{3-\cos^2 x}}$; 6) $\int \frac{3\cos^3 xdx}{\sin^4 x}$;

7) $\int \frac{\cos^3 x dx}{1+\sin^2 x}$;

8) $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x} dx$;

9) $\int \sin^2 x \cos^4 x dx$;

10) $\int \frac{dx}{\sin x \cos^3 x}$;

11) $\int \frac{dx}{2+3\sin^2 x - 7\cos^2 x}$;

12) $\int ctg^3 2x dx$;

13) $\int \frac{\sin^2 x dx}{1+\cos^2 x}$;

14) $\int \cos 2x \cos 5x dx$;

15) $\int \sin^2 x \cos 3x dx$;

16) $\int \cos x \cos 2x \cos 3x dx$.

7.4. IRRATSIONAL IFODALARNI INTEGRALLASH

Irratsional ifodalarni o'z ichga olgan ayrim integrallarni ko'rib chiqamiz.

7.4.1. $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \ldots\right)dx$ ko'rinishidagi integrallar

$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \ldots\right)dx$ (R – ratsional funksiya, m_1, n_1, m_2, n_2,...– butun sonlar) ko'rinishdagi integrallar $\frac{ax+b}{cx+d} = t^s$ o'rniga qo'yish yordamida ratsional funksiyaning integraliga keltiriladi, bunda $s = EKUK(n_1, n_2, \ldots)$.

Xususan, $\int R\left(x, (ax+b)^{\frac{m_1}{n}}, (ax+b)^{\frac{m_2}{n_2}}, \ldots\right)dx$ integrallar $ax+b=t^s$ o'rniga qo'yish yordamida, $\int R\left(x, x^{\frac{m_1}{n_1}}, x^{\frac{m_2}{n_2}}, \ldots\right)dx$ integrallar esa $x = t^s$ o'rniga qo'yish yordamida t o'zgaruvchili ratsional funksiyaga keltiriladi.

1-misol. $I = \int \frac{x^2 + \sqrt[3]{1+x}}{\sqrt{1+x}} dx$ integralni toping.

Yechish. Bu yerda $EKUK(2,3) = 6$ bo'lgani uchun $1+x=t^6$ tarzda belgilash kiritamiz.

U holda

$$\sqrt{1+x}=t^3,\ \ \sqrt[3]{1+x}=t^2,\ \ dx=6t^5dt.$$

Demak,

$$I=\int\frac{(t^6-1)^2+t^2}{t^3}\cdot 6t^5dt=6\int t^2(t^{12}-2t^6+t^2+1)dt=$$

$$=6\left(\frac{t^{15}}{15}-2\frac{t^9}{9}+\frac{t^5}{5}+\frac{t^3}{3}\right)+C=\frac{2t^3}{15}(3t^{12}-10t^6+9t^2+15)+C=$$

$$=\frac{2\sqrt{1+x}}{15}\left(3(1+x)^2-10(1+x)+9\sqrt[3]{1+x}+15\right)+C.$$

7.4.2. $\int R(x,\sqrt{ax^2+bx+c})dx$ ko‘rinishidagi integrallar

$\int R(x,\sqrt{ax^2+bx+c})dx$ ko‘rinishidagi integrallar *Eylerning uchta o‘rniga qo‘yish usuli* orqali ratsional funksiyalardan olingan integrallarga keltiriladi:

a) $a>0$ bo‘lganida $\sqrt{ax^2+bx+c}=t\pm\sqrt{a}x$ almashtirish orqali integral ostidagi funksiya ratsionallashtiriladi (*Eylerning birinchi o‘rniga qo‘yishi*);

b) $c>0$ bo‘lganida $\sqrt{ax^2+bx+c}=tx\pm\sqrt{c}$ almashtirish yordamida integral ostidagi funksiya ratsionallashtiriladi (*Eylerning ikkinchi o‘rniga qo‘yishi)*;

c) ax^2+bx+c kvadrat uchhad $a(x-x_1)(x-x_2)$ ko‘rinishda ko‘paytuvchilarga ajralganida integral ostidagi funksiya $\sqrt{ax^2+bx+c}=t(x-x_1)$ almashtirish bilan ratsionallashtiriladi (*Eylerning uchinchi o‘rniga qo‘yishi*).

2-misol. $I=\int\frac{dx}{1+\sqrt{x^2+2x+2}}$ integralni toping.

Yechish. Bunda $a>0$. Shu sababli $\sqrt{x^2+2x+2}=t-x$ ko‘rinishdagi o‘rniga qo‘yish bajaramiz.

U holda

$$x^2+2x+2=t^2-2tx+x^2,\ 2x+2tx=t^2-2.$$

Bundan

$$x=\frac{t^2-2}{2(1+t)},\ dx=\frac{t^2+2t+2}{2(1+t)^2},\ 1+\sqrt{x^2+2x+2}=1+t-\frac{t^2-2}{2(1+t)}=\frac{t^2+4t+4}{2(1+t)}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$I=\int\frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)2(1+t)^2}dt=\int\frac{t^2+2t+2}{(1+t)(2+t)^2}dt.$$

Integral ostidagi to'g'ri kasrni sodda kasrlarga yoyamiz:

$$\frac{t^2+2t+2}{(1+t)(2+t)^2}=\frac{A}{1+t}+\frac{B}{2+t}+\frac{C}{(2+t)^2}.$$

Koeffitsiyentlarni tenglashtirish usulini qo'llaymiz: $A=1,\ B=0,\ C=-2.$

Bundan

$$I=\int\frac{dt}{1+t}-2\int\frac{dt}{(2+t)^2}=\ln|1+t|+\frac{2}{2+t}+C.$$

x o'zgaruvchiga qaytamiz:

$$I=\ln\left|1+x+\sqrt{x^2+2x+2}\right|+\frac{2}{x+2+\sqrt{x^2+2x+2}}+C.$$

3-misol. $I=\int\frac{dx}{\sqrt{x^2-3x+2}}$ integralni toping.

Yechish. $x^2-3x+2=(x-1)(x-2)$ bo'lgani uchun

$$\sqrt{(x-1)(x-2)}=(x-1)t$$

shaklda o'rniga qo'yish bajaramiz.

U holda

$$(x-1)(x-2)=(x-1)^2t^2,\ t=\sqrt{\frac{x-2}{x-1}}.$$

Bundan

$$x=\frac{t^2-2}{t^2-1},\ dx=\frac{2tdt}{(t^2-1)^2},\ \sqrt{x^2-3x+2}=\left(\frac{t^2-2}{t^2-1}-1\right)t=-\frac{t}{t^2-1}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$I=\int\frac{-(t^2-1)2tdt}{(t^2-1)^2t}=-2\int\frac{dt}{t^2-1}=-\ln\left|\frac{t-1}{t+1}\right|+C=-\ln\left|\frac{1-2t+t^2}{t^2-1}\right|+C.$$

Dastlabki o‘zgaruvchiga qaytamiz:

$$I=-\ln\left|\frac{1-2\sqrt{\frac{x-2}{x-1}}+\frac{x-2}{x-1}}{\frac{x-2}{x-1}-1}\right|+C=-\ln|3-2x+2\sqrt{x^2-3x+2}|+C.$$

Eyler o‘rniga qo‘yishlari ayrim integrallarda murakkab hisoblashlarga olib kelishi mumkin. Bunday hollarda integrallashning quyidagi usullaridan foydalanilsa bo‘ladi.

1. $\int R(x,\sqrt{ax^2+bx+c})dx$ ko‘rinishidagi integrallarni hisoblashning *kvadrat uchhaddan to‘la kvadrat ajratish* usulida berilgan integrallar ax^2+bx+c kvadrat uchhaddan to‘la kvadrat ajratish yo‘li bilan ushbu integrallardan biriga keltiriladi:

a) agar $a>0$ va $b^2-4ac<0$ bo‘lsa, u holda $\int R(t,\sqrt{m^2+n^2t^2})dt$, bu yerda $n^2=a,\ m^2=-\frac{b^2-4ac}{4a},\ t=x+\frac{b}{2a}$;

b) agar $a>0$ va $b^2-4ac>0$ bo‘lsa, u holda $\int R(t,\sqrt{n^2t^2-m^2})dt$, bu yerda $n^2=a,\ m^2=\frac{b^2-4ac}{4a},\ t=x+\frac{b}{2a}$;

c) agar $a<0$ va $b^2-4ac>0$ bo‘lsa, u holda $\int R(t,\sqrt{m^2-n^2t^2})dt$, bu yerda $n^2=-a,\ m^2=-\frac{b^2-4ac}{4a},\ t=x+\frac{b}{2a}$.

Hosil qilingan integrallar mos ravishda $t=\frac{m}{n}tgz,\ t=\frac{m}{n\sin z},\ t=\frac{m}{n}\sin z$

o‘rniga qo‘yishlar orqali $\int R(\sin z,\cos z)dz$ ko‘rinishga keltiriladi.

4-misol. $\int\sqrt{5+4x-x^2}dx$ integralni toping.

Yechish. Kvadrat uchhaddan to‘la kvadrat ajratamiz, yangi t o‘zgaruvchi kiritamiz va trigonometrik o‘rniga qo‘yishdan foydalanib, topamiz:

$$\int\sqrt{5+4x-x^2}dx=\int\sqrt{9-(x-2)^2}dx=\left|\begin{array}{c}x-2=t,\\ dx=dt\end{array}\right|=\int\sqrt{9-t^2}dt=$$

$$=\left| \begin{array}{l} t=3\sin z, \\ dt=\cos z dz \end{array} \right| = \int \sqrt{9-9\sin^2 z}\, 3\cos z dz = \int 9\cos^2 z dz =$$

$$=\frac{9}{2}\int (1+\cos 2z)dz = \frac{9}{2}\left(z+\frac{\sin 2z}{2}\right)+C=\frac{9}{2}(z+\sin z\sqrt{1-\sin^2 z})+C=$$

$$=\left|z=\arcsin\frac{t}{3}\right|=\frac{9}{2}\left(\arcsin\frac{t}{3}+\frac{t}{3}\sqrt{1-\frac{t^2}{9}}\right)+C=\frac{9}{2}\arcsin\frac{t}{3}+\frac{t}{2}\sqrt{9-t^2}+C=$$

$$=\frac{9}{2}\arcsin\frac{x-2}{3}+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C.$$

2. $\int R(x,\sqrt{ax^2+bx+c})dx$ ko'rinishidagi ayrim integrallarni hisoblashning boshqa usullarini keltiramiz.

a) $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$ ko'rinishidagi integrallar, bu yerda $P_n(x)-$ n- darajali ko'phad:

1) $n=0$ da $\int \frac{Adx}{\sqrt{ax^2+bx+c}}$ ko'rinishda bo'ladi; bu integral $a>0$ bo'lganda jadvaldagi 14- integralga, $a<0$ bo'lganda jadvaldagi 13- integralga keltiriladi;

2) $n=1$ da $\int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}}$ ko'rinishda bo'ladi; bu integral suratda kvadrat uchhadning hosilasini ajratish natijasida ikkita, biri jadvaldagi 1- integralga va ikkinchisi 1) banddagi integralga keltiriladi;

3) $n\geq 2$ bo'lganda berilgan integraldan keltirish formulalari yordamida

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c}+M\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

ko'rinishdagi ifoda hosil qilinadi, bu yerda $Q_{n-1}(x)-$ koeffitsiyentlari noma'lum bo'lgan $n-1$-darajali ko'phad, $M-$ qandaydir o'zgarmas son. Bunda ko'phadning noma'lum koeffitsiyentlari va M soni oxirgi tenglikni differensiallash hamda x ning chap va o'ng tomondagi bir xil darajalari oldidagi koeffitsiyentlarni tenglashtirish orqali topiladi.

b) $\int \frac{dx}{(\alpha x+\beta)\sqrt{ax^2+bx+c}}$ ko'rinishdagi integral $\alpha x+\beta=\frac{1}{t}$

almashtirish yordamida 1) banddagi integralga keltiriladi;

c) $\int \frac{dx}{(\alpha x+\beta)^n\sqrt{ax^2+bx+c}}$ $(n\in Z, n>1)$ koʻrinishdagi integral $\alpha x+\beta=\frac{1}{t}$ oʻrniga qoʻyish orqali 3) banddagi integralga keltiriladi.

5- misol. $\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}}$ integralni toping.

Yechish. $x-2=\frac{1}{t}$ deymiz. U holda $dx=-\frac{dt}{t^2}$, $x^2-4x+5=\frac{1}{t^2}+1$.

Bundan

$$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}}=-\int \frac{\frac{dt}{t^2}}{\frac{1}{t^3}\sqrt{\frac{1}{t^2}+1}}=-\int \frac{t^2dt}{\sqrt{t^2+1}}.$$

Demak, b) banddagi integral hosil qilindi. Bunda $n=2$ boʻlgani uchun

$$\int \frac{t^2dt}{\sqrt{t^2+1}}=(At+B)\sqrt{t^2+1}+M\int \frac{dt}{\sqrt{t^2+1}}.$$

Tenglikning har ikkala tomonini differensiallaymiz:

$$\frac{t^2}{\sqrt{t^2+1}}=A\sqrt{1+t^2}+\frac{(At+B)t}{\sqrt{t^2+1}}+\frac{M}{\sqrt{t^2+1}}$$

yoki

$$t^2=A(1+t^2)+(At+B)t+M.$$

t ning bir xil darajalari oldidagi koeffitsiyentlarni tenglab, topamiz:

$$A=\frac{1}{2},\ b=0,\ M=-\frac{1}{2}.$$

U holda

$$\int \frac{t^2dt}{\sqrt{1+t^2}}=\frac{t\sqrt{1+t^2}}{2}-\frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}=\frac{t\sqrt{1+t^2}}{2}-\frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right|+C.$$

Dastlabki oʻzgaruvchiga qaytamiz:

$$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}}=-\frac{\sqrt{x^2-4x+5}}{2(x-2)^2}+\frac{1}{2}\ln\left|\frac{1+\sqrt{x^2-4x+5}}{x-2}\right|+C.$$

7.4.3. $\int x^m(a+bx^n)^p\,dx$ binominal differensialni integrallash

$\int x^m(a+bx^n)^p\,dx$ ko'rinishidagi integral *binominal differensial integrali* deyiladi. Bunda integral ostidagi ifoda $x^m(a+bx^n)^p$ ga *binominal differensial* deyiladi, bu yerda $m,\ n,\ p-$ ratsional sonlar.

Binominal differensial integrali uchta holdagina ratsional funksiyalarni integrallashga keltiriladi:

a) p butun son bo'lganida integral $x=t^s$ (bu yerda $s=EKUK(m,n)$) o'rniga qo'yish orqali ratsionallashtiriladi;

b) $\dfrac{m+1}{n}$ butun son bo'lganida integral $a+bx^n=t^s$(bu yerda $s-p$ sonning maxraji) o'rniga qo'yish yordamida ratsionallashtiriladi;

c) $\dfrac{m+1}{n}+p$ butun son bo'lganida integralda $a+bx^n=t^s x^n$ (bu yerda $s-p$ sonning maxraji) almashtirish bajariladi.

Agar yuqorida keltirilgan shartlar bajarilmasa binominal differensial elementar funksiyalar orqali ifodalanmaydi, ya'ni integrallanmaydi.

Masalan, $\int\sqrt{1+x^3}\,dx$ integralning integral osti funksiyasi binominal differensial: $m=0,\ n=3,\ p=\frac{1}{2}$. Bunda $p=\frac{1}{2},\ \frac{m+1}{n}=\frac{1}{3},\ \frac{m+1}{n}+p=\frac{5}{6}$ sonlardan birortasi butun son emas. Shu sababli bu integral elementar funksiyalar orqali ifodalanmaydi.

6-misol. $I=\int\dfrac{dx}{x^3\sqrt{1+x^4}}$ integralni toping.

Yechich. Shartga ko'ra,

$$m=-3,\ n=4,\ p=-\frac{1}{2},\quad \frac{m+1}{n}+p=\frac{-3+1}{4}-\frac{1}{2}=-1.$$

c) bandda aytilganidek, almashtirish bajaramiz:

$$1+x^4=t^2x^4,\ x=(t^2-1)^{-\frac{1}{4}},\ dx=-\frac{1}{2}t(t^2-1)^{-\frac{5}{4}},\ t=\frac{\sqrt{1+x^4}}{x^2}.$$

U holda

$$I=\int x^{-3}(1+x^4)^{-\frac{1}{2}}dx=-\frac{1}{2}\int(t^2-1)^{-\frac{1}{4}(-3)}(t^2)^{-\frac{1}{2}}\left((t^2-1)^{-\frac{1}{4}\cdot 4}\right)^{-\frac{1}{2}}t(t^2-1)^{-\frac{5}{4}}dt=$$

$$=-\frac{1}{2}\int(t^2-1)^{\frac{3}{4}+\frac{1}{2}-\frac{5}{4}}\cdot t^{-1+1}dt=-\frac{1}{2}\int dt=-\frac{1}{2}t+C=-\frac{\sqrt{1+x^4}}{2x^2}+C.$$

7.4.4. Elementar funksiyalarda ifodalanmaydigan integrallar

Biz integrallashning elementar funksiyalarning keng sinfini qamrab olgan muhim usullarini ko'rib chiqdik. Bu usullar ko'pchilik hollarda aniqmas integralni topish, ya'ni boshlang'ich funksiyalarni aniqlash imkonini beradi.

Ma'lumki, har qanday uzluksiz funksiya boshlang'ich funksiyaga ega bo'ladi. Agar biror $f(x)$ elementar funksiyaning boshlang'ich funksiyasi ham elementar funksiya bo'lsa, u holda $\int f(x)dx$ *integral elementar funksiyalarda ifodalanadi* deyiladi.

Adabiyotlarda keltirilishicha, $\int\sqrt{x}\cdot\cos x dx$ integral elementar funksiyalarda ifodalanmaydi, chunki hosilasi $\sqrt{x}\cdot\cos x$ ga teng bo'lgan elementar funksiya mavjud emas. Amaliy tatbiqda muhim ahamiyatga ega bo'lgan elementar funksiyalarda ifodalanmaydigan integrallarga misollar keltiramiz:

$\int e^{-x^2}dx$ – Puasson integrali (ehtimollar nazariyasi);

$\int\frac{dx}{\ln x}$ – integralli logarifm (sonlar nazariyasi);

$\int\cos x^2dx$, $\int\sin x^2dx$ – Frenel integrallari (fizika);

$\int\frac{\sin x}{x}dx$, $\int\frac{\cos x}{x}dx$ – integralli sinus va kosinus;

$\int\frac{e^x}{x}dx$ – integralli ko'rsatkichli funksiya.

Elementar funksiyalarda ifodanlanmasa-da, e^{-x^2}, $\frac{1}{\ln x}$, $\cos x^2$, $\sin x^2$, $\frac{\sin x}{x}$, $\frac{\cos x}{x}$, $\frac{e^x}{x}$ funksiyalarning boshlang'ich funksiyalari yetarlicha o'rganilgan, x argumentning turli qiymatlarida ularning qiymatlari uchun mufassal jadvallar tuzilgan.

7.4.5. Mashqlar

1. Berilgan integrallarni toping:

1) $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$; 2) $\int\frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3}$;

3) $\int\frac{x^2+\sqrt[3]{1+x}}{\sqrt{1+x}}dx$ 4) $\int\frac{x-\sqrt{x+1}}{\sqrt[3]{x+1}}dx$;

5) $\int \frac{dx}{\sqrt{2x-1}+\sqrt[3]{(2x-1)^2}}$;

6) $\int \left(\sqrt[3]{\left(\frac{x+1}{x-1}\right)^2} - \sqrt[6]{\left(\frac{x+1}{x-1}\right)^5} \right) \frac{dx}{1-x^2}$;

7) $\int \frac{dx}{\sqrt{x^2-3x+2}}$;

8) $\int \frac{dx}{\sqrt{x^2+2x+5}}$;

9) $\int \frac{dx}{x\sqrt{x^2+x+1}}$;

10) $\int \frac{dx}{x\sqrt{4-2x-x^2}}$;

11) $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$;

12) $\int \frac{dx}{1+\sqrt{x^2+2x+2}}$;

13) $\int \sqrt{5+4x-x^2}\,dx$;

14) $\int \sqrt{x^2-4}\,dx$;

15) $\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}}$;

16) $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$;

17) $\int \frac{xdx}{\sqrt{3-2x-x^2}}$;

18) $\int \frac{(2x+3)dx}{\sqrt{6x-x^2-8}}$;

19) $\int \frac{dx}{x(1+\sqrt[3]{x})^2}$;

20) $\int \frac{dx}{x^3\sqrt[3]{2-x^3}}$;

21) $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$;

22) $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$;

23) $\int \frac{dx}{x^3\sqrt{1+x^4}}$;

24) $\int \frac{\sqrt{1+\sqrt[3]{x}}}{x\sqrt{x}}dx$.

7.5. ANIQ INTEGRAL

7.5.1. Aniq integral tushunchasiga olib keluvchi masalalar

Aniq integral tabiat va texnikaning bir qancha masalalarini yechishda, xususan, har xil geometrik va fizik kattaliklarni hisoblashda keng qo‘llaniladi.

Egri chiziqli trapetsiyaning yuzasi haqidagi masalasi

Tekislikda Oxy to‘g‘ri burchakli dekart koordinatalar sistemasi kiritilgan va $[a;b]$ kesmada uzluksiz va manfiy bo‘lmagan $y=f(x)$ funksiya aniqlangan bo‘lsin.

Yuqoridan $y=f(x)$ funksiya grafigi bilan, quyidan Ox o'qi bilan, yon tomonlaridan $x=a$ va $x=b$ to‘g‘ri chiziqlar bilan chegaralangan figuraga *egri chiziqli trapetsiya* deyiladi (2-shaklda bu figura $-aABb$).

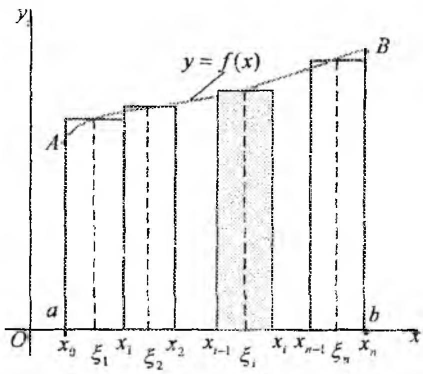
$aABb$ egri chiziqli trapetsiyaning S yuzasiga ta’rif beramiz.

$[a;b]$ kesmani n ta kichik kesmalarga bo‘lamiz: bo‘linish nuqtalarining abssissalarini $a=x_0<x_1<...<x_{i-1}<x_i<...<x_{n-1}<x_n=b$ bilan belgilaymiz. $\{x_i\}=\{x_0,x_1,...,x_n\}$ bo‘linish nuqtalari to‘plamini $[a;b]$ kesmaning bo‘linishi deymiz. x_i bo‘linish nuqtalari orqali Oy o‘qqa parallel $x=x_i$ to‘g‘ri chiziq o'tkazamiz. Bu to‘g‘ri chiziqlar $aABb$ trapetsiyaning asoslari $[x_{i-1};x_i]$ bo‘lgan n ta bo‘lakka bo‘ladi. $aABb$ trapetsiyaning S yuzasi n ta tasma yuzalarining yig‘indisiga teng bo‘ladi. n yetarlicha katta va barcha $[x_{i-1};x_i]$ kesmalar kichik bo‘lganida har bir n ta tasmaning yuzasini hisoblash oson bo‘lgan mos to‘g‘ri to‘trburchakning yuzasi bilan almashtirish mumkin bo‘ladi. Har bir $[x_{i-1};x_i]$ kesmada biror ξ_i nuqtani tanlaymiz, $f(x)$ funksiyaning bu nuqtadagi qiymati $f(\xi_i)$ ni hisoblaymiz va uni to‘g‘ri to‘rtburchakning balandligi deb qabul qilamiz. $[x_{i-1};x_i]$ kesma kichik bo‘lganida $f(x)$ uzluksiz funksiya bu kesmada kichik o‘zgarishga ega bo‘ladi. Shu sababli bu kesmalarda funksiyani va taqriban $f(\xi_i)$ ga teng deyish mumkin. Bitta tasmaning yuzasi $f(\xi_i)(x_i-x_{i-1})$ ga teng bo‘lganidan $aABb$ egri chiziqli trapetsiyaning S ga yuzasi taqriban S_n teng bo‘ladi:

$$S\approx S_n=\sum_{i=1}^{n}f(\xi_i)\Delta x_i,\quad \Delta x_i=x_i-x_{i-1} \tag{5.1}$$

(5.1) taqribiy qiymat $d=\max\limits_i \Delta x_i\ (i=\overline{1,n})$ kattalik qancha kichik bo‘lsa, shuncha aniq bo‘ladi. d kattalikka $\{x_i\}$ bo‘linishning diametri deyiladi. Bunda $n\to\infty$ da $d\to 0$.

Shunday qilib, *egri chiziqli trapetsiyaning S yuzasi* deb, S_n to‘g‘ri to‘rtburchaklar yuzasining bo‘linish diametri nolga intilgandagi limitiga



2-shakl.

aytiladi, ya'ni

$$S = \lim_{d \to 0} S_n = \lim_{d \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i. \qquad (5.2)$$

Demak, egri chiziqli trapetsiyaning yuzasini hisoblash masalasi (5.2) ko'rinishdagi limitni hisoblashga keltiriladi.

Bosib o'tilgan yo'l masalasi

Agar moddiy nuqtaning harakat qonuni $s = f(t)$ (bunda $t-$ vaqt, $s-$ bosib o'tilgan yo'l) tenglama bilan berilgan bo'lsa, $f(t)$ funksiyaning $f'(t)$ hosilasi moddiy nuqtaning berilgan vaqtdagi harakat tezligi $v(t)$ ga teng, ya'ni $v(t) = f'(t)$ bo'ladi. Fizikada quyidagi masalani yechishga to'g'ri keladi. Moddiy nuqta to'g'ri chiziq bo'ylab v tezlik bilan harakat qilayotgan va v tezlik t vaqtning uzluksiz funksiyasi bo'lsin deymiz. Moddiy nuqta vaqtning $t = a$ dan $t = b$ gacha bo'lgan biror $[a;b]$ oralig'ida bosib o'tgan yo'l s ni topamiz. $[a;b]$ kesmani $a = t_0 < t_1 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$ nuqtalar bilan vaqtning n ta yetarlicha kichik oraliqlariga bo'lamiz. Vaqtning kichik $[t_{i-1};t_i]$ oralig'ida $v(t)$ tezlik «deyarli» o'zgarmaydi. Uni bu vaqt oralig'ida o'zgarmas va taqriban $v(\xi_i)$ $(\xi_i \in [t_{i-1};t_i])$ ga teng deyish mumkin. Bunda harakat $[t_{i-1};t_i]$ kesmada tekis bo'ladi. U holda bosib o'tilgan yo'l bu vaqt oralig'ida $v(\xi_i)(t_i - t_{i-1})$ ga, $[a;b]$ vaqt oralig'ida $s \approx s_n = \sum_{i=1}^{n} v(\xi_i)\Delta t_i$ $(\Delta t_i = t_i - t_{i-1})$ ga teng bo'ladi. Bu taqribiy qiymat $d = \max_i \Delta t_i \ (i = \overline{1,n})$ kattalik qancha kichik bo'lsa, shuncha aniq bo'ladi.

Shunday qilib, s *bosib o'tilgan yo'l* deb, s_n yig'indining $d \to 0$dagi limitiga aytiladi, ya'ni

$$s = \lim_{d \to 0} s_n = \lim_{d \to 0} \sum_{i=1}^{n} v(\xi_i)\Delta t_i. \qquad (5.3)$$

Demak, bosib o'tilgan yo'lni hisoblash masalasi (5.3) ko'rinishdagi limitni hisoblashga keltiriladi.

Qaralgan har ikki masalada biror ko'rinishdagi yig'indining limitini topishga olib keluvchi bir xil usul qo'llanildi. Tabiat va texnikaning bir qancha masalalari yuqoridagi kabi yig'indining limitini topishga keltiriladi.

7.5.2. Integral yig'indi va aniq integral

$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan bo'lsin.

$[a;b]$ kesmani ixtiyoriy ravishda

$$a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{n-1} < x_n = b$$

nuqtalar bilan n ta qismga bo'lamiz, bunda $\{x_i\}$ ga $[a;b]$ kesmaning *bo'linishi*, $d = \max\limits_{1 \le i \le n}(x_i - x_{i-1}), (i = \overline{1,n})$ kattalikka *bo'linish diametri* deymiz.

Har bir $[x_{i-1};x_i]$ kesmada ixtiyoriy ξ_i nuqtani tanlaymiz. Bunday nuqtalarni *belgilangan nuqtalar* deb ataymiz. Funksiyaning $f(\xi_i)$ qiymatini mos $\Delta x_i = x_i - x_{i-1}$ uzunlikka ko'paytirib, bu ko'paytmalardan

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (5.4)$$

yig'indini tuzamiz. (5.4) yig'indiga $f(x)$ funksiya uchun $[a;b]$ kesmaning $\{x_i\}$ bo'linishidagi *Riman integral yig'indisi* deyiladi.

w_n yig'indining $d \to 0$dagi limiti tushunchasini kiritamiz.

1-ta'rif. Agar $\forall\ \varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsa va $|I - w_n| < \varepsilon$ tengsizlik $[a;b]$ kesmaning diametri $d < \delta$ bo'lgan istalgan $\{x_i\}$ bo'linishida ξ_i belgilangan nuqtalarning tanlanishiga bog'liq bo'lmagan holda bajarilsa, I soniga w_n Riman integral yig'indisining limiti deyiladi va u $I = \lim\limits_{d \to 0} w_n$ deb yoziladi.

2-ta'rif. Agar (5.4) Riman integral yig'indisi $d \to 0$da chekli limitga ega bo'lsa, u holda bu limitga $[a;b]$ kesmada $f(x)$ funksiyadan olingan *aniq* (bir karrali) *integral* (Riman integrali) deyiladi va $\int\limits_a^b f(x)dx$ kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\int\limits_a^b f(x)dx = \lim_{d \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \qquad (5.5)$$

bu yerda $f(x)$– integral ostidagi funksiya, x– integrallash o'zgaruvchisi, a, b– integralning quyi va yuqori chegarasi, $[a;b]$–integrallash sohasi (kesmasi) deyiladi.

$[a;b]$ kesmada $\int\limits_a^b f(x)dx$ aniq integral mavjud bo'lsa, $y = f(x)$ funksiya shu kesmada *integrallanuvchi* (Riman bo'yicha integrallanuvchi) deyiladi.

Izoh. Oliy matematika kursida boshqa aniq integrallar qaralmagani sababli bundan keyin «Riman integrali» va «Riman bo‘yicha integrallanuvchi» iboralarini mos ravishda «integral» va «integrallanuvchi» deb ishlatamiz.

Keltirilgan ta’riflarda $a<b$ bo‘lsin deb faraz qilindi. Aniq integral tushunchasini $a=b$ va $a>b$ bo‘lgan hollar uchun umumlashtiramiz.

$a>b$ bo‘lganida 2-ta’rifga ko‘ra,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx. \tag{5.6}$$

2-ta’rifga ko‘ra, $a=b$ bo‘lganida ((5.5) ga qarang).

$$\int_a^a f(x)dx = 0. \tag{5.7}$$

(5.4) integral yig‘indi berilgan funksiyaning argumenti qanday harf bilan belgilanishiga bog‘liq bo‘lmagani sababli, uning limiti va shuningdek, aniq integral integrallash o‘zgaruvchisining belgilanishiga bog‘liq bo‘lmaydi:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$$

1-misol. $\int_0^1 x^2 dx$ integralni aniq integralning ta’rifidan foydalanib hisoblang.

Yechish. $[0;1]$ kesmada $y=x^2$ funksiya uzluksiz.

$[0;1]$ kesmani $0=x_0<x_1<...<x_{i-1}<x_i<...<x_{n-1}<x_n=1$ nuqtalar bilan uzunliklari $\Delta x_i = \frac{1}{n} (i=\overline{1,n})$ bo‘lgan n ta bo‘lakka bo‘lamiz. Bunda $d=\max\limits_{1\le i\le n}\Delta x_i$. Demak, $d\to 0$ da $n\to\infty$.

ξ_i nuqta sifatida qismiy kesmalarning oxirlarini olamiz:

$$\xi_i = x_i = \frac{i}{n}.$$

Tegishli integral yig‘indini tuzamiz:

$$w_n = \sum_{i=1}^n f(\xi_i)\Delta x_i = \sum_{i=1}^n \frac{i^2}{n^2}\cdot\frac{1}{n} = \frac{1}{n^3}(1^2+2^2+...+n^2) =$$

$$= \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

Bundan

$$\lim_{d\to 0} w_n = \lim_{n\to\infty} w_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

Demak, ta'rifga ko'ra,

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

Endi ξ_i nuqta sifatida qismiy kesmalarning boshlarini olamiz:

$$\xi_i = x_{i-1} = \frac{i-1}{n}.$$

Bundan

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n}\frac{(i-1)^2}{n^2}\cdot\frac{1}{n} = \frac{(n-1)n(2n-1)}{6n^3} = \frac{(n-1)(2n-1)}{6n^2}$$

yoki

$$\int_0^1 x^2 dx = \lim_{d\to 0\ (n\to\infty)} w_n = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}.$$

Demak, berilgan integralning qiymati $[0;1]$ kesmani bo'lish usuliga va bu kesmada ξ_i nuqtani tanlash usuliga bog'liq emas va $\int_0^1 x^2 dx = \frac{1}{3}.$

Aniq integral mavjud bo'lishi haqidagi teoremani isbotsiz keltiramiz.

1-teorema. (*Koshi teoremasi*). Agar $y = f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda $\int_a^b f(x)dx$ aniq integral mavjud bo'ladi.

Funksiyaning uzluksiz bo'lishi uning integrallanuvchi bo'lishining yetarli sharti bo'ladi. Boshqacha aytganda, $[a;b]$ kesmada uzilishga ega bo'lgan, ammo bu kesmada integrallanuvchi funksiyalar mavjud bo'lishi ham mumkin.

2-teorema. $[a;b]$ kesmada chekli sondagi birinchi tur uzilish nuqtalariga ega bo'lgan funksiya bu kesmada integrallanuvchi bo'ladi.

7.5.3. Aniq integralning geometrik va mexanik ma'nolari

Egri chiziqli trapetsiyaning yuzasi masalasiga qaytamiz. (5.2) tenglikning o'ng tomoni integral yig'indidan iborat. U holda

(5.5) formuladan *aniq integralning geometrik ma'nosi* kelib chiqadi: agar $f(x)$ funksiya $[a;b]$ kesmada integrallanuvchi va manfiy bo'lmasa, u holda $[a;b]$ kesmada $f(x)$ funksiyadan olingan aniq integral $y=f(x)$, $y=0$, $x=a$ va $x=b$ chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasiga teng.

2- misol. $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ integralni uning geometrik ma'nosiga tayanib hisoblang.

Yechish. Bunda x ning -3 dan 3 gacha o'zgarishida tenglamasi $y=\sqrt{9-x^2}$ bo'lgan chiziq $x^2+y^2=9$ aylananing Ox o'qidan yuqorida joylashgan bo'lagidan iborat bo'ladi. Shu sababli $x=-3$, $x=3$, $y=0$ va $y=\sqrt{9-x^2}$ chiziqlar bilan chegaralangan egri chiziqli trapetsiya $x^2+y^2=9$ doiraning yarmidan tashkil topadi.

Uning yuzi $S=\dfrac{9\pi}{2}$ ga teng.

Demak,

$$\int_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{9\pi}{2}.$$

Endi bosib o'tilgan yo'l masalasiga o'tamiz. (5.3) tenglikning o'ng tomoni integral yig'indidan iborat bo'lgani uchun (5.5) formuladan ushbu xulosaga kelamiz: agar $v(t)$ funksiya $[a;b]$ kesmada integrallanuvchi va manfiy bo'lmasa, u holda $v(t)$ tezlikdan $[a;b]$ vaqt oralig'ida olingan aniq integral moddiy nuqtaning $t=a$ dan $t=b$ gacha vaqt oralig'ida bosib o'tgan yo'liga teng. Bu jumla *aniq integralning mexanik ma'nosini* anglatadi.

7.5.4. Aniq integralning xossalari

1°. Agar integral ostidagi funksiya birga teng bo'lsa, u holda

$$\int_a^b dx=b-a$$

bo'ladi.

Isboti Aniq integralning ta'rifiga ko'ra,

$$\int_a^b dx=\lim_{d\to 0}\sum_{i=1}^{n}1\cdot\Delta x_i=\lim_{d\to 0}\sum_{i=1}^{n}(x_i-x_{i-1})=\lim_{d\to 0}(b-a)=b-a.$$

2°. O‘zgarmas ko‘paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, ya’ni

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx\,,\ k = const\,.$$

Isboti. $\int_a^b kf(x)dx = \lim_{d\to 0}\sum_{i=1}^{n} kf(\xi_i)\Delta x_i = k\lim_{d\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x = k\int_a^b f(x)dx.$

3°. Chekli sondagi funksiyalar algebraik yig‘indisining aniq integrali qo‘shiluvchilar aniq integrallarining algebraik yig‘indisiga teng, ya’ni

$$\int_a^b (f(x)\pm\varphi(x))dx = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx\,.$$

Isboti. $\int_a^b (f(x)\pm\varphi(x))dx = \lim_{\lambda\to 0}\sum_{i=1}^{n}(f(\xi_i)\pm\varphi(\xi_i))\Delta x_i =$

$$= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \pm \lim_{\lambda\to 0}\sum_{i=1}^{n}\varphi(\xi_i)\Delta x_i = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx\,.$$

4°. Agar $[a;b]$ kesma bir necha qismga bo‘lingan bo‘lsa, u holda $[a;b]$ kesma bo‘yicha olingan aniq integral har bir qism bo‘yicha olingan aniq integrallar yig‘indisiga teng bo‘ladi.

Masalan,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\,,\quad c\in[a;b].$$

Isboti. $a<c<b$ bo‘lsin deylik. Integral yig‘indi $[a;b]$ kesmani bo‘lish usuliga bog‘liq emas. Shu sababli c ni $[a;b]$ kesmani bo‘lish nuqtasi qilib olamiz. Masalan, agar $c = x_m$ deb olsak, u holda w_n ni ikki yig‘indiga ajratish mumkin:

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{m} f(\xi_i)\Delta x_i + \sum_{i=m+1}^{n} f(\xi_i)\Delta x_i\,.$$

Bunda $d\to 0$ da limitga o‘tamiz:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

a, *b*, *c* nuqtalarning boshqacha joylashishida ham xossa shu kabi isbotlanadi.

Masalan, $a<b<c$ bo'lsa, $\int_a^c f(x)dx=\int_a^b f(x)dx+\int_b^c f(x)dx$ bo'ladi.

Bundan

$$\int_a^b f(x)dx=\int_a^c f(x)dx-\int_b^c f(x)dx$$

yoki integrallash chegaralarining almashtirilishi xossaga ko'ra, ((5.6) ga qarang)

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

5°. Agar $[a;b]$ kesmada funksiya o'z ishorasini o'zgartirmasa, u holda funksiya aniq integralining ishorasi funksiya ishorasi bilan bir xil bo'ladi, ya'ni:

$$[a;b] \text{ da } f(x)\geq 0 \text{ bo'lganda, } \int_a^b f(x)dx\geq 0 \text{ bo'ladi;}$$

$$[a;b] \text{ da } f(x)\leq 0 \text{ bo'lganda, } \int_a^b f(x)dx\leq 0 \text{ bo'ladi.}$$

Isboti. $f(x)\geq 0$ funksiya uchun integral yig'indi $w_n\geq 0$ bo'ladi, chunki $f(\xi_u)\geq 0$ va $\Delta x_i>0$. Bundan $\int_a^b f(x)dx\geq 0$. Shu kabi $\Delta x_i>0$, $f(x)\leq 0$ ekanidan $w_n\leq 0$ va $\int_a^b f(x)dx\leq 0$ kelib chiqadi.

6°. Agar $[a;b]$ kesmada $f(x)\geq \varphi(x)$ bo'lsa, u holda

$$\int_a^b f(x)dx\geq \int_a^b \varphi(x)dx$$

bo'ladi.

Isboti. $f(x)\geq\varphi(x)$ dan $f(x)-\varphi(x)\geq 0$ bo'ladi. U holda 5-xossaga ko'ra $\int_a^b (f(x)-\varphi(x))dx\geq 0$ yoki 3-xossaga ko'ra, $\int_a^b f(x)dx-\int_a^b\varphi(x)dx\geq 0$.

Bundan

$$\int_a^b f(x)dx\geq\int_a^b\varphi(x)dx.$$

7°. Agar m va M sonlar $f(x)$ funksiyaning $[a;b]$ kesmadagi eng kichik va eng katta qiymatlari bo'lsa, u holda

$$m(b-a)\leq\int_a^b f(x)dx\leq M(b-a)$$

bo'ladi.

Isboti. Shartga ko'ra, $m \le f(x) \le M$. U holda 6-xossaga ko'ra,

$$\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx .$$

Bunda

$$\int_a^b m dx = m\int_a^b dx = m(b-a), \qquad \int_a^b M dx = M\int_a^b dx = M(b-a).$$

U holda

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Bu xossa aniq integralni *baholash haqidagi teorema* deb yuritiladi.

8°. Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda shunday $c \in [a;b]$ nuqta topiladiki,

$$\int_a^b f(x)dx = f(c)(b-a) \qquad (5.8)$$

bo'ladi.

Isboti. 7-xossaga ko'ra,

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Bundan

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M .$$

$\dfrac{\int_a^b f(x)dx}{b-a} = \mu$ deymiz. U holda $m \le \mu \le M$ bo'lgani uchun Bolsano-Koshining ikkinchi teoremasiga ko'ra, biror $c \in [a;b]$ nuqta uchun $f(c) = \mu$ bo'ladi.

Shu sababli

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$

yoki

$$\int_a^b f(x)dx = f(c)(b-a).$$

8-xossa *o'rta qiymat haqidagi teorema* deb ataladi.

(5.8) formulaga *o'rta qiymat formulasi,* $f(c)$ ga $f(x)$ *funksiyaning* $[a;b]$ *kesmadagi o'rtacha qiymati* deyiladi.

O'rta qiymat haqidagi teorema quyidagi geometrik talqinga ega: agar $f(x)>0$ bo'lsa, u holda $\int_a^b f(x)dx$ integralning qiymati balandligi $f(c)$ ga va asosi $(b-a)$ ga teng bo'lgan to'g'ri to'rtburchakning yuzasiga teng bo'ladi.

Aniq integralning xossalaridan quyidagi natijalar kelib chiqadi.

1-natija. $[a;b]$ kesmada aniqlangan $f(x)$ funksiya uchun

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$$

bo'ladi.

2-natija. Agar $[a;b]$ kesmada $|f(x)| \le k$ bo'lsa, u holda

$$\left|\int_a^b f(x)dx\right| \le k(b-a), \ (k = const.)$$

bo'ladi.

3- misol. $y = 2x+2$ funksiyaning $[-1;2]$ kesmadagi o'rtacha qiymatini toping.

Yechish. O'rta qiymat haqidagi teoremadan topamiz:

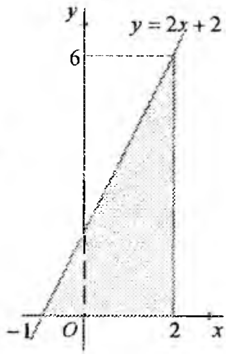
$$f_{o'rt} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Aniq integralning geometrik ma'nosiga ko'ra, $\int_{-1}^{2}(2x+2)dx$ integralning qiymati 3-shaklda keltirilgan uchburchakning yuzasiga teng, ya'ni

$$S = \frac{1}{2}\cdot(2+1)\cdot 6 = 9.$$

Bundan

$$f_{o'rt} = \frac{1}{2-(-1)}\cdot 9 = 3.$$



3-shakl.

7.5.4. Mashqlar

1. Integrallarni aniq integralning geometrik ma'nosiga tayanib hisoblang:

1) $\int_0^{\pi} \cos x dx$; 2) $\int_0^{2} (3+x) dx$;

3) $\int_0^{4} \sqrt{16-x^2} dx$; 4) $\int_{-2}^{2} f(x) dx$, $f(x) = \begin{cases} -x, agar\ -2 \le x \le 0, \\ x, agar\ 0 \le x \le 2. \end{cases}$

2. Integrallarni taqqoslang:

1) $I_1 = \int_0^{\frac{\pi}{4}} \cos x dx$, $I_2 = \int_0^{\frac{\pi}{4}} \sin x dx$; 2) $I_1 = \int_{-1}^{1} \sqrt{2-x^2} dx$, $I_2 = \int_{-1}^{1} x^2 dx$.

3) $I_1 = \int_{-2}^{0} \sqrt{1-x^3} dx$, $I_2 = \int_{-2}^{0} (1-x) dx$; 4) $I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx$, $I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x dx$.

3. Integrallarni baholang:

1) $I_1 = \int_0^{\pi} \frac{dx}{3-2\cos x}$; 2) $I_2 = \int_1^{3} \sqrt{1+3x^2} dx$;

3) $I_3 = \int_0^{2} \sqrt{1+x^3} dx$; 4) $I_4 = \int_0^{2} \frac{dx}{4-2x-x^2}$.

4. Funksiyalarning berilgan kesmalardagi o'rtacha qiymatini toping:

1) $y = \sqrt{4-x^2}$, $[-2;2]$; 2) $y = |x|$, $[-1;1]$;

3) $y = 3x+2$, $[1;3]$; 4) $y = x^2 e^x$, $[0;1]$.

7.6. ANIQ INTEGRALNI HISOBLASH

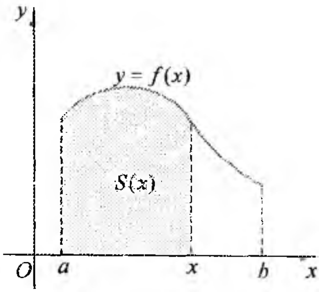
7.6.1. Yuqori chegarasi o'zgaruvchi aniq integral

$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsin. U holda u ixtiyoriy $[a;x]$ $(a \le x \le b)$ kesmada integrallanuvchi bo'ladi, ya'ni istalgan $x \in [a;b]$ uchun

$$F(x) = \int_a^x f(t) dt \tag{6.1}$$

integral mavjud bo'ladi.

Agar ixtiyoriy $t \in [a;b]$da $f(t) > 0$ bo'lsa, u holda $F(x) = \int_a^x f(t)dt$ integral asosi $[a;x]$ kesmadan iborat bo'lgan egri chiziqli trapetsiyaning o'zgaruvchi yuzasi $S(x)$ ni ifodalaydi (4-shakl).



4-shakl.

$[a;b]$ kesmada (6.1) tenglik bilan aniqlangan $F(x)$ funksiyaga *yuqori chegarasi o'zgaruvchi aniq integral* deyiladi.

$F(x)$ funksiya $[a;b]$ kesmada uzluksiz va differensiallanuvchi bo'ladi. Shuningdek, bunda $F(x)$ funksiya uchun quyidagi fundamental teorema o'rinli bo'ladi.

1-teorema (*Nyuton-Leybnis teoremasi*). $[a;b]$ kesmada uzluksiz $f(x)$ funksiyaning yuqori chegarasi o'zgaruvchi integrali $F(x)$ dan yuqori chegara bo'yicha olingan hosila mavjud va u integral ostidagi funksiyaning yuqori chegaradagi qiymatiga teng bo'ladi, ya'ni

$$F'(x) = \left(\int_a^x f(t)dt\right)'_x = f(x), \quad x \in [a;b]. \tag{6.2}$$

Isboti. $x \in [a;b]$ va $x + \Delta x \in [a;b]$ bo'lsin.
U holda aniq integralning 4-xossasini qo'llab, topamiz:

$$F(x+\Delta x) = \int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt.$$

Bundan (6.1) tenglik va o'rta qiymat haqidagi teoremaga ko'ra,

$$\Delta F = F(x+\Delta x) - F(x) = \int_x^{x+\Delta x} f(t)dt = f(c)\Delta x, \text{ bu yerda } c \in [x, x+\Delta x].$$

$F(x)$ funksiyaning hosilasini aniqlaymiz:

$$F'(x) = \lim_{\Delta x \to 0}\frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(c)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(c).$$

$\Delta x \to 0$ da $x + \Delta x \to x$ va $c \to x$, chunki $c \in [x, x+\Delta x]$.

U holda $f(x)$ funksiyaning uzluksizligidan

$$F'(x) = \lim_{\Delta x \to 0} f(c) = f(x)$$

bo'ladi.

Nyuton-Leybnis teoremasi matematik analizning asosiy teoremalaridan biri hisoblanadi. Bu teorema differensial bilan aniq integral tushunchalari orasidagi munosabatni ochib beradi.

Bu teoremadan $[a;b]$ kesmada uzluksiz har qanday $f(x)$ funksiya shu kesmada boshlang'ich funksiyaga ega bo'ladi va uning boshlang'ich funksiyalaridan biri yuqori chegarasi o'zgaruvchi $F(x)$ integral bo'ladi degan xulosa kelib chiqadi.

$f(x)$ funksiyaning boshqa bir boshlang'ich funksiyasi $F(x)$ funksiyadan o'zgarmas C songa farq qilgani uchun aniqmas va aniq integrallar orasidagi ushbu bog'lanish kelib chiqadi:

$$\int f(x)dx = \int_a^x f(t)dt + C,\ x \in [a;b]. \tag{6.3}$$

7.6.2. Nyuton-Leybnis formulasi

Aniq integralni integral yig'indining limiti sifatida hisoblash, hatto, oddiy funksiyalar uchun ham ancha qiyinchiliklar tug'diradi. Shu sababli aniq integralni hisoblashning (6.3) formulaga asoslangan, amaliy jihatdan qulay bo'lgan hamda keng qo'llaniladigan usuli bilan tanishamiz.

2-teorema (*integral hisobning asosiy teoremasi*). Agar $F(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lgan $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda

$$\int_a^b f(x)dx = F(b) - F(a). \tag{6.4}$$

Isboti. $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyalaridan biri bo'lsin. U holda 1-teoremaga asosan, $\int_a^x f(t)dt$ funksiya ham $f(x)$ funksiyaning $[a;b]$ kesmadagi boshlang'ich funksiyasi bo'ladi.

Boshlang'ich funksiyalar o'zgarmas C songa farq qilganidan

$$\int_a^x f(t)dt = F(x) + C.$$

Bu tenglikka $x = a$ ni qo'yamiz va chegaralari teng bo'lgan aniq integralning xossasini qo'llaymiz:

$$\int_a^a f(t)dt = F(a) + C = 0.$$

Bundan $C=-F(a)$.

U holda istalgan $x\in[a;b]$ uchun

$$\int_a^x f(t)dt=F(x)-F(a)$$

bo'ladi.

Oxirgi tenglikda $x=b$ deymiz va t o'zgaruvchini x bilan almashtiramiz. Natijada (6.4) formula kelib chiqadi.

(6.4) formulaga *Nyuton-Leybnis formulasi* deyiladi.

$F(b)-F(a)$ ayirmani shartli ravishda $F(x)\Big|_a^b$ deb yozish kelishilgan. Bu kelishuv natijasida Nuyton-Leybnis formulasi

$$\int_a^b f(x)dx=F(x)\Big|_a^b \tag{6.5}$$

ko'inishda ifodalanadi.

1-misol. $\int_0^3 \frac{dx}{\sqrt{1+x^2}}$ integralni hisoblang.

Yechish. $\int_0^3 \frac{dx}{\sqrt{1+x^2}}=\ln\left|x+\sqrt{1+x^2}\right|\Big|_0^3=\ln\left|3+\sqrt{10}\right|-\ln 1=\ln\left|3+\sqrt{10}\right|$.

2-misol. $\int_1^4 \frac{dx}{x^2-2x+10}$ integralni hisoblang.

Yechish.

$$\int_1^4 \frac{dx}{x^2-2x+10}=\int_1^4 \frac{dx}{(x-1)^2+3^2}=\frac{1}{3}arctg\frac{x-1}{3}\Big|_1^4=\frac{1}{3}(arctg1-arctg0)=\frac{\pi}{12}.$$

7.6.3. Aniq integralda o'zgaruvchini almashtirish

3-teorema. $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsin. Agar: 1) $x=\varphi(t)$ funksiya $[\alpha;\beta]$ kesmada differensiallanuvchi va $\varphi'(t)$ funksiya $[\alpha;\beta]$ kesmada uzluksiz; 2) $x=\varphi(t)$ funksiyaning qiymatlar sohasi $[a;b]$ kesmadan iborat; 3) $\varphi(\alpha)=a$ va $\varphi(\beta)=b$ bo'lsa, u holda

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt \tag{6.6}$$

bo'ladi.

Isboti. Nyuton-Leybnis formulasiga ko‘ra,

$$\int_a^b f(x)dx = F(b) - F(a),$$

bu yerda $F(x)$ funksiya $f(x)$ funksiyaning $[a;b]$ kesmadagi boshlang‘ich funksiyalaridan biri. $\Phi(t) = F(\varphi(t))$ murakkab funksiyani qaraymiz.

Murakkab funksiyani differensiallash qoidasiga asosan,

$$\Phi'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t).$$

Demak, $\Phi(t)$ funksiya $[\alpha;\beta]$ kesmada $f(\varphi(t))\varphi'(t)$ uzluksiz funksiya uchun boshlang‘ich funksiya bo‘ladi. Nyuton-Leybnis formulasi bilan topamiz:

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \Phi(\beta) - \Phi(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) =$$

$$= F(b) - F(a) = \int_a^b f(x)dx.$$

(6.6) formula *aniq integralda o‘zgaruvchini almashtirish* formulasi deb yuritiladi. Aniq integralni hisoblashning bu usulida *aniq integralda o‘rniga qo‘yish* usuli deyiladi.

Izoh. Aniq integralni (6.6) formula bilan hisoblashda dastlabki eski o‘zgaruvchiga qaytish shart emas, chunki integrallash chegarasi o‘rniga qo‘yishga mos tarzda o‘zgaradi.

3-misol. $\int_0^{\sqrt{3}} \sqrt{4 - x^2}\,dx$ integralni hisoblang.

Yechish. $x = 2\sin t,\ 0 \le t \le \frac{\pi}{3}$ belgilash kiritamiz. Bu o‘zgaruvchini almashtirish 3-teoremaning barcha shartlarini qanoatlantiradi: birinchidan, $f(x) = \sqrt{4 - x^2}$ funksiya $[0;\sqrt{3}]$ kesmada uzluksiz, ikkinchidan, $x = 2\sin t$ funksiya $\left[0;\frac{\pi}{3}\right]$ kesmada differensiallanuvchi va $x' = 2\cos t$ bu kesmada uzluksiz, uchinchidan t o‘zgaruvchi 0 dan $\frac{\pi}{3}$ gacha o‘zgarganda $x = 2\sin t$ funksiya 0 dan $\sqrt{3}$ gacha o‘sadi va bunda $\varphi(0) = 0$ va $\varphi\left(\frac{\pi}{3}\right) = \sqrt{3}$. Bunda $dx = 2\cos t dt$.

(6.6) formuladan topamiz:

$$\int_0^{\sqrt{3}}\sqrt{4-x^2}dx = 4\int_0^{\frac{\pi}{3}}\cos^2 tdt = 2\int_0^{\frac{\pi}{3}}(1+\cos 2t)dt = 2\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{3}} = \frac{2\pi}{3}+\frac{\sqrt{3}}{2}.$$

4-misol. $\int_0^1 x\sqrt{1+x^2}dx$ integralni hisoblang.

Yechish. $t=\sqrt{1+x^2}$ o'rniga qo'yish bajaramiz.

U holda

$$x=\sqrt{t^2-1},\ dx=\frac{tdt}{\sqrt{t^2-1}},\ x=0 \text{ da } t=1,\ x= \text{ da } t=\sqrt{2}.$$

$[1;\sqrt{2}]$ kesmada $\sqrt{t^2-1}$ funksiya monoton o'sadi, demak o'rniga qo'yish to'g'ri bajarilgan.

Bundan

$$\int_0^1 x\sqrt{1+x^2}dx = \int_1^{\sqrt{2}}\sqrt{t^2-1}\cdot t\cdot\frac{tdt}{\sqrt{t^2-1}} = \int_1^{\sqrt{2}}t^2dt = \frac{t^3}{3}\Big|_1^{\sqrt{2}} = \frac{2\sqrt{2}-1}{3}.$$

Izoh. (6.6) formulani qo'llashda teoremada sanab o'tilgan shartlarning bajarilishini tekshirish lozim. Agar bu shartlar buzilsa, keltirilgan formula bo'yicha o'zgaruvchini almashtirish xato natijaga olib kelishi mumkin.

7.6.4. Aniq integralni bo'laklab integrallash

4-teorema. Agar $u(x)$ va $v(x)$ funksiyalar $u'(x)$ va $v'(x)$ hosilalari bilan $[a;b]$ kesmada uzluksiz bo'lsa, u holda

$$\int_a^b udv = uv\big|_a^b - \int_a^b vdu \qquad (6.7)$$

bo'ladi.

Isboti. Teoremaning shartiga ko'ra, $u(x)$ va $v(x)$ funksiyalar hosilaga ega.

U holda ko'paytmani differensiallash qoidasiga binoan,

$$(u(x)v(x))' = u(x)v'(x) + v(x)u'(x).$$

Bundan $u(x)v(x)$ funksiya $u(x)v'(x)+v(x)u'(x)$ funksiya uchun boshlang'ich funksiya bo'lishi kelib chiqadi. $u(x)v'(x)+v(x)u'(x)$

funksiya $[a;b]$ kesmada uzluksiz bo'lgani uchun u bu kesmada integrallanuvchi bo'ladi.

U holda aniq integralning 3-xossasiga va Nyuton-Leybnis formulasiga ko'ra

$$\int_a^b u(x)v'(x)dx+\int_a^b v(x)u'(x)dx=u(x)v(x)\Big|_a^b.$$

Bundan, $u'(x)dx=du(x)$ va $v'(x)dx=dv(x)$ bo'lgani uchun

$$\int_a^b udv=uv\Big|_a^b-\int_a^b vdu.$$

(6.7) formula *aniq integralni bo'laklab integrallash formulasi* deb ataladi.

5- misol. $\int_0^1 xe^{-x}dx$ integralni hisoblang.

Yechish.

$$\int_0^1 xe^{-x}dx=\left|\begin{array}{l} u=x,\ dv=e^{-x}dx \\ du=dx,\ v=-e^{-x}\end{array}\right|=-xe^{-x}\Big|_0^1+\int_0^1 e^{-x}dx=$$

$$=-\frac{1}{e}-e^{-x}\Big|_0^1=-\frac{1}{e}-\frac{1}{e}+1=1-\frac{2}{e}.$$

7.6.5. Mashqlar

Berilgan integrallarni hisoblang:

1) $\int_{-1}^{2}(x^2+2x+1)dx;$

2) $\int_0^{\frac{\pi}{4}}\sin 4xdx;$

3) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos xdx;$

4) $\int_1^e \frac{dx}{x};$

5) $\int_0^{\frac{\pi}{2}}\cos^2 xdx;$

6) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{dx}{\sin^2 x};$

7) $\int_1^2 \frac{dx}{x+x^2};$

8) $\int_0^1 (2x^3+1)x^2dx;$

9) $\int_0^1 x\sqrt{1+x^2}dx;$

10) $\int_0^{\frac{\pi}{2}}\cos x\sin^3 xdx;$

11) $\int\limits_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x dx}{1+\cos x}$;

12) $\int\limits_{\frac{\sqrt{2}}{3}}^{\frac{\sqrt{3}}{3}} \frac{dx}{\sqrt{4-9x^2}}$;

13) $\int\limits_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{3+4x^2}$;

14) $\int\limits_0^{\frac{\pi}{4}} \sin^3 x dx$;

15) $\int\limits_0^{\frac{\pi}{2}} \frac{\cos x dx}{6-5\sin x+\sin^2 x}$;

16) $\int\limits_{\frac{\sqrt{2}}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}dx$;

17) $\int\limits_0^1 \arcsin x dx$;

18) $\int\limits_1^e \ln^2 x dx$;

19) $\int\limits_0^{\pi} x\sin\frac{x}{2}dx$;

20) $\int\limits_0^{\frac{\pi}{4}} e^x \sin 2x dx$;

21) $\int\limits_0^1 x^2 e^{3x} dx$;

22) $\int\limits_1^{\sqrt{e}} x\ln x dx$;

23) $\int\limits_0^{\frac{\pi^2}{4}} \sin\sqrt{x}dx$;

24) $\int\limits_0^{e^{\frac{\pi}{2}}} \cos(\ln x)dx$.

7.7. XOSMAS INTEGRALLAR

Aniq integral qaralganida $\int\limits_a^b f(x)dx$ integral mavjud bo‘lishi uchun ikkita shartning bajarilishi talab qilingan edi: 1) integrallash chegarasi chekli $[a;b]$ kesmadan iborat bo‘lishi; 2) integral ostidagi funksiya $[a;b]$ kesmada aniqlangan va chegaralangan bo‘lishi.

Aniq integral uchun keltirilgan shartlardan biri bajarilmaganda u *xosmas integral* deb ataladi: 1) faqat birinchi shart bajarilmasa, cheksiz chegarali xosmas integral (yoki I tur xosmas integral) deyiladi; 2) faqat ikkinchi shart bajarilmasa, uzilishga ega bo‘lgan funksiyaning xosmas integrali (yoki II tur xosmas integral) deyiladi.

7.7.1. Cheksiz chegarali xosmas integrallar (I tur xosmas integrallar)

1-ta’rif. $f(x)$ funksiya $[a;+\infty)$ oraliqda uzluksiz bo‘lsin. Agar $\lim\limits_{b\to+\infty}\int\limits_a^b f(x)dx$ limit mavjud va chekli bo‘lsa, bu limitga *yuqori chegarasi* *cheksiz xosmas integral* deyiladi va $\int\limits_a^{+\infty} f(x)dx$ kabi belgilanadi:

$$\int\limits_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int\limits_a^b f(x)dx. \qquad (7.1)$$

Bu holda $\int\limits_a^{+\infty} f(x)dx$ integralga *yaqinlashuvchi integral* deyiladi.

Agar $\lim\limits_{b\to+\infty}\int\limits_a^b f(x)dx$ limit mavjud bo'lmasa yoki cheksiz bo'lsa, $\int\limits_a^{+\infty} f(x)dx$ integralga *uzoqlashuvchi integral* deyiladi.

Quyi chegarasi cheksiz va har ikkala chegarasi cheksiz xosmas integrallar shu kabi ta'riflanadi:

$$\int\limits_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int\limits_a^b f(x)dx, \qquad (7.2)$$

$$\int\limits_{-\infty}^{+\infty} f(x)dx = \lim_{a\to-\infty}\int\limits_a^c f(x)dx + \lim_{b\to+\infty}\int\limits_c^b f(x)dx, \qquad (7.3)$$

bu yerda $c-$ sonlar o'qining biror fiksirlangan nuqtasi. Bunda (7.3) tenglikning chap tomonidagi xosmas integral, o'ng tomondagi har ikkala xosmas integral yaqinlashgandagina yaqinlashadi.

1- misol. $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}(\alpha>0)$ integralni yaqinlashishga tekshiring.

Yechish. $\alpha\neq1$ bo'lsin.

U holda

$$\int\limits_1^{+\infty}\frac{dx}{x^\alpha} = \lim_{b\to+\infty}\int\limits_1^b\frac{dx}{x^\alpha} = \lim_{b\to+\infty}\left.\frac{x^{1-\alpha}}{1-\alpha}\right|_1^b = \frac{1}{1-\alpha}(\lim_{b\to+\infty}b^{1-\alpha}-1).$$

Bunda: 1) $\alpha<1$ bo'lganda, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha} = \frac{1}{1-\alpha}(\lim\limits_{b\to+\infty}b^{1-\alpha}-1)=+\infty$,

2) $\alpha>1$ bo'lganda, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha} = \frac{1}{1-\alpha}(\lim\limits_{b\to+\infty}b^{1-\alpha}-1)=\frac{1}{1-\alpha}$,

3) $\alpha=1$ bo'lganda, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha} = \lim\limits_{b\to+\infty}\int\limits_1^b\frac{dx}{x} = \lim\limits_{b\to+\infty}\ln x|_1^b = \lim\limits_{b\to+\infty}\ln b=+\infty$.

Demak, $\int_1^{+\infty}\frac{dx}{x^\alpha}$ xosmas integral $\alpha>1$ da yaqinlashadi va $0<\alpha\le 1$ da uzoqlashadi.

2- misol. $\int_{-\infty}^{0} x\cos x dx$ integralni yaqinlashishga tekshiring.

Yechish. $\int_{-\infty}^{0} x\cos x dx=\lim_{a\to-\infty}\int_a^0 x\cos x dx=\lim_{a\to-\infty}\left(x\sin x\Big|_a^0-\int_a^0 \sin x dx\right)=$

$=\lim_{a\to-\infty}(-a\sin a+\cos x\Big|_a^0=\lim_{a\to-\infty}(-a\sin a-\cos a+1).$

Bu limit mavjud emas. Shu sababli $\int_{-\infty}^{0} x\cos x dx$ integral uzoqlashadi.

3- misol. $\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}$ integralni yaqinlashishga tekshiring.

Yechish. Oraliq nuqtani $c=0$ deymiz.

U holda (7.3) tenglikga ko‘ra,

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}=\int_{-\infty}^{0}\frac{dx}{x^2+6x+10}+\int_{0}^{+\infty}\frac{dx}{x^2+6x+10}.$$

Bundan

$$\int_{-\infty}^{0}\frac{dx}{x^2+6x+10}=\lim_{a\to-\infty}\int_a^0\frac{dx}{(x+3)^2+1}=\lim_{a\to-\infty}arctg(x+3)\Big|_a^0=$$

$$=arctg3-\lim_{a\to-\infty}arctg(a+3)=arctg3+\frac{\pi}{2},$$

$$\int_{0}^{+\infty}\frac{dx}{x^2+6x+10}=\lim_{b\to+\infty}\int_0^b\frac{dx}{(x+3)^2+1}=\lim_{b\to+\infty}arctg(x+3)\Big|_0^b=$$

$=\lim_{b\to+\infty}arctg(b+3)-arctg3=\frac{\pi}{2}-arctg3.$

U holda

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}=\int_{-\infty}^{0}\frac{dx}{x^2+6x+10}+\int_{0}^{+\infty}\frac{dx}{x^2+6x+10}=$$

$$=arctg3+\frac{\pi}{2}+\frac{\pi}{2}-arctg3=\pi.$$

Demak, xosmas integral yaqinlashadi.

7.7.2. Uzulishga ega bo'lgan funksiyalarning xosmas integrallari (II tur xosmas integrallar)

2-ta'rif. $f(x)$ funksiya $[a;b)$ oraliqda aniqlangan va uzluksiz bo'lib, $x=b$ da cheksiz uzulishga ega bo'lsin. Agar $\lim\limits_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon} f(x)dx$ limit mavjud va chekli bo'lsa, bu limitga *uzulishga ega bo'lgan funksiyaning xosmas integrali* deyiladi va $\int\limits_a^b f(x)dx$ kabi belgilanadi:

$$\int\limits_a^b f(x)dx=\lim_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon} f(x)dx. \tag{7.4}$$

Shu kabi: 1) $f(x)$ funksiya x ning a ga o'ngdan yaqinlashishida uzilishga ega bo'lsa,

$$\int\limits_a^b f(x)dx=\lim_{\varepsilon\to 0}\int\limits_{a+\varepsilon}^{b} f(x)dx; \tag{7.5}$$

bo'ladi;

2) agar $f(x)$ funksiya $c\in[a;b]$ da uzilishga ega bo'lsa,

$$\int\limits_a^b f(x)dx=\lim_{\varepsilon\to 0}\int\limits_a^{c-\varepsilon} f(x)dx+\lim_{\varepsilon\to 0}\int\limits_{c+\varepsilon}^{b} f(x)dx \tag{7.6}$$

bo'ladi.

II tur xosmas integrallar uchun yaqinlashish (uzoqlashish) tushunchalari I tur integrallardagi kabi kiritiladi.

4- misol. $\int\limits_{-1}^{1}\dfrac{dx}{x\sqrt[3]{x}}$ integralni yaqinlashishga tekshiring.

Yechish. $x=0$ da integral ostidagi funksiya uzilishga ega. U holda (7.6) tenglikka ko'ra,

$$\int\limits_{-1}^{1}\frac{dx}{x\sqrt[3]{x}}=\lim_{\varepsilon\to 0}\int\limits_{-1}^{-\varepsilon}\frac{dx}{x\sqrt[3]{x}}+\lim_{\varepsilon\to 0}\int\limits_{+\varepsilon}^{1}\frac{dx}{x\sqrt[3]{x}}=-3\lim_{\varepsilon\to 0}x^{-\frac{1}{3}}\Big|_{-1}^{-\varepsilon}-3\lim_{\varepsilon\to 0}x^{-\frac{1}{3}}\Big|_{+\varepsilon}^{1}=$$

$$=3\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{3}}-1-1+3\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{3}}=6\left(\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{3}}-1\right)=+\infty.$$

Demak, xosmas integral uzoqlashadi. Berilgan integralga Nyuton-Leybnis formulasi formal qo'llanilishi xato natijaga olib keladi:

$$\int\limits_{-1}^{1}\frac{dx}{x\sqrt[3]{x}}=-\frac{3}{\sqrt[3]{x}}\Bigg|_{-1}^{1}=-6.$$

7.7.3. Xosmas integrallarning yaqinlashish alomatlari

Ko'pincha, xosmas integralni (7.1) - (7.6) formulalar orqali hisoblash shart bo'lmasdan, faqat uning yaqinlashuvchi yoyi uzoqlashuvchi bo'lishini bilish yetarli bo'ladi. Bunday hollarda berilgan integral yaqinlashishga yaqinlashuvchi yoki uzoqlashuvchiligi oldindan ma'lum bo'lgan boshqa xosmas integral bilan taqqoslash orqali tekshiriladi. Xosmas integrallarning taqqoslash alomatlarini ifodalovchi teoremalarni isbotsiz keltiramiz.

1-teorema *(I tur xosmas integralning yaqinlashish alomati).* $[a;+\infty)$ oraliqda $f(x)$ va $\varphi(x)$ funksiyalar uzluksiz va $0 \le f(x) \le \varphi(x)$ bo'lsin. U holda:

1) agar $\int_a^{+\infty} \varphi(x)dx$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi;

2) agar $\int_a^{+\infty} f(x)dx$ integral uzoqlashsa, $\int_a^{+\infty} \varphi(x)dx$ integral ham uzoqlashadi.

5-misol. $\int_0^{+\infty} e^{-x^2} dx$ integralni yaqinlashishga tekshiring.

Yechish. Puasson integrali deb ataluvchi bu integral boshlang'ich funksiyaga ega emas. Uni ikkita integralning yig'indisi ko'rinishida ifodalaymiz:

$$\int_0^{+\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{+\infty} e^{-x^2} dx.$$

Tenglikning chap qismidagi integrallardan birinchisi chekli qiymatga ega bo'lgan aniq integral. Ikkinchi xosmas $\int_1^{+\infty} e^{-x^2} dx$ integralni yaqinlashishga tekshiramiz.

$[1;+\infty)$ oraliqda $0 < e^{-x^2} \le e^{-x}$ bo'ladi, e^{-x^2} va e^{-x} funksiyalar uzluksiz.

Bunda

$$\int_1^{+\infty} e^{-x} dx = \lim_{b \to +\infty} \int_1^b e^{-x} dx = \lim_{b \to +\infty} (-e^{-x})\Big|_1^b = \frac{1}{e} - \lim_{b \to +\infty} \frac{1}{e^b} = \frac{1}{e}.$$

Demak, bu integral yaqinlashadi va 1-teoremaning 1) bandiga asosan, Puasson integrali ham yaqinlashadi.

6-misol. $\int_0^1 \frac{dx}{e^x - \cos x}$ integralni yaqinlashishga tekshiring.

Yechish. Integral ostidagi funksiya $x = 0$ da uzulishga ega.

$x \in (0;1]$ da $\frac{1}{e^x - \cos x} \geq \frac{1}{xe}$.

Bundan

$$\int_0^1 \frac{dx}{xe} = \frac{1}{e}\lim_{\varepsilon \to 0}\int_{+\varepsilon}^1 \frac{dx}{x} = \frac{1}{e}\lim_{\varepsilon \to 0} \ln x\Big|_{+\varepsilon}^1 = \frac{1}{e}(0 - \lim_{\varepsilon \to 0} \ln|\varepsilon|) = +\infty.$$

Demak, $\int_0^1 \frac{dx}{xe}$ integral uzoqlashadi va 1-teoremaning 2) bandiga asosan berilgan integral ham uzoqlashadi.

2-teorema (*II tur xosmas integralning yaqinlashish alomati*). $[a;b)$ oraliqda $f(x)$ va $\varphi(x)$ funksiyalar uzluksiz bo'lsin va $0 \leq f(x) \leq \varphi(x)$ tengsizlikni qanoatlantirsin, $x = b$ da $f(x)$ va $\varphi(x)$ funksiyalar cheksiz uzilishga ega bo'lsin. U holda:

1) agar $\int_a^b \varphi(x)dx$ integral yaqinlashsa, $\int_a^b f(x)dx$ integral ham yaqinlashadi;

2) agar $\int_a^b f(x)dx$ integral uzoqlashsa, $\int_a^b \varphi(x)dx$ integral ham uzoqlashadi.

Taqqoslash teoremalari integral ostidagi funksiya bir xil ishorali bo'lganida o'rinli bo'ladi. Ishorasi almashinadigan funksiyalarning xosmas integrallari uchun quyidagi alomat o'rinli bo'ladi.

3-teorema. Agar $\int_a^{+\infty} |f(x)|dx \left(\int_a^b |f(x)|dx\right)$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx \left(\int_a^b f(x)dx\right)$ integral ham yaqinlashadi.

3-ta'rif. Agar $\int_a^{+\infty} |f(x)|dx \left(\int_a^b |f(x)|dx\right)$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx \left(\int_a^b f(x)dx\right)$ integralga *absolut yaqinlashuvchi xosmas integral* deyiladi.

4-ta'rif. Agar $\int\limits_a^{+\infty} f(x)dx \left(\int\limits_a^b f(x)dx\right)$ integral yaqinlashsa va $\int\limits_a^{+\infty} |f(x)|dx \left(\int\limits_a^b |f(x)|dx\right)$ integral uzoqlashsa, $\int\limits_a^{+\infty} f(x)dx \left(\int\limits_a^b f(x)dx\right)$ integralga *shartli yaqinlashuvchi xosmas integral* deyiladi.

7 misol. $\int\limits_1^{+\infty} \frac{\cos x}{x^2}dx$ integralni yaqinlashishga tekshiring.

Yechish. Integral ostidagi funksiya $[1;+\infty)$ oraliqda ishorasini almashtiradi.

Ma'lumki, $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$.

1-misolga ko'ra, $\int\limits_1^{+\infty} \frac{dx}{x^2}$ integral yaqinlashuvchi.

U holda 1-teoremaga asosan, $\int\limits_1^{+\infty} \left|\frac{\cos x}{x^2}\right|dx$ integral yaqinlashuvchi va 3-teorema va 3-ta'rifga ko'ra, $\int\limits_1^{+\infty} \frac{\cos x}{x^2}dx$ integral absolut yaqinlashuvchi bo'ladi.

(7.2), (7.3) ko'rinishdagi ((7.5),(7.6) ko'rinishdagi) xosmas integrallar uchun taqqoslash alomatlari hamda absolut va shartli yaqinlashish tushunchalari yuqorida (7.1) ko'rinishdagi ((7.4) ko'rinishdagi) integrallar uchun keltirilgandagi
kabi kiritiladi.

7.7.4. Mashqlar

1. Berilgan integrallarni hisoblang yoki uzoqlashuvchi ekanini aniqlang:

1) $\int\limits_1^{+\infty} \frac{dx}{1+x^2}$; 2) $\int\limits_0^{+\infty} xe^{-\frac{x}{2}}dx$;

3) $\int\limits_{-\infty}^{0} x\cos x dx$; 4) $\int\limits_2^{+\infty} \frac{\ln x dx}{x}$;

5) $\int\limits_2^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$; 6) $\int\limits_1^{+\infty} \frac{arctgx dx}{x^2}$;

7) $\int\limits_0^{+\infty} e^{-x}\sin x dx$;

8) $\int\limits_1^{e} \frac{dx}{x\sqrt{\ln x}}$;

9) $\int\limits_0^{1} \frac{dx}{\sqrt{1-x^2}}$;

10) $\int\limits_1^{3} \frac{xdx}{\sqrt{(x-1)}}$;

11) $\int\limits_{-1}^{1} \frac{3x^2+2}{\sqrt[3]{x^2}}dx$;

12) $\int\limits_0^{2} \frac{dx}{x^2-4x+3}$;

13) $\int\limits_{-1}^{1} \frac{dx}{x\sqrt[3]{x}}$;

14) $\int\limits_{-\infty}^{+\infty} \frac{dx}{x^2+6x+10}$.

2. Integrallarni yaqinlashishga tekshiring:

1) $\int\limits_1^{+\infty} \frac{dx}{x^\alpha}$;

2) $\int\limits_0^{+\infty} \frac{dx}{\sqrt{1+x^3}}$;

3) $\int\limits_0^{+\infty} \sqrt{x}e^{-x}dx$;

4) $\int\limits_1^{+\infty} \frac{\sin xdx}{x^2}$;

5) $\int\limits_1^{+\infty} \frac{x^3+1}{x^4}dx$;

6) $\int\limits_0^{1} \frac{dx}{e^{\sqrt{x}}-1}$;

7) $\int\limits_0^{1} \frac{e^x dx}{\sqrt{1-\cos x}}$;

8) $\int\limits_0^{1} \frac{dx}{e^x-\cos x}$;

9) $\int\limits_1^{2} \frac{3+\sin x}{(x-1)^3}dx$;

10) $\int\limits_0^{1} \frac{\sqrt{x}dx}{\sqrt{1-x^4}}$;

11) $\int\limits_1^{+\infty} \frac{\cos x}{x^2}dx$;

12) $\int\limits_0^{+\infty} e^{-x}\sin x dx$.

7.8. ANIQ INTEGRALNING TATBIQLARI

7.8.1. Aniq integralning qo‘llanilish sxemalari

x o‘zgaruvchi aniqlangan $[a;b]$ kesmaga bog‘liq biror geometrik yoki fizik A kattalikning qiymatini (tekis shakl yuzasini, jism hajmini, kuchning bajargan ishini va hokazo) hisoblash talab qilingan bo‘lsin. Bunda A kattalik additiv deb faraz qilinadi, ya’ni $[a;b]$ kesmaning $c\in[a;b]$ nuqta bilan $[a;c]$ va $[c;b]$ qismlarga bo‘linishida A kattalikning $[a;b]$ kesmaga mos qiymati uning $[a;c]$ va $[c;b]$ kesmalarga mos qiymatlarining yig‘indisiga teng deb qaraladi.

A kattalikning qiymatini hisoblash ma'lum tartibda (sxema asosida) bajariladi. Bunda ikki sxemadan biriga amal qilish mumkin: *I sxema* (*integral yig'indilar usuli*) va *II sxema* (*differensial usuli*).

I sxema aniq integralning ta'rifiga asoslanadi. Bunda:

1°. $[a;b]$ kesma $a = x_0 < x_1 < ... < x_{i-1} < x_i < ... < x_{n-1} < x_n = b$ nuqtalar bilan n ta kichik kesmalarga bo'linadi. Bunda A kattalik mos n ta ΔA_i, $(i = \overline{1,n})$ «elementar qo'shiluvchilar» ga bo'linadi:

$$A = \Delta A_1 + \Delta A_2 + ... + \Delta A_n;$$

2°. Har bir «elementar qo'shiluvchi» masalaning shartidan aniqlanuvchi funksiyaning mos kesma istalgan nuqtasida hisoblangan qiymati bilan kesmaning uzunligi ko'paytmasi ko'rinishiga keltiriladi:

$$\Delta A_i \approx f(\xi_i)\Delta x_i;$$

ΔA_i ning taqribiy qiymatini topishda ayrim soddalashtirishlar qilish mumkin: kichik bo'lakda yoyni uning chekkalarini tortib turuvchi vatar bilan almashtirish mumkin; kichik bo'lakda o'zgaruvchi tezlikni o'zgarmas deyish mumkin va hokazo.

Bunda A kattalikning taqribiy qiymati integral yig'indidan iborat bo'ladi:

$$A \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i .$$

3°. A kattalikning haqiqiy qiymati bu integral yig'indining $n \to \infty$ dagi (bunda $\lambda = \max\limits_{a \le x \le b} \Delta x_i$) limitiga teng bo'ladi:

$$A = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx.$$

II sxema *I* sxemaning o'zgargan ko'rinishi hisoblanadi va «differensial usul» yoki «yuqori tartibli cheksiz kichiklarni tashlab yuborish usuli» deb ataladi.

Bunda:

1°. $[a;b]$ kesmada ixtiyoriy x qiymatni tanlaymiz va o'zgaruvchi $[a;x]$ kesmani qaraymiz. Bu kesmada A kattalik x ning funksiyasi bo'ladi: $A = A(x)$, ya'ni A kattalikning bo'lagi noma'lum $A(x)$ funksiya bo'ladi;

2°. x ning kichik $\Delta x = dx$ kattalikka o'zgarishida ΔA orttirmaning bosh qismini topamiz: $dA = f(x)dx$, bu yerda $f(x) -$ x

o'zgaruvchining masala shartidan kelib chiquvchi funksiyasi (bunda mumkin bo'lgan soddalashtirishlar qilinishi mumkin);

3°. $\Delta A \approx dA$ deb, dA ni a dan b gacha integrallash orqali A ning izlanayotgan qiymati topiladi:

$$A = \int_a^b f(x)dx.$$

7.8.2. Tekis shakl yuzasini hisoblash

Yuzani dekart koordinatalarida hisoblash

Aniq integralning geometrik ma'nosiga asosan, abssissalar o'qidan yuqorida yotgan, ya'ni yuqoridan $y = f(x)$ ($f(x) \geq 0$) funksiya grafigi bilan, quyidan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

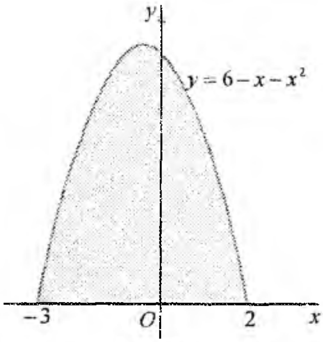
$$S = \int_a^b f(x)dx \qquad (8.1)$$

integralga teng bo'ladi.

Shu kabi, abssissalar o'qidan pastda yotgan, ya'ni quyidan $y = f(x)$ ($f(x) \leq 0$) funksiya grafigi bilan, yuqoridan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

$$S = -\int_a^b f(x)dx \qquad (8.2)$$

integtegralga teng bo'ladi.



5-shakl.

1-misol. Ox o'q va $y = 6 - x - x^2$ parabola bilan chegaralangan tekis shakl yuzasini toping.

Yechish. Parabolaning Ox o'q bilan kesishish nuqtasini topamiz (5-shakl):

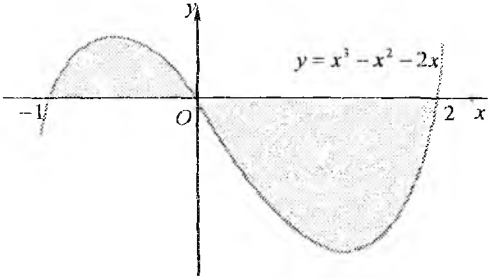
$$y = 0 = 6 - x - x^2 = (3 + x)(2 - x), \; x_1 = -3, \; x_2 = 2.$$

Tekis shakl yuzasini (8.1) formula bilan topamiz:

$$s = \int_{-3}^{2} (6 - x - x^2)dx = \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-3}^{2} =$$

$$=\left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+27\right)=20\frac{5}{6}.$$

Yuzani hisoblashga oid murakkabroq masalalar yuzaning additivlik xossasiga asoslangan holda yechiladi. Bunda tekis shakl kesishmaydigan qismlarga ajratiladi va aniq integralning 4-xossasiga ko'ra tekis shaklning yuzasi qismlar yuzalarining yig'indisiga teng bo'ladi.



6-shakl.

2-misol. $y=x^3-x^2-2x$ va $y=0$ chiziqlar bilan chegaralangan tekis shakl yuzasini hisoblang (6-shakl).

Yechish Berilgan tekis shaklni yuzalari S_1 va S_2 bo'lgan kesishmaydigan qismlarga ajratamiz. U holda yuzaning additivlik xossasiga asosan, berilgan tekis shaklning yuzasi qismlar yuzalarining yig'indisiga teng bo'ladi.

Demak,

$$S=S_1+S_2=\int_{-1}^{0}(x^3-x^2-2x)dx-\int_{0}^{2}(x^3-x^2-2x)dx=$$

$$=\left(\frac{x^4}{4}-\frac{x^3}{3}-x^2\right)\Bigg|_{-1}^{0}-\left(\frac{x^4}{4}-\frac{x^3}{3}-x^2\right)\Bigg|_{0}^{2}=$$

$$=-\left(\frac{1}{4}+\frac{1}{3}-1\right)+\left(4-\frac{8}{3}-4\right)=\frac{37}{12}.$$

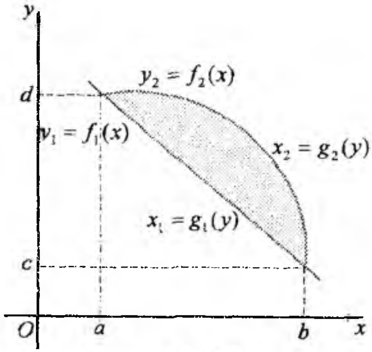
$[a;b]$ kesmada ikkita $y_1=f_1(x)$ va $y_2=f_2(x)$ uzliksiz funksiyalar berilgan va $x\in[a;b]$ da $f_2(x)\geq f_1(x)$ bo'lsin. Bu funksiyalarning grafiklari va $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan tekis shaklni qaraymiz.

Bu tekis shaklning yuzasi yuqoridan $y_2=f_2(x)$ va $y_1=f_1(x)$ funksiyalar grafiklari bilan, quyidan Ox o'q bilan, yon tomonlardan $x=a$ va $x=b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyalar yuzalarining ayirmasiga teng bo'ladi:

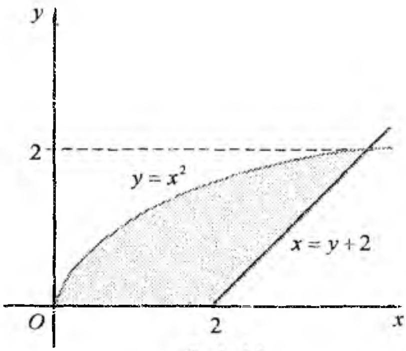
$$S=\int_a^b f_2(x)dx-\int_a^b f_1(x)dx=\int_a^b (f_2(x)-f_1(x))dx. \qquad (8.3)$$

Ayrim hollarda yuzani hisoblashga oid masalalar yuzaning ko'chishga nisbatan invariantlik xossasidan foydalangan holda soddalashtiriladi. Bunda tekis shakl yuzasi (8.3) formulada x va y o'zgaruvchilar (Ox va Oy o'qlar) ning o'rnini almashtirish yo'li bilan hisoblanadi, ya'ni (7-shakl).

$$S=\int_a^b (f_2(x)-f_1(x))dx=\int_c^d (g_2(y)-g_1(y))dy. \quad (8.4)$$



7-shakl.



8-shakl.

3-misol. $x=y^2$ va $x=y+2$ chiziqlar bilan chegaralangan tekis shaklning abssissalar o'qidan yuqorida yotgan qismining yuzasini hisoblang (8-shakl).

Yechish. Berilgan chiziqlarning kesishish nuqtalarini topamiz:

$$y^2=y+2,\ y^2-y-2=0,\ y_1=0,\ y=2.$$

Tekis shakl yuzasini (8.4) formula bilan topamiz:

$$s=\int_0^2 (y+2-y^2)dy=\left(\frac{y^2}{2}+2y-\frac{y^3}{3}\right)\Bigg|_0^2=2+4-\frac{8}{3}=\frac{10}{3}.$$

Agar egri chiziqli trapetsiya yuqoridan $x=\varphi(t)$, $y=\psi(t)$, $\alpha\le t\le\beta$ parametrik tenglamalar bilan berilgan funksiya grafigi bilan chegaralangan va $a=\varphi(\alpha)$, $b=\varphi(b)$ bo'lsa, (8.1) formulada $x=\varphi(t)$, $dx=\varphi'(t)dt$ ko'rinishdagi o'rniga qo'yish orqali o'zgaruvchi almashtiriladi.

U holda

$$S = \int_{\alpha}^{\beta} \psi(t)\varphi'(t)dt \tag{8.5}$$

bo'ladi, bu yerda, $a = \varphi(\alpha)$ va $b = \varphi(\beta)$.

4-misol. Radiusi R ga teng doira yuzasini hisoblang.

Yechish. Koordinatalar boshini doiraning markaziga joylashtiramiz. Bu doiraning aylanasi $x = R\cos t$, $y = R\sin t$ parametrik tenglamalar bilan aniqlanadi va doira koordinata o'qlariga nisbatan simmetrik bo'ladi. Shu sababli uning birinchi chorakdagi yuzasini hisoblaymiz (bunda x o'zgaruvchi 0 dan R gacha o'zgarganda t parametr $\frac{\pi}{2}$ dan 0 gacha o'zgaradi) va natijani to'rtga ko'paytiramiz. U holda (8.5) formulaga ko'ra:

$$S = 4S_1 = 4\int_{\frac{\pi}{2}}^{0} R\sin t(-R\sin t)dt = 4R^2\int_{0}^{\frac{\pi}{2}} \sin^2 t dt =$$

$$= 2R^2\int_{0}^{\frac{\pi}{2}} (1-\cos 2t)dt = 2R^2\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{2}} = \pi R^2.$$

Yuzani qutb koordinatalarida hisoblash

Qutb koordinatalar ($r-$ qutb radiusi, $\varphi-$ qutb burchagi) sistemasida berilgan $r = r(\varphi)$ funksiya $\varphi \in [\alpha; \beta]$ kesmada uzluksiz bo'lsin.

$r = r(\varphi)$ funksiya grafigi hamda O qutbdan chiqqan $\varphi = \alpha$ va $\varphi = \beta$ nurlar bilan chegaralangan tekis shaklga *egri chiziqli sektor* deyiladi.

AOB egri chiziqli sektor yuzasini hisoblaymiz. Bunda II sxemadan foydalanamiz.

1°. Qutbdan chiqqan φ va α burchaklarga mos nurlar bilan chegaralangan egri chiziqli sektor yuzi φ burchakning $S = S(\varphi)$ funksiyasi bo'lsin, bu yerda $\alpha \le \varphi \le \beta$ ($\varphi = \alpha$ bo'lganda $S(\alpha) = 0$, $\varphi = \beta$ bo'lganda $S(\beta) = S$).

2°. Joriy φ qutb burchak $\Delta\varphi = d\varphi$ orttirma olganida ΔS yuza OAB «elementar egri chuziqli sektor» yuziga teng orttirma oladi.

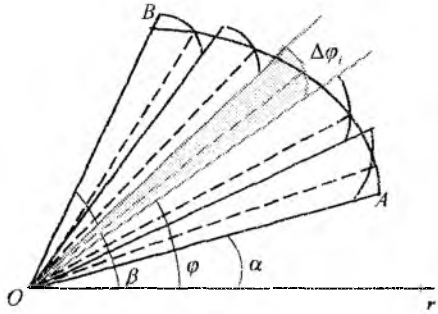
Bunda dS differensial ΔS orttirmaning $d\varphi \to 0$ dagi orttirmasining bosh qismini ifodalaydi va radiusi r ga, markaziy burchagi $d\varphi$ ga teng bo'lgan OAC doiraviy sektor yuziga teng bo'ladi (9-shakl).

Shu sababli

$$dS = \frac{1}{2} r^2 d\varphi.$$

3°. Oxirgi ifodani $\varphi = \alpha$ dan $\varphi = \beta$ gacha integrallab izlanayotgan yuzani topamiz:

$$S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi) d\varphi. \qquad (8.6)$$



9-shakl.

5-misol. $r = 2\cos 3\varphi$ egri chiziq bilan chegaralangan figuraning yuzasini hisoblang.

Yechish. $r = 2\cos 3\varphi$ egri chiziq bilan chegaralangan figuraga uch yaproqli gul deyiladi (1-ilovaga qarang).

Uning oltidan bir qismi yuzasini hisoblaymiz:

$$\frac{1}{6}S = \frac{1}{2}\int_0^{\frac{\pi}{6}} 4\cos^2 3\varphi d\varphi = \int_0^{\frac{\pi}{6}} (1 + \cos 6\varphi) d\varphi = \left(\varphi + \frac{\sin 6\varphi}{6}\right)\Bigg|_0^{\frac{\pi}{6}} = \frac{\pi}{6}.$$

Bundan $S = \pi$.

Agar egri chiziqli sektor $r_1 = r_1(\varphi)$ va $r_2 = r_2(\varphi)$ ($\varphi \in [\alpha;\beta]$ da $r_2(\varphi) > r_1(\varphi)$) funksiyalar grafiklari bilan chegaralangan bo'lsa,

$$S = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2(\varphi) - r_1^2(\varphi)) d\varphi \qquad (8.7)$$

bo'ladi.

7.8.3. Yassi egri chiziq yoyi uzunligini hisoblash

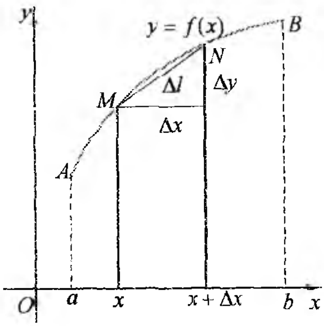
AB egri chiziq $y = f(x) \ge 0$ funksiyaning grafigi bo'lsin. Bunda $x \in [a;b]$, $y = f(x)$ funksiya va $y' = f'(x)$ hosila bu kesmada uluksiz bo'lsin.

AB egri chiziq uzunligi l ni *II* sxemadan foydalangan holda topamiz.

1°. $[a;b]$ kesmada ixtiyoriy x nuqtani tanlaymiz va o'zgaruvchi $[a;x]$ kesmani qaraymiziz. $(x, f(x))$ nuqta M bo'lsin.

$\overset{\frown}{AM}$ yoy uzunligi l_{AM} x ning funksiyasi bo‘ladi: $l = l(x)$ ($l(a) = 0$ va $l(b) = l$).

2°. x ning kichik $\Delta x = dx$ kattalikka o‘zgarishida dl differensialni topamiz: $dl = l'(x)dx$. $\overset{\frown}{MN}$ yoyni uni tortib turuvchi vatar bilan almashtiramiz va $l'(x)$ ni topamiz (10-shakl):



10-shakl.

$$l'(x) = \lim_{\Delta x \to 0} \frac{\Delta l}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} = \sqrt{1 + (y'_x)^2}.$$

Demak,

$$dl = \sqrt{1 + (y'_x)^2}\, dx.$$

3°. dl ni a dan b gacha integrallaymiz:

$$l = \int_a^b \sqrt{1 + (y'_x)^2}\, dx. \qquad (8.8)$$

(8.7) tenglikka *yoy differensialining* to‘g‘ri burchakli koordinatalardagi formulasi deyiladi.

Agar egri chiziq $x = g(y)$, $y \in [c; d]$, tenglama bilan berilgan bo‘lsa, yuqorida keltirilganlarni takrorlab, AB yoy uzunligini hisoblashning quyidagi formulasini hosil qilamiz:

$$l = \int_c^d \sqrt{1 + (x'_y)^2}\, dy. \qquad (8.9)$$

6-misol. $y = \frac{3}{8} x \sqrt[3]{x} - \frac{3}{4} \sqrt[3]{x^2}$ egri chiziq yoyining Ox o‘q bilan kesishish nuqtalari orasidagi uzunligini hisoblang.

Yechish. $y = 0$ deb egri chiziqning Ox oq bilan kesishish nuqtalarini aniqlaymiz: $x_1 = 0$, $x_2 = 2\sqrt{2}$.

Hosilani topamiz:

$$y' = \frac{3}{8} \cdot \frac{4}{3} x^{\frac{1}{3}} - \frac{3}{4} \cdot \frac{2}{3} x^{-\frac{1}{3}} = \frac{1}{2}\left(x^{\frac{1}{3}} - x^{-\frac{1}{3}}\right).$$

Yoy uzunligini hisoblaymiz:

$$l=\int_0^{2\sqrt{2}}\sqrt{1+\frac{1}{4}\left(x^{\frac{1}{3}}-x^{-\frac{1}{3}}\right)^2}dx=\frac{1}{2}\int_0^{2\sqrt{2}}\sqrt{\left(x^{\frac{1}{3}}+x^{-\frac{1}{3}}\right)^2}dx=$$

$$=\frac{1}{2}\int_0^{2\sqrt{2}}\left(x^{\frac{1}{3}}+x^{-\frac{1}{3}}\right)dx=\frac{1}{2}\left(\frac{3}{4}x^{\frac{4}{3}}+\frac{3}{2}x^{\frac{2}{3}}\right)\Bigg|_0^{2\sqrt{2}}=3.$$

Agar egri chiziq $x=\varphi(t),\ \ y=\psi(t),\ \ \alpha\le t\le\beta$, parametrik tenglamalar bilan berilgan bo'lsa, (8.8) formulada $x=\varphi(t),\quad y=\psi(x), dx=\varphi'(t)dt$ o'rniga qo'yish orqali o'zgaruvchi almashtiriladi.

Bunda

$$l=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \tag{8.10}$$

kelib chiqadi, bu yerda $a=\varphi(\alpha)$ va $b=\varphi(\beta)$.

Egri chiziq qutb koordinatalar sistemasida $r=r(\varphi),\quad \alpha\le\varphi\le\beta$, tenglama bilan berilgan bo'lsin, bunda $r(\varphi),\ r'(\varphi)$ funksiyalar $[\alpha;\beta]$ kesmada uzluksiz va A,B nuqtalarga qutb koordinatalarida α,β burchaklar mos keladi.

$x=r\cos\varphi,\ \ y=r\sin\varphi$ ekanligidan

$$x'(\varphi)=r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi,$$

$$y'(\varphi)=r'(\varphi)\sin\varphi-r(\varphi)\cos\varphi.$$

(8.10) formulaga $x'(\varphi)$ va $y'(x)$ hosilalarni qo'yamiz va almashtirishlar bajarib, topamiz:

$$l=\int_\alpha^\beta\sqrt{r^2(\varphi)+r'^2(\varphi)}d\varphi\ . \tag{8.11}$$

7-misol. $r=a(1+\cos\varphi),\ a>0$ kardioida uzunligini toping.

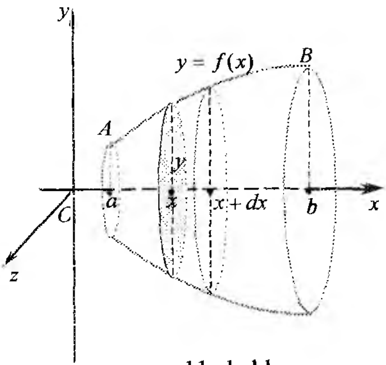
Yechish. Egri chiziqning simmetrikligini (1-ilovaga qarang) hisobga olib, (8.11) formula bilan topamiz:

$$l=2\int_0^\pi\sqrt{a^2(1+\cos\varphi)^2+a^2(-\sin\varphi)^2}d\varphi=4a\int_0^\pi\sqrt{\frac{1+\cos\varphi}{2}}d\varphi=$$

$$=4a\int_0^\pi\cos\frac{\varphi}{2}d\varphi=8a\sin\frac{\varphi}{2}\Bigg|_0^\pi=8a.$$

7.8.4. Aylanish sirti yuzini hisoblash

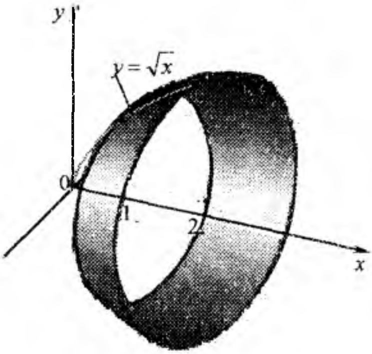
AB egri chiziq $y=f(x)\geq 0$ funksiyaning grafigi bo'lsin. Bunda $x\in[a;b]$, $y=f(x)$ funksiya va $y'=f'(x)$ hosila bu kesmada uzluksiz bo'lsin.



11-shakl.

AB egri chiziqning Ox o'q atrofida aylanishidan hosil bo'lgan jism sirti yuzini hisoblaymiz. Buning uchun II sxemani qo'llaymiz.

1°. Istalgan $x\in[a;b]$ nuqta orqali Ox o'qqa perpendikular tekislik o'tkazamiz. Bu tekislik aylanish sirtini radiusi $y=f(x)$ bo'lgan aylana bo'ylab kesadi (11-shakl). Bunda aylanish sirtidan Ox o'qidagi a va x nuqtalardan zOy tekisligiga parallel ravishda o'tkazilgan tekisliklar ajratgan sirt yuzi x ning funksiyasi bo'ladi: $S=S(x)$ $(S(a)=0$ va $S(b)=S)$.



12-shakl.

2°. x argumentga $\Delta x=dx$ orttirma beramiz va $x+\Delta x\in[a;b]$ nuqta orqali Ox o'qqa perpendikular tekislik o'tkazamiz. Bunda $S=S(x)$ funksiya «belbog'» ko'rinishida ΔS orttirma oladi.

Kesimlar orasidagi jismni yasovchisi dl bo'lgan va asoslarining radiuslari y va $y+dy$ bo'lgan kesik konus bilan almashtiramiz. Bu kesik konusning yon sirti $dS=\pi(y+y+dy)dl=2\pi ydl+\pi dydl$ ga teng. $dydl$ ko'paytmani dS ga nisbatan yuqori tartibli cheksiz kichik sifatida tashlab yuboramiz: $dS=2\pi ydl$. Bunda $dl=\sqrt{1+(y'_x)^2}dx$ ekanini hisobga olamiz: $dS=2\pi y\sqrt{1+(y'_x)^2}dx$.

3°. dS ni a dan b gacha integrallab, topamiz:

$$S=2\pi\int_a^b y\sqrt{1+(y'_x)^2}dx \tag{8.12}$$

Shu kabi $x=g(y)$, $y\in[c;d]$ funksiya grafigining Oy o‘q atrofida aylantirshdan hosil bo‘lgan jism sirtining yuzi ushbu

$$S=2\pi\int_c^d x\sqrt{1+(x_y')^2}dy \qquad (8.13)$$

formula bilan hisoblanadi.

5-misol. $y=2\sqrt{x},\ 1\le x\le 2$ egri chiziqning Ox o‘qi atrofida aylanishidan hosil bo‘lgan sirt (12-shakl) yuzini toping.

Yechish. Misol shartidan topamiz: $y'=\dfrac{1}{\sqrt{x}}$.

(8.12) formula bilan topamiz:

$$S=2\pi\int_1^2 2\sqrt{x}\cdot\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx=4\pi\int_1^2\sqrt{x+1}dx=4\pi\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_1^2=\frac{8\pi}{3}(3\sqrt{3}-2\sqrt{2}).$$

Agar AB egri chiziq $x=\varphi(t)$, $y=\psi(t)$, $\alpha\le t\le\beta$, parametrik tenglamalar bilan berilgan bo‘lsa, u holda AB egri chiziqning Ox (Oy) o‘q atrofida aylanishidan hosil bo‘lgan jism sirti yuzi quyidagicha hisoblanadi:

$$S=2\pi\int_\alpha^\beta\psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \quad \left(S=2\pi\int_{\alpha_1}^{\beta_1}\varphi(t)\sqrt{\psi'^2(t)+\varphi'^2(t)}dt\right), \qquad (8.14)$$

bu yerda $a=\varphi(\alpha)$ va $b=\varphi(\beta)$ $(c=\psi(\alpha_1)$ va $d=\psi(\beta_1))$.

AB egri chiziq qutb koordinatalar sistemasida $r=r(\varphi)$, $\alpha\le\varphi\le\beta$ tenglama bilan berilgan bo‘lganida quyidagi formulalar o‘rinli bo‘ladi:

$$Ox:\ S=2\pi\int_\alpha^\beta r\sin\varphi\sqrt{r^2+r'^2}d\varphi,\quad Oy:\ S=2\pi\int_\alpha^\beta r\cos\varphi\sqrt{r^2+r'^2}d\varphi. \qquad (8.15)$$

6- misol. Radiusi R ga teng bo‘lgan shar sirti yuzini hisoblang.

Yechish. Shar parametrik tenglamasi $x=R\cos t$, $y=R\sin t$ bo‘lgan yarim aylananing Ox o‘q atrofida aylanishidan hosil bo‘ladi. Sharning koordinata o‘qlariga simmetrik bo‘lishini inobatga olib, hisoblaymiz:

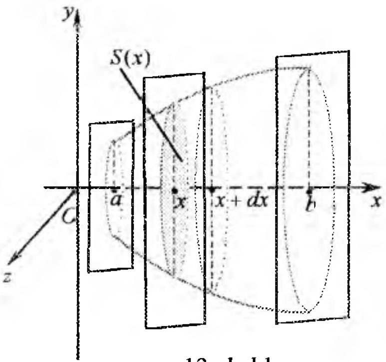
$$S=2\cdot 2\pi\int_0^{\frac{\pi}{2}}R\sin t\sqrt{(-R\sin t)^2+(R\cos t)^2}dt=$$

$$=4\pi R^2\int_0^{\frac{\pi}{2}}\sin t dt=-4\pi R^2\cos t\Big|_0^{\frac{\pi}{2}}=4\pi R^2.$$

7.8.5. Hajmlarni hisoblash

Hajmni ko'ndalang kesim yuzasi bo'yicha hisoblash

Hajmi hisoblanishi lozim bo'lgan qandaydir jism (13-shakl) uchun uning istalgan ko'ndalang kesim yuzasi S ma'lum bo'lsin. Bu yuza ko'ndalang kesim joylashishiga bog'liq bo'ladi: $S = S(x)$, $x \in [a;b]$, bu yerda $S(x)$–$[a;b]$ kesmada uzluksiz funksiya.



13-shakl.

Izlanayotgan hajmni II sxema asosida topamiz.

1°. Istalgan $x \in [a;b]$ nuqta orqali Ox o'qqa perpendikular tekislik o'tkazamiz. Jismning bu tekislik bilan kesimi yuzasini $S(x)$ bilan va jismning bu tekislikdan chapda yotgan bo'lagining hajmini $V(x)$ bilan belgilaymiz (13-shakl). Bunda V kattalik x ning funksiyasi bo'ladi: $V = V(x)$ $(V(a) = 0$ va $V(b) = V)$.

2°. $V(x)$ funksiyaning dV differensialini topamiz. Bu differensial Ox o'q bilan x va $x + \Delta x$ nuqtalarda kesishuvchi parallel tekisliklar orasidagi «elementar qatlam»dan iborat bo'ladi. Bu differensialni asosi $S(x)$ga va balandligi dx ga teng silindr bilan taqriban almashtirish mumkin. Demak, $dV = S(x)dx$.

3°. dV ni a dan b gacha integrallab, izlanayotgan hajmni topamiz:

$$V = \int_a^b S(x)dx. \tag{8.16}$$

7-misol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidning hajmini hisoblang.

Yechish. Ellipsoidning Ox o'qqa perpendikulyar va koordinatalar boshidan x $(-a \le x \le a)$ masofada o'tuvchi tekislik bilan kesamiz.

Kesimda yarim o'qlari $b(x) = b\sqrt{1 - \frac{x^2}{a^2}}$ va $c(x) = c\sqrt{1 - \frac{x^2}{a^2}}$ bo'lgan ellips hosil bo'ladi.

Uning yuzasi

$$S(x) = \pi b(x)c(x) = \pi bc\left(1 - \frac{x^2}{a^2}\right).$$

U holda

$$V = \int_{-a}^{a} \pi bc\left(1 - \frac{x^2}{a^2}\right)dx = \pi bc\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a} = \frac{4}{3}\pi abc.$$

Aylanish jismlarining hajmini hisoblash

Yuqoridan $y = f(x)$ uzluksiz funksiya grafigi bilan, quyidan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Ox o'q atrofida aylantirishdan hosil bo'lgan jism hajmini hisoblaymiz. Bu jismning ixtiyoriy ko'ndalang kesimi doiradan iborat. Shu sababli jismning $X = x$ tekislik bilan kesimining yuzasi $S(x) = \pi y^2$ bo'ladi.

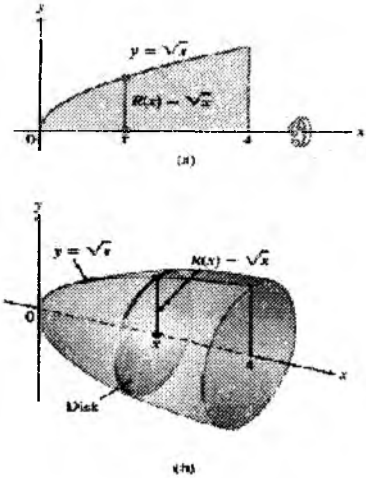
U holda (8.16) formulaga ko'ra,

$$V = \pi\int_{a}^{b} y^2 dx. \tag{8.17}$$

Shu kabi yuqoridan $y = f(x)$ uzluksiz funksiya grafigi bilan, quyidan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyani Oy o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmi quyidagi
formula bilan hisoblanadi:

$$V = 2\pi\int_{a}^{b} yxdx. \tag{8.18}$$

8-misol. $y = \sqrt{x},\ y = 0,\ x = 0,\ x = 4$ chiziqlar bilan chegaralangan yassi shaklning Ox o'qi atrofida aylanishidan hosil bo'lgan jism hajmini toping (14-shakl).



14-shakl.

Yechish. Hajmni (8.17) formula bilan topamiz:

$$S=\pi\int_0^4 xdx=\pi\cdot\frac{x^2}{2}\Big|_0^4=8\pi.$$

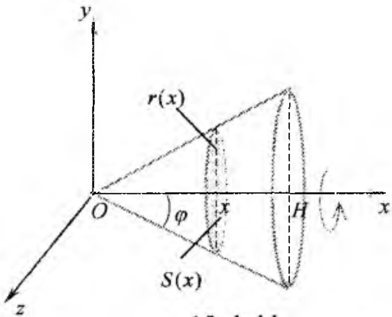
Agar egri chiziqli trapetsiya $x=\varphi(y)$ uzluksiz funksiya grafigi, Oy o‘qi, $y=c$ va $y=d$ to‘g‘ri chiziqlar bilan chegaralangan bo‘lsa,

$$V=\pi\int_c^d x^2dy\;(Oy)\quad\left(V=2\pi\int_c^d xydy\;(Ox)\right) \tag{8.19}$$

bo‘ladi.

9-misol. Radiusi R ga va balandligi H ga teng bo‘lgan konusning hajmini hisoblang.

Yechish. Konusni katetlari R va H bo‘lgan to‘g‘ri burchakli uchburchakning balandlik bo‘ylab yo‘nalgan Ox o‘q atrofida aylanishidan hosil bo‘lgan jism deyish mumkin (15-shakl). Gipotenuza tenglamasi $y=kx$ bo‘lsin.



15-shakl.

U holda

$$y=kx,\; k=tg\varphi=\frac{R}{H},\quad y=\frac{R}{H}x.$$

Bundan

$$V=\pi\int_0^H y^2dx=\pi\int_0^H\frac{R^2}{H^2}x^2dx=\frac{\pi R^2}{H^2}\cdot\frac{x^3}{3}\Big|_0^H=\frac{1}{3}\pi R^2H.$$

7.8.6. Momentlar va og‘irlik markazini hisoblash

Tekis shakl va aylanish sirti yuzalarini, yassi egri chiziq yoyi uzunligini, hajmlarini hisoblashga oid yuqorida keltirilgan masalalarni yechishda aynan bir xil usul qo‘llanildi. Bunda izlanilayotgan Q kattalikni hisoblash integral yig‘indi limitini topishga keltirildi. Barcha masalalarda Q kattalik biror $[a;b]$ kesma va bu kesmadagi uzluksiz funksiyalarga bog‘liq holda o‘rganildi.

Shu kabi Q kattalik quyidagi xossalarga ega deb faraz qilamiz:

1°. *Additivlik xossasi.* $[a;b]$ kesma istalgancha bo‘laklarga bo‘linganida ham $Q=\sum_{i=1}^{n}Q_i$ bo‘ladi, bu yerda Q_i – Qning i-bo‘lakdagi qiymati;

2°. *Kichiklikdagi chiziqlilik xossasi.* $[a;b]$ kesmadagi istalgan kichik $[x_{i-1};x_i]$ kesmada $Q_i \approx k\Delta x_i$ bo‘ladi, bu yerda $\Delta x_i = x_i - x_{i-1}$.

Quyida keltiriladigan momentlarni, og‘irlik markazi va kuchning bajargan ishini hisoblash formulalari ham yuqoridagi singari hosil qilinadi. Shu sababli bu formulalarni keltirib chiqarmaymiz va ulardan masalalarni yechishda foydalanamiz.

Oxy tekislikda massalari mos ravishda $m_1, m_2, \ldots, m_n$ bo‘lgan $A_1(x_1;y_1), A_2(x_2;y_2), \ldots, A_n(x_n;y_n)$ nuqtalar sistemasi berilgan bo‘lsin.

Sistemaning Ox (Oy) o‘qqa nisbatan *statik momenti* M_x (M_y) deb nuqtalar massalarini ularning ordinatalariga (absissalariga) ko‘paytmalari yig‘indisiga aytiladi, ya’ni

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \left(M_y = \sum_{i=1}^{n} m_i x_i\right).$$

Sistemaning Ox (Oy) o‘qqa nisbatan *inersiya momenti* J_x (J_y) deb nuqtalar massalarini ularning ordinatalari (absissalari) kvadratiga ko‘paytmalari yig‘indisiga aytiladi, ya’ni

$$J_x = \sum_{i=1}^{n} m_i y_i^2 \quad \left(J_y = \sum_{i=1}^{n} m_i x_i^2\right).$$

Sistemaning *og‘irlik markazi* deb koordinatalari $\left(\frac{M_y}{m};\frac{M_x}{m}\right)$ bo‘lgan nuqtalarga aytiladi, bu yerda $m = \sum_{i=1}^{n} m_i$.

Yassi egri chiziqning momentlari va og‘irlik markazi

Oxy tekislikda AB egri chiziq $y = f(x)$ $(a \le x \le b)$ tenglama bilan berilgan bo‘lib, egri chiziqning har bir $(x, f(x))$ nuqtasidagi zichlik $\gamma = \gamma(x)$ ga teng va $f(x)$ funksiya $f'(x)$ hosilasi bilan birga uzluksiz bo‘lsin.

U holda AB egri chiziqning momentlari va og‘irlik markazining koordinatalari quyidagi formulalar bilan aniqlanadi:

$$M_x = \int_a^b \gamma y dl, \quad M_y = \int_a^b \gamma x dl; \tag{8.20}$$

$$J_x = \int_a^b \gamma y^2 dl, \quad J_y = \int_a^b \gamma x^2 dl; \tag{8.21}$$

$$x_c = \frac{\int_a^b \gamma x dl}{\int_a^b \gamma dl}, \quad y_c = \frac{\int_a^b \gamma y dl}{\int_a^b \gamma dl}, \tag{8.22}$$

bu yerda $y = f(x)$, $\gamma = \gamma(x)$, $dl = \sqrt{1+y'^2}\,dx$, $a \le x \le b$.

10-misol. Zichligi $\gamma = 1$ ga teng bo'lgan $y = \sqrt{R^2 - x^2}$, $|x| \le R$ yarim aylananing momentlari va og'irlik markazini toping.

Yechish. $y' = -\dfrac{x}{\sqrt{R^2 - x^2}}$ bo'lgani uchun $dl = \dfrac{Rdx}{\sqrt{R^2 - x^2}}$.

U holda (8.20) - (8.22) formulalardan topamiz:

$$M_x = \int_{-R}^{R} \sqrt{R^2 - x^2}\,\frac{Rdx}{\sqrt{R^2 - x^2}} = R\int_{-R}^{R} dx = Rx\Big|_{-R}^{R} = 2R^2,$$

$$M_y = \int_{-R}^{R} \frac{xRdx}{\sqrt{R^2 - x^2}} = -R\sqrt{R^2 - x^2}\Big|_{-R}^{R} = 0.$$

$$J_x = \int_{-R}^{R} (R^2 - x^2)\frac{Rdx}{\sqrt{R^2 - x^2}} = R\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx = R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2\cos^2 t dt =$$

$$= R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt = \frac{R^3}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi R^3}{2},$$

$$J_y = \int_{-R}^{R} \frac{x^2 Rdx}{\sqrt{R^2 - x^2}} = R\left(-x\sqrt{R^2 - x^2}\Big|_{-R}^{R} + \int_{-R}^{R}\sqrt{R^2 - x^2}\,dx\right) = R\left(0 + \frac{\pi R^2}{2}\right) = \frac{\pi R^3}{2}.$$

$$\int_{-R}^{R} dl = \int_{-R}^{R} \frac{Rdx}{\sqrt{R^2 - x^2}} = R\arcsin\frac{x}{R}\Big|_{-R}^{R} = \pi R,$$

$$x_c = 0, \qquad y_c = \frac{2R^2}{\pi R} = \frac{2R}{\pi}.$$

Oxy tekislikda $[a;b]$ kesmada uzluksiz bo'lgan $y = f(x)$ funksiya grafigi, *Ox* o'q, $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya (tekis shakl) berilgan va tekis shaklning har bir nuqtasida $\gamma = \gamma(x)$ zichlik uzluksiz bo'lsin. U holda tekis shaklning momentlari va og'irlik markazining koordinatalari quyidagi formulalar orqali topiladi:

$$M_x = \frac{1}{2}\int_a^b \gamma y^2 dx, \quad M_y = \int_a^b \gamma xy dx; \tag{8.23}$$

$$J_x = \frac{1}{2}\int_a^b \gamma y^3 dx, \quad J_y = \int_a^b \gamma x^2 y dx; \tag{8.24}$$

$$x_c = \frac{\int_a^b \gamma xy dx}{\int_a^b \gamma y dx}, \quad y_c = \frac{\frac{1}{2}\int_a^b \gamma y^2 dx}{\int_a^b \gamma y dx}, \tag{8.25}$$

bu yerda $\gamma = \gamma(x)$, $y = y(x)$, , $a \le x \le b$.

11-misol. $y = \sin x$ sinusoida yoyi va *Ox* o'qining $0 \le x \le \pi$ bo'lagi bilan chegaralangan, zichligi $\gamma = 1$ ga teng tekis shaklning og'irlik markazini toping.

Yechish. Sinusoidaning simmetrikligidan $x_c = \frac{\pi}{2}$ bo'ladi. U holda

$$M_x = \frac{1}{2}\int_0^\pi y^2 dx = \frac{1}{2}\int_0^\pi \sin^2 x dx = \frac{1}{2}\int_0^\pi \frac{1-\cos 2x}{2} dx = \frac{1}{4}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^\pi = \frac{\pi}{4}.$$

$$\int_0^\pi y dx = \int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = 2, \qquad y_c = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}.$$

Demak,

$$x_c = \frac{\pi}{2}, \ y_c = \frac{\pi}{8}.$$

7.8.7. Kuchning bajargan ishini hisoblash

Moddiy nuqta o'zgaruvchan $\vec{F}$ kuch ta'sirida *Ox* o'qi bo'ylab harakatlanayotgan bo'lsin va bunda kuchning yo'nalishi harakat yo'nalishi bilan bir xil bo'lsin. U holda $\vec{F}$ kuchning moddiy nuqtani Ox o'qi bo'ylab $x=a$ nuqtadan $x=b$ $(a<b)$ nuqtaga ko'chirishda bajargan ishi quyidagi formula bilan hisoblanadi:

$$A=\int_a^b F(x)dx, \tag{8.26}$$

bu yerda $F(x)$ funksiya $[a;b]$ kesmada uzluksiz.

12-misol. m massali kosmik kemani yerdan h masofaga uchirish uchun qancha ish bajarish kerak?

Yechish. Butun olam tortishish qonuniga ko'ra, yerning jismni tortish kuchi $F=k\dfrac{mM}{x^2}$ ga teng bo'ladi, bu yerda $M-$ yerning massasi, $x-$ yer markazidan kosmik kemagacha bo'lgan masofa, $k-$ gravitasiya doimiyligi. Yer sirtida, ya'ni $x=R$ da $F=mg$ ga teng, bu yerda g – erkin tushish tezlanishi.

U holda

$$mg=k\frac{mM}{R^2}.$$

Bundan

$$kM=gR^2, \quad F=mg\frac{R^2}{x^2}.$$

Izlanayotgan ishni (8.26) formula bilan topamiz:

$$A=\int_R^{R+h} mg\frac{R^2}{x^2}dx=-mgR^2\frac{1}{x}\Big|_R^{R+h}=-mgR^2\left(\frac{1}{R+h}-\frac{1}{R}\right)=mgR\frac{h}{R+h}.$$

7.8.8. Jismning bosib o'tgan yo'li

Moddiy nuqta (jism) to'g'ri chiziq bo'ylab o'zgaruvchan $v=v(t)$ tezlik bilan harakatlanayotgan bo'lsin. Bu nuqtaning t_1 dan t_2 gacha vaqt oralig'ida bosib o'tgan yo'lini topamiz.

Hosilaning fizik ma'nosiga ko'ra, nuqtaning bir tomonga harakatida «to'g'ri chiziqli harakat tezligi yo'ldan vaqt bo'yicha olingan hosilaga teng», ya'ni $v(t)=\dfrac{dS}{dt}$. Bundan $dS=v(t)dt$. Bu tenglikni t_1 dan t_2 gacha integrallaymiz:

$$S=\int_{t_1}^{t_2} v(t)dt. \tag{8.27}$$

Izoh. Bu formulani aniq integralni qo'llash sxemalari bilan ham topish mumkin.

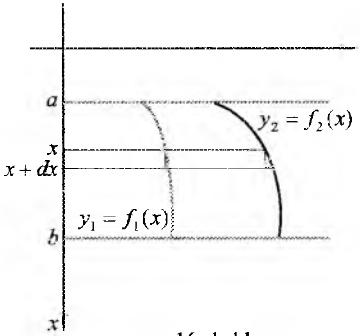
7.8.9. Suyuqlikning vertikal plastinkaga bosimi

Paskal qonuniga ko'ra, suyuqlikning gorizontal plastinkaga bosimi

$$P = g \cdot \gamma \cdot S \cdot h$$

formula bilan topiladi, bu yerda $g-$ erkin tushish tezlanishi, $\gamma-$ suyuqlik zichligi, $S-$ plastinkaning yuzasi, $h-$ plastinkaning botish chuqurligi.

Plastinkaning vertikal botishida suyuqlikning plastinkaga bosimini bu formula bilan topib bo'lmaydi, chunki plastinkaning har xil nuqtalari turli chuqurlikda yotadi.

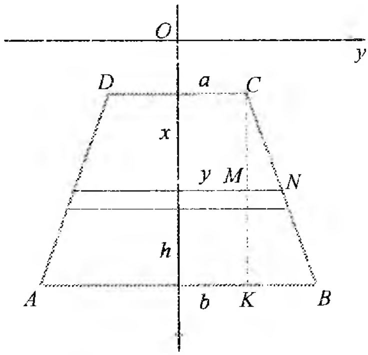


16-shakl.

Suyuqlikka $x = a$, $x = b$, $y_1 = f_1(x)$, $y_2 = f_2(x)$ chiziqlar bilan chegaralangan plastinka vertikal botirilayotgan bo'lsin. Bunda koordinatalar sistemasi 16-shaklda ko'rsatilganidek tanlangan bo'lsin. Suyuqlikning plastinkaga P bosimini topish uchun *II* sxemadan foydalanamiz.

Bunda:

1°. Izlanayotgan P kattalikning bir qismi $x$ning funksiyasi bo'lsin: $p = p(x)$, ya'ni $p = p(x)$ – plastinkaning x o'zgaruvchi $[a;x]$ kesmasiga mos qismiga suyuqlik bosimi, bu yerda $x \in [a;b]$ ($p(a) = 0$ va $p(b) = P$).



17-shakl.

2°. x argumentga $\Delta x = dx$ orttirma beramiz. Bunda $p(x)$ funksiya Δp orttirma oladi (16-shaklda dx qalinlikdagi tasma). Funksiyaning dp differensialni topamiz. dx kichik ekanidan tasmani barcha nuqtalari bitta chuqurlikda yotuvchi to'g'ri to'rtburchak deb hisoblaymiz, ya'ni plastinka gorizontal bo'lsin deymiz.

U holda Paskal qonuniga ko'ra,

$$dp = g \cdot \gamma \cdot \underbrace{(y_2 - y_1) \cdot dx}_{S} \cdot \underbrace{x}_{h}.$$

3^o. dp ni $x = a$ dan $x = b$ gacha integrallaymiz:

$$P = g \cdot \gamma \cdot \int_a^b (y_2 - y_1) x dx$$

yoki

$$P = g \cdot \gamma \cdot \int_a^b (f_2(x) - f_1(x)) x dx. \qquad (8.28)$$

13-misol. Asoslari a va b ga, balandligi h ga teng bo'lgan teng yonli trapetsiya shaklidagi plastinka suyuqlikka c chuqurlikda botirilgan (17-shakl). Suyuqlikning plastinkaga bosimini toping.

Yechish. Izlanayotgan bosim (8.28) formulaga ko'ra,

$$P = g\gamma \int_c^{c+h} y x dx$$

bo'ladi.

y o'zgaruvchini x o'zgaruvchi orqali ifodalash uchun CMN va CKB uchburchaklarning o'xshashligidan foydalanamiz:

$$\frac{y - a}{b - a} = \frac{x - c}{h}.$$

Bundan $y = a + \dfrac{b - a}{h}(x - c)$.

U holda

$$P = g\gamma \int_c^{c+h} \left(a + \frac{b - a}{h}(x - c) \right) x dx = g\gamma \left(\frac{ax^2}{2} + \frac{b - a}{h}\left(\frac{x^3}{3} - \frac{cx^2}{2} \right) \right) \Bigg|_c^{c+h} =$$

$$= g\gamma \left(\frac{a + b}{2} ch + \frac{h^2}{6}(a + 2b) \right).$$

7.8.10. Mashqlar

1 Berilgan chiziqlar bilan chegaralangan tekis shakl yuzalarini hisoblang:

1) $y = 9 - x^2$, $y = 0$; 2) $y = -x$, $y = 2x - x^2$;

3) $y = \ln(x + 6)$, $y = 3\ln x$, $y = 0$, $x = 0$; 4) $y = \ln x$, $y = 0$, $x = e^2$;

5) $x = y^2$, $x = |y+2|$;

6) $xy = 4$, $x = 5 - y$;

7) $y = x^2$, $y^2 = -x$;

8) $y = x^2$, $y = x^3$, $x = -1$, $x = 1$;

9) $x = 4\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$;

10) $x = 3(t - \sin t)$, $y = 3(1 - \cos t)$ (sikloida bitta arkasi)

11) $r = 3\sqrt{\cos 2\varphi}$;

12) $r = 3\sin 2\varphi$.

13) $r = 2 + 3\cos\varphi$;

14) $r = 2\varphi$, bir o'rami.

2. Berilgan egri chiziqlarning argumentning ko'rsatilgan oralig'iga mos yoylari uzunliklarini toping:

1) $y = \dfrac{x^2}{2}$, $x = 0$ dan $x = \sqrt{3}$ gacha;

2) $y = chx$, $x = 0$ dan $x = 1$ gacha;

3) $y^2 = x^3$, $x = 0$ dan $x = 5$ gacha;

4) $y = \arccos\sqrt{x} - \sqrt{x - x^2}$, $x = 0$ dan $x = 1$ gacha;

5) $x = \dfrac{1}{4}y^2 - \dfrac{1}{2}\ln y$, $y = 1$ dan $y = 2$ gacha;

6) $x = 1 - \ln(y^2 - 1)$, $y = 3$ dan $y = 4$ gacha;

7) $x = t^2$, $y = \dfrac{t^3}{3} - t$, koordinata o'qlari bilan kesishish nuqtalari orasidagi;

8) $x = t^2$, $y = t^3$, $t = 0$ dan $t = 1$ gacha;

9) $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ (sikloida bitta arkasi);

10) $x = 3(2\cos t - \cos 2t)$, $y = 3(2\sin t - \sin 2t)$;

11) $r = a(1 - \cos\varphi)$, $r \le \dfrac{a}{2}$ kardioida bo'lagining;

12) $r = 8\cos^3\dfrac{\varphi}{3}$, $\varphi = 0$ dan $\varphi = \dfrac{\pi}{2}$ gacha.

3. Chiziqlarning berilgan o'q atrofida aylanishidan hosil bo'lgan sirt yuzini hisoblang:

1) $y^2 = 4x$, $x = 0$ dan $x = 3$ gacha, Ox o'q;

2) $x^2 + y^2 = 9$, Oy o'q;

3) $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$, bitta arkasi, Ox o'q;

4) $x=\sqrt{2}\cos t,\ \ y=\sin t$, Ox o‘q;

4. R radiusli shar hajmini hisoblang.

5. Asosi $\frac{x^2}{16}+\frac{y^2}{9}=1$ ellipsdan iborat bo‘lgan va balandligi $h=3$ ga teng elliptik konusning hajmini hisoblang.

6. $x^2+y^2+z^2=16$ shar hamda $x=2$ va $x=3$ tekisliklar bilan chegaralangan jism hajmini hisoblang.

7. $\frac{y^2}{4}+\frac{z^2}{9}-x^2=1$ bir pallali giperboloid hamda $x=-1$ va $x=2$ tekisliklar bilan chegaralangan jism hajmini hisoblang.

8. Berilgan chiziqlar bilan chegaralangan tekis shaklning berilgan o‘q atrofida aylanishidan hosil bo‘lgan jism hajmini hisoblang:

1) $x^2=4-y,\ \ y=0$, Ox o‘qi;

2) $x^2+y^2=4$ yarim aylana $(x\geq 0)$ va $y^2=3x$ parabola, Ox o‘qi;

3) $y=\arcsin x,\ \ y=0,\ \ x=1$, Oy o‘qi;

4) $y^2=x^3,\ \ x=1,\ \ y=0$, Oy o‘qi;

5) $x^2=4y,\ \ x=0,\ \ y=1$, Oy o‘qi;

6) $\frac{x^2}{25}+\frac{y^2}{9}=1$, Oy o‘qi;

7) $x=2(t-\sin t),\ \ y=2(1-\cos t)$, bitta arkasi, Ox o‘qi;

8) $x=t^2,\ \ y=t^3,\ \ x=0,\ \ y=1$, Oy o‘qi;

9) $r=3(1+\cos\varphi)$, qutb o‘qi;

10) $r=2R\cos\varphi$, yarim aylana, qutb o‘qi;

9. $r=2R\sin\varphi$ bir jinsli aylananing og‘irlik markazini toping.

10. $x=a\cos^3 t,\ \ y=a\sin^3 t$ bir jinsli astroidaning Ox o‘qdan yuqorida yotgan yoyining og‘irlik markazini toping.

11. $4x+3y-12=0$ bir jinsli to‘g‘ri chiziqning koordinata o‘qlari orasida joylashgan kesmasining koordinata o‘qlariga nisbatan statik momentlarini toping.

12. $x=0,\ \ y=0,\ \ x+y=2$ chiziqlar bilan chegaralangan bir jinsli tekis shaklning koordinata o‘qlariga nisbatan statik va inersiya momentlarini, og‘irlik markazini toping.

13. $y=4-x^2$ va $y=0$ bir jinsli chiziqlar bilan chegaralangan tekis shaklning og'irlik markazini toping.

14. Yarim o'qlari $a=5$ va $b=4$ bo'lgan bir jinsli ellipsning koordinata o'qlariga nisbatan inersiya momentini toping.

15. $x^2+y^2=R^2$ aylananing birinchi chorakda joylashgan bo'lagining o'girlik markazini toping. Bunda aylananing har bir nuqtasidagi chiziqli zichligi shu nuqta koordinatalarining ko'paytmasiga proporsional.

16. $x=8\cos^3 t,\ 8=4\sin^3 t$ astroida birinchi chorakda yotgan yoyining koordinata o'qlariga nisbatan statik momentlarini va massasini toping. Bunda astroidaning har bir nuqtasidagi chiziqli zichligi x ga teng.

17. Prujinani $4\ sm$ ga cho'zish uchun $24\ J$ ish bajariladi. $150\ J$ ish bajarilsa, prujina qanday uzunlikka cho'ziladi?

18. Agar prujinani $1\ sm$ ga siqish uchun $1kG$ kuch sarf qilinsa, prujinaning $8\ sm$ ga siqishda sarf bo'ladigan F kuch bajargan ishni toping.

19. Uzunligi $0,5\ m$ va radiusi $4\ mm$ bo'lgan mis simni $2\ mm$ cho'zish uchun qancha ish bajarish kerak? Bunda $F=E\dfrac{Sx}{l}$, $E=12\cdot 10^4\ N/mm^2$.

20. Og'irligi $P=1,5\ T$ bo'lgan kosmik kemani yer sirtidan $h=2000\ km$ masofaga uchirish uchun bajarilishi kerak bo'ladigan ishni toping.

21. Jismning to'g'ri chiziqli harakat tezligi $v=2t+3t^2\ (m/s)$ formula bilan ifodalanadi. Jismning harakat boshlanishidan $5\ s$ davomida bosib o'tgan yo'lini toping.

22. Nuqtaning harakat tezligi $v=0,1t^3\ (m/s)$ ga teng. Nuqtaning $10\ s$ davomidagi o'rtacha tezligini toping.

23. Sportchining parashutdan tushish tezligi $v=\dfrac{mg}{k}\left(1-e^{-\frac{kt}{m}}\right)$ formula bilan ifodalanadi, bu yerda g – erkin tushish tezlanishi, m – sportchining massasi, k – proporsionallik koeffitsiyenti. Agar parashutdan tushish $3\ \text{min}$ davom etgan bo'lsa, sportchi qanday balandlikdan sakragan?

24. Nuqtaning harakat tezligi $v=0,1e^{-0,01t}\ (m/s)$ ga teng. Nuqtaning harakat boshlanishidan harakat to'xtaguncha bosib o'tgan yo'lini toping.

25. Suyuqlikka vertikal botirilgan asoslari a va $b\ (b>a)$ ga, balandligi h ga teng bo'lgan teng yonli trapetsiya shaklidagi plastinkaga suyuqlikning bosimini toping.

26. Asosi 18 m va balandligi $6\ m$ bo'lgan to'rtburchakli shlyuzga suv bosimini toping.

27. Diametri 6 m bo'lgan va suv sathida joylashgan yarim doira shaklidagi vertikal devorga suv bosimini toping. Suv zichligi $\gamma = 1000\ kg/m^3$.

7.9. ANIQ INTEGRALNI TAQRIBIY HISOBLASH

Ma'lumki, agar $f(x)$ funksiyaning $[a;b]$ kesmada boshlang'ich funksiyasi $F(x)$ mavjud bo'lsa, $f(x)$ funksiyaning aniq integrali Nyuton-Leybnis formulasi bilan topiladi. Elementar funksiyalarda olinmaydigan aniq integrallar amalda taqribiy hisoblash usullari bilan topiladi. Bunday usullardan aniq integralning integral yig'indining limiti haqidagi ta'rifiga va geometrik ma'nosiga asoslangan usullarni ko'rib chiqamiz.

7.9.1. To'g'ri to'rtburchaklar formulasi

$[a;b]$ kesmada uzluksiz $y = f(x)$ funksiya uchun $\int_a^b f(x)dx$ integralni hisoblash talab qilingan bo'lsin. Aniqlik uchun barcha $x \in [a;b]$ da $f(x)$ funksiya musbat va monoton o'suvchi deb faraz qilamiz.

$[a;b]$ kesmani $a = x_0 < x_1 < ... < x_{i-1} < x_i < ... < x_{n-1} < x_n = b$ nuqtalar bilan uzunliklari $\Delta x = \dfrac{b-a}{n}$ bo'lgan n ta teng kesmalarga ajratamiz. Funksiyaning $x_0,\ x_1, ..., x_i, ..., x_n$ nuqtalardagi qiymatlarini $y_0,\ y_1, ..., y_i, ..., y_n$ bilan belgilab, $y_0 = f(x_0),\ y_1 = f(x_1), ..., y_i = f(x_i), ..., y_n = f(x_n)$ larni hisoblaymiz va quyidagi integral yig'indilarni tuzamiz:

$$y_0\Delta x + y_1\Delta x + ... + y_i\Delta x + ... + y_{n-1}\Delta x = \sum_{i=0}^{n-1} y_i\Delta x,$$

$$y_1\Delta x + y_2\Delta x + ... + y_i\Delta x + ... + y_n\Delta x = \sum_{i=1}^{n} y_i\Delta x.$$

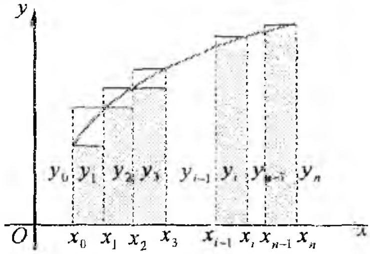
U holda 18-shaklga ko'ra,

$$\int_a^b f(x)dx \approx \frac{b-a}{n}(y_0 + y_1 + ... + y_i + ... + y_{n-1}) = \frac{b-a}{n}\sum_{i=0}^{n-1} y_i, \tag{9.1}$$

$$\int_a^b f(x)dx \approx \frac{b-a}{n}(y_1 + y_2 + ... + y_i + ... + y_n) = \frac{b-a}{n}\sum_{i=1}^{n} y_i. \tag{9.2}$$

(9.1) va (9.2) formulalar aniq integralni taqribiy hisoblashning *to'g'ri to'rtburchaklar formulalari* deyiladi.

18-shakldan ko'rinadiki, agar $f(x)$ funksiya $[a;b]$ kesmada musbat va o'suvchi bo'lsa, (9.1) formula berilgan integralning qiymatini kami bilan, (9.2) formula esa ortig'i bilan beradi.



18-shakl.

Agar $[a;b]$ kesmada $f'(x)$ chekli hosila mavjud bo'lsa, to'g'ri to'rtburchaklar formulalarining absolut xatoliklari $|\delta_n| \le \frac{M_1(b-a)^2}{2n}$ tengsizlik bilan baholanadi, bu yerda $M_1 = \max\limits_{a\le x\le b}|f'(x)|$.

7.9.2. Trapetsiyalar formulasi

$[a;b]$ kesmani bo'lishlar sonini avvalgidek qoldiramiz va Δx intervalga mos keladigan $y = f(x)$ funksiyaning har bir yoyini bu yoyning chetki nuqtalarini tutashtiruvchi N_0N_1, $N_1N_2,...,N_{i-1}N_i,...,N_{n-1}N_n$ vatarlar bilan almashtiramiz. Bunda berilgan egri chiziqli trapetsiya n ta to'g'ri chiziqli trapetsiya bilan almashtiriladi (19-shakl).

Bu to'g'ri chiziqli trapetsiyalar har birining yuzasi $\frac{y_{i-1}+y_i}{2}\Delta x$ $(i=\overline{1,n})$ ga teng. Bu yuzalarning barchasini qo'shib, *trapetsiyalar formulasi*ni hosil qilamiz:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n}\frac{y_{i-1}+y_i}{2}\Delta x = \frac{b-a}{n}\left(\frac{y_0+y_n}{2}+y_1+y_2+...+y_{n-1}\right). \qquad (9.3)$$

Bu formulaning xatoligi $|\delta_n| \le \frac{M_2(b-a)^3}{12n^2}$ tengsizlik bilan baholanadi, bu yerda $M_2 = \max\limits_{a\le x\le b}|f''(x)|$.

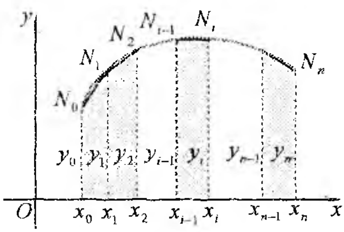
7.9.3. Simpson formulasi (parabolalar usuli)

$[a;b]$ kesmani

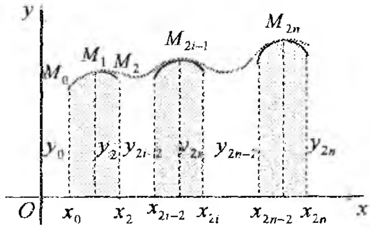
$$a = x_0 < x_1 < x_2 < ... < x_{2i-2} < x_{2i-1} < x_{2i} < ... < x_{2m-2} < x_{2m-1} < x_{2m} = b$$

nuqtalar bilan uzunliklari $h = \frac{b-a}{2m}$ bo'lgan $n = 2m$ ta teng kesmalarga ajratamiz va har bir ketma-ket kelgan M_{2m-2}, M_{2m-1}, M_{2m} nuqtalar orqali parabolalar o'tkazamiz. Egri chiziqli $aM_0M_{2m}b$ trapetsiyaning yuzasini parabolalarning M_{2i-2}, M_{2i-1} va M_{2i-1} $(i=\overline{1,m})$ nuqtalarini tutashtiruvchi yoylari bilan chegaralangan $2m$ ta parabolik trapetsiyalar yuzalarining yig'indisi bilan almashtiramiz.

M_{2m-2}, M_{2m-1}, M_{2m} nuqtalar orqali o'tkazilgan parabola tenglamasini $y = Ax^2 + Bx + C$ ko'rinishda izlaymiz. A, B, C koeffitsiyentlarni parabolaning berilgan uchta nuqtadan o'tishi shartidan topamiz. Hisoblashlar qulay bo'lishi uchun koordinatalar boshini o'qlarning yo'nalishini o'zgartirmasdan $[x_{2i-2}; x_{2i}]$ kesmaning o'rtasiga joylashtiramiz (20-shakl).



19-shakl.



20-shakl.

Parabolaning $(-h; y_{2i-2})$, $(0; y_{2i-1})$, $(h; y_{2i})$ nuqtalardan o'tishi shartidan $y_{2i-2} = Ah^2 - Bh + C$, $y_{2i-1} = C$, $y_{2i} = Ah^2 + Bh + C$ kelib chiqadi.

Bundan

$$A = \frac{1}{2h^2}(y_{2i-2} - 2y_{2i-1} + y_{2i}),\ B = \frac{1}{2h}(y_{2i} - y_{2i-2}),\ C = y_{2i-1}.$$

Endi $2i$-parabolik trapetsiya yuzini aniq integral orqali hisoblaymiz:

$$S_{2i} = \int_{-h}^{h}(Ax^2 + Bx + C)dx = \left(A\frac{x^3}{3} + B\frac{x^2}{2} + Cx\right)\Bigg|_{-h}^{h} = \frac{h}{3}(2Ah^2 + 6C).$$

Formulaga A, B, C koeffitsiyentlarning qiymatlarini qo'yib, topamiz:

$$S_{2i} = \frac{b-a}{6m}(y_{2i-2} + 4y_{2i-1} + y_{2i}),\ i = \overline{1,n}.$$

Parabolik trapetsiyalarning yuzlarini qo‘shib, izlanayotgan integralning taqribiy qiymatini beruvchi formulani hosil qilamiz:

$$\int_a^b f(x)dx \approx \frac{b-a}{6m}\left(y_0 + y_{2m} + 4(y_1 + y_3 + \ldots + y_{2m-1}) + 2(y_2 + y_4 + \ldots + y_{2m-2})\right). \quad (9.4)$$

(9.4) formulaga *Simpson formulasi* (yoki *parabolalar formulasi*) deyiladi.

Bu formulaning xatoligi $|\delta_n| \le \frac{M_4(b-a)^5}{2880n^4}$ tengsizlik bilan baholanadi, bu yerda $M_4 = \max\limits_{a\le x\le b}\left|f^{IV}(x)\right|$.

1-misol. $\int_0^1 \frac{dx}{1+x^2}$ integralni taqribiy hisoblash usullari bilan ($n = 10$ da) aniqlang.

Yechish. Quyidagi jadvalni tuzamiz:

x	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
y	1	0,9901	0,9615	0,9174	0,8621	0,8	0,7353	0,6711	0,6098	0,5526	0,5

1) (9.1) va (9.2) formulalar yordamida to‘g‘ri to‘rtburchaklar usuli bilan topamiz:

$$\int_0^1 \frac{dx}{1+x^2} \approx 0{,}1(1 + 0{,}9901 + 0{,}9615 + 0{.}9174 + 0{,}8621 + 0{,}8 + 0{,}7353 + 0{,}6711 +$$

$$+ 0{,}6098 + 0{,}5525) = 0{,}1 \cdot 8{,}0998 = 0{,}80998 \ (\textit{ortig'i bilan}),$$

$$\int_0^1 \frac{dx}{1+x^2} \approx 0{,}1(0{,}9901 + 0{,}9615 + 0{,}9174 + 0{,}8621 + 0{,}8 + 0{,}7353 + 0{,}6711 +$$

$$+ 0{,}6098 + 0{,}5525 + 0{,}5) = 0{,}1 \cdot 7{,}5998 = 0{,}75998 \ (\textit{kami bilan}).$$

2) Trapetsiyalar formulasi bilan hisoblaymiz:

$$\int_0^1 \frac{dx}{1+x^2} \approx 0{,}1\left(\frac{1+0{,}5}{2} + 0{,}9901 + 0{,}9615 + 0{,}9174 + 0{,}8621 + 0{,}8 + 0{,}7353 +\right.$$

$$+ 0{,}6711 + 0{,}6098 + 0{,}5525 = 0.1 \cdot 7{,}8498 = 0{,}78498.$$

3) Simpson formulasi bilan hisoblaymiz:

$$\int_0^1 \frac{dx}{1+x^2} = (1+0,5+4(0,9901+0,9174+0,8+0,6711+0,5525)+$$

$$+290,9615+0,8621+0,7353+0,6098)) = 0,78539$$

Berilgan integralni aniq qiymatini topamiz:

$$\int_0^1 \frac{dx}{1+x^2} = arctgx\Big|_0^1 = \frac{\pi}{4} = 0,787571.$$

Taqribiy hisoblashlarning nisbiy va absolut xatoliklarini aniqlaymiz:

To'g'ri to'rtburchaklar (kami bilan)		**To'g'ri to'rtburchaklar (ortig'i bilan)**		**Trapetsiyalar**		**Parabolalar**	
3,3%	**0.026**	**3,1%**	**0.024**	**0,09%**	**0.0003**	**0,04%**	**0,0003**

Berilgan integralning *Maple* paketida yechimini beramiz.

1) To'g'ri to'rtburchaklar usuli bo'yicha:

> $restart:$

> $with(Student[Calculus1]):$

> $RiemannSum\left(\frac{1}{(1+x^2)}, x=0..1, method=left\right)$

$$\frac{1579799420518583}{1950414208136225}$$

> $evalf(\%);$

$$0.8099814972$$

> $RiemannSum\left(\frac{1}{(1+x^2)}, x=0..1, method=right\right);$

$$\frac{5929114840447087}{7801656832544900}$$

> $evalf(\%);$

$$0.7599814972$$

2) Trapetsiyalar usuli bo'yicha:

>$with(Student[Calculus1]):$

> $ApproximateInt\left(\frac{1}{(1+x^2)}, x=0..1, method=trapezoid\right);$

$$\frac{12248312522521419}{15603313665089800}$$

> $evalf(\%);$

$$0.7849814972$$

3) Simpson formulasi bo'yicha:

> $with(Student[Calculus1]):$

> $ApproximateInt\left(\frac{1}{(1+x^2)}, x=0..1, method=simpson\right);$

$$\frac{59885853158383117749014845366768364 63}{762490364265046352030169414165528 3000}$$

> $evalf(\%);$

$$0.7853981632$$

7.9.4. Mashqlar

1. $\int_0^1 e^{-x^2}dx$ aniq integralni 0,001 aniqlikda taqribiy hisoblang: 1) to'g'ri to'rtburchaklar usullari bilan; 2) trapetsiyalar usuli bilan; 3) Simpson formulasi bilan.

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JAVOBLAR

Matritsalar

2. $A=3X+4Y+5Z$. **3.** $a=2, b=-4$ **4.** $1\times 30, 30\times 1, 2\times 15, 15\times 2, 3\times 10, 10\times 3, 5\times 6, 6\times 5$.

5. $\begin{pmatrix} 3 & 7 & -1 \\ -4 & 3 & 4 \end{pmatrix}$. **6.** $\begin{pmatrix} 3 & -12 \\ -13 & 5 \\ -4 & 23 \end{pmatrix}$. **7.** $\begin{pmatrix} 0 & 1 & -2 \\ 3 & -7 & 6 \\ 2 & -3 & -7 \end{pmatrix}$. **8.** $\begin{pmatrix} 2-\nu & -1 & 2 \\ 5 & -3-\nu & 3 \\ -1 & 0 & -2-\nu \end{pmatrix}$.

10. $x=3, y=0$. **11.** 3×5. **12.** $\begin{pmatrix} 0 & 0 \\ b_{21} & b_{22} \end{pmatrix}$. **13.** $\begin{pmatrix} 2 & -2 & -4 \\ 8 & 7 & 2 \end{pmatrix}$. **14.** $\begin{pmatrix} 10 & -1 \\ -2 & -3 \\ 16 & 0 \end{pmatrix}$.

15. $\begin{pmatrix} 7 & 6 \\ -1 & 10 \\ -2 & 5 \end{pmatrix}$. **16.** $\begin{pmatrix} 2 & -1 & 4 \\ -8 & -3 & 13 \\ 2 & 1 & -2 \end{pmatrix}$. **17.** $\begin{pmatrix} -8 & 20 \\ -38 & 30 \end{pmatrix}$. **18.** $\begin{pmatrix} 35 & 67 \\ 154 & 166 \end{pmatrix}$.

19. $\begin{pmatrix} -12 & 0 \\ -9 & 0 \\ 2 & 5 \end{pmatrix}$; $\begin{pmatrix} 13 & 6 \\ 6 & 5 \end{pmatrix}$; $\begin{pmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{pmatrix}$. **20.** $\begin{pmatrix} 4 & -34 \\ 0 & 38 \end{pmatrix}$. **21.** $\begin{pmatrix} 1 & 0 \\ 20 & 1 \end{pmatrix}$.

22. $I, -I, \begin{pmatrix} 1 & a_{12} \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & a_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_{21} & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ a_{21} & 1 \end{pmatrix}$.

Determinantlar

1. $\lambda^n \det A$. **5.** 1. **6.** 1) -2; 2) -125; 3) 1; 4) 8. **7.** 1) -12; 2) -243; 3) 64; 4) 4. **8.** $-x^2$. **9** $-b(a+b)$. **10.** $\sin(\alpha-\beta)\sin(\alpha+\beta)$. **11.** $2\sin\alpha$. **12.** -47. **13.** -18. **14.** $b^2(b-2)$. **15.** $4x$. **16.** $-2\sin\alpha\sin\beta\sin\gamma$. **17.** $-tg\alpha - tg\beta$. **18.** $(a-b)(a-c)(b-c)$. **19.** $a(x-y)(x-z)(z-y)$. **20.** $a^2(a+3b)$. **21.** $-xyz$ **22.** 0. **23.** $\cos 2\alpha$. **4.** 63. **25.** 100. **26.** $2a-8b+c+5d$. **27.** -6.

Matritsalar ustida almashtirishlar

3. $ad-bc\neq 0$. **7.** B, D. **8.** $A^{-1}=\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. **9.** 1);2)3). **10.** $\frac{1}{3}\begin{pmatrix} -5 & -6 \\ -2 & -3 \end{pmatrix}$.

11. $\frac{1}{22}\begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix}$. **12.** 1) $A=\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; 2) $A=\frac{1}{10}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$; 3) $A=\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$;

4) $A=\frac{1}{2}\begin{pmatrix} -9 & 14 & -3 \\ -4 & 8 & -2 \\ 3 & -4 & 1 \end{pmatrix}$. **13.** $\begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. **14.** 0. **15.** -10.

16. $\frac{1}{2}\begin{pmatrix} 10 & -2 & -3 \\ -6 & 2 & 2 \\ -2 & 0 & 1 \end{pmatrix}$. **17.** $\frac{1}{6}\begin{pmatrix} 8 & 14 & -10 \\ -4 & -1 & 2 \\ 2 & -4 & 2 \end{pmatrix}$. **18.** $\frac{1}{8}\begin{pmatrix} 0 & 0 & 4 & -4 \\ -4 & 6 & -3 & 5 \\ 0 & -4 & 6 & -2 \\ 4 & 2 & -5 & 3 \end{pmatrix}$.

19. $\begin{pmatrix} 2 & -1 & -1 & 1 \\ -4 & 0 & 2 & -1 \\ 2 & 0 & -1 & 1 \\ -5 & 1 & 3 & -2 \end{pmatrix}$. **20.** 1) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$; 2) $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix}$;

3) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$; 4) mavjud emas; 5) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 5 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 2 & 7 \end{pmatrix}$;

6) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{pmatrix}$. **21.** 3 . **22.** 2. **23.** 2. **24.** 3.

Chiziqli tenglamalar sistemasi

1. Birgalikda emas. **2.** Birgalikda, aniqmas. **3.** Birgalikda, aniq. **4.** Birgalikda emas.
5. $x_1 = -1,\ x_2 = -2,\ x_3 = -3$. **6.** $x_1 = 2,\ x_2 = -2,\ x_3 = 1$.

7. $x_1 = 1,\ x_2 = -1,\ x_3 = -1,\ x_4 = 1$. **8.** $x_1 = 2,\ x_2 = -1,\ x_3 = -2,\ x_4 = 1$.

9. $x_1 = 2,\ x_2 = c + 1,\ x_3 = 2c - 1,\ x_4 = c$. **10.** $x_1 = 5c_2 - 13c_1 - 3,\ x_2 = 5c_2 - 8c_1 - 1,$

$x_3 = c_1,\ x_4 = c_2$. **11.** $x_1 = 3,\ x_2 = 0,\ x_3 = 6$. **12.** $x_1 = 1,\ x_2 = -1,\ x_3 = 0$.

13. $x_1 = 3,\ x_2 = 4,\ x_3 = -6$. **4.14.** $x_1 = -5,\ x_2 = -4,\ x_3 = 0$. **15.** $x_1 = -5,\ x_2 = 1,\ x_3 = 3$.

16. $x_1 = -11,\ x_2 = -6,\ x_3 = -3$. **17.** $x_1 = -1,\ x_2 = 3,\ x_3 = 2$. **18.** $x_1 = 3,\ x_2 = -3,\ x_3 = 1$.

19. $x_1 = 3,\ x_2 = 2,\ x_3 = 1$. **20.** $x_1 = 1,\ x_2 = 1,\ x_3 = -1$. **21.** $x_1 = 3,\ x_2 = -2$.

22. $x_1 = 3,\ x_2 = -2$. **23.** $x_1 = 1,\ x_2 = 2,\ x_3 = 0$. **24.** $x_1 = 1,\ x_2 = -2,\ x_3 = 2$.

25. $x_1 = -2,\ x_2 = 1,\ x_3 = 2$. **26.** $x_1 = 0,\ x_2 = \frac{1}{a},\ x_3 = 0,\ a(a-1)(a+2) \neq 0$.

27. $x_1 = -c,\ x_2 = 0,\ x_3 = c$. **28.** $x_1 = -15c,\ x_2 = 11c,\ x_3 = 14c$.

29. $x_1 = 7c,\ x_2 = -11c,\ x_3 = -5c$. **30.** $x_1 = x_2 = x_3 = 0$. **31.** $x_1 = x_2 = x_3 = x_4 = 0$.

32. $x_1 = -2c, x_2 = 7c, x_3 = 0, x_4 = 3c$. **33.** $\left(1, 0 - \frac{3}{2}, -\frac{1}{2}\right)$, $(0,1,0,-1)$. **34.** $(1,0-2,3), (0,1,1,2)$.

Vektorlar

1. $\vec{a}\perp\vec{b}$. **2.** 22. **5.** $\overrightarrow{AN}=\dfrac{\vec{a}+2\vec{b}}{3}$. **6.** $\overrightarrow{NM}=\vec{n}-2\vec{m}$. **7.** $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}$.

8. $-\dfrac{1}{2}(\vec{a}+\vec{b})$, $\dfrac{1}{2}(\vec{a}-\vec{b})$. **9.** $m=2\sqrt{3}$. **10.** $m=1$, $n=-3$. **11.** $\Pi p_l\overrightarrow{AB}=2\sqrt{2}$,

$\Pi p_l\overrightarrow{AD}=-\sqrt{2}$, $\Pi p_l\overrightarrow{DC}=\sqrt{2}$, $\Pi p_l\overrightarrow{AC}=0$. **12.** $\Pi p_l\overrightarrow{AB}=3$, $\Pi p_l\overrightarrow{BC}=0$, $\Pi p_l\overrightarrow{CA}=-3$,

$\Pi p_l\overrightarrow{AD}=3$, $\Pi p_l\overrightarrow{BF}=-\dfrac{3}{2}$, $\Pi p_l\overrightarrow{CE}=-\dfrac{3}{2}$. **1.13.** $\left\{\dfrac{1}{2};-\dfrac{1}{2}\right\}$. **14.** $\left\{\dfrac{2}{3};\dfrac{2}{3}\right\}$. **15.** $\left\{\dfrac{1}{2};\dfrac{1}{2};-\dfrac{1}{2}\right\}$.

16. $\left\{\dfrac{1}{3};\dfrac{1}{3};\dfrac{1}{3}\right\}$. **17.** $\vec{a}=2\vec{b}+\vec{c}$, $\vec{b}=\dfrac{\vec{a}-\vec{c}}{2}$, $\vec{c}=\vec{a}-2\vec{b}$. **1.18.** $\vec{d}=2\vec{a}-3\vec{b}+\vec{c}$.

19.) $\{-7;17;-12\}$; 2) $\left\{\dfrac{5}{3};-\dfrac{7}{3};\dfrac{8}{3}\right\}$. **20.** 1) $\{-8;-4;0\}$; 2) $\{7;-3;2\}$. **21.** $|\vec{a}+\vec{b}|=6$, $|\vec{a}-\vec{b}|=14$.

22. $\sqrt{10}$. **23.** $M(\pm\sqrt{3};\pm\sqrt{3};\pm\sqrt{3})$. **24.** $\vec{a}=\{2;\pm2\sqrt{2};-2\}$ **25.** $B(5;-3;-3\}$. **26.** $A(-3;-1;-3\}$.

27. 1) $|\overrightarrow{AB}|^{o}=\left\{\dfrac{12}{25};\dfrac{3}{5};-\dfrac{16}{25}\right\}$; 2) $\overrightarrow{AB}|^{o}=\left\{-\dfrac{4}{13};-\dfrac{3}{13};-\dfrac{12}{13}\right\}$. **28.** $\vec{a}^{o}=\left\{-\dfrac{2}{7};\dfrac{6}{7};\dfrac{3}{7}\right\}$. **29.** $\alpha=-3$.

30. $\vec{b}=\left\{\dfrac{48}{5};-\dfrac{36}{5};9\right\}$. **31.** $p=12$. **33.** $O\left(\dfrac{1}{2};\dfrac{13}{2}\right)$, $R=\dfrac{\sqrt{130}}{2}$. **34.** $M(-3;-5)$.

35. $(-2;1)$. **36.** $D(-3;-7)$ **.37.** $AD=7b$.

Vektorlarni ko'paytirish

1. $-\dfrac{3}{2}$. **2.** -19. **3.** 1)112; 2)252. **4.** 1) -89; 2)86. **5.** 1) $m=1$; 2) $m=2$, $m=3$.

6. $-\dfrac{3}{2}$. **7.** $\dfrac{\pi}{2}$. **8.** $\dfrac{\pi}{4}$. **9.** $\dfrac{\pi}{3}$. **10.** 1) $\dfrac{\pi}{3}$; 2) π. **11.** 1) $\dfrac{21}{13}$; 2) $\dfrac{261}{13}$. **12.** 10 (*ish.b.*).

13. $\vec{x}=2\vec{i}-3\vec{j}$. **14.** $\vec{x}=7\vec{i}+5\vec{j}+\vec{k}$. **15.** 1) $12\vec{e}^{\,0}$; 2) 132. **16.** 1) $\dfrac{3}{2}$(*y.b.*); 2) $66\sqrt{3}$(*y.b.*).

17. $25\sqrt{3}$. **18.** ±15. **19.** 1) $\{9;9;-3\}$; 2) $\{63;63;-21\}$. **20.** 1) $9\sqrt{2}$; 2) $\dfrac{49}{2}$.

21. $h=\dfrac{14}{\sqrt{13}}$(*u.b.*). **22.** $\vec{M}=\{-8;-9;-4\}$; $\vec{M}=\{1;-4;-7\}$. **23.** 28, $\cos\alpha=-\dfrac{3}{7}$, $\cos\beta=-\dfrac{6}{7}$,

$\cos\gamma=\dfrac{2}{7}$. **24.** $\alpha=-9$. **25.** $\alpha=\dfrac{3}{2}$, $\beta=2$. **26.** 1) $\{3;2\}$; 2) $\{-2;3\}$. **27.** $\{-6;-24;8\}$.

28. 1) yo'q; 2) ha. **29.** 1) $\alpha=\dfrac{1}{3}$; 2) $\alpha=-3$. **30.** 1) $V=14$(*h.b.*), $h=\sqrt{14}$(*u.b.*); 2) $V=2$(*h.b.*), $h=3\sqrt{2}$(*u.b.*). **31.** 1) o'ng uchlik, $V=12$(*h.b*); 2) chap uchlik, $V=27$(*h.b*).

Tekislikdagi to'g'ri chiziq

1. 1) $k=-\dfrac{3}{4}, a=4, b=3$; 2) $k=\dfrac{1}{3}, a=-2, b=\dfrac{2}{3}$; 3) $k=\dfrac{1}{2}, a=5, b=-\dfrac{5}{2}$;

4) $k=-\frac{3}{5}, a=\frac{5}{2}, b=\frac{3}{2}$. **2.** 1) $3x+4y+6=0$; 2) $3x+y+9=0$; 3) $x+2=0$;

4) $x+y-5=0$. **3.** 2va3. .**4.** 1) $M_0(1;2), \varphi=45^0$; 2) $M_0(2;-1), \varphi=90^0$;

3) $M_0 \in \varnothing, \varphi=0$; 4) $M_0(2;2), \varphi=45^0$. **5.** 1) $m=-6, n\neq 3$ va $m=6$, $n\neq -3$;

2) $m=-6, n=3$ va $m=6$, $n=-3$; 3) $m=0$, $n-$chekli son. **6.** 1) $m=-\frac{3}{2}$ *da* $\|$.

$m=\frac{2}{3}$ *da* $\perp$; 2) $m=4$ *da* $\|$, $m=-9$ *da* $\perp$. **7.** (1;6). **8.** $x-y-2=0$ va $x-4y+4=0$.

9. $3x+2y-11=0$. **10.** $x-5y+2=0$. **11.** $12x+9y-17=0$. **12.** $5x-y+3=0$, $x+5y+11=0$. **13.** $3x+y-4=0$, $x+5y+8=0$, $3x+y+10=0$, $x+5y-6=0$.

14. $M(4;4), \varphi=\frac{\pi}{2}$. **15.** $3x-3y-8=0$. **16.** $3x+4y-12=0$.

17. $x+2y-7=0$, $7x+2y-37=0$, $5x-2y+1=0$. **18.** $y=2x$.

19. $x-y+7=0$, $7x+4y-6=0$, $6x+5y+9=0$. **20.** $2x+y+9=0$, $x-y-3=0$.

21. $29x-2y+33=0$. **22.** $2x-3y+8=0$. **23.** $29(y.b)$. **24.** $\frac{23}{10}(u.b)$. **25.** $6\sqrt{2}(u.b)$.

26. $x-2=0$, $3x-4y+10=0$. **27.** $y-3=0$, $4x+y+5=0$. **8.** $(-12;5)$.

Tekislikdagi ikkinchi tartibli chiziqlar

1. 1) $(x+1)^2+(y-3)^2=36$; 2) $(x+3)^2+(y-5)^2=50$; 3) $(x+2)^2+(y-4)^2=2$;

4) $(x-4)^2+(y+4)^2=16$, $(x-20)^2+(y+20)^2=400$; 5) $(x-2)^2+(y+1)^2=1$.

2. 1) $M_0(-4;7)$, $R=7$; 2) $M_0(-2;3)$, $R=4$; 3) $M_0\left(\frac{1}{2};-1\right)$, $R=\frac{3}{2}$; 4) $M_0\left(-\frac{3}{2};\frac{7}{2}\right)$, $R=4$. **3.** 1) aylanada; 2) aylana ichkarisida; 3) aylanada; 4) aylana ichkarisida.

4. $5\sqrt{2}(u.b)$. **5.** $(x-2)^2+\left(y-\frac{3}{2}\right)^2=\frac{25}{4}$. **6.** $(x-5)^2+(y-1)^2=13$. **7.** $M_0(3;2)$, $R=5$. **8.** $0<k<\frac{8}{15}, k_1=0$ va $k_2=\frac{8}{15}$. **9.** $y=0$ va $4x-3y=0$.

10. 1) $\begin{cases} x=8(1+\cos t), \\ y=8\sin t, t\in[0;2\pi]; \end{cases}$ 2) $\begin{cases} x=2\cos t, \\ y=2(1+\sin t), t\in[0;2\pi]; \end{cases}$ 3) $\begin{cases} x=1+\sqrt{2}\cos tt, \\ y=1+\sqrt{2}\sin t, t\in[0;2\pi]. \end{cases}$

11. 1) $\frac{x^2}{36}+\frac{y^2}{100}=1$; 2) $\frac{x^2}{24}+\frac{y^2}{49}=1$; 3) $\frac{x^2}{36}+\frac{y^2}{81}=1$; 4) $\frac{x^2}{16}+\frac{y^2}{25}=1$.

12. $12(kv.b)$. **13.** $x+y+5=0$ va $x+y-5=0$. **14.** $\frac{32}{5}(u.b)$.

15. $M_1\left(-\frac{15\sqrt{2}}{4};\frac{\sqrt{126}}{4}\right)$, $M_2\left(-\frac{15\sqrt{2}}{4};-\frac{\sqrt{126}}{4}\right)$. **16.** $M(3;0)$. **17.** $16x^2+25y^2=400$.

18. 1) $\begin{cases} x=5\cos t, \\ y=4\sin t, t\in[0;2\pi]; \end{cases}$ 2) $\begin{cases} x=5\cos t, \\ y=12\sin t, t\in[0;2\pi]. \end{cases}$ **19.** $\left(4;\frac{3}{2}\right)$, $(3;2)$..

20. 1) $\frac{y^2}{9}-\frac{x^2}{16}=1$; 2) $\frac{y^2}{144}-\frac{x^2}{25}=1$; 3) $\frac{y^2}{16}-\frac{x^2}{9}=1$; 4) $\frac{y^2}{25}-\frac{x^2}{24}=1$.

21. 1) $\frac{x^2}{24}-\frac{y^2}{8}=1$; 2) $\frac{x^2}{8}-\frac{y^2}{4}=1$; 3) $\frac{x^2}{12}-\frac{y^2}{27}=1$; 4) $\frac{x^2}{24}-\frac{y^2}{18}=1$. **22.** $\frac{2\pi}{3}$.

23. $\sqrt{2}$. **24.** $x^2-y^2=6$. **25.** $\frac{x^2}{4}-\frac{y^2}{12}=1$. **26.** $y=\frac{-3x+10}{x-2}$. **27.** $y'=-\frac{29}{9x'}$.

28.) $x=-\frac{1}{16}y^2+\frac{1}{2}y$; 2) $y=\frac{1}{10}x^2-x+3$. **29.** 1) $A(-4;1)$, $y=1$; 2) $A(2;3)$, $x=2$.

30. 1) $x^2-y^2=1-$giperbola; 2) $y^2=\frac{9}{2}x-$ parabola; 3)giperbolaning pastgi yarim tekislikdagi tarmog‘i; 4) giperbolaning chap yarim tekislikdagi tarmog‘i.

33. $\frac{x'^2}{4}+\frac{y'^2}{9}=1$. **34.** $\frac{x'^2}{2}-y'^2=1$. **36.** $x-3y+12=0$. (Ko‘rsatma. urinma tenglamalari: $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1-ellips$, $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1-giperbola$, $yy_0=p(x+x_0)-parabola$). **37.** $4x+y-4=0$. **37.** $2\sqrt{5}x-3y+4=0$. **38.** $2x-y-16=0$.

Qutb koordinatalarda chiziqlar

1. $A\left(2;\frac{\pi}{6}\right)$, $B\left(2;\frac{5\pi}{6}\right)$. **2.** $A(1;-\sqrt{3})$, $B\left(-\frac{1}{2};\frac{\sqrt{3}}{2}\right)$. **3.** 7 b. **4.** $S=\frac{1}{2}r_1r_2\sin(\varphi_2-\varphi_1)$. **5.** 4 $y.b$.

6. 64 $y.b$. **7.** 1) $x^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{4}$; 2) $\left(x-\frac{5}{2}\right)^2+y^2=\frac{25}{4}$; 3) $(x^2+y^2)^2=2xy$;

4) $(x^2+y^2)^3=4(x^2-y^2)$; 5) $x^2=4(1-y)$; 6) $y^2=3(3+2x)$; 7) $\frac{\left(x+\frac{3}{8}\right)^2}{25}+\frac{y^2}{16}=\frac{1}{64}$;

8) $\frac{x^2}{16}-\frac{(y-5)^2}{9}=\frac{13}{3}$. **8.** 1) $r=\frac{5}{\sin\varphi}$; 2) $r=\frac{1}{2\cos\varphi-\sin\varphi}$; 3) $r=\frac{4\sin\varphi}{\cos^2\varphi}$; 4) $r^2=\frac{a^2}{\cos 2\varphi}$; 5) $r=2a\cos\varphi$; 6) $r^2=\frac{2}{\sin 2\varphi}$; 7) $r=\frac{16}{1-\frac{9}{25}\cos^2\varphi}$; 9) $r=\frac{-4}{1-\frac{10}{9}\cos^2\varphi}$.

9. 1) ellips; 2) giperbola; 3) parabola; 4) ellips.

Tekislik

1. $2x-y+3z-14=0$. **2.** $2x-3y+4z+20=0$. **3.** $2x-3y+5z+10=0$.

4. $x+3y-4z-21=0$. **5.** 1) $2y+3z=0$; 2) $y+1=0$; 3) $z-4=0$;

4) $x+z-3=0$; 5) $22x+14y-5z=0$. **6.** 1) $2x-z=0$; 2) $x-3=0$; 3) $y+3=0$;

4) $x+3z+4=0$; 5) $8x-y+6z=0$. **7.** $A(-3;0;0), B(0;-6;0), C(0;0;2)$. 1).

8. $a=-15, b=-10, c=6$. **9.** $x+y+z-4=0$. **10.** $x+3y+z-15=0$.

11. 1) $x+3y-z-6=0$; 2) $x-y-z=0$. **12.** $x+y+z-6=0$.

13. $x+y-2z=0$. **14.** $x-2y-3z+3=0$. **15.** 1) $2x-5y+z-15=0$;

2) $2x+4y+9z-21=0$. **16.** $x-y-2z+5=0$. **17.** $\dfrac{x}{\frac{11}{9}}+\dfrac{y}{-\frac{11}{2}}+\dfrac{z}{\frac{11}{6}}=1$;

$\dfrac{9}{11}x-\dfrac{2}{11}y+\dfrac{6}{11}z-1=0$. **18.** $\dfrac{x}{-6}+\dfrac{y}{7}+\dfrac{z}{7}=1$; $-\dfrac{7}{11}x+\dfrac{6}{11}y+\dfrac{6}{11}z-\dfrac{42}{11}=0$.

19. 1) 45°; 2) 90°; 3) 90°; 4) $\arccos(0{,}4)$. **20.**) $m=-\dfrac{6}{5}, n=-\dfrac{15}{2}$; 2) $m=3, n=-4$.

21. 1) $m=13$; 2) $m=1$. **22.** 1) a) $x-2y-3z-4=0$; b) $2x+3y+z-8=0$;

2) a) $2x+3y+4z-3=0$; b) $4x+y-7z+19=0$; 3) a) $5x+7y+3=0$; b) $y-z+7=0$;

c) $5x+7z-46=0$. **23.** $7x+14y-2z+6=0$. **24.** $x-y+z+1=0$.

25. $x+2y+\sqrt{5}z-2=0$ va $x+2y-\sqrt{5}z-2=0$. **26.** 1) $M(-2;1;2)$; 2) $M(2;-1;1)$. **27.** $4(b)$.

28. $M(-15;0;0)$ va $M(1;0;0)$. **29.** $2x-y-2z=0$ va $2x-y-2z-18=0$. **30.** $8(h.b.)$.

31. -2. **32.** -3. **33.** $\dfrac{\sqrt{3}}{6}$.

Fazodagi to'g'ri chiziq

1. 1) $\dfrac{x-1}{2}=\dfrac{y-1}{3}=\dfrac{z+2}{-1}$; 2) $\dfrac{x-2}{0}=\dfrac{y+3}{1}=\dfrac{z+1}{0}$; 3) $\dfrac{x-2}{3}=\dfrac{y+1}{1}=\dfrac{z+2}{-2}$.

2. 1) $\dfrac{x-2}{4}=\dfrac{y+3}{1}=\dfrac{z-2}{3}$; 2) $\dfrac{x-2}{2}=\dfrac{y+3}{3}=\dfrac{z-2}{-1}$; 3) $\dfrac{x-2}{-11}=\dfrac{y+3}{6}=\dfrac{z-2}{-7}$.

3. $\dfrac{x}{-1}=\dfrac{y}{2}=\dfrac{z-2}{0}$. **4.** $\dfrac{x+1}{1}=\dfrac{y-2}{\sqrt{2}}=\dfrac{z+3}{-1}$. **5.** 1) $\begin{cases} x+4y-7=0, \\ x+z-1=0; \end{cases}$ 2) $\begin{cases} 3x-2y-7=0, \\ 2y+3z+1=0. \end{cases}$

6. $\dfrac{x-2}{-5}=\dfrac{y-2}{-3}=\dfrac{z+1}{4}$. **7.** 1) $x=13t, y=1+19t, z=2+28t$; 2) $x=t, y=1-3t, z=-2t$.

8. 1) $\frac{x}{1}=\frac{y-7}{2}=\frac{z-5}{2}$; 2) $\frac{x}{1}=\frac{y+1}{2}=\frac{z-2}{3}$. **9.** $\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z-3}{0}$. **10.** $\frac{x+3}{-5}=\frac{y}{1}=\frac{z-4}{-3}$

11. 1) $\varphi=\frac{\pi}{4}$; 2) $\varphi=\frac{\pi}{2}$. **12.** 1) $\frac{x+2}{-5}=\frac{y-3}{1}=\frac{z+1}{3}$; 2) $\frac{x+2}{6}=\frac{y-3}{16}=\frac{z+1}{17}$

13. $\frac{x-1}{1}=\frac{y+1}{3}=\frac{z-2}{2}$. **14.** 1) parallel; 2) ayqash. **15.** 1) $\varphi=\frac{\pi}{4}$; 2) $\varphi=\frac{\pi}{6}$.

16. 1) parallel; 2) to'g'ri chiziq tekisligida yotadi. **17.** 1) $M(3;2;1)$; 2) $M(2;4;6)$.

18. 1) $\frac{x-4}{1}=\frac{y-5}{-2}=\frac{z+6}{0}$; 2) $\frac{x-4}{1}=\frac{y-5}{-1}=\frac{z+6}{1}$. **19.** 1) $m=3$, $n=-23$; 2) $m=12$, $n=-12$. **20.** 1) $2x-3y+4z-1=0$; 2) $4x-y-2z-7=0$; 3) $z+1=0$.

21. $3x+5y+2z-9=0$. **22.** $(-2;0;-3)$. **23.** $M(2;3;4)$. **24.** $5x-2y+2z+5=0$.

25. $8x+2y-9z+12=0$. **26.** 1) $\frac{\sqrt{102}}{10}(u.b.)$; 2) $\frac{\sqrt{41}}{3}(u.b.)$.

Ikkinchi tartibli sirtlar

1. 1) $(x-3)^2+(y+1)^2+(z-1)^2=21$; 2) $(x-3)^2+(y+5)^2+(z+2)^2=56$.

2. 1) $M\left(-\frac{1}{2};\frac{1}{2};-\frac{1}{2}\right), R=\frac{\sqrt{3}}{2}$; 2) $M(3;-4;-5), R=5$. **3.** 1) $x^2+y^2+z^2=9$.

4. $x^2+z^2=10y$. **5.** $\frac{x^2+y^2}{9}+\frac{z^2}{25}=1$. **6.** $\frac{y^2+z^2}{16}-\frac{x^2}{9}=-1$. **7.** 1) $4y^2-x^4+4z^2=0$, $z=-\frac{x^2+y^2}{2}$; 2) $\frac{x^2}{16}-\frac{y^2+z^2}{25}=1$, $\frac{y^2}{25}-\frac{x^2+z^2}{16}=-1$; 3) $\frac{y^2}{64}+\frac{x^2+z^2}{16}=1$, $\frac{x^2+y^2}{64}+\frac{z^2}{16}=1$. **8.** $x^2+y^2-z^2=0$. **9.** 1) $m\neq 0$ va $m\geq -\frac{1}{4}$; 2) $m=0$. **10.** 1) ellips; 2) giperbola; 3) parabola; 4) nuqta. **11.** $y^2+z^2-2x^2=-6$ (ikki pallali giperboloid).

12. 1) $M_1(3;4;-2), M_2(6;-2;2)$; 2) $M(4;-3;2)$. **13.** 1) ikki pallali giperboloid; 2) sfera; 3) elliptik paraboloid; 4) aylanish ellipsoidi; 5) giperbolik silindr; 6) giperbolik paraboloid; 7) ikki pallali giperboloid; 8) doiraviy silindr.

Haqiqiy sonlar

1. 1) $A\cap B=\{4\}$, $A\cup B=\{1,2,3,4,5,6\}$, $A/B=\{1,2,3\}$, $B/A=\{5,6\}$;
2) $A\cap B=\{1,4,8\}$, $A\cup B=\{1,2,3,4,5,6,7,8,9\}$, $A/B=\{3\}$, $B/A=\{2,5,7,9\}$;
3) $A\cap B=\{4\}$, $A\cup B=\{-5;3,4\}$, $A/B=\{-5\}$, $B/A=\{3\}$.
2. 1) $A\cap B=\emptyset$, $A\cup B=N$. **3.** $A\cap B-$barcha 10 ga bo'linadigan sonlar.
6. 1) $A=\{1,2,3,4\}$; 2) $A=(1,2,3)$. **7.** 1) $\left\{\frac{7}{6},\frac{1}{3}\right\}$; 2) $\emptyset$; 3) $\{-1,0\}$; 4) $\{-\infty,2\}$.

8. 1) $(-\infty;1]\cup[3;\infty)$; 2) $(3,4)$; 3) $[-\infty;-1]$; 4) $\{-1,0\}$; 4) $\{-\infty,2\}$. **9.** 1) $0,-5$;
2) 0, mavjud emas.

Sonli ketma-ketliklar

1. 1) $x_n=\dfrac{1}{3n-1}$; 2) $x_n=\dfrac{5^n}{n!}$; 3) $x_n=\cos n\pi$; 4) $x_n=3+2(-1)^n$. **2.** 1); 2); 4); 6).

3. 2), 5)-monoton, 1),3),4),6)-qat'iy monoton. **6.** 1) $-\dfrac{1}{2}$; 2) 0; 3) ∞; 4) 8; 5) 4;

6) 2; 7) $\dfrac{1}{5}$; 8) $-\dfrac{5}{2}$; 9) 0; 10) 1; 11) $-\dfrac{5}{2}$; 12) $-\dfrac{4}{3}$; 13) ∞; 14) ∞; 15) 1; 16) 0;

17) -3; 18) $\dfrac{1}{2}$; 19) $\dfrac{1}{6}$; 20) $\dfrac{1}{4}$; 21) 0; 22) $-\dfrac{3}{2}$; 23) $\dfrac{4}{3}$; 24) $\dfrac{1}{36}$; 25) $-\dfrac{1}{2}$; 26) 2;

27) $\dfrac{1}{e}$; 28) $\dfrac{1}{e^4}$; 29) e^3; 30) e^2.

Bir o'zgaruvchining funksiyasi

1. 1) $(-\infty;-2)\cup(-2;+\infty)$; 2) $(-\infty;-3)\cup(-3;-2)\cup(-2;+\infty)$; 3) $[-2;2]$; 4) $(-2;1)\cup(1;+\infty)$;

5) 4) $(-\infty;2)\cup(9;10]$; 6) $\left[-1;-\dfrac{1}{2}\right)\cup\left(-\dfrac{1}{2};\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2};1\right]$; 7) $[7;10]$; 8) $\left[-\dfrac{1}{2};+\infty\right)$; 9) $\{2\}$;

10) $(2;+\infty)$; 11) $\varnothing$; 12) $(2;3]$; 13) $(10;+\infty)$; 14) $(2n\pi;(2n+1)\pi), n\in Z$; 15) $\left[0;\dfrac{2}{3}\right)$;

16) $[3;6)\cup(6;7]$; 17) $\left[-\dfrac{3}{4};\dfrac{3}{4}\right]$; 18) $[-5;0)\cup(0;1]$; 19) $(-\infty;1)\cup(1;2)\cup(2;+\infty)$;

20) $(-3;2)$. **2.** 1) $[-2;+\infty)$; 2) $[2;+\infty)$; 3) $[-7;-3]$; 4) $\left[-\sqrt{2};\sqrt{2}\right]$; 5) $[0,+\infty)$; 6) $(1;3]$;

7) $[0;3]$; 8) $\left(-\dfrac{1}{2};\dfrac{1}{2}\right)$; 9) $\left[-\dfrac{1}{5};+\infty\right)$; 10) $\{-1\}\cup\{1\}$; 11) $(0;3]$; 12) $(0;2]$. **3.** 1) 3;

1) $-\dfrac{4}{3\sqrt[3]{4}}$; 3) $-\dfrac{x^3}{3^x}$; 4) $\dfrac{3^{\frac{1}{x}}}{x^3}$. **4.** 1) $\left(-\infty;\dfrac{5}{2}\right)$ da kamayadi, $\left(\dfrac{5}{2};+\infty\right)$ da o'sadi;

2) $(-\infty;+\infty)$ da o'sadi; 3) $(-\infty;0)\cup(0;+\infty)$ da kamayadi; 4) $(-\infty;+\infty)$ da kamayadi. **5.** 1) toq; 2) juft; 3) juft; 4) umumiy ko'rinishda; 5) toq; 6) toq; 7) juft; 8) toq; 9) toq; 10) juft. **6.** 1) $M=n, m=k$; 2) $M=4, m=-4$;
3) $M=\sqrt{2}, m=-\sqrt{2}$; 4) $M=\sqrt{5}, m=-\sqrt{5}$; 5) $M=1, m=\dfrac{1}{2}$; 6) $M=1, m=0$.

7. 1) chegaralangan; 2) qat'iy monoton; 3) qat'iy monoton; 4) monoton.

8. 1) 6π; 2) $\frac{\pi}{2}$; 3) 4π; 4) $\frac{\pi}{2}$; 5) π; 6) 2π; 7) $\frac{\pi}{2}$; 8) $\frac{\pi}{3}$; 9) 12π; 10) 6π.

9. 1) $y=\frac{x-5}{3}$; 2) $y=\frac{x}{1-x}$; 3) $y=3^{x-4}$; 4) $y=\frac{1}{3}\arcsin\frac{x}{2}$.

10. 1) $f(g(x))=3x^3+1$, $g(f(x))=(3x+1)^3$; 2) $f(g(x))=\sin|x|$, $g(f(x))=|\sin x|$; 3) $f(g(x))=5-x$, $g(f(x))=\frac{x}{3x-1}$; 4) $f(g(x))=x^3$, $g(f(x))=3x$.

13. $A;C;D$. **14.** $A;B$. **15.** 1) $y=x^2+1$; 2) $\frac{x^2}{9}+\frac{y^2}{4}=1$.

Funksiyaning limiti

2. $f(x_0-0)=2$, $f(x_0+0)=3$; 2) $f(x_0-0)=0$, $f(x_0+0)=+\infty$; 3) $f(x_0-0)=2$, $f(x_0+0)=0$; 4) $f(x_0-0)=\frac{1}{5}$, $f(x_0+0)=1$. **5.** 1) 8; 2) 0; 3) $\frac{3}{2}$; 4) $\frac{1}{3}$; 5) $\frac{4}{3}$; 6) 2; 7) $-\frac{1}{12}$; 8) $\frac{1}{3}$; 9) -1; 10) $+\infty$; 11) -2; 12) -1; 13) $-\frac{4}{3}$; 14) -3; 15) 0; 16) $+\infty$; 17) $-\frac{1}{4}$; 18) 2; 19) 0; 20) $\frac{2}{25}$; 21) 2; 22) 0; 23) 1; 24) $-\frac{3}{2}$; 25) $\frac{3}{4}$; 26) $\frac{1}{2}$; 27) $6\sqrt{2}$; 28) $\frac{\sqrt{2}}{8}$; 29) 0; 30) 0; 31) $\frac{1}{\pi}$; 32) $\frac{1}{\pi}$; 33) -1; 34) $\frac{1}{2}$; 35) e^{-3}; 36) e; 37) $+\infty$; 38) 0; 39) e^2; 40) e^{-1}; 41) e; 42) e^{-2}; 43) e; 44) e; 45) 1; 37) 3; 46) $\frac{1}{2}$; 47) 4; 48) 1.

6. 1) $\frac{2}{3}$; 2) $\frac{1}{2}$; 3) -1; 4) $\ln 3$; 5) 1; 6) 5; 7) $\frac{\ln 3}{2}$; 8) 2; 9) $\frac{2}{3}$; 10) $\frac{1}{6}$; 11) 1; 12) 2; 13) $\frac{1}{2}$; 14) $\frac{1}{2}\ln\frac{9}{5}$; 15) $-\frac{1}{4}$; 16) $-\frac{1}{2}$; 17) $\frac{1}{2}$; 18) -9; 19) 3; 20) $\frac{2}{\pi}$; 21) 0; 22) $\ln 2$; 23) -1; 24) $\frac{3}{2}$.

Funksiyaning uzluksizligi

4. 1) $-3{,}3$; 2) -1. **5.** 1) ikkinchi tur uzilish nuqtasi; 2) birinchi tur (bartaraf qilinadigan) uzilish nuqtasi; 3) birinchi tur uzilish (sakrash) nuqtasi; 4) ikkinchi tur uzilish nuqtasi; **6.** 1) $x=0$ birinchi tur (bartaraf qilinadigan) uzilish nuqtasi; 2) $x=\frac{\pi}{2}+n\pi (n\in z)$ birinchi tur (bartaraf qilinadigan) uzilish nuqtasi. **7.** 1) $x=-3$ da ikkinchi tur uzilishga ega; 2) uzluksiz. **8.** 1) $[4{,}5]$ da uzluksiz, $[0;2]$ da

$x=1$ – ikkinchi tur uzilishga ega, $[-3;1]$ da $x=-3$, $x=1$ – ikkinchi tur uzilishga ega; 2) hech bir kesmada aniqlanmagan. **10.** 1) $\ln a$; 2) m.

Funksiyaning hosilasi va differensiali

1. 1) $f'(x)=\frac{3}{2\sqrt{3x-1}}$; 2) $f'(x)=\frac{5}{(1-5x)^2}$; 3) $f'(x)=-\frac{2}{\sin^2 2x}$; 4) $f'(x)=2sh2x$.

2. 1) -3; 2) -4; 3) 4; 4) $-\frac{1}{2}$. **3.** 1) -3, 3; 2) 0, 2; 3) 1, $-2x+3$; 4) -1, 1.

4. $t_1=1$, $t_2=3$. **5.** 1) $t=2c$; 2) $t=1c$. **6.** $I=12a$.

Differensiallash qoidalari va formulalari

1. 1) $y'=12x^3-x^2$; 2) $y'=x^5+12x^3-2$; 3) $y'=-\frac{1}{x\sqrt{x}}+7x\sqrt[3]{x}+\frac{4}{x\sqrt[3]{x^2}}$;

4) $y'=\frac{1}{2\sqrt{x}}+\frac{3}{x^2}-\frac{1}{x^4}$; 5) $y'=\frac{xe^x(x-1)+e^{-x}(x+2)}{x^3}$; 6) $y'=\frac{2\cdot 6^x\ln\frac{3}{2}}{(2^x-3^x)^2}$;

7) $y'=\frac{\ln^2 x-\ln x-1}{(\ln x-1)^2}$; 8) $y'=\frac{2e^x(x\ln x-1)}{x(\ln x-e^x)^2}$; 9) $y'=-\frac{2\sin x}{(1-\cos x)^2}$; 10) $y'=\frac{2}{1-\sin 2x}$;

11) $y'=\frac{4}{\sin^2 2x}$; 12) $y'=\frac{x^2+2}{(x\cos x+\sin x)^2}$; 13) $y'=-\left(\frac{x}{xshx-chx}\right)^2$; 14) $y'=-\frac{4}{sh^2 2x}$;

15) $y'=-\frac{1}{x\ln^2 x}$; 16) $y'=-\frac{3}{x\ln 10}$; 17) $y'=-\frac{3x}{\sqrt{4-3x^2}}$; 18) $y'=\frac{1}{2\sqrt{x-x^2}}$;

19) $y'=-2\sin 2x$; 20) $y'=\frac{1}{x^2-9}$; 21) $y'=\arcsin x$; 22) $y'=\frac{2e^x(e^x-1)}{e^{2x}+1}$;

23) $y'=\frac{3^x\ln 3}{1-9^x}$; 24) $y'=\frac{1}{3}$; 25) $y'=(1-tg3x)^2$; 26) $y'=-6e^{-3x}\sin 3x$; 27) $y'=\frac{\sqrt{e^x-1}}{2}$;

28) $y'=-\frac{1}{\cos x}$; 29) $y'=-\frac{x}{\sqrt{6x-4-x^2}}$; 30) $y'=\frac{x^3+x-1}{(x^2+2)^2}$. **2.** 1) $y'=-\frac{2}{(1+x)^2}$; 2) $y'=-\frac{1}{x}$; 3) $y'=\frac{1}{\sqrt{4-x^2}}$; 4) $y'=-\frac{3}{x^2+9}$. **3.** 1) 2,0125; 2) 1,009; 3) 0,9942; 4) 27,351. **4.** 1) 2,03; 2) 0,97; 3) 0,31; 4) 1,01. **5.** 1) $dy=\ln x dx$; 2) $dy=\frac{1-\ln x}{x^2}dx$;

3) $dy=-2\sin 4x dx$; 4) $dy=3a\sin^2 x\cos x dx$; 5) $dy=-\sin x 3^{\cos x}\ln 3 dx$;

6) $dy=-3tgx\ln^2\cos x dx$. **6.** 1) $y'''=24x(5x^2-3)$; 2) $y'''=e^{2x}(2\cos x-11\sin x)$;

3) $y''' = \frac{4}{(1+x^2)^2}$; 4) $y''' = \frac{2}{x}$. **7.** 1) $\sin\frac{n\pi}{2}$; 2) $n\sin\frac{n\pi}{2}$; 3) $-n(n-1)\sin\frac{n\pi}{2}$; 4) $n(n-1)$.

8. 1) $y' = -\frac{b^2x}{a^2y}$; 2) $y' = \frac{x^2+y}{y^2-x}$; 3) $y' = \frac{y(1-x)}{x(y-1)}$; 4) $y' = -\frac{2x+y\sin(xy)}{x\sin(xy)}$;

5) $y' = -\frac{y}{e^y+x}$; 6) $y' = -\frac{y\cos x+\sin y}{x\cos y+\sin x}$. **9.** 1) $\frac{3}{4t}$; 2) $-\frac{1}{a\sin^3 t}$; 3) $\frac{1+t^2}{4t}$; 4) $-\sqrt{1-t^2}$.

10. 1) $3x-3y+2=0,\ 3x+3y+4=0$; 2) $x+y-\pi=0$,

$x-y-\pi=0$; 3) $5x-y-4=0,\ x+5y-6=0$; 4) $5x+4y-25=0,\ 20x-25y+64=0$;

5) $x-y=0,\ x+y-4=0$; 6) $4x+2y-3=0,\ 2x-4y+1=0$.

Differensial hisobining asosiy teoremalari

1. 1) $c=\frac{2\sqrt{3}}{3}$; 2) $c=\frac{3\pi}{4}$; 3) yo‘q; 4) yo‘q. **2.** 1) $c=\frac{\sqrt{3}}{3}$; 2) $c=\ln(e-1)$; 3) $c=e-1$;

4) $c=\frac{1}{2}$. **3.** 1) $\left(-\frac{1}{2};-\frac{5}{4}\right)$; 2) $\left(\frac{5}{4};\frac{3}{2}\right)$. **4.** 1) $c=\frac{\pi}{8}$; 2) $c=\frac{3}{2}$. **6.** 1) $-\pi$; 2) $\frac{1}{3}$; 3) 1;

4) 0; 5) 0; 6) 2; 7) $\frac{1}{2}$; 8) $-\frac{1}{4}$; 9) 3; 10) -3; 11) 0; 12) 0; 13) 1; 14) e; 15) $e^{\frac{2}{\pi}}$;

16) e^{-9}; 17) 1; 18) $3e$. **7.** 1) $P(x)=19-11(x+2)-(x+2)^2+(x+2)^3$;

2) $P(x)=4+13(x-2)+12(x-2)^2+6(x-2)^3+(x-2)^4$;

8. 1) $2+\frac{1}{4}(x-3)-\frac{1}{64}(x-3)^2+\frac{1}{512}(x-3)^3-\frac{5(x-3)^4}{128\sqrt{(1+c)^7}}$, $c=x_0+\theta(x-x_0)$, $0<\theta<1$;

2) $-\frac{1}{2}-\frac{(x+2)}{4}-\frac{(x+2)^2}{8}-\frac{(x+2)^3}{16}+\frac{(x+2)^4}{c^5}$, $c=x_0+\theta(x-x_0)$, $0<\theta<1$.

9. 1) $f(x)=x+\frac{x^2}{1!}+\frac{x^3}{2!}+\ldots+\frac{x^n}{(n-1)!}+\frac{x^{n+1}}{n!}(\theta x+n+1)e^{\theta x}$, $0<\theta<1$;

2) $f(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\frac{x^{2n+1}}{(2n+1)!}\cdot\frac{e^{\theta x}-e^{-\theta x}}{2}$, $0<\theta<1$. **10.** 1) 0,587; 2) 0,868;

3) 1,395; 4) 1,004. **11.** 1) 1; 2) $\frac{1}{24}$; 3) 1; 4) -1.

Funksiyalarni tekshirish va grafiklarini chizish

1. 1) $(-\infty;1)\cup(5;+\infty)$ intervalda o‘sadi, $(1;5)$ intervalda kamayadi, $f_{max}=f(1)=7$, $f_{min}=f(5)=-25$; 2) $(-\infty;-1)\cup(2;+\infty)$ intervalda o‘sadi, $(-1;2)$ intervalda kamayadi, $f_{max}=f(-1)=\frac{7}{6}$, $f_{min}=f(2)=-\frac{10}{3}$; 3) $(0;2)\cup(2;+\infty)$ intervalda o‘sadi, $(-\infty;-2)\cup(2;0)$

intervalda kamayadi, $f_{\min}=f(0)=0$; 4) $(-2;2)$ intervalda o‘sadi, $(-\infty;-2)\cup(2;+\infty)$ intervalda kamayadi, $f_{\max}=f(2)=1$, $f_{\min}=f(-2)=-1$; 5) $\left(-\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$ intervalda o‘sadi, $\left(-1;-\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}};1\right)$ intervalda kamayadi, $f_{\max}=f\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$, $f_{\min}=f\left(-\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}$; 6) $(-\infty;-1)\cup(0;1)$ intervalda o‘sadi, $(-1;0)\cup(1;+\infty)$ intervalda kamayadi, $f_{\max 1}=f(-1)=2$, $f_{\max 2}=f(1)=2$, $f_{\min}=f(0)=0$;) $(-\infty;1)$ intervalda o‘sadi, $(1;+\infty)$ intervalda kamayadi, $f_{\max}=f(1)=\frac{1}{e}$; 8) $(0;+\infty)$ intervalda o‘sadi, $(-\infty;0)$ intervalda kamayadi, $f_{\min}=f(0)=1$; 9) $(0;+\infty)$ intervalda o‘sadi, $(-\infty;0)$ intervalda kamayadi, $f_{\min}=f(0)=0$; 10) $(e;+\infty)$ intervalda o‘sadi, $(0;1)\cup(1;e)$ intervalda kamayadi, $f_{\min}=f(e)=e$; 11) $\left(\frac{\pi}{3};\frac{5\pi}{3}\right)$ intervalda o‘sadi, $\left(0;\frac{\pi}{3}\right)\cup\left(\frac{5\pi}{3};2\pi\right)$ intervalda kamayadi, $f_{\max}=f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sqrt{3}$,

$f_{\min}=f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-\sqrt{3}$; 12) $\left(0;\frac{\pi}{12}\right)\cup\left(\frac{5\pi}{12};\pi\right)$ intervalda o‘sadi, $\left(\frac{\pi}{12};\frac{5\pi}{12}\right)$ intervalda kamayadi, $f_{\max}=f\left(\frac{\pi}{12}\right)=\frac{\pi+6\sqrt{3}+12}{12}$, $f_{\min}=f\left(\frac{5\pi}{12}\right)=\frac{5\pi-6\sqrt{3}+12}{12}$.

2. 1) $M=2$, $m=-2$; 2) $M=17$, $m=-10$; 3) $M=\frac{\pi+6\sqrt{3}}{12}$, $m=\frac{2\pi-3}{6}$;

4) $M=e^3$, $m=0$. **3.** $v=24$ (tez.birl.). **4.** $\frac{\sqrt{3}}{3}D$ (eni), $\sqrt{\frac{2}{3}}D$ (bo‘yi). **5.** $\frac{l}{4},\frac{l}{4}$.

6. $S=24$ (yuz birl.). **7.** $H=R\sqrt{2}$. **8.** $H=\sqrt[3]{\frac{4V_0}{\pi}}$. **9.** $\alpha=\frac{\pi}{3}$. **10.** 13,5 so‘m.

11. 1) $(-\infty;0)\cup(2;+\infty)$ intervalda botiq, $(0;2)$ intervalda qavariq, $M_1(0;0)$, $M_2(2;-4)$ egilish nuqtalari; 2) $(5;+\infty)$ intervalda botiq, $(-\infty;5)$ intervalda qavariq, $M(5;7)$ egilish nuqtasi; 3) $(-\infty;0)\cup(0;+\infty)$ intervalda botiq, egilish nuqtasi yo‘q;
4) $(3;+\infty)$ intervalda botiq, $(-\infty;3)$ intervalda qavariq, $M(3;1)$ egilish nuqtasi;
5) $(-1;+\infty)$ intervalda botiq, egilish nuqtasi yo‘q; 6) $(-1;1)$ intervalda botiq, $(-\infty;-1)\cup(1;+\infty)$ intervalda qavariq, $M_1(-1;\ln 2)$, $M_2(1;\ln 2)$ egilish nuqtalari;
7) $\left(-\infty;-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}};+\infty\right)$ intervalda botiq, $\left(-\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right)$ intervalda qavariq, $M_1\left(-\frac{1}{\sqrt{3}};\frac{3}{4}\right)$, $M_2\left(\frac{1}{\sqrt{3}};\frac{3}{4}\right)$ egilish nuqtalari; 8) $(-1;0)\cup(1;+\infty)$ intervalda botiq, $(-\infty;-1)\cup(0;1)$ intervalda qavariq, $M_1(-1;2)$, $M_2(1;-2)$ egilish nuqtalari.
12. 1) $x=\pm1$, $y=0$; 2) $x=0$, $y=1$ $(x\to+\infty\ da)$, $y=-1$ $(x\to-\infty\ da)$, 3) $y=x$;

4) $y=-\frac{\pi}{2}x+1$ $(x\to+\infty$ *da*$)$, $y=\frac{\pi}{2}x+1$ $(x\to-\infty$ *da*$)$. 5) $x=-2$, $y=0$ $(x\to-\infty$ *da*$)$; 6) $x=0$, $y=0$;

7) $x=0$, $y=3x$. 8) $x=1$, $y=x+\frac{1}{2}$ $(x\to+\infty$ *da*$)$, $y=-x-\frac{1}{2}$ $(x\to-\infty$ *da*$)$.

Hosilalarning geometrik tatbiqlari

1. 1) 4. 2) 2; 3) $(1+8\sin^2 t)^{-\frac{3}{2}}$; 4) $\frac{1}{16\left|\sin\frac{t}{2}\right|}$; 5) $\frac{3}{8}$; 6) $\frac{3}{a}$. **2.** 1) $\frac{25}{3}$; 2) $\frac{\sqrt{(145)^3}}{12}$;

3) 2; 4) $\frac{a\sqrt{(1+\theta^2)^3}}{2+\theta^2}$. **3.** 1) (2;2); 2) $\left(\frac{\pi}{2}+1;-1\right)$. **4.** 1) $\eta^2=\frac{8}{27}\xi^3$;2) $\eta^2=\frac{8}{27}(\xi-1)^3$;

3) $\xi^2=\frac{4}{27}\eta^3$; 4) $\xi^2+\eta^2=a^2$. **5.** 1) $\frac{x+1}{2}=\frac{y-2}{-3}=\frac{z}{4}$; 2) $\frac{x^2}{16}-\frac{z^2}{9}=1, y=-1$;

3) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=2^{\frac{2}{3}}, z=1$; 4) $\frac{x^2}{36}+\frac{y^2}{25}=1, z=3$. **6.** 1) $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-1}{6}$,

$2x+3y+6z-11=0$; 2) $\frac{x-2}{1}=\frac{y-\frac{8}{3}}{2}=\frac{z-4}{4}$, $3x+6y+12z-70=0$;

3) $\frac{x-\frac{1}{2}}{1}=\frac{y-1}{0}=\frac{z-\frac{3}{2}}{-3}$, $x-3z+4=0$; 4) $\frac{x-1}{2}=\frac{y+1}{0}=\frac{z-1}{1}$, $2x+z-3=0$.

7. 1)$5dt$; 2)$13dt$.

Tenglamalarni taqribiy yechish

1. 1) $-2{,}2583$; 2) 0,6823; 3) 2,5062; 4)$-0{,}8879$.

Kompleks sonlar

1. $x=2$, $y=-1$. **2.** $x=5$, $y=10$. **3.** $x=-1$, $y=21$. **4.** $x=\frac{1}{5}$, $y=\frac{1}{3}$.

5. 1) $z_1=-3+4i$, $z_2=-3-4i$; 2) $z_1=\frac{1}{2}i$, $z_2=-i$; 3) $z_1=i$, $z_2=-3i$; 4) $z_1=5i$, $z_2=i$.

6. 1) $x=a$ to‘g‘ri chiziq; 2) $y=b$ to‘g‘ri chiziq; 3) markazi $O(0;0)$ nuqtada bo‘lgan va r, R radiusli aylanalar orasidagi halqa; 4) φ va ψ nurlar orasidagi sektor; 5) 3) markazi $O(0;0)$ nuqtada bo‘lgan va r, R radiusli aylanalar orasidagi halqaning φ va ψ nurlar orasidagi bo‘lagi. **7.** 1) $-2+2\sqrt{3}i=4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=4e^{\frac{2\pi}{3}i}$;

2) $\sqrt{3}-i=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=2e^{-\frac{\pi}{6}i}$; 3) $-\frac{\sqrt{6}}{2}+\frac{\sqrt{6}}{2}i=\sqrt{3}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{3}e^{\frac{3\pi}{4}i}$;

4) $2+2i=2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=2\sqrt{2}e^{\frac{\pi}{4}i}$; 5) $1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)=\sqrt{2}e^{-\frac{\pi}{4}i}$;

6) $-3-2i=\sqrt{13}\left(\cos\left(arctg\frac{2}{3}-\pi\right)+i\sin\left(arctg\frac{2}{3}-\pi\right)\right)=\sqrt{13}e^{i\left(arctg\frac{2}{3}-\pi\right)}$,

7) $1-\sqrt{3}i=2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=2e^{-\frac{\pi}{3}i}$;

8) $-\sqrt{2}-\sqrt{2}i=2\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)=2e^{-\frac{3\pi}{4}i}$.

8. 1) $z_1+z_2=-3-i,\ z_1-z_2=-7+7i,\ z_1\cdot z_2=2+26i,\ \frac{z_1}{z_2}=-\frac{11}{10}-\frac{7}{10}i$;

2) $z_1+z_2=-1-i,\ z_1-z_2=-5-7i,\ z_1\cdot z_2=6-17i,\ \frac{z_1}{z_2}=-\frac{18}{13}+\frac{1}{13}i$. **9.** 1) $5\sqrt{2}$; 2) $5\sqrt{2}$;

3) $\sqrt{5}$; 4) 10. **10.** 1) $\frac{6}{5}-\frac{17}{5}i$; 2) $-\frac{9}{5}-\frac{2}{5}i$; 3) $24i$; 4) $48i$. **11.** 1) $\operatorname{Re} z=\frac{1}{2},\operatorname{Im} z=\frac{\sqrt{3}}{2}$;

2) $\operatorname{Re} z=0,\ \operatorname{Im} z=\frac{1}{8}$; 3) $\operatorname{Re} z=\frac{4}{5},\operatorname{Im} z=\frac{3}{5}$; 4) $\operatorname{Re} z=-\frac{37}{5},\operatorname{Im} z=-\frac{29}{5}$.

12. 1) $z_1\cdot z_2=-4+4\sqrt{3}i,\ \frac{z_1}{z_2}=\sqrt{3}-i$; 2) $z_1\cdot z_2=3\sqrt{3}+3i,\ \frac{z_1}{z_2}=-6i$; 3) $z_1\cdot z_2=-16,$

$\frac{z_1}{z_2}=4i$; 4) $z_1\cdot z_2=-4+4\sqrt{3}i,\ \frac{z_1}{z_2}=4+4\sqrt{3}i$. **13.** 1) $16(1+i)$; 2) -1; 3) $2^{13}(1-i)$; 4)

$-32(1+\sqrt{3}i)$. **14.** 1) $\pm(\sqrt{3}-i)$; 2) $i,\ -\frac{\sqrt{3}}{2}-\frac{1}{2}i,\ \frac{\sqrt{3}}{2}-\frac{1}{2}i$; 3) $\pm(\sqrt{3}+i),\pm(1-\sqrt{3}i)$;

4) $\sqrt[10]{2}\left(\cos\frac{\pi+8k\pi}{20}+i\sin\frac{\pi+8k\pi}{20}\right),\ k=0,1,2,3,4.$

Ko'phadlar

1. 1) $P+Q=x^5+5x^3-x^2-x+4,\ P-Q=-x^5-x^3-x^2+x+4,$

$P\cdot Q=2x^8-x^7+6x^6+x^5-2x^4+13x^3-4x$; 2) $P+Q=4x^4+x^3-5,$

$P-Q=-2x^4-x^3+2x^2-5,\ P\cdot Q=3x^8+x^7+2x^6+x^5-16x^4-5x^3+5x^2.$

2. 1) $R=x+1,\ r=-x+2$; 2) $R=x+3,\ r=-6x^2-5x+8$;

3) $R=3x^2-5,\ r=-21x^2+17x+34$; 4) $R=2x^2-3x+5,\ r=19x-29$. **3.** 1) 73; 2) -4;

3) 96; 4) 84. **4.** $a=3, b=-1, c=1$. **5.** 1) $a=2, b=1, c=1$. **6.** 1) $(x-2)(x+2)(x^2+4)$;

2) $x(x-9)(x+9)$; 3) $5(x-1)(x-2)^2(x-3)$; 4) $(x-3)^2(x-1)(x^2+x+1)$.

7. 1) $\frac{3}{x}+\frac{1}{x^2}-\frac{2}{x+1}$; 2) $\frac{2}{x}-\frac{1}{x^2}+\frac{x-4}{x^2+4}$; 3) $\frac{1}{x}+\frac{1}{3(x-1)}-\frac{4}{3(x+2)}$;

4) $\frac{3}{x^2-x+1}-\frac{2}{x^2+x+1}$. **8.** 1) $\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x^3}-\frac{2}{x+1}$; 2) $\frac{1}{x}-\frac{2}{x-2}+\frac{3}{x+3}$;

3) $\frac{2}{x}-\frac{7}{x^2}+\frac{3}{x-1}-\frac{2}{x+1}$; 4) $-\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x^2-x+1}$.

Integrallashning asosiy usullari

1. 1) $-5\ln\cos x-2arctgx+\frac{x^5}{5}+C$; 2) $\frac{(x-3)^2}{2}+2\ln|x+3|+C$; 3) $-\frac{3}{2\sqrt[3]{x^2}}-e^x-\ln|x|+C$;

4) $3arctgx-2\arcsin x+C$; 5) $2x+\frac{3\cdot 2^x}{3^x(\ln 3-\ln 2)}+C$; 6) $x-\cos x+C$; 7) $e^x+tgx+C$;

8) $-ctgx+\cos x+C$; 9) $-ctgx-x+C$; 10) $-ctgx+C$; 11) $\frac{1}{10}arctg\frac{2x}{5}+C$;

12) $\frac{1}{\sqrt{2}}\arcsin\frac{\sqrt{2}(x-1)}{\sqrt{5}}+C$. **2.** 1) $\frac{1}{4}tg^2x+C$; 2) $-\frac{1}{3}\cos^3 x+C$; 3 $\frac{3}{16}\sqrt[3]{arctg^8 2x}+C$;)

4) $\frac{7}{10}\sqrt[7]{\ln^{10}(x+5)}+C$; 5) $e^{\sin x}+C$; 6) $-\frac{1}{3}e^{-x^3}+C$; 7) $-\frac{1}{4\sin^4 x}+C$; 8) $-2\cos\sqrt{x}+C$;

9) $\arcsin\frac{e^x}{2}+C$; 10) $-\frac{3}{4}\sqrt[3]{ctg4x}+C$. **3.** 1) $2\ln(1+e^x)-x+C$; 2) $\frac{1}{6}\ln(x^6+1)+C$;

3) $8\arcsin\frac{x}{4}+\frac{1}{2}x\sqrt{16-x^2}+C$; 4) $\frac{3}{8}\sqrt[3]{(x^4+4)^2}+C$; 5) $\frac{2}{9}(x^3+3)\sqrt{x^3+3}+C$;

6) $\ln|2+\sin 2x|+C$; 7) $-\frac{1}{2(\arcsin x)^2}+C$; 8) $2\ln|x^2+5|-\sqrt{5}arctg\frac{x}{\sqrt{5}}+C$; 9) $\arcsin\frac{x+2}{3}+C$; 10) $\frac{1}{\sqrt{3}}\ln\left|3x-1+\sqrt{9x^2-6x-3}\right|+C$; 11) $\frac{1}{4}\left(\frac{(2x+7)^{12}}{12}-\frac{7(2x+7)^{11}}{11}\right)+C$;

12) $2\arcsin\sqrt{x}+C$; 13) $\frac{1}{12}\ln\left|\frac{e^{2x}-3}{e^{2x}+3}\right|$; 14) $\ln x-\ln 2\cdot|\ln x+2\ln 2|+C$.

4. 1) $\frac{x^2+1}{2}arctgx-\frac{x}{2}+C$; 2) $x\arcsin x+\sqrt{1-x^2}+C$; 3) $\frac{x^2}{2}\ln|x|-\frac{x^2}{4}+C$;

4) $e^x(x^2-2x+2)+C$; 5) $\frac{3^x}{\ln^2 3}(x\ln 3-1)+C$;

6) $\frac{1}{4}\sin 2x-\frac{1}{2}x\cos 2x+C$; 7) $\frac{1}{2}(x^2-1)\ln|x+1|-\frac{1}{4}x(x-2)+C$; 8) $\frac{x}{2\cos^2 x}-\frac{1}{2}tgx+C$;

9) $\frac{x}{2}(\sin\ln|x|-\cos\ln|x|)+C$; 10) $\frac{1}{5}e^{4x}(\sin 4x-\cos 4x)+C$; 11) $tgx(\ln tgx-1)+C$;

12) $arctgx(\ln arctgx-1)+C$. **5.** 1) $\frac{3}{14}(1+x^2)^2\sqrt[3]{1+x^2}-\frac{3}{8}(1+x^2)\sqrt[3]{1+x^2}+C$;

2) $\frac{1}{4}\sin 2x-\frac{1}{16}\sin 8x+C$; 3) $\frac{e^x}{4}(2+\sin(2e^x)+C$; 4) $-\frac{1}{9}e^{-3x}(3x+1)+C$;

5) $\ln|\sin x+\cos x|+C$; 6) $-\frac{1}{2}\ln|1-\ln^2|x||+C$; 7) $\frac{1}{5}\ln\left|\frac{2x-3}{x+1}\right|+C$;

8) $\frac{1}{2}arctg\frac{\ln x}{2}+C$; 9) $xtgx+\ln|\cos x|+C$; 10) $\frac{2}{3}arctg\frac{\sqrt{2x-9}}{3}+C$; 11) $e^{arctgx}+C$;

12) $\sqrt{3+e^{2x}}+C$; 13) $\frac{1}{2}x-\frac{1}{6}\sin 3x+C$; 14) $\frac{1}{2}tgx^2-\frac{1}{2}x^2+C$;

15) $\frac{1}{27}x^3(9\ln^2 x-6\ln x+2)+C$; 16) $\frac{2-\cos x}{\sin x}+C$.

Ratsional funksiyalarni integrallash

1. 1) $\ln|x^2+3x-10|+C$; 2) $\ln\frac{|x+1|}{\sqrt{2x+1}}+C$; 3) $\frac{1}{2}\ln\frac{(x+2)^4}{|(x+1)(x+3)^3|}+C$;

4) $\frac{5}{2}\ln\left|\frac{x+1}{x+5}\right|+\frac{2}{x+1}+C$; 5) $\ln\left|\frac{x^3(x-1)}{x+1}\right|+C$;

6) $\frac{x}{4}+\ln|x|-\frac{7}{16}\ln|2x-1|-\frac{9}{16}\ln|2x+1|+C$;

7) $\frac{2x^2-3}{x}+\ln\left|\frac{x}{x+1}\right|+C$; 8) $5x+\frac{1}{2}\ln|x|-\frac{7}{3}\ln|x-1|+\frac{161}{6}\ln|x-4|+C$;

9) $x+\frac{1}{3}\ln\frac{|(x-2)^5(x-1)^3|}{(x+2)^2}+C$; 10) $-\frac{1}{x}-arctgx+C$; 11) $\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right)+C$;

12) $\frac{1}{6}\ln\left(\frac{(x+1)^2}{x^2-x+1}\right)+\frac{1}{\sqrt{3}}arctg\frac{2x-1}{\sqrt{3}}+C$; 13) $\frac{x^3}{3}+x^2-x+\frac{4}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}+C$;

14) $\frac{1}{4}\left(\frac{2x^6+6x^2}{3}+\ln\left(\frac{x^2-1}{x^2+1}\right)\right)+C$; 15) $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}arctgx+C$;

16) $\frac{1}{54}\left(\frac{3x}{x^2+9}+arctg\frac{x}{3}\right)+C$; 17) $\frac{2x-1}{2(x^2+2x+2)}+arctg(x+1)+C$;

18) $\ln|x-1|+\frac{1}{2}\left(\frac{x}{x^2+1}+arctgx\right)+C$; 19) $\frac{1}{8}arctg(x+2)-\frac{1}{24}arctg\frac{x+2}{3}+C$,

20) $\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}-\frac{1}{2(x+1)}+C$; 21) $\frac{1}{8}\left(\frac{x(3x^2+5)}{(x^2+1)^2}+3arctgx\right)+C$;

22) $\frac{x-9}{8(x^2-2x-5)}+\frac{1}{16}arctg\frac{x-1}{2}+C$; 23) $\frac{4x-7}{x^2-3x+3}+\frac{8\sqrt{3}}{3}arctg\frac{2x-3}{\sqrt{3}}+C$;

24) $\ln\left(\frac{x^2+9}{x^2+4}\right)+arctg\frac{x}{3}+C.$

Trigonometrik funksiyalarni integrallash

1. 1) $\frac{2}{3}arctg\frac{5tg\frac{x}{2}-4}{3}+C$; 2) $\frac{1}{8}\left(tg^2\frac{x}{2}+2\ln\left|tg\frac{x}{2}\right|\right)+C$; 3) $\frac{1}{5}\ln\left|5tg\frac{x}{2}+3\right|+C$;

4) $\frac{2}{\sqrt{3}}arctg\frac{tg\frac{x}{2}+2}{\sqrt{3}}+C$; 5) $\arccos\frac{\cos x}{\sqrt{3}}+C$; 6) $\frac{3}{\sin x}-\frac{1}{\sin^3 x}+C$; 7) $2arctg(\sin x)-\sin x+C$;

8) $\frac{1}{4}\ln\left|\frac{1+tgx}{1-tgx}\right|+\frac{1}{4}\sin 2x+C$; 9) $\frac{1}{16}\left(x-\frac{1}{4}\sin 4x+\frac{1}{3}\sin^3 2x\right)+C$;

10) $\ln|tgx|+\frac{1}{2}tg^2x+C$; 11) $\frac{1}{10}\ln\left|\frac{tgx-1}{tgx+1}\right|+C$; 12) $-\frac{1}{4}\left(2\ln|\sin 2x|+ctg^2 2x\right)+C$;

13) $\sqrt{2}arctg\left(\frac{shx}{\sqrt{2}}\right)-x+C$; 14) $\frac{1}{42}(3\sin 7x+7\sin 3x)+C$;

15) $\frac{1}{2}\left(\frac{1}{3}\sin 3x-\frac{1}{10}\sin 5x-\frac{1}{2}\sin x\right)+C$; 16) $\frac{x}{4}+\frac{\sin 2x}{8}+\frac{\sin 4x}{16}+\frac{\sin 6x}{24}+C.$

Irratsional funksiyalarni integrallash

1. 1) $2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(1+\sqrt[6]{x})+C$; 2) $\frac{2}{(1+\sqrt[4]{x})^2}-\frac{4}{1+\sqrt[4]{x}}+C$;

3) $\frac{2\sqrt{1+x}}{15}\cdot(3(1+x)^2-10(1+x)+9\sqrt[3]{1+x}+15)+C$; 4) $6\sqrt[3]{(x+1)^2}\cdot\left(\frac{2x-3}{20}-\frac{\sqrt{x+1}}{7}\right)+C$;

5) $\sqrt[3]{2x-1}-3\sqrt[6]{2x-1}+3\ln(\sqrt[6]{2x-1}+1)+C$; 6) $\frac{3}{4}\sqrt[3]{\left(\frac{x+1}{x-1}\right)^2}-\frac{3}{5}\sqrt[6]{\left(\frac{x+1}{x-1}\right)^5}+C$;

7) $\ln|2x-3+2\sqrt{x^2-3x+2}|+C$; 8) $\ln|x+1+\sqrt{x^2 2x+5}|+C$; 9) $\ln\left|\frac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}\right|+C$;

10) $\frac{1}{2}\ln\left|\frac{x}{4-x+2\sqrt{4-2x-x^2}}\right|+C$; 11) $\ln\left|\frac{1-x+\sqrt{1-2x-x^2}}{1+\sqrt{1-2x-x^2}}\right|-2arctg\left(\frac{1+\sqrt{1-2x-x^2}}{x}\right)+C$;

12) $\ln|1+x+\sqrt{x^2+x+1}|+\dfrac{2}{x+2+\sqrt{x^2+2x+2}}+C$;

13) $\dfrac{9}{2}\arcsin\left(\dfrac{x-2}{3}\right)+\dfrac{1}{2}(x-2)\sqrt{5+4x-x^2}+C$; 14) $\dfrac{x}{2}\sqrt{x^2-4}-2\ln|x+\sqrt{x^2-4}|+C$;

15) $-2\sqrt{\dfrac{2-x}{x-1}}+C$, 16) $-\arcsin\left(\dfrac{1}{x-1}\right)+C$; 17) $-\sqrt{3-2x-x^2}-\arcsin\left(\dfrac{x+1}{2}\right)+C$;

18) $-2\sqrt{6x-x^2-8}+9\arcsin(x-3)+C$; 19) $2\,3\left(\ln\left|\dfrac{\sqrt{x}}{1+\sqrt{x}}\right|+\dfrac{2}{1+\sqrt{x}}\right)+C$;

20) $-\dfrac{\sqrt[3]{(2-x^3)^2}}{4x^2}+C$; 21) $\sqrt[3]{(1+x^3)^5}\cdot\left(\dfrac{x^3+1}{8}-\dfrac{1}{5}\right)+C$; 22) $\dfrac{3}{7}(\sqrt[4]{x}-3)\cdot\sqrt[3]{1+\sqrt[4]{x}}+C$;

23) $-\dfrac{\sqrt{1+x^4}}{2x^2}+C$; 24) $-2\sqrt{\left(1+\dfrac{1}{\sqrt[3]{x}}\right)^3}$.

Aniq integral

1. 1) 0; 2) 8; 3) 4π; 4) 4. **2.** 1) $I_1>I_2$; 2) $I_1>I_2$; 3) $I_1<I_2$; 4) $I_1<I_2$. **3.** 1) $\dfrac{\pi}{5}\le I_2\le\pi$; 2) $4\le I_3\le 4\sqrt{7}$; 3) $2\le I_3\le 6$; 4) $\dfrac{3}{5}\le I_4\le 3$. **4.** 1) $\dfrac{\pi}{2}$; 2) $\dfrac{1}{2}$; 3) 8; 4) $e-2$.

Aniq integralni hisoblash

1) 9; 2) $\dfrac{1}{2}$; 3) $\dfrac{1}{2}$; 4) 1; 5) $\dfrac{\pi}{4}$; 6) $\dfrac{3-\sqrt{3}}{3}$; 7) $\ln\dfrac{4}{3}$; 8) $\dfrac{2}{3}$; 9) $\dfrac{2\sqrt{2}-1}{3}$; 10) $\dfrac{1}{4}$; 11) $\ln\dfrac{3}{2}$;

12) $\dfrac{\pi}{36}$; 13) $\dfrac{\pi\sqrt{3}}{36}$; 14) $\dfrac{1}{12}(8-5\sqrt{2})$; 15) $\ln\dfrac{4}{3}$; 16) $1-\dfrac{\pi}{4}$; 17) $\dfrac{\pi}{2}-1$; 18) $e-2$; 19) 4;

20) $\dfrac{1}{5}\left(e^{\frac{\pi}{4}}+2\right)$; 21) $\dfrac{5e^3-2}{27}$, 22) $\dfrac{e+1}{4}$, 23) 2; 24) $\dfrac{e^{\frac{\pi}{2}}-1}{2}$.

Xosmas integrallar

1. 1) $\dfrac{\pi}{4}$; 2) -4; 3) uzoqlashadi; 4) uzoqlashadi; 5) $\dfrac{\pi}{6}$; 6) $\dfrac{\pi}{4}+\dfrac{1}{2}\ln 2$; 7) $\dfrac{1}{2}$; 8) 2;

9) $\dfrac{\pi}{2}$; 10) $\dfrac{8}{3}$; 11) $14\dfrac{4}{7}$; 12) uzoqlashadi; 13) uzoqlashadi; 14) π . **2.** 1) $\alpha>1$ da

yaqinlashadi va $0<\alpha\le 1$ da uzoqlashadi; 2) yaqinlashadi; 3) yaqinlashadi;

4) yaqinlashadi; 5) uzoqlashadi; 6) yaqinlashadi; 7) uzoqlashadi;
8) uzoqlashadi; 9) yaqinlashadi; 10) yaqinlashadi;
11) absolut yaqinlashadi; 12) absolut yaqinlashadi.

1. 1) 36; 2)$\frac{9}{2}$; 3)$1+6\ln\frac{4}{3}$, 4)e^2+1; 5)$\frac{9}{2}$; 6)$\frac{15}{2}-8\ln 2$; 7)$\frac{1}{3}$; 8)$\frac{2}{3}$; 9)12π; 10)27π; 11)9; 12) $\frac{9\pi}{2}$; 13) $\frac{1}{3}(17\pi+48)$, 14) $\frac{16\pi^3}{3}$. **2.** 1)$\sqrt{3}+\frac{1}{2}\ln(2+\sqrt{3})$; 2)$\frac{1}{2}\left(e-\frac{1}{e}\right)$; 3) $\frac{335}{27}$; 4) 2; 5)$\frac{3}{4}+\frac{1}{2}\ln 2$; 6)$1+\ln\frac{6}{5}$; 7)$4\sqrt{3}$; 8)$\frac{1}{27}(13\sqrt{13}-8)$ 9)16; 10)48, 11) $4a(2-\sqrt{3})$; 12) $2\pi+3\sqrt{3}$. **3.** 1)$\frac{56}{3}\pi$; 2)36π; 3)$\frac{256}{3}\pi$; 4) $\pi(2+\pi)$. **4.** $\frac{4}{3}\pi R^3$. **5.** 12π. **6.** $\frac{29}{3}\pi$. **7.** 36π. **8.** 1)$\frac{512}{15}\pi$; 2)$\frac{19}{6}\pi$; 3)$\frac{\pi^2}{4}$; 4)$\frac{4}{7}\pi$; 5)32π; 6)100π; 7)$40\pi^2$; 8)$\frac{3}{7}\pi$; 9)72π; 10) $\frac{4}{3}\pi R^3$. **9.** $C(0;R)$. **10.** $C\left(0;\frac{2a}{5}\right)$. **11.** $M_x=10\gamma$, $M_y=\frac{15}{2}\gamma$, $C\left(\frac{3}{2};2\right)$

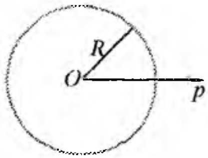
12. $M_x=M_y=\frac{4}{3}$, $I_x=I_y=\frac{4}{3}$, $C\left(\frac{2}{3};\frac{2}{3}\right)$. **13.** $C\left(0;\frac{8}{5}\right)$. **14.** $I_x=80\pi; I_y=125\pi$.

15. $C\left(\frac{2}{3}R;\frac{2}{3}R\right)$. **16.** $C\left(5;\frac{15}{32}\pi\right)$ **17.** 5 *sm*; **18.** 0,32 *kGm*.. **19.** $7{,}68\pi$. **20.** $22\cdot 10^9$ *J*.

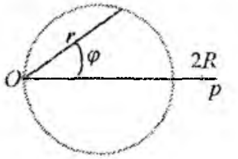
21. 150 *m*. **22.** 25 *m/s*. **23.** $\frac{mg}{t}\left(180+\left(\frac{m}{k}e^{-\frac{180}{m}}-1\right)\right)$. **24.** 1000 *m*. **25.** $g\gamma\frac{h^2(b+2a)}{6}$.

26. 324 *T*. **27.** 175.4 *kN*.

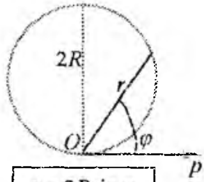
Ayrim chiziqlarning grafiklari va tenglamalari



$r = R$

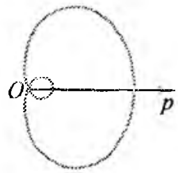


$r = 2R\cos\varphi$



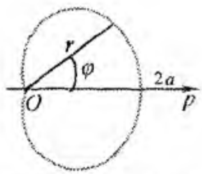
$r = 2R\sin\varphi$

R radiusli aylana



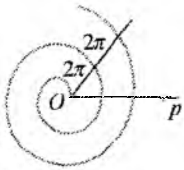
$r = b + a\cos\varphi \ (a > b)$

Paskal chig'anog'i



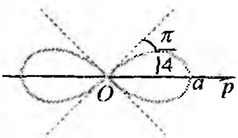
$r = a(1 + \cos\varphi) \ (a > 0)$

Kardioida



$r = a\varphi \ (a > 0)$

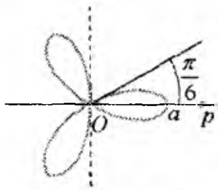
Arximed spirali



$r = a\sqrt{\cos 2\varphi} \ (a > 0)$

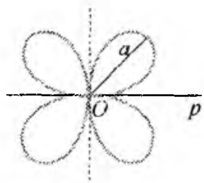
$\left(x^2 + y^2\right)^2 - a^2\left(x^2 - y^2\right) = 0$

Bernulli limniskatasi



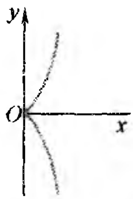
$r = a\cos 3\varphi \ (a > 0)$

Uch yaproqli gul



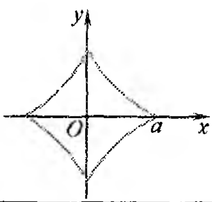
$r = a\sin 2\varphi \ (a > 0)$

To'rt yaproqli gul



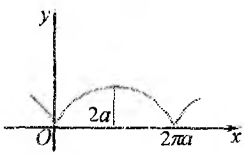
$y^2 = x^3 \ yoki \ \begin{cases} x = t^2, \\ y = t^3 \end{cases}$

Yarimkubik parabola



$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ yoki \ \begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t \end{cases}$

Astroida



$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \ a > 0 \end{cases}$

Sikloida

MUNDARIJA

3. ANALITIK GEOMETRIYA

4. MATEMATIK ANALIZGA KIRISH

5. BIR O'ZGARUVCHI FUNKSIYASINING DIFFERENSIAL HISOBI

6. OLIY ALGEBRA ELEMENTLARI

7. BIR O'ZGARUVCHI FUNKSIYASINING INTEGRAL HISOBI

Shavkat Rahmatullayevich XURRAMOV

OLIY MATEMATIKA

Uch jildlik

I jild

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi barcha texnika yo'nalishlari uchun darslik sifatida tavsiya etgan

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Badiiy muharrir ***Maftuna Vaxxobova***
Texnik muharrir ***Yelena Tolochko***
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Sahifalovchi ***Gulchehra Azizova***

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