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XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARDAN MISOL VA MASALALAR TO'PLAMI

O‘ZBEKISTON RESPUBLIKASI OLIY VA
O‘RTA MAXSUS TA’LIM VAZIRLIGI

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(O‘QUV QO‘LLANMA)

Ushbu qo‘llanma matematika, amaliy matematika va informatika, fizika ta’lim yo‘nalishlari talabalari va magistrlari uchun mo‘ljallangan.

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So‘z boshi

Xususiy hosilali differensial tenglamalar fani nazariy va amaliy ahamiyatga ega. Ushbu fanda, asosan, ikkinchi tartibli xususiy hosilali differensial tenglamalar va ularga qo‘yilgan masalalar o‘rganiladi. Ikkinchi tartibli xususiy hosilali differensial tenglamalar matematik fizika tenglamalari deb ham yuritiladi, chunki bu tenglamalar fizikaning turli sohalarida uchraydigan jarayonlarning matematik modellarini tuzishda ishlatiladi. Fanning maqsadi matematik fizikaning klassik tenglamalari deb ataluvchi to‘lqin, Laplas hamda issiqlik tarqalish tenglamalarini tekshirish va ularga qo‘yiladigan asosiy masalalarni yechishdan iborat. Bu tenglamalarni o‘rganish talabalarda tegishli jarayonlar haqida tasavvurga ega bo‘lishlariga imkon beradi. Ayni paytda ularni mantiqiy fikrlashga, to‘g‘ri xulosalar chiqarishga o‘rgatadi.

Xususiy hosilali differensial tenglamalar hozirgi zamon matematikasining muhim sohalaridan bo‘lib, u matematikaning bir necha sohalari, jumladan, matematik analiz, funksiyalar nazariyasi, integral va differensial tenglamalar nazariyasi, funksional analiz, fizika, texnika fanlari bilan uzviy bog‘liq. Matematik fizika tenglamalari so‘ngi yillarda keng rivoj topib kelyapti. Endigi kunda matematik fizikaning klassik tenglamalaridan tashqari aralash turdagi xususiy hosilali differensial tenglamalar ham o‘rganilib, fizikaning ko‘pgina masalalarini hal qilish uchun keng tatbiq qilinmoqda.

Matematik fizika tenglamalar fani 2018–2019 o‘quv yilidan boshlab xususiy hosilali differensial tenglamalar fani deb yuritila boshlandi. Xususiy hosilali differensial tenglamalar fanining asosiy vazifalariga xususiy hosilali tenglamalar haqida umumiy tushuncha berish, ikkinchi tartibli kvazichiziqli tenglamalarning turlarini aniqlab ularni kanonik ko‘rinishga keltirish va matematik fizikaning klassik tenglamalari va integral tenglamalarni o'rganish, har bir turdagi tenglamalarga korrekt masalalarning qo'yilishi va bu masalalarni yechish usullarini o'rganishdan iborat.

Ushbu qo'llanmada xususiy hosilali differensial tenglamalarning yechimlarini analitik ravishda olish, bu tenglamalarga qo'yilgan turli masalalarni, integral tenglamalarni yechish usullariga bag'ishlangan bo'lib, bu usullar imkon qadar keng yoritishga harakat qilingan. Ko'plab misol va masalalar javoblar bilan ta'minlangan.

O'quv qo'llanma mualliflarning Buxoro Davlat universitetida ko'p yillar davomida matematik fizika tenglamalari va xususiy hosilali differensial tenglamalar fanlaridan olib borgan amaliy mashg'ulotlarida o'rganilgan misol va masalalar asosida yozildi.

1-BOB. XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR HAQIDA ASOSIY TUSHUNCHALAR. BIRINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR

Ushbu bobda xususiy hosilali differensial tenglamalar haqida umumiy ma'lumotlar berilib, birinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimlarini topish, ularga qo'yilgan Koshi masalasini yechish o'rganilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

1.1. Birinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimini topish

Erkli o'zgaruvchi, noma'lum funksiya va uning hosilalari orasidagi funksional bog'lanishga **differensial tenglama** deyiladi.

Agar tenglamada noma'lum funksiya ko'p o'zgaruvchining (o'zgaruvchilar 2 va undan ortiq) funksiyasi bo'lsa, bunday tenglama **xususiy hosilali differensial tenglama** deyiladi.

n o'lchovli R^n Evklid fazosida nuqtaning dekart koordinatalarini $x_1, x_2, ..., x_n$, $n \geq 2$ orqali belgilaymiz. Tartiblangan manfiy bo'lmagan n ta butun sonning $\alpha = (\alpha_1, \alpha_2, ... \alpha_n)$ ketma-ketligi n - **tartibli multiindeks**, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ soniga **multiindeks uzunligi** deyiladi. Q - R^n fazodagi biror soha (ochiq, bog'langan to'plam) bo'lsin. $u(x) = u(x_1, x_2, ... x_n)$ funksiyaning $x \in Q$ nuqtadagi $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ tartibli hosilasini

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, \; D^0 u = u(x)$$

ko'rinishda yozamiz. Masalan, $\alpha = \alpha_i$ xususiy hol uchun

$$D^\alpha u = \frac{\partial^{\alpha_i} u}{\partial x_i^{\alpha_i}} = D_i^{\alpha_i} u, \; D_i u = \frac{\partial u}{\partial x_i} = u_{x_i}, \; D_i^2 u = \frac{\partial^2 u}{\partial x_i^2} = u_{x_i x_i}.$$

$F=F(x,...q_\alpha,...)$ funksiya Q soha x nuqtalarining va $q_\alpha=q_{\alpha_1\alpha_2...\alpha_n}=D^\alpha u$, $\alpha_i=0,1,...$ haqiqiy o'zgaruvchilarning berilgan funksiyasi bo'lsin.

Ta'rif. Ushbu

$$F(x,...,D^\alpha u,...)=0 \qquad (1)$$

tenglik noma'lum $u(x)=u(x_1,x_2,...x_n)$ funksiyaga nisbatan **xususiy hosilali differensial tenglama** deyiladi.

(1) da qatnashayotgan hosilaning eng yuqori tartibiga tenglamaning tartibi deyiladi.

Agar F barcha q_α, $(|\alpha|=0,1,...,m)$ o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, (1) tenglama **chiziqli differensial tenglama** deyiladi.

Agar differensial tenglamaning tartibi m bo'lib, F barcha q_α, $|\alpha|=m$ o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, (1) tenglama **kvazichiziqli differensial tenglama** deyiladi.

Ta'rif. Q sohada aniqlangan $u(x)$ funksiya (1) tenglamada ishtirok etuvchi barcha hosilalari bilan uzluksiz bo'lib, uni ayniyatga aylantirsa, $u(x)$ ga (1) tenglamaning **klassik yechimi** deyiladi.

Xususiy hosilali m - tartibli chiziqli differensial tenglamani ushbu

$$Lu \equiv \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x) \qquad (2)$$

ko'rinishda yozish mumkin, bu yerda $a_\alpha(x)$ lar tenglama koeffitsiyentlari,

$$L \equiv \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

esa xususiy hosilali m - tartibli differensial operator.

Barcha $x\in Q$ lar uchun (2) tenglamaning o'ng tomoni $f(x)$ nolga teng bo'lsa, (2) tenglama **bir jinsli**, $f(x)$ nolga teng bo'lmasa, **bir jinsli bo'lmagan** tenglama deyiladi.

Agar $u(x)$ va $v(x)$ funksiyalar bir jinsli bo‘lmagan (2) tenglamaning yechimlari bo‘lsa, ravshanki, (tenglama chiziqli bo‘lgani sababli) $w(x)=u(x)-v(x)$ ayirma bir jinsli ($f=0$) tenglamaning yechimi bo‘ladi.

Agarda $u_i(x)$, $i=1,...,k$ funksiyalar bir jinsli ($f=0$) tenglamaning yechimlari bo‘lsa, $u(x)=\sum_{i=1}^{k} C_i u_i(x)$ funksiya ham, bu yerda C_i – haqiqiy o‘zgarmaslar, shu tenglamaning yechimi bo‘ladi.

Eslatib o‘tamiz, Q sohada aniqlangan va k- tartibgacha xususiy hosilalari bilan uzluksiz bo‘lgan haqiqiy $u(x)$ funksiyalar sinfi $C^k(Q)$ orqali belgilanadi, $C(Q)$- Q sohada uzluksiz funksiyalar sinfi. $g(x)\in C^k(Q)$ funksiyaning normasi

$$\|g\|=\sum_{i=0}^{k}\max_{x\in Q}\left|D^{\alpha}g(x)\right|$$

kabi aniqlanadi.

Ushbu

$$F\left(x_1,x_2,...,x_n,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...,\frac{\partial u}{\partial x_n}\right)=0 \qquad (1)$$

ko`rinishdagi ifoda **birinchi tartibli xususiy hosilali** tenglama deyiladi.

Agar (1) da F funksiya xususiy hosilalarga chiziqli bo‘liq bo‘lsa, u holda

$$X_1(x_1,x_2,...,x_n,u)\frac{\partial u}{\partial x_1}+...+X_n(x_1,x_2,...,x_n,u)\frac{\partial u}{\partial x_n}=R(x_1,x_2,...,x_n,u) \qquad (2)$$

ko`rinishdagi tenglama kvazichiziqli tenglama deyiladi.

(2) tenglama bir jinli bo‘lmagan tenglama bo‘lib, uning simmetrik formasini quyidagicha

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=...=\frac{dx_n}{X_n}=\frac{du}{R} \qquad (3)$$

yozish mumkin. Ushbu sistema xarakteristik tenglamalar sistemasi ham deyiladi. Bu sistemaning n ta erkli integralini

$$\left.\begin{array}{l}\psi_1(x_1,x_2,...,x_n,u)=C_1\\ \psi_2(x_1,x_2,...,x_n,u)=C_2\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \psi_n(x_1,x_2,...,x_n,u)=C_n\end{array}\right\} \qquad (4)$$

topamiz. U holda (2) ning umumiy yechimi

$$\Phi(\psi_1(x_1,x_2,...x_n,u),\psi_2(x_1,x_2,...x_n,u),...\psi_n(x_1,x_2,...x_n,u))=0 \qquad (5)$$

ko`rinishda bo`ladi.

Bir jinsli chiziqli birinchi tartibli xususiy hosilali differensial tenglama quyidagi umumiy ko'rinishga ega:

$$X_1(x_1,x_2,...,x_n)\frac{\partial u}{\partial x_1}+...+X_n(x_1,x_2,...,x_n)\frac{\partial u}{\partial x_n}=0 \qquad (6)$$

Shuni aytish lozimki, $u=const$ har doim (6) tenglamaning yechimi. Biz trivial bo`lmagan yechimni qidiramiz.

(6) ga mos oddiy differensial tenglamalar sistemasining simmetrik formasi ushbu

$$\frac{dx_1}{X_1(x_1,x_2,...,x_n)}=\frac{dx_2}{X_2(x_1,x_2,...,x_n)}=...=\frac{dx_n}{X_n(x_1,x_2,...,x_n)} \qquad (7)$$

ko'rinishda bo'ladi.

(7) sistemaga (6) tenglamaga mos bo'lgan, oddiy differensial tenglamalar sistemasi yoki xarakteristik tenglamalar sistemasi deyiladi. Ushbu sistemaning yechimlari esa (6) tenglamaning xarakteristikalari deyiladi.

Eslatma. Ba'zan 1-tartibli xususiy hosilali differensial tenglamaning umumiy yechimini topishda xarakteristik tenglamalar sistemasi integrallarini topish jarayonida

$$\frac{dx_1}{b_1(x_1,\cdots,x_n)}=\frac{dx_2}{b_2(x_1,\cdots,x_n)}=\cdots=\frac{dx_n}{b_n(x_1,\cdots,x_n)}=k$$

munosabatning o'rinli ekanligidan

$$\frac{a_1dx_1+a_2dx_2+\cdots+a_mdx_m}{a_1b_1+ab+\cdots+a_mb_m}=k \qquad (8)$$

tenglikning bajarilishidan ham foydalanish mumkin. Bunda $a_i=a_i(x_1,x_2,\cdots,x_n), i=1,2,\cdots,m, m\in N$ biror bir funksiyalar.

Misol. Quyidagi tenglamaning umumiy yechimini toping:

$$xz\frac{\partial u}{\partial x}+yz\frac{\partial u}{\partial y}-\left(x^2+y^2\right)\frac{\partial u}{\partial z}=0.$$

Yechish: Berilgan tenglamaning xarakteristik tenglamalar sistemasini tuzamiz:

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}$$

Sistemaning birinchi integrallarini topamiz.

$$\frac{dx}{xz}=\frac{dy}{yz} \Rightarrow \frac{dx}{x}=\frac{dy}{y} \Rightarrow \ln|y|=\ln|x|+\ln|C_1|$$

$$\Rightarrow C_1=\frac{y}{x} \Rightarrow \psi_1=\frac{y}{x},$$

$$\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}=\frac{xdx+ydy+zdz}{0} \Rightarrow$$

$$d(x^2+y^2+z^2)=0 \Rightarrow x^2+y^2+z^2=C_2$$

$$\Rightarrow \psi_2=(x^2+y^2+z^2)$$

bu yerda (8) tenglikdan foydalandik, ya'ni

$$\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}=\frac{xdx+ydy+zdz}{x\cdot xz+y\cdot yz+z\cdot\left(-\left(x^2+y^2\right)\right)},$$

$$\frac{xdx+ydy+zdz}{x\cdot xz+y\cdot yz+z\cdot\left(-\left(x^2+y^2\right)\right)}=\frac{xdx+ydy+zdz}{0}.$$

U holda umumiy yechim

$$u=\Phi\left(\frac{y}{x},x^2+y^2+z^2\right)$$

ko'rinishda bo'ladi.

Tekshirish:

$$\frac{\partial u}{\partial x}=\frac{\partial\Phi}{\partial\psi_1}\cdot\frac{\partial\psi_1}{\partial x}+\frac{\partial\Phi}{\partial\psi_2}\cdot\frac{\partial\psi_2}{\partial x}=-\frac{y}{x^2}\frac{\partial\Phi}{\partial\psi_1}+2x\frac{\partial\Phi}{\partial\psi_2}$$

$$\frac{\partial u}{\partial x}=\frac{\partial\Phi}{\partial\psi_1}\frac{\partial\psi_1}{\partial y}+\frac{\partial\Phi}{\partial\psi_2}\cdot\frac{\partial\psi_2}{\partial y}=\frac{1}{x}\cdot\frac{\partial\Phi}{\partial\psi_1}+2y\frac{\partial\Phi}{\partial\psi_2}$$

$$\frac{\partial u}{oz}=\frac{\partial\Phi}{\partial\psi_1}\cdot\frac{\partial\psi_1}{\partial z}+\frac{\partial\Phi}{\partial\psi_2}\cdot\frac{\partial\psi_2}{\partial z}=0\cdot\frac{\partial\Phi}{\partial\psi_1}+2z\frac{\partial\Phi}{\partial\psi_2}$$

Topilgan ifodalarni tenglamaga qo'yib, uning ayniyatga aylanishiga ishonch hosil qilish mumkin.

Misol. Quyidagi tenglamaning umumiy yechimini toping:

$$xy\frac{\partial z}{\partial x}+(x-2z)\frac{\partial z}{\partial y}=yz.$$

Yechish: Berilgan tenglamaning xarakteristik tenglamalar sistemasini tuzamiz:

$$\frac{dx}{xy}=\frac{dy}{x-2z}=\frac{dz}{yz}$$

Sistemaning birinchi integrallarini topamiz:

$$\frac{dx}{xy}=\frac{dz}{yz}\Rightarrow\frac{dx}{x}=\frac{dz}{z}\Rightarrow\ln|z|=\ln|x|+\ln|C_1|$$

$$\Rightarrow C_1=\frac{z}{x}\Rightarrow\psi_1=\frac{z}{x},$$

$$\frac{dy}{x-2z}=\frac{dx}{xy}\Rightarrow\frac{dy}{x-2C_1x}=\frac{dx}{xy}\Rightarrow ydy=(1-2C_1)dx$$

$$\Rightarrow\frac{y^2}{2}=(1-2C_1)x+C_2\Rightarrow C_2=\frac{y^2}{2}-\left(1-2\frac{z}{x}\right)x\Rightarrow\psi_2=\frac{y^2}{2}-x+2z.$$

Natijada umumiy yechim quyidagicha aniqlanadi:

$$\Phi\left(\frac{z}{x},\frac{y^2}{2}-x+2z\right)=0.$$

Mustaqil bajarish uchun misollar

Tenglamalarning umumiy yechimini toping.

1. $y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0.$
2. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$
3. $yz\frac{\partial u}{\partial x}+xz\frac{\partial u}{\partial y}+xy\frac{\partial u}{\partial z}=0.$
4. $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x-y.$
5. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=xy+u.$
6. $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=u.$
7. $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$
8. $(x+2y)\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0.$

9. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$

10. $(x-z)\frac{\partial u}{\partial x}+(y-z)\frac{\partial u}{\partial y}+2z\frac{\partial u}{\partial z}=0.$

11. $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=x-y.$

12. $e^x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=ye^x.$

13. $2x\frac{\partial z}{\partial x}+(y-x)\frac{\partial z}{\partial y}-x^2=0.$

14. $xy\frac{\partial z}{\partial x}-x^2\frac{\partial z}{\partial y}=yz.$

15. $x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}=x^2y+z.$

16. $(x^2+y^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}+z^2=0.$

17. $2y^4\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=x\sqrt{z^2+1}.$

18. $x^2z\frac{\partial z}{\partial x}+y^2z\frac{\partial z}{\partial y}=y+x.$

19. $yz\frac{\partial z}{\partial x}-xz\frac{\partial z}{\partial y}=e^z.$

20. $(z-y)^2\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=xy.$

21. $xy\frac{\partial z}{\partial x}+(x-2z)\frac{\partial z}{\partial y}=yz.$

22. $y\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}=\frac{y}{x}.$

23. $\sin^2 x\frac{\partial z}{\partial x}+tgz\frac{\partial z}{\partial y}=\cos^2 z.$

24. $(x+z)\frac{\partial z}{\partial x}+(y+z)\frac{\partial z}{\partial y}=x+y.$

25. $(xz+y)\frac{\partial z}{\partial x}+(x+yz)\frac{\partial z}{\partial y}=1-z^2.$

26. $(y+z)\frac{\partial u}{\partial x}+(z+x)\frac{\partial u}{\partial y}+(x+y)\frac{\partial u}{\partial z}=u.$

27. $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+(z+u)\frac{\partial u}{\partial z}=xy.$$

28. $$(u-x)\frac{\partial u}{\partial x}+(u-y)\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=x+y.$$

1.2. Koshi masalalarini yechish

Birinchi tartibli xususiy hosilali differensial tenglama uchun Koshi masalasi quyidagicha qo`yiladi. (2) tenglamaning yechimlari ichidan shunday

$$u=f(x_1,x_2,\dots,x_n)$$

yechimni topingki, u $x_n=x_n^0$ da

$$u=\varphi(x_1,x_2,\dots,x_{n-1}) \qquad (9)$$

funksiyaga teng bo`lsin, bunda φ – berilgan funksiya.

Koshi masalasini yechish ushbu tartibda amalga oshiriladi:

1. Tenglamaning simmetrik (3) formasini tuzib, (4) n ta integral topiladi.

2. (4) dagi x o'rniga x_n^0 ni qo'yamiz:

$$\left.\begin{aligned}&\psi_1(x_1,x_2,\dots,x_{n-1},x_n^0,u)=\overline{\psi}_1\\&\psi_2(x_1,x_2,\dots,x_{n-1},x_n^0,u)=\overline{\psi}_2\\&\dots\dots\dots\dots\dots\dots\dots\dots\\&\psi_n(x_1,x_2,\dots,x_{n-1},x_n^0,u)=\overline{\psi}_n\end{aligned}\right\}$$

va sistema $x_1,x_2,\dots,x_{n-1},u$ ga nisbatan yechiladi.

$$\left.\begin{aligned}&x_1=\omega_1(\overline{\psi}_1,\overline{\psi}_2,\dots,\overline{\psi}_n)\\&\dots\dots\dots\dots\dots\dots\\&x_{n-1}=\omega_{n-1}(\overline{\psi}_1,\overline{\psi}_2,\dots,\overline{\psi}_n)\\&u=\omega(\overline{\psi}_1,\overline{\psi}_2,\dots,\overline{\psi}_n)\end{aligned}\right\}$$

3. Ushbu funksiyalardan

$$\omega(\psi_1,\psi_2,\dots,\psi_n)=\varphi(\omega_1(\psi_1,\psi_2,\dots\psi_n),\omega_2(\psi_1,\psi_2,\dots\psi_n),\dots\omega_{n-1}(\psi_1,\psi_2,\dots\psi_n)), \qquad (10)$$

munosabatni tuzamiz. (10) ga Koshi masalasining oshkormas ko`rinishdagi yechimi deyiladi. Agar (10) ni $u$ funksiyaga nisbatan yechsak, oshkor ko`rinishida Koshi masalasining yechimini olamiz.

Masala. Ushbu $\left(1+\sqrt{z-x-y}\right)\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=2$ tenglamaning $y=0$ da $z=2x$ shartni qanoatlantiruvchi yechimini toping.

Yechish: Berilgan tenglamaning simmetrik formasi

$$\frac{dx}{1+\sqrt{z-x-y}}=\frac{dy}{1}=\frac{dz}{2}$$

ko'rinishdan iborat. Bu sistemani yechib,

$\psi_1=z-2y,\quad \psi_2=2\sqrt{z-x-y}+y$ larni hosil qilamiz,

bunda $y=0$ ni qo'yib,

$$z=\psi_1,$$
$$2\sqrt{z-x}=\psi_2$$

larga ega bo'lamiz. Bu sistemadan x va z ni topamiz:

$$x=\psi_1-\frac{\psi_2^2}{4},$$
$$z=\psi_1.$$

(10) formulaga ko'ra

$$\psi_1-2\left(\psi_1-\frac{\psi_2^2}{4}\right)=0,\ 2\psi_1-\psi_2^2=0,$$

bunda ψ_1 va ψ_2 larning ko'rinishidan foydalansak,

$$2z-4y-\left(2\sqrt{z-x-y}+y\right)^2=0$$

Koshi masalasining yechimini hosil qilamiz.

Masala. Quyidagi tenglamaning

$$(4y-z)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

$u|_{x=0}=y^2+z^2$ shartni qanoatlantiruvchi yechimini toping.

Yechish: Berilgan tenglamaning simmetrik formasi $\frac{dx}{4y-z}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{0}$ ni yozib olib, uni yechish natijasida

$\psi_1=\frac{z}{y},\ \psi_2=x-4y+z,\ \psi_3=u$ ifodalarga ega bo'lamiz.

Bu yerda $x=0$ deb,

$$\frac{z}{y}=\psi_1,\quad -4y+z=\psi_2,\quad y^2+z^2=\psi_3$$

tengliklarni hosil qilamiz hamda ulardan y va z larni topamiz.

(10) formulaga ko‘ra

$$\psi_3=\left(\frac{\psi_2}{\psi_1-4}\right)^2\left(1+\psi_1^2\right)^2,$$

bunda ψ_1, ψ_2 va ψ_3 larning ko`rinishidan foydalansak,

$$u=\frac{(x-4y+z)^2}{(z-4y)^2}\left(y^2+z^2\right)$$

Koshi masalasining yechimi bo`ladi.

Masala. Quyidagi tenglamaning

$$xz\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}=-xy$$

$y=x^2$, $z=x^2$ shartni qanoatlantiruvchi yechimini toping.

Yechish: Berilgan tenglamaning simmetrik formasidan iborat

$$\frac{dx}{xz}=\frac{dy}{yz}=-\frac{dz}{xy}$$

sistemani yechib,

$\psi_1=\frac{x}{y}$, $\psi_2=z^2+xy$ larni hosil qilamiz.

Bunda berilgan shartlardan foydalanib,

$$x=\frac{1}{\psi_1},$$

$$x^2+x\cdot\frac{x}{\psi_1}=\psi_2$$

tengliklarni va quyidagi funksional bog‘lanishni olamiz:

$$\psi_2=\frac{1}{\psi_1^2}+\frac{1}{\psi_1^3}.$$

Bu yerda ψ_1 va ψ_2 larning ko`rinishidan foydalansak,

$$z^2+xy=\left(\frac{y}{x}\right)^2+\left(\frac{y}{x}\right)^3$$

Koshi masalasining yechimini topamiz.

Mustaqil bajarish uchun masalalar

Quyidagi Koshi masalalarini yeching:

4. $(4y-z)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0,\ u|_{x=0}=y^2+z^2$.

5. $xz\frac{\partial u}{\partial x}+yz\frac{\partial u}{\partial y}+xy\frac{\partial u}{\partial z}=0,\ u|_{z=0}=xy.$

6. $x(z-y)\frac{\partial u}{\partial x}+y(y-x)\frac{\partial u}{\partial y}+(y^2-xz)\frac{\partial u}{\partial z}=0,\ u|_{x=1}=\frac{z}{y}.$

7. $x\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=0,\ u|_{x=1}=-y.$

8. $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0,\quad z=2x,\ y=1.$

9. $\frac{\partial z}{\partial x}+(2e^x-y)\frac{\partial z}{\partial y}=0,\ z=y,\ x=0.$

10. $2\sqrt{x}\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0,\quad z=y^2,\ x=1.$

11. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+2\frac{\partial u}{\partial z}=0,\ u=yz,\quad x=1.$

12. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+xy\frac{\partial u}{\partial z}=0,\ u=x^2+y^2,\ z=0.$

13. $y^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=x,\ x=0,\ z=y^2.$

14. $x\frac{\partial z}{\partial x}-2y\frac{\partial z}{\partial y}=x^2+y^2,\ y=1,\ z=x^2.$

15. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy,\quad x=2,\quad z=y^2+1.$

16. $tgx\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z,\quad y=x,\ z=x^3.$

17. $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=z^2(x-3y),\quad x=1,\ yz+1=0.$

18. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-x^2-y^2,\quad y=-2,\ z=x-x^2.$

19. $yz\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=xy,\ x=a,\quad y^2+z^2=a^2.$

20. $z\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=2xy,\ x+y=2,\quad yz=1.$

21. $z\frac{\partial z}{\partial x}+(z^2-x^2)\frac{\partial z}{\partial y}+x=0,\ y=x^2,\ z=2x.$

22. $(y-z)\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y,\ z=y=-x.$

23. $x\frac{\partial z}{\partial x}+(xz+y)\frac{\partial z}{\partial y}=z$, $x+y=2z$, $xz=1$.

24. $y^2\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}+z^2=0$, $x-y=0$, $x-zy=1$.

25. $x\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}=y$, $y=2x$, $x+2y=z$.

26. $(y+2z^2)\frac{\partial z}{\partial x}-2x^2z\frac{\partial z}{\partial y}=x^2$, $x=z$, $y=x^2$.

27. $(x-z)\frac{\partial z}{\partial x}+(y-z)\frac{\partial z}{\partial y}=2z$, $x-y=2$, $z+2x=1$.

28. $xy^3\frac{\partial z}{\partial x}+x^2z^2\frac{\partial z}{\partial y}=y^3z$, $x=-z^3$, $y=z^2$.

29. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xy$, $y=x$, $z=x^2$.

2-BOB. IKKINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR HAQIDA ASOSIY TUSHUNCHALAR. IKKINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING KLASSIFIKATSIYASI. KANONIK KO'RINISHGA KELTIRISH

Ushbu bobda ikkinchi tartibli xususiy hosilali differensial tenglamalar haqida umumiy ma'lumotlar berilgan bo'lib, ikkinchi tartibli xususiy hosilali differensial tenglamalarning klassifikatsiyasi, ko'p erkli o'zgaruvchili funksiyalar, n=2 va n>2 bo'lgan hollar uchun ikkinchi tartibli xususiy hosilali differensial tenglamalarni kanonik ko'rinishga keltirish bayon etilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

2.1. Ikkinchi tartibli xususiy hosilali differensial tenglamalarni turi saqlanadigan sohada kanonik ko'rinishga keltirish

Ta'rif. x,y erkli o'zgaruvchilarning $u(x,y)$ noma'lum funksiyasi va funksiyaning ikkinchi tartibigacha xususiy hosilalari orasidagi bog'lanishga, ikkinchi tartibli xususiy hosilali differensial tenglamalar deyiladi.

Ta'rif. R^2 fazoda ikkinchi tartibgacha xususiy hosilalari mavjud qandaydir $u(x,y)$ funksiya berilgan bo'lsin ($u_{xy}=u_{yx}$). U holda

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=0 \qquad (1)$$

tenglama umumiy holda berilgan ikkinchi tartibli xususiy hosilali differensial tenglama deyiladi, bu yerda F – berilgan biror-bir funksiya.

Xuddi shunga o'xshash ko'p erkli o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglama quyidagi ko'rinishda ifodalanadi:

$$F(x_1, x_2, ..., x_n, u, u_{x_1}, u_{x_2}, ..., u_{x_n}, ..., u_{x_i x_j}, ...) = 0.$$

(2)

Ta'rif. Agarda ikkinchi tartibli ikki o'zgaruvchili xususiy hosilali differensial tenglama yuqori tartibli hosilalarga nisbatan ushbu ko'rinishga

$$a_{11}(x,y) \cdot u_{xx} + 2a_{12}(x,y) \cdot u_{xy} + a_{22}(x,y) \cdot u_{yy} + F(x, y, u, u_x, u_y) = 0 \quad (3)$$

ega bo'lsa, unda ushbu tenglamaga yuqori tartibli hosilalarga nisbatan chiziqli deyiladi.

Ta'rif. Quyidagi ko'rinishdagi tenglamalarga ikkinchi tartibli ikki o'zgaruvchili kvazichiziqli xususiy hosilali differensial tenglamalar deyiladi:

$$a_{11}(x, y, u, u_x, u_y) \cdot u_{xx} + 2a_{12}(x, y, u, u_x, u_y) \cdot u_{xy} + a_{22}(x, y, u, u_x, u_y) \cdot u_{yy} + F(x, y, u, u_x, u_y) = 0.$$

(4)

Ta'rif. Agarda ikkinchi tartibli ikki o'zgaruvchili xususiy hosilali differensial tenglama barcha xususiy hosilalariga va noma'lum funksiyaning o'ziga nisbatan ham chiziqli bo'lsa, ya'ni quyidagi ko'rinishga

$$a_{11}(x,y) \cdot u_{xx} + 2a_{12}(x,y) \cdot u_{xy} + a_{22}(x,y) \cdot u_{yy} + b_1(x,y) \cdot u_x + b_2(x,y) \cdot u_y + c(x,y) \cdot u + f(x,y) = 0.$$

(5)

ega bo'lsa, unda ushbu tenglamaga chiziqli tenglama deyiladi.

(5) tenglamada $a_{11}(x,y)$, $a_{12}(x,y)$, $a_{22}(x,y)$, $b_1(x,y)$, $b_2(x,y)$, $c(x,y)$larga (5) tenglamaning koeffitsiyentlari, $f(x,y)$ ga (5) tenglamaning ozod hadi deyiladi va ular oldindan berilgan deb hisoblanadi.

Ta'rif. Agar (5) tenglamada $f(x,y) \equiv 0$ bo'lsa, u holda bu tenglama bir jinsli tenglama deyiladi. Aks holda, ya'ni $f(x,y) \neq 0$ bo'lsa, (5) tenglama bir jinsli bo'lmagan differensial tenglama deyiladi.

(3) (yoki (5)) tenglamada o'zgaruvchilarni ixtiyoriy (o'zaro bir qiymatli) almashtiramiz. Bu uchun biz x va y erkli o'zgaruvchilarni teskari almashtirish natijasida, ya'ni

$$\xi = \varphi(x,y), \quad \eta = \psi(x,y) \tag{6}$$

berilgan chiziqli tenglamaga ekvivalent bo'lgan va soddaroq ko'rinishga ega bo'lgan tenglamaga ega bo'lishimiz mumkin.

Buning uchun (3) tenglamada x va y erkli o'zgaruvchilardan yangi ξ va η o'zgaruvchilarga o'tamiz:

$$\left.\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y, \\ u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}, \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}. \end{aligned}\right\} \tag{7}$$

(7) ifodalarni (3) tenglamaga keltirib qo'yib, ξ va η o'zgaruvchilarga nisbatan (3) tenglamaga ekvivalent bo'lgan quyidagi tenglamani olamiz:

$$\overline{a_{11}}(\xi,\eta)\cdot u_{\xi\xi} + 2\overline{a_{12}}(\xi,\eta)\cdot u_{\xi\eta} + \overline{a_{22}}(\xi,\eta)\cdot u_{\eta\eta} + \overline{F}(\xi,\eta,u,u_\xi,u_\eta) = 0, \tag{8}$$

bu yerda

$$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2,$$

$$\bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \eta_x\xi_y) + a_{22}\xi_y\eta_y,$$

$$\bar{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2,$$

Ta'rif.

$$a_{11}dy^2 - 2a_{12}dxdy + a_{22}dx^2 = 0 \tag{9}$$

oddiy differensial tenglama, (3) tenglamaning xarakteristik tenglamasi deyiladi.

Ta'rif. (9) tenglamaning integral chiziqlari esa (3) tenglamaning xarakteristikalari deyiladi.

(9) tenglama quyidagi ikkita tenglamaga ajraladi:

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}\cdot a_{22}}}{a_{11}}, \tag{10}$$

$$\frac{dy}{dx} = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}\cdot a_{22}}}{a_{11}}. \tag{11}$$

(9) yoki (10) va (11) oddiy differensial tenglama yordamida berilgan (3)-tenglamaning xarakteristikalari topiladi.

Ta'rif. Agar qandaydir D sohada $a_{12}^2-a_{11}\cdot a_{22}>0$ bo'lsa, (3) tenglama giperbolik turga qarashli, agar D sohada $a_{12}^2-a_{11}\cdot a_{22}<0$ bo'lsa, (3) tenglama elliptik turga qarashli, agar D sohada $a_{12}^2-a_{11}\cdot a_{22}=0$ bo'lsa, (3) tenglama parabolik turga qarashli deyiladi.

Shunday qilib, $a_{12}^2-a_{11}\cdot a_{22}$ ifodaning ishorasiga qarab (3) tenglama quyidagi kanonik ko'rinishlarga keltirilishi mumkin ekan:

$a_{12}^2-a_{11}\cdot a_{22}>0$ (giperbolik tur), $u_{xx}-u_{yy}=\Phi(x,y,u,u_x,u_y)$ yoki $u_{xy}=\Phi(x,y,u,u_x,u_y)$.

$a_{12}^2-a_{11}\cdot a_{22}<0$ (elliptik tur), $u_{xx}+u_{yy}=\Phi(x,y,u,u_x,u_y)$.

$a_{12}^2-a_{11}\cdot a_{22}=0$ (parabolik tur) $u_{xx}=\Phi(x,y,u,u_x,u_y)$.

Bu yerda $\Phi(x,y,u,u_x,u_y)$ soddalashtirish natijasida hosil bo'lgan funksiya.

<u>**Misol.**</u> Quyidagi tenglamani kanonik ko'rinishga keltiraylik:

$$u_{xx}-2u_{xy}-3u_{yy}+u_y=0.$$

<u>**Yechish:**</u> $a_{12}=-1$, $a_{11}=1$, $a_{22}=-3$ – tenglama koeffitsiyentlari. $\Delta=a_{12}^2-a_{11}\cdot a_{22}$ ifodaning qiymatini hisoblaymiz. $\Delta=4>0$, demak tenglama giperbolik turga tegishli. (9) xarakteristik tenglamani tuzib, uni yechamiz:

$$\frac{dy}{dx}=\frac{-1+2}{1}\Rightarrow\frac{dy}{dx}=1\Rightarrow x-y=C\ ,$$

$$\frac{dy}{dx}=\frac{-1-2}{1}\Rightarrow\frac{dy}{dx}=-3\Rightarrow 3x+y=C\ .$$

Umumiy integrallardan birini ξ va ikkinchisini η bilan belgilab, (7) formulalardan foydalanib hisoblashlarning natijalarini berilgan tenglamaga keltirib qo'yib, soddalashtirishlardan so'ng tenglamaning quyidagi kanonik ko'rinishini hosil qilamiz: $u_{\xi\eta}-\frac{1}{16}(u_\xi-u_\eta)=0$.

<u>**Misol.**</u> Quyidagi tenglamani kanonik ko'rinishga keltiraylik: $y^2u_{xx}+2yu_{xy}+u_{yy}=0$.

Yechish: $a_{12}=y$, $a_{11}=y^2$, $a_{22}=1$ – tenglama koeffitsiyentlari. $\Delta=a_{12}^2-a_{11}\cdot a_{22}$ ifodaning qiymatini hisoblaymiz. $\Delta=0$, demak tenglama parabolik turga tegishli. (9) xarakteristik tenglamani tuzib, uni yechamiz:

$$\frac{dy}{dx}=\frac{y}{y^2}\Rightarrow\frac{dy}{dx}=\frac{1}{y}\Rightarrow x-\frac{y^2}{2}=C.$$

Natijada olingan integralni ξ orqali, η orqali esa ixtiyoriy funksiyani, masalan, $\eta=y$ deb belgilab, (7) formulalardan foydalanib hisoblashlarning natijalarini berilgan tenglamaga keltirib qo'yib, soddalashtirishlardan so'ng tenglamaning quyidagi kanonik ko'rinishini hosil qilamiz: $u_{\eta\eta}=u_\xi$.

Misol. Quyidagi tenglamani kanonik ko'rinishga keltiraylik:

$$(1+x^2)u_{xx}+(1+y^2)u_{yy}+yu_y=0.$$

Yechish: $a_{12}=0$, $a_{11}=1+x^2$, $a_{22}=1+y^2$ – tenglama koeffitsiyentlari. $\Delta=a_{12}^2-a_{11}\cdot a_{22}$ ifodaning qiymatini hisoblaymiz. $\Delta=-(1+x^2)(1+y^2)$, demak tenglama elliptik turga tegishli. (9) xarakteristik tenglamani tuzib, uni yechamiz:

$$\frac{dy}{dx}=\frac{0\pm i\sqrt{(1+x^2)(1+y^2)}}{1+x^2}\Rightarrow\frac{dy}{dx}=\pm i\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}\Rightarrow,$$

$$\ln\left(y+\sqrt{1+y^2}\right)\mp i\ln\left(x+\sqrt{1+x^2}\right)=C$$

Umumiy nazariyaga asosan, olingan integralning haqiqiy qismini ξ $\left(\xi=\mathrm{Re}\left(\ln\left(y+\sqrt{1+y^2}\right)\mp i\ln\left(x+\sqrt{1+x^2}\right)\right)=\ln\left(y+\sqrt{1+y^2}\right)\right)$ orqali, mavhum qismini esa η $\left(\eta=\mathrm{Im}\left(\ln\left(y+\sqrt{1+y^2}\right)\mp i\ln\left(x+\sqrt{1+x^2}\right)\right)=\ln\left(x+\sqrt{1+x^2}\right)\right)$ orqali belgilab, (7) formulalardan foydalanib hisoblashlarning natijalarini berilgan tenglamaga keltirib qo'yib, soddalashtirishlardan so'ng tenglamaning quyidagi kanonik ko'rinishini hosil qilamiz: $u_{\xi\xi}+u_{\eta\eta}-th\eta u_\eta=0$.

Mustaqil bajarish uchun misollar

Quyidagi tenglamalarning turini aniqlang:

1. $(y+1)\frac{\partial^2u}{\partial x^2}-2\frac{\partial^2u}{\partial x\partial y}+x\frac{\partial^2u}{\partial y^2}-\frac{\partial u}{\partial y}=0,\ 1<x<3,\qquad 0<y<1.$

2. $y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x^2}+2(x+y)\frac{\partial^2 u}{\partial x\partial y}=0,\ x^2+(y-6)^2<1.$

3. $2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}+y^2\frac{\partial^2 u}{\partial x^2}-x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial x}=0,\ |x|<1,\qquad |y|<1.$

4. $(x+y)\frac{\partial^2 u}{\partial x^2}+(x-y)\frac{\partial^2 u}{\partial y^2}+xu=0,\ (x+5)^2+y^2<1.$

5. $(y+1)\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}=0,\ 1<x<3,\qquad 0<y<1.$

6. $4\frac{\partial^2 u}{\partial x^2}-2(x-y)\frac{\partial^2 u}{\partial x\partial y}+(1-xy)\frac{\partial^2 u}{\partial y^2}=0,\ 2<x+y<5.$

7. $x^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0,\ 1<x^2+y^2<7.$

8. $x\frac{\partial^2 u}{\partial x^2}+6\frac{\partial u}{\partial x}+(x+y)\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}=0,\ 0<x<2,\qquad 0<y<2.$

9. $6\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=0,\ 1<x<2,\qquad 2<y<3.$

10. $2x\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x^2}-(x^2-2)\frac{\partial^2 u}{\partial y^2}-2y\frac{\partial^2 u}{\partial x\partial y}=0,\ x^2+y^2<1.$

11. $5x\frac{\partial^2 u}{\partial x^2}-4x\frac{\partial^2 u}{\partial x\partial y}+2y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-u=0,\ 1<x<3,\qquad 4<y<8.$

Quyidagi tenglamalarni kanonik ko‘rinishga keltiring:

12. $u_{xx}-6u_{xy}+10u_{yy}+u_x-3u_y=0.$

13. $4u_{xx}+4u_{xy}+u_{yy}-2u_y=0.$

14. $u_{xx}-xu_{yy}=0.$

15. $u_{xx}-yu_{yy}=0.$

16. $xu_{xx}-yu_{yy}=0.$

17. $yu_{xx}-xu_{yy}=0.$

18. $x^2u_{xx}+y^2u_{yy}=0.$

19. $y^2u_{xx}+x^2u_{yy}=0.$

20. $y^2u_{xx}-x^2u_{yy}=0.$

21. $(1+x^2)u_{xx}+(1+y^2)u_{yy}+yu_y=0.$

22. $4y^2u_{xx}-e^{2x}u_{yy}=0.$

23. $u_{xx}-2\sin x\cdot u_{xy}+\left(2-\cos^2 x\right)u_{yy}=0$.

24. $y^2u_{xx}+2yu_{xy}+u_{yy}=0$.

25. $x^2u_{xx}-xu_{xy}+u_{yy}=0$.

26. $2\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$.

27. $2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}-6\frac{\partial^2 u}{\partial y^2}=0$.

28. $\frac{\partial^2 u}{\partial x^2}-10\frac{\partial^2 u}{\partial x\partial y}+25\frac{\partial^2 u}{\partial y^2}=0$.

29. $\frac{\partial^2 u}{\partial x^2}+e^{2x}\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}-x\frac{\partial u}{\partial x}=0$.

30. $e^{2y}\frac{\partial^2 u}{\partial x^2}+2xe^{y}\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=0$.

31. $y\frac{\partial^2 u}{\partial x^2}+x(2y-1)\frac{\partial^2 u}{\partial x\partial y}-2x^2\frac{\partial^2 u}{\partial y^2}=0$.

32. $9y^4\frac{\partial^2 u}{\partial x^2}+6y^2\sin x\frac{\partial^2 u}{\partial x\partial y}+\sin^2 x\frac{\partial^2 u}{\partial y^2}=0$.

33. $x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+(4+y^2)\frac{\partial^2 u}{\partial y^2}=0$.

34. $y\frac{\partial^2 u}{\partial x^2}+(e^x-y)\frac{\partial^2 u}{\partial x\partial y}-e^x\frac{\partial^2 u}{\partial y^2}=0$.

35. $x\frac{\partial^2 u}{\partial x^2}+(1+x\,tgx)\frac{\partial^2 u}{\partial x\partial y}+tgx\frac{\partial^2 u}{\partial y^2}=0$.

36. $\cos^2 y\frac{\partial^2 u}{\partial x^2}-2\sin x\cdot\cos y\frac{\partial^2 u}{\partial x\partial y}+\sin^2 x\frac{\partial^2 u}{\partial y^2}=0$.

37. $x^2\frac{\partial^2 u}{\partial x^2}+(2x^2-y^2)\frac{\partial^2 u}{\partial x\partial y}-2y^2\frac{\partial^2 u}{\partial y^2}=0$.

38. $\frac{\partial^2 u}{\partial x^2}+2\cos^2 y\frac{\partial^2 u}{\partial x\partial y}+\cos^4 y\frac{\partial^2 u}{\partial y^2}=0$.

39. $\sin^2 y\frac{\partial^2 u}{\partial x^2}+\cos^2 x\frac{\partial^2 u}{\partial y^2}=0$.

40. $x^4\frac{\partial^2 u}{\partial x^2}-2x^2y\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial x}=0$.

41. $\sin^4 x\frac{\partial^2 u}{\partial x^2}+2\sin^2 x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}+\sin x\frac{\partial u}{\partial x}=0$.

42. $$e^{2x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+2e^{-2x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}=0.$$

43. $$\cos^4 x\frac{\partial^2 u}{\partial x^2}+\sin^4 y\frac{\partial^2 u}{\partial y^2}-3\frac{\partial u}{\partial y}=0.$$

44. $$tg^2x\frac{\partial^2 u}{\partial x^2}-2ytgx\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}=0.$$

45. $$e^{2y}\frac{\partial^2 u}{\partial x^2}+3e^y\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}+e^y\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0.$$

46. $$x^4\frac{\partial^2 u}{\partial x^2}+4x^2\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}+\frac{1}{x}\frac{\partial u}{\partial y}=0.$$

47. $$\sin^2 y\frac{\partial^2 u}{\partial x^2}-4\sin y\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+2\cos y\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0.$$

48. $$\frac{\partial^2 u}{\partial x^2}+2ctgx\frac{\partial^2 u}{\partial x\partial y}+ctg^2x\frac{\partial^2 u}{\partial y^2}-\cos x\frac{\partial u}{\partial x}=0.$$

49. $$tg^2x\frac{\partial^2 u}{\partial x^2}+ctg^2y\frac{\partial^2 u}{\partial y^2}-\sin x\frac{\partial u}{\partial x}+2\cos y\frac{\partial u}{\partial y}=0.$$

50. $$(x+y)\frac{\partial^2 u}{\partial x^2}+2y\frac{\partial^2 u}{\partial x\partial y}+(y-x)\frac{\partial^2 u}{\partial y^2}=0.$$

51. $$(x^2+9)\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

52. $$x\frac{\partial^2 u}{\partial x^2}+(x+x^2+2y)\frac{\partial^2 u}{\partial x\partial y}+(x^2+2y)\frac{\partial^2 u}{\partial y^2}=0.$$

53. $$x^2\frac{\partial^2 u}{\partial x^2}-(1+xy+x^2)\frac{\partial^2 u}{\partial x\partial y}+(xy+1)\frac{\partial^2 u}{\partial y^2}=0.$$

54. $$\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 u}{\partial t^2}=0.$$

55. $$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0.$$

56. $$4\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$$

57. $$\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+8\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y}=0.$$

58. $$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0.$$

59. $$3\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$$

60. $5\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$

61. $9\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0.$

62. $4\frac{\partial^2 u}{\partial x^2}-8\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0.$

63. $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+9\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=0.$

64. $2\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$

65. $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0.$

66. $5\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$

67. $5\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}+6(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y})=0.$

68. $9\frac{\partial^2 u}{\partial x^2}-12\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y}=0.$

69. $5\frac{\partial^2 u}{\partial x^2}-8\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}+3(\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y})=0.$

70. $3\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}+7(\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y})=0.$

71. $\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}+cu=0.$

72. $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0.$

73. $3\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-3\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$

74. $\frac{x}{y}\frac{\partial^2 u}{\partial x^2}-\frac{y}{x}\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial x}-\frac{1}{x}\frac{\partial u}{\partial y}=0.$

75. $(1+x^2)\frac{\partial^2 u}{\partial x^2}+(1+y^2)\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

76. $x\frac{\partial^2 u}{\partial x^2}-4x^3\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}=0.$

77. $x^2\frac{\partial^2 u}{\partial x^2}-6xy\frac{\partial^2 u}{\partial x\partial y}+9y^2\frac{\partial^2 u}{\partial y^2}+12y\frac{\partial u}{\partial y}=0.$

78. $4y^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}=0.$

79. $e^y\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+(1+e^y)\frac{\partial u}{\partial y}=0.$

80. $4y^2\frac{\partial^2 u}{\partial x^2}-4y\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{y}\frac{\partial u}{\partial y}=0.$

81. $y^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2x^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=0.$

82. $\cos^2 y\frac{\partial^2 u}{\partial x^2}-2\cos y\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-x\cos^2 y\frac{\partial u}{\partial x}+(tgx-x\cos y)\frac{\partial u}{\partial y}=0.$

83. $\frac{\partial^2 u}{\partial x^2}+2\sin x\frac{\partial^2 u}{\partial x\partial y}-\cos^2 x\frac{\partial^2 u}{\partial y^2}=0.$

84. $\sin y\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\left(\frac{\sin y}{x}-ctgy\right)\frac{\partial u}{\partial y}=0.$

85. $9x^2\frac{\partial^2 u}{\partial x^2}-6xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=0.$

86. $x^2\frac{\partial^2 u}{\partial x^2}-2x\sin y\frac{\partial^2 u}{\partial x\partial y}+\sin^2 y\frac{\partial^2 u}{\partial y^2}=0.$

87. $x^2\frac{\partial^2 u}{\partial x^2}+\cos^4 y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}=0.$

88. $\sin^2 y\frac{\partial^2 u}{\partial x^2}+2\sin y\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\cos y\frac{\partial u}{\partial x}=0.$

89. $e^{2y}\frac{\partial^2 u}{\partial x^2}+3e^y\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial y}=0.$

90. $y^2\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial y^2}+\frac{4}{y}\frac{\partial u}{\partial y}=0.$

91. $y^2\frac{\partial^2 u}{\partial x^2}-2ye^x\frac{\partial^2 u}{\partial x\partial y}+e^{2x}\frac{\partial^2 u}{\partial y^2}-y^2\frac{\partial u}{\partial x}-\frac{e^{2x}}{y}\frac{\partial u}{\partial y}=0.$

92. $\frac{\partial^2 u}{\partial x^2}-4x\frac{\partial^2 u}{\partial x\partial y}+8x^2\frac{\partial^2 u}{\partial y^2}-\frac{1}{x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0.$

93. $y^2\frac{\partial^2 u}{\partial x^2}+4yx^2\frac{\partial^2 u}{\partial x\partial y}+4x^4\frac{\partial^2 u}{\partial y^2}+2x^2\frac{\partial u}{\partial x}+4xy\frac{\partial u}{\partial y}=0.$

94. $\cos^2 y\frac{\partial^2 u}{\partial x^2}-4\cos y\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+2\sin y\frac{\partial u}{\partial x}=0.$

95. $\frac{\partial^2 u}{\partial x^2}+e^y\frac{\partial^2 u}{\partial x\partial y}+\frac{5}{4}e^{2y}\frac{\partial^2 u}{\partial y^2}+\frac{5}{4}e^{2y}\frac{\partial u}{\partial y}=0.$

96. $\frac{\partial^2 u}{\partial x^2}-2\cos x\frac{\partial^2 u}{\partial x\partial y}-\sin^2 x\frac{\partial^2 u}{\partial y^2}=0.$

97. $\sin^2 x\frac{\partial^2 u}{\partial x^2}-2y\sin x\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$

98. $cth^2 x\frac{\partial^2 u}{\partial x^2}-2ycthx\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2y\frac{\partial u}{\partial y}=0.$

99. $tg^2 x\frac{\partial^2 u}{\partial x^2}-2ytgx\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+tg^3 x\frac{\partial u}{\partial x}=0.$

100. $y\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

101. $\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+\alpha\frac{\partial u}{\partial y}=0.\qquad \alpha=const$

102. $y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}=0.$

103. $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}=0.$

2.2. Ko'p erkli o'zgaruvchili funksiyalar (n>2) bo'lgan hol uchun ikkinchi tartibli xususiy hosilali differensial tenglamalarni kanonik ko'rinishga keltirish

Ko'p erkli o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglama qanday kanonik ko'rinishga keltiriladi? Shu masalani qarab chiqaylik. Ko'p o'zgaruvchili chiziqli ikkinchi tartibli xususiy hosilali differensial tenglama umumiy holda quyidagicha berilgan bo'lsin:

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i}+Cu=f, \qquad (12)$$

bu yerda A_{ij},B_i,C – tenglamaning koeffitsiyentlari, f – ozod hadi.

Ushbu tenglamaga mos keluvchi xarakteristik tenglama:

$$Q(\lambda_1,...,\lambda_n)=\sum_{i,j=1}^{n} A_{ij}(x)\lambda_i\lambda_j,$$

kvadratik formaga ega bo'ladi.

Chiziqli algebra kursidan ma'lumki, har bir tayin x nuqtada Q kvadratik formani uncha qiyin bo'lmagan affin almashtirishlari yordamida kanonik ko'rinishga keltirish mumkin:

$$Q=\sum_{i=1}^{n}\alpha_i\xi_i^2 \qquad (13)$$

Bu yerda α_i lar 1, -1, 0 qiymatlarni qabul qiladi. (13) dagi manfiy va nol koeffitsiyentlar Q ni kanonik ko'rinishga keltirish usuliga bog'liq emas. Shunga asosan (12) tenglama klassifikatsiyalanadi.

Ta'rif. Agar har bir $x\in D$ nuqtada (13) dagi α_i koeffitsiyentlar mos ravishda: hammasi noldan farqli va bir xil ishorali; hammasi noldan farqli va har xil ishorali; va nihoyat hech bo'lmaganda bittasi (hammasi emas) nol bo'lsa, (12) chiziqli tenglama D sohada mos ravishda elliptik, giperbolik yoki parabolik deyiladi.

Ko'p erkli o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalardan bittasini kanonik ko'rinishga keltirish usulini qarab chiqaylik.

Misol. Quyidagi tenglama berilgan bo'lsin:

$$u_{xx}+2u_{xy}+2u_{yy}+4u_{yz}+5u_{zz}=0.$$

Uning turini aniqlaymiz va kanonik ko'rinishga keltiramiz.

Yechish: Ushbu tenglamaga mos xarakteristik kvadratik forma $Q=\lambda_1^2+2\lambda_1\lambda_2+2\lambda_2^2+4\lambda_2\lambda_3+5\lambda_3^2$ ko'rinishda bo'ladi. Bu kvadratik formani, masalan, Lagranj usulidan foydalanib kanonik ko'rinishga keltiramiz: $Q=(\lambda_1+\lambda_2)^2+(\lambda_2+2\lambda_3)^2+\lambda_3^2$. Quyidagi belgilashlar kiritamiz:

$$\mu_1=\lambda_1+\lambda_2;\ \mu_2=\lambda_2+2\lambda_3;\ \mu_3=\lambda_3 \qquad (14)$$

va natijada Q formani kanonik ko'rinishga keltiramiz: $Q=\mu_1^2+\mu_2^2+\mu_3^2$.

(14) tengliklardan λ larni topib olamiz. Shunday qilib, $M=\begin{pmatrix}1&-1&2\\0&1&-2\\0&0&1\end{pmatrix}$

matrisali quyidagi xosmas affin almashtirishlari: $\lambda_1=\mu_1-\mu_2+2\mu_3$,

$\lambda_2=\mu_2-2\mu_3$, $\lambda_3=\mu_3$lar Q formani kanonik ko‘rinishga keltiradi: $Q=\mu_1^2+\mu_2^2+\mu_3^2$.

Berilgan differensial tenglamani kanonik ko‘rinishga keltiradigan xosmas affin almashtirishining matrisasi M matrisaga simmetrik bo‘lgan matrisa bo‘ladi: $M^*=\begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -2 & 1\end{pmatrix}$, bu almashtirish quyidagi ko‘rinishga ega: $\xi=x$; $\eta=-x+y$; $\zeta=2x-2y+z$.

Shulardan va $u(x,y,z)=v(\xi,\eta,\zeta)$ belgilashdan foydalanib, quyidagilarni topamiz:

$$u_{xx}=v_{\xi\xi}+v_{\eta\eta}+4v_{\zeta\zeta}-2v_{\xi\eta}+4v_{\xi\zeta}-4v_{\eta\zeta};$$

$$u_{yy}=v_{\eta\eta}+4v_{\zeta\zeta}-4v_{\eta\zeta}; \quad u_{zz}=v_{\zeta\zeta};$$

$$u_{xy}=-v_{\eta\eta}-4v_{\zeta\zeta}+v_{\xi\eta}-2v_{\xi\zeta}+4v_{\eta\zeta}; \; u_{yz}=-2v_{\zeta\zeta}+v_{\eta\zeta}.$$

Topilgan ifodalarni tenglamaga qo‘yib, soddalashtirishlar bajarilgandan so‘ng, berilgan tenglamaning kanonik ko‘rinishiga ega bo‘lamiz: $v_{\xi\xi}+v_{\eta\eta}+v_{\zeta\zeta}=0$.

Mustaqil bajarish uchun misollar

Quyidagi tenglamalarni kanonik ko‘rinishga keltiring:

104. $u_{xx}+2u_{xy}-2u_{xz}+2u_{yy}+6u_{zz}=0$.
105. $4u_{xx}-4u_{xy}-2u_{xy}+u_y+u_z=0$.
106. $u_{xy}-u_{xz}+u_x+u_y-u_z=0$.
107. $u_{xx}+2u_{xy}-2u_{xz}+2u_{yy}+2u_{zz}=0$.
108. $u_{xx}+2u_{xy}-2u_{xz}-6u_{yz}-u_{zz}=0$.
109. $u_{xx}+2u_{xy}+2u_{yy}+2u_{yt}+2u_{zz}+3u_{tt}=0$.
110. $u_{xy}-u_{xt}+u_{xz}-2u_{zt}+2u_{tt}=0$.
111. $u_{xy}+u_{xz}+u_{xt}+u_{zt}=0$.
112. $u_{xx}+2u_{xy}-2u_{yy}-4u_{yz}+2u_{yt}+u_{xz}=0$.
113. $u_{xx}+2u_{xz}-2u_{xt}+u_{yy}+2u_{yz}+2u_{yt}+2u_{zz}+2u_{tt}=0$.

114. $u_{x_1x_1} + 2\sum_{k=2}^{n} u_{x_kx_k} - 2\sum_{k=2}^{n} u_{x_kx_{k+1}} = 0$

115. $u_{x_1x_1} - 2\sum_{k=2}^{n} (-1)^k u_{x_{k-1}x_k} = 0$

116. $\sum_{k=2}^{n} k u_{x_kx_k} + 2\sum_{l<k} l u_{x_lx_k} = 0$

117. $\sum_{k=1}^{n} u_{x_kx_k} + \sum_{l<k} u_{x_lx_k} = 0$

118. $\sum_{l<k} u_{x_lx_k} = 0.$

3-BOB. XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING UMUMIY YECHIMINI TOPISH

Ushbu bobda ikkinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimini topish o'rganilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

3.1. O'zgarmas koeffitsiyentli xususiy hosilali differensial tenglamalarning umumiy yechimini topish

Oddiy differensial tenglamalar kursidan ma'lumki, $n-$tartibli

$$F(x, y, y', ..., y^{(n)}) = 0$$

tenglamaning yechimi n ta ixtiyoriy o'zgarmasga bog'liqdir, ya'ni $y = \varphi(x, c_1, ..., c_n)$. Bu o'zgarmaslarni aniqlash uchun noma'lum funksiya $y(x)$ qo'shimcha shartlarni qanoatlantirishi kerak.

Xususiy hosilali differensial tenglamalar uchun bu masala murakkabroqdir. Bu tenglamalarning yechimi ixtiyoriy o'zgarmaslarga emas, balki ixtiyoriy funksiyalarga bog'liq bo'lib, bu funksiyalar soni tenglamalar tartibiga teng bo'ladi va ixtiyoriy funksiyalar argumentlarining soni yechim argumentlari sonidan bitta kam bo'ladi.

Misol. Quyidagi tenglamaning $u(x, y)$ umumiy yechimini toping: $u_{xy} = 0$.

Yechish: Dastlab x bo'yicha, so'ngra y bo'yicha integrallaymiz, natijada $u(x, y) = f_1(x) + f_2(y)$ yechimni olamiz. Ko'rib turganingizdek, xususiy hosilali differensial tenglamaning yechimida tenglama tartibiga teng miqdorda, ya'ni ikkita funksiya qatnashayapti, bu funksiyalar argumenti esa yechim argumentlari sonidan bitta kam.

Misol. Quyidagi tenglamaning ham $u(x,y)$ umumiy yechimini topaylik:

$$u_{xyy}=0.$$

Yechish: Yuqoridagidek mulohaza yuritsak, umumiy yechim:

$$u(x,y)=f_1(x)y+f_2(x)+f_3(y).$$

Misol. Quyidagi tenglamaning ham $u(x,y,z)$ umumiy yechimini topaylik:

$$u_{xyz}=0.$$

Yechish: Yuqoridagidek mulohaza yuritsak, umumiy yechim:

$$u(x,y,z)=x\cdot y\cdot f_1(x,y)+x\cdot f_2(x,z)+f_3(y,z)$$

ifodaga teng bo'ladi.

Oxirgi misolda, ko'rib turganingizdek yechimda tenglama tartibiga mos uchta funksiya qatnashayapti, yechim uch o'zgaruvchili bo'lgani uchun ixtiyoriy funksiyalar argumenti ikki o'zgaruvchilidir.

Mustaqil bajarish uchun misollar

Quyida berilgan tenglamalarning umumiy yechimini toping:

1. $u_{xx}-a^2u_{yy}=0.$
2. $u_{xx}-2u_{xy}-3u_{yy}=0.$
3. $u_{xy}+au_x=0.$
4. $3u_{xx}-5u_{xy}-2u_{yy}+3u_x+u_y=2.$
5. $u_{xy}+au_x+bu_y+abu=0.$
6. $u_{xy}-2u_x-3u_y+6u=2e^{x+y}.$
7. $u_{xx}+2au_{xy}+a^2u_{yy}+u_x+au_y=0.$

3.2. Xususiy hosilali differensial tenglamalarning turi saqlanadigan sohada umumiy yechimini topish

Ta'rif. Xususiy hosilali differensial tenglamaning umumiy yechimi deb, shu tenglamani qanoatlantiradigan funksiyaga aytiladi.

Misol. Quyidagi tenglamaning turi saqlanadigan sohani topib, umumiy yechimini aniqlang: $x^2u_{xx}-y^2u_{yy}=0$.

Yechish: $a_{11}=x^2$, $a_{12}=0$, $a_{22}=-y^2$ – tenglama koeffitsiyentlari. $\Delta=a_{12}^2-a_{11}a_{22}$ ifodaning qiymatini hisoblaymiz. $\Delta=(xy)^2$, $x\neq 0$ va $y\neq 0$ bo'lganda, tenglamamiz giperbolik ekan. Yangi ξ va η o'zgaruvchilarga o'tamiz : $\xi=xy$, $\eta=\dfrac{x}{y}$ almashtirish yordamida berilgan tenglamani kanonik ko'rinishga keltiramiz. Qiyin bo'lmagan hisoblashlarni bajarib, tenglamaning kanonik ko'rinishini topamiz:

$$u_{\xi\eta}-\frac{1}{2\xi}u_{\eta}=0.$$

Endi bu tenglamaning umumiy yechimini topamiz. $u_\eta=v$ almashtirish bajarib tenglamani yechamiz, natijada

$$\begin{cases}\ln v=\frac{1}{2}\ln\xi-\ln f(\eta)\\ \quad v=\sqrt{\xi}f(\eta) \qquad \Rightarrow u=\sqrt{\xi}f(\eta)+g(\xi)\\ \quad u_\eta=\sqrt{\xi}f(\eta)\end{cases}$$

yechimni olamiz. Dastlabki o'zgaruvchilarga qaytsak, biz izlayotgan umumiy yechim

$$u(x,y)=\sqrt{|xy|}\cdot f\left(\frac{x}{y}\right)+g(xy)$$

ko'rinishda bo'ladi.

Mustaqil bajarish uchun misollar

Quyidagi tenglamalarning umumiy yechimini toping.

8. $yu_{xx}+(x-y)u_{xy}-xu_{yy}=0$.

9. $x^2u_{xx}+2xyu_{xy}-3y^2u_{yy}-2xu_x=0$.

10. $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0$.

11. $xyu_{xy}-xu_x+u=0$.

12. $u_{xy}+2xyu_y-2xu=0$.

13. $u_{xy}+u_x+yu_y+(x-1)u=0$.

14. $u_{xy}+xu_x+2yu_y+2xyu=0$.

15. $\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 u}{\partial t^2}=0$.

16. $\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$.

17. $4\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$.

18. $\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+8\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y}=0$.

19. $3\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$.

20. $9\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$.

21. $4\frac{\partial^2 u}{\partial x^2}-8\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$.

22. $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+9\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=0$.

23. $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$.

24. $5\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$.

25. $9\frac{\partial^2 u}{\partial x^2}-12\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y}=0$.

26. $3\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}+7\left(\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}\right)=0$.

27. $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0$.

28. $3\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-3\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$.

29. $e^y \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + (1+e^y)\frac{\partial u}{\partial y} = 0.$

30. $\sin y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{\sin y}{x} - ctgy\right)\frac{\partial u}{\partial y} = 0.$

31. $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0.$

32. $\frac{\partial^2 u}{\partial x^2} - \sin x \frac{\partial^2 u}{\partial x \partial y} + (\sin x - ctgx)\frac{\partial u}{\partial x} = 0.$

33. $4x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} + 7\frac{\partial u}{\partial x} = 0.$

34. $\frac{\partial^2 u}{\partial x^2} - 4x\frac{\partial^2 u}{\partial x \partial y} + 4x^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2(x-1)\frac{\partial u}{\partial y} = 0.$

35. $x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial x} = 0.$

36. $2x\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y} = 0.$

37. $\frac{\partial^2 u}{\partial x^2} - 4x\frac{\partial^2 u}{\partial x \partial y} + 4x^2\frac{\partial^2 u}{\partial y^2} - \frac{1}{x}\frac{\partial u}{\partial x} = 0.$

38. $x\frac{\partial^2 u}{\partial x \partial y} - 3y\frac{\partial^2 u}{\partial y^2} - 5\frac{\partial u}{\partial y} = 0.$

39. $x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial u}{\partial x} = 0.$

40. $\frac{\partial^2 u}{\partial x^2} - 2\cos x\frac{\partial^2 u}{\partial x \partial y} + \cos^2 x\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x} + (2\cos x + \sin x)\frac{\partial u}{\partial y} = 0.$

41. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 0.$

42. $4x\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial y} = 0.$

43. $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

44. $t^2\frac{\partial^2 u}{\partial t^2} - x^2\frac{\partial^2 u}{\partial x^2} = 0.$

45. $3x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial u}{\partial x} = 0.$

46. $3\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} = 0.$

47. $2x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial u}{\partial x}=0$.

48. $\frac{\partial^2 u}{\partial x^2}+2\sin x\frac{\partial^2 u}{\partial x\partial y}-\cos^2 x\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+(\sin x+\cos x+1)\frac{\partial u}{\partial y}=0$.

49. $\frac{\partial^2 u}{\partial x^2}+2\sin x\frac{\partial^2 u}{\partial x\partial y}-\cos^2 x\frac{\partial^2 u}{\partial y^2}+\cos x\frac{\partial u}{\partial y}=0$.

50. $\frac{\partial^2 u}{\partial x^2}-2\sin x\frac{\partial^2 u}{\partial x\partial y}-(3+\cos^2 x)\frac{\partial^2 u}{\partial y^2}-\cos x\frac{\partial u}{\partial y}=0$.

51. $\frac{\partial^2 u}{\partial x^2}-2\sin x\frac{\partial^2 u}{\partial x\partial y}-(3+\cos^2 x)\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+(2-\sin x-\cos x)\frac{\partial u}{\partial y}=0$.

52. $\frac{\partial^2 u}{\partial x^2}-3xy\frac{\partial^2 u}{\partial x\partial y}+2y^2\frac{\partial^2 u}{\partial y^2}+3y\frac{\partial u}{\partial y}=0$.

53. $3x^2\frac{\partial^2 u}{\partial x^2}-16xy\frac{\partial^2 u}{\partial x\partial y}+16y^2\frac{\partial^2 u}{\partial y^2}+15x\frac{\partial u}{\partial x}=0$.

54. $\frac{\partial^2 u}{\partial x^2}-2\frac{x}{y}\frac{\partial^2 u}{\partial x\partial y}+\frac{x^2}{y^2}\frac{\partial^2 u}{\partial y^2}-\frac{2}{x}\frac{\partial u}{\partial x}+\frac{y^2-x^2}{y^3}\frac{\partial u}{\partial y}-x^3=0$.

55. $\frac{\partial^2 u}{\partial x\partial y}-2x\frac{\partial^2 u}{\partial y^2}+\frac{1}{x^2+y}\left(\frac{\partial u}{\partial x}-2x\frac{\partial u}{\partial y}\right)+1=0$.

4-BOB. IKKINCHI TARTIBLI GIPERBOLIK TURDAGI DIFFERENSIAL TENGLAMALARGA QO‘YILGAN KOSHI MASALASI

Biror fizik jarayonni to‘la o‘rganish uchun, bu jarayonni tasvirlayotgan tenglamalardan tashqari, uning boshlang‘ich holatini (boshlang‘ich shartlarni) va jarayon sodir bo‘ladigan sohaning chegarasidagi holatini (chegaraviy shartlarni) berish zarurdir. Ushbu bobda ikkinchi tartibli xususiy hosilali differensial tenglamalarga qo‘yilgan Koshi va Gursa masalasini yechish o‘rganilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

4.1. Koshi masalalarini yechish

Shunday qilib, aniq fizik jarayonni ifodalovchi yechimni ajratib olish uchun qo‘shimcha shartlarni berish zarur. Bunday qo‘shimcha shartlar boshlang‘ich va chegaraviy shartlardan iborat.

Jarayon sodir bo‘layotgan soha $G \subset R^n$ bo‘lib, S uning chegarasi bo‘lsin. S ni bo‘laklari silliq sirt deb hisoblaymiz.

Differensial tenglamalar uchun, asosan, 3 turdagi masalalar bir-biridan farq qiladi.

a) **Koshi masalasi.** Bu masala, asosan giperbolik va parabolik turdagi tenglamalar uchun qo‘yiladi; G soha butun R^n fazo bilan ustma-ust tushadi, bu holda chegaraviy shartlar bo‘lmaydi.

b) **Chegaraviy masala** elliptik turdagi tenglamalar uchun qo‘yiladi; S da chegaraviy shartlar beriladi, bu holda jarayon statsionar bo‘lgani sababli boshlang‘ich shartlar tabiiy ravishda bo‘lmaydi.

c) **Aralash masala** giperbolik va parabolik turdagi tenglamalar uchun qo‘yiladi; $G \subset R^n$ bo‘lib, boshlang‘ich va chegaraviy shartlar beriladi.

Har qanday masalaning mohiyati berilgan $\varphi \in E_\varphi$ funksiyalarga asosan uning $u \in E_u$ yechimini topishdan iboratdir, bu yerda E_u va E_φ - metrikalari ρ_u va ρ_φ bo'lgan qandaydir metrik fazolardir. Bu fazolar masalaning qo'yilishi bilan aniqlanadi.

Masalaning yechimi tushunchasi aniqlangan bo'lib, har bir $\varphi \in E_\varphi$ funksiyalarga yagona $u = R(\varphi) \in E_u$ yechim mos kelsin.

Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta(\varepsilon) > 0$ sonni ko'rsatish mumkin bo'lib, $\rho_\varphi(\varphi_1, \varphi_2) \le \delta(\varepsilon)$ tengsizlikdan $\rho_u(u_1, u_2) \le \varepsilon$ tengsizlik kelib chiqsa, masala (E_u, E_φ) fazolar juftida turg'un masala deyiladi.

Bunda $u_i = R(\varphi_i)$, $u_i \in E_u$, $\varphi_i \in E_\varphi$, $i = 1,2,...$ masalaning yechimi berilgan shartlar (boshlang'ich va chegaraviy shartlar, tenglamaning koeffitsiyentlari, ozod hadi va h.k.) ga uzluksiz bog'liq bo'ladi.

Agar tekshirilayotgan masala uchun ushbu

1) ixtiyoriy $\varphi \in E_\varphi$ uchun $u \in E_u$ yechim mavjud;
2) u yechim yagona;
3) masala (E_u, E_φ) fazolar juftligida turg'unlik shartlar bajarilsa, masala (E_u, E_φ) fazolar juftligida korrekt (to'g'ri) qo'yilgan yoki to'g'ridan-to'g'ri korrekt masala deyiladi.

Aks holda masala korrekt qo'yilmagan masala deyiladi. Yuqoridagi talablardan kamida bittasi bajarilmay qolsa, yechim boshlang'ich va chegaraviy shartlarga uzluksiz bog'liq bo'lmasligi ham mumkin.

<u>Masala.</u> Quyidagi Koshi masalasini yeching:

$$xu_{xx} - u_{yy} + \frac{1}{2}u_x = 0;$$

$$u\big|_{y=0} = x, \qquad u_y\big|_{y=0} = 0, \qquad x > 0.$$

<u>Yechish:</u> Dastlab, tenglamani kanonik ko'rinishga keltiramiz. $\Delta = a_{12}^2 - a_{11}a_{22}$ ifodaninig qiymatini hisoblaylik. $\Delta = x$, $x > 0$ bo'lgani uchun tenglama giperbolik. Yangi ξ va η o'zgaruvchilarga o'tamiz: $\xi = 2\sqrt{x} + y$, $\eta = 2\sqrt{x} - y$ almashtirish yordamida berilgan tenglamani

kanonik ko‘rinishga keltiramiz. U quyidagi kanonik ko‘rinishga ega: $u_{\xi\eta}=0$. Berilgan tenglamaninig umumiy yechimi $u(x,y)=f(2\sqrt{x}+y)+g(2\sqrt{x}-y)$ ko‘rinishda bo‘ladi.

Bu yechimlar orasidan Koshi shartlarini qanoatlantiruvchi yechimni topamiz. Buning uchun quyidagi tenglamalar sistemasini topamiz:

$$\begin{cases} f(2\sqrt{x})+g(2\sqrt{x})=x \\ f_y'(2\sqrt{x})-g_y'(2\sqrt{x})=0 \end{cases}.$$

Natijada, $f(2\sqrt{x})=g(2\sqrt{x})=\frac{x}{2}$ yechimlarni olamiz, bu natijalarni keltirib umumiy yechimga qo‘ysak, Koshi masalasining yechimi hosil bo‘ladi: $u(x,y)=x+\frac{y^2}{4}$, $x>0$, $|y|<2\sqrt{x}$.

Masala. Xarakteristikada berilgan quyidagi masalani yeching:

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$; $y+x=0$ da $u(x,y)=\varphi(x)$ va $y-x=0$ da $u(x,y)=\psi(x)$,

$\varphi(0)=\psi(0)$.

(Eslatma. Giperbolik turdagi tenglamaga xarakteristikada qo‘yilgan masala Gursa masalasi deyiladi.)

Yechish: Dastlab, tenglamani kanonik ko‘rinishga keltiramiz. $\Delta=a_{12}^2-a_{11}a_{22}$ ifodaninig qiymatini hisoblaylik. $\Delta=1$, bo‘lgani uchun tenglama giperbolik. Yangi ξ va η o‘zgaruvchilarga o‘tamiz: $\xi=x+y$, $\eta=x-y$ almashtirish yordamida berilgan tenglamani kanonik ko‘rinishga keltiramiz. U quyidagi kanonik ko‘rinishga ega: $u_{\xi\eta}=0$.

Berilgan tenglamaninig umumiy yechimi $u(x,y)=f(x+y)+g(x-y)$ ko‘rinishda bo‘ladi.

Bu yechimlar orasidan xarakteristikada berilgan shartlarini qanoatlantiruvchi yechimni topamiz. Buning uchun quyidagi tenglamalar sistemasini topamiz:

$$\begin{cases} f(0)+g(2x)=\varphi(x) \\ f(2x)+g(0)=\psi(x) \end{cases}$$

Natijada, $f(2x)=\psi(x)-g(0)$ va $g(2x)=\varphi(x)-f(0)$ yechimlarni olamiz. Muvofiqlik shartidan esa $f(0)+g(0)=\varphi(0)(=\psi(0))$ tenglikni olamiz. Bundan $f(x+y)=\psi\left(\frac{x+y}{2}\right)-g(0)$ va $g(x-y)=\varphi\left(\frac{x-y}{2}\right)-f(0)$ funksiyalarni aniqlab, natijalarni keltirib umumiy yechimga qo'ysak, masalaning yechimi hosil bo'ladi : $u(x,y)=\varphi\left(\frac{x-y}{2}\right)+\psi\left(\frac{x+y}{2}\right)-\varphi(0)$.

Mustaqil bajarish uchun mashqlar

Quyidagi Koshi masalalarini yeching:

1. $u_{xy}=0;$

   $u\big|_{y=x^2}=0,\quad u_y\big|_{y=x^2}=\sqrt{|x|},\quad |x|<1.$

2. $u_{xy}+u_x=0;$

   $u\big|_{y=x}=\sin x,\quad u_x\big|_{y=x}=1\quad |x|<\infty.$

3. $u_{xx}-u_{yy}+2u_x+2u_y=0;$

   $u\big|_{y=0}=x,\quad u_y\big|_{y=0}=0,\quad |x|<\infty.$

4. $u_{xx}-u_{yy}-2u_x-2u_y=4;$

   $u\big|_{x=0}=-y,\quad u_x\big|_{x=0}=y-1,\quad |y|<\infty.$

5. $u_{xx}+2u_{xy}-u_{yy}=2;$

   $u\big|_{y=0}=0,\quad u_y\big|_{y=0}=x+\cos x,\quad |x|<\infty.$

6. $u_{xy}+yu_x+xu_y+xyu=0;$

   $u\big|_{y=3x}=0,\quad u_y\big|_{y=3x}=e^{-5x^2},\quad x<1.$

7. $xu_{xx}+(x+y)u_{xy}+yu_{yy}=0;$

   $u\big|_{y=\frac{1}{x}}=x^3,\quad u_x\big|_{y=\frac{1}{x}}=2x^2,\quad x>0.$

8. $u_{xx}+2(1+2x)u_{xy}+4x(1+x)u_{yy}+2u_y=0;$

   $u\big|_{x=0}=y,\quad u_x\big|_{x=0}=2,\quad |y|<\infty$

9. $x^2u_{xx}-y^2u_{yy}-2yu_y=0;$

   $u\big|_{x=1}=y,\quad u_x\big|_{x=1}=y,\quad y<0.$

10. $x^2u_{xx}-2xyu_{xy}-3y^2u_{yy}=0;$

$u|_{y=1}=0, \quad u_y|_{y=1}=\sqrt[4]{x^7}, \quad x>0.$

11. $yu_{xx}+x(2y-1)u_{xy}-2x^2u_{yy}-\frac{y}{x}u_x=0;$

$u|_{y=0}=x^2, \quad u_y|_{y=0}=1, \qquad x>0.$

12. $yu_{xx}-(x+y)u_{xy}+xu_{yy}=0;$

$u|_{y=0}=x^2, \quad u_y|_{y=0}=x, \quad x>0$

13. $u_{xy}+2u_x+u_y+2u=1, \; x>0, \; y<1;$

$u|_{x+y=1}=x, \quad u_x|_{x+y=1}=x.$

14. $xyu_{xy}+xu_x-yu_y+u=2y, \quad x>0, \; y<\infty \;;$

$u|_{xy=1}=1-y, \quad u_y|_{xy=1}=x-1.$

15. $u_{xy}+\dfrac{1}{x+y}(u_x+u_y)=2, \quad 0<x, \quad y<\infty$

$u|_{y=x}=x^2, \quad u_x|_{y=x}=1+x.$

16. $u_{xx}-u_{yy}+\dfrac{2}{x}u_x-\dfrac{2}{y}u_y=0, \quad |x-y|<1, \qquad |x+y-2|<1$

$u|_{y=1}=u_0(x) \qquad u_y|_{y=1}=u_1(x), \quad u_0\in C^2(0,2), \quad u_1\in C^1(0,2).$

17. $2u_{xy}-e^{-x}u_{yy}=4x, \quad -\infty<x, \quad y<\infty$

$u|_{y=x}=x^5\cos x, \quad u_y|_{y=x}=x^2+1.$

18. $\dfrac{\partial^2 u}{\partial x\partial t}+\dfrac{\partial^2 u}{\partial t^2}=0, \; u|_{t=0}=0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=-x-1.$

19. $3\dfrac{\partial^2 u}{\partial x^2}+5\dfrac{\partial^2 u}{\partial x\partial y}-2\dfrac{\partial^2 u}{\partial y^2}+7\left(\dfrac{\partial u}{\partial x}+2\dfrac{\partial u}{\partial y}\right)=0, \; u|_{x=0}=1, \left.\dfrac{\partial u}{\partial x}\right|_{x=0}=3y.$

20. $5\dfrac{\partial^2 u}{\partial x^2}+6\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=0, \; u|_{x=0}=2y, \left.\dfrac{\partial u}{\partial x}\right|_{x=0}=5y.$

21. $3\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=0, \; u|_{x=0}=2y, \left.\dfrac{\partial u}{\partial x}\right|_{x=0}=4y.$

22. $\dfrac{\partial^2 u}{\partial x^2}-6\dfrac{\partial^2 u}{\partial x\partial y}+8\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial u}{\partial x}-2\dfrac{\partial u}{\partial y}=0, \; u|_{y=0}=2x, \left.\dfrac{\partial u}{\partial y}\right|_{y=0}=3x+1.$

23. $4\frac{\partial^2 u}{\partial x^2}-8\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$, $u|_{y=0}=3x$, $\frac{\partial u}{\partial y}\Big|_{y=0}=2x+6$.

24. $3\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}=0$, $u|_{y=0}=f(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=F(x)$.

25. $3\frac{\partial^2 u}{\partial x^2}-5\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}=0$, $u|_{y=0}=f(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=F(x)$.

26. $2\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial x\partial y}-5\frac{\partial^2 u}{\partial y^2}=0$, $u|_{y=0}=f(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=F(x)$.

27. $3\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}-7\frac{\partial^2 u}{\partial y^2}=0$, $u|_{y=0}=f(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=F(x)$.

28. $\frac{\partial^2 u}{\partial x^2}-5\frac{\partial^2 u}{\partial x\partial y}+6\frac{\partial^2 u}{\partial y^2}=0$, $u|_{y=0}=f(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=F(x)$.

29. $3\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-3\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$, $u|_{y=0}=\varphi(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=\psi(x)$.

30. $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}-5\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$, $u|_{y=0}=f(x)$, $\frac{\partial u}{\partial y}\Big|_{y=0}=F(x)$.

31. $x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial u}{\partial x}=0$, $u|_{x=1}=y$, $\frac{\partial u}{\partial x}\Big|_{x=1}=2y^3$.

32. $2x\frac{\partial^2 u}{\partial x\partial y}-y\frac{\partial^2 u}{\partial y^2}-3\frac{\partial u}{\partial y}=0$, $u|_{y=1}=x^4$, $\frac{\partial u}{\partial y}\Big|_{y=1}=3x^3$

33. $x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial u}{\partial x}=0$, $u|_{x=1}=3y^4$, $\frac{\partial u}{\partial x}\Big|_{y=0}=2y^5$.

34. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}=0$, $u|_{x=1}=2y+1$, $\frac{\partial u}{\partial x}\Big|_{x=1}=y$.

35. $4x\frac{\partial^2 u}{\partial x\partial y}-y\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial y}=0$, $u|_{y=1}=4x^3$, $\frac{\partial u}{\partial y}\Big|_{y=1}=8x$.

36. $3\frac{\partial^2 u}{\partial x\partial y}-y\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial y}=0$, $u|_{y=1}=x$, $\frac{\partial u}{\partial y}\Big|_{y=1}=15x^2$.

37. $4x\frac{\partial^2 u}{\partial x\partial y}-y\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial y}=0$, $u|_{y=1}=x^2+1$, $\frac{\partial u}{\partial y}\Big|_{y=1}=4$.

38. $3\frac{\partial^2 u}{\partial x\partial y}-y\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial y}=0$, $u|_{y=1}=0$, $\frac{\partial u}{\partial y}\Big|_{y=1}=6x^2\sqrt[3]{x}$.

39. $4x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+7\frac{\partial u}{\partial x}=0,\ u\big|_{x=1}=3y^5,\ \frac{\partial u}{\partial x}\Big|_{x=1}=2y^{11}.$

40. $x\frac{\partial^2 u}{\partial x\partial y}-3y\frac{\partial^2 u}{\partial y^2}-5\frac{\partial u}{\partial y}=0,\ u\big|_{y=1}=4x^4,\ \frac{\partial u}{\partial y}\Big|_{y=1}=2x^8.$

41. $3x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial u}{\partial x}=0,\ u\big|_{x=1}=4y^3,\ \frac{\partial u}{\partial x}\Big|_{x=1}=y^7.$

42. $2x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial u}{\partial x}=0,\ u\big|_{x=1}=y^2,\ \frac{\partial u}{\partial x}\Big|_{x=1}=2y^7.$

43. $3x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial u}{\partial x}=0,\ u\big|_{x=1}=1+y^4,\ \frac{\partial u}{\partial x}\Big|_{x=1}=y^4.$

44. $2x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial u}{\partial x}=0\ u\big|_{x=1}=0,\ \frac{\partial u}{\partial x}\Big|_{x=1}=y^5.$

45. $2x\frac{\partial^2 u}{\partial x^2}-3y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=0;\ u\big|_{y=1}=2+3x^2,\ \frac{\partial u}{\partial y}\Big|_{y=1}=x^4$

46. $3x\frac{\partial^2 u}{\partial x^2}-2y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}=0;\ u\big|_{x=1}=y^5+3,\ \frac{\partial u}{\partial x}\Big|_{x=1}=2y^2-y.$

47. $3x\frac{\partial^2 u}{\partial x\partial y}-4y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=0;\ u\big|_{y=1}=3x^2+2x,\ \frac{\partial u}{\partial y}\Big|_{y=1}=1-2x.$

48. $2x\frac{\partial^2 u}{\partial x\partial y}-5y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=0;\ u\big|_{y=1}=3x^2+1,\ \frac{\partial u}{\partial y}\Big|_{y=1}=5x+2.$

49. $4x\frac{\partial^2 u}{\partial x^2}-3y\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}=0;\ u\big|_{x=1}=1,\ \frac{\partial u}{\partial x}\Big|_{x=1}=3y^3.$

50. $3x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}=0;\ u\big|_{x=1}=3y^2,\ \frac{\partial u}{\partial x}\Big|_{x=1}=1-y.$

51. $3x\frac{\partial^2 u}{\partial x\partial y}-2y\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial y}=0;\ u\big|_{y=1}=3x^2,\ \frac{\partial u}{\partial y}\Big|_{y=1}=4-x^2.$

52. $4x\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\partial y}+7\frac{\partial u}{\partial x}=0;\ u\big|_{x=1}=3y,\ \frac{\partial u}{\partial x}\Big|_{x=1}=2y^2.$

53. $t^2\frac{\partial^2 u}{\partial t^2}-x^2\frac{\partial^2 u}{\partial x^2}=0;\ u\big|_{t=1}=2x^2,\ \frac{\partial u}{\partial t}\Big|_{t=1}=x^2.$

54. $\frac{\partial^2 u}{\partial x^2}-3xy\frac{\partial^2 u}{\partial x\partial y}+2y^2\frac{\partial^2 u}{\partial y^2}+3y\frac{\partial u}{\partial y}=0;\ u\big|_{y=1}=1+2x^2,\ \frac{\partial u}{\partial y}\Big|_{y=1}=4x^2.$

55. $3x^2\frac{\partial^2 u}{\partial x^2}-16xy\frac{\partial^2 u}{\partial x\partial y}+16y^2\frac{\partial^2 u}{\partial y^2}+15x\frac{\partial u}{\partial x}=0$; $u\big|_{y=1}=5x^4-3x^2$,

$\frac{\partial u}{\partial y}\Big|_{y=1}=10x^4-9x^2$.

56. $t^2\frac{\partial^2 u}{\partial t^2}-x^2\frac{\partial^2 u}{\partial x^2}=0$; $u\big|_{t=1}=2\sqrt{x}$, $\frac{\partial u}{\partial t}\Big|_{t=1}=\sqrt{x}$.

57. $4x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}+8x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$; $u\big|_{x=1}=2y$, $\frac{\partial u}{\partial x}\Big|_{x=1}=0$.

58. $x^2\frac{\partial^2 u}{\partial x^2}-9y^2\frac{\partial^2 u}{\partial y^2}+6x\frac{\partial u}{\partial x}+6y\frac{\partial u}{\partial y}=0$; $u\big|_{y=1}=3x$, $\frac{\partial u}{\partial y}\Big|_{y=1}=x^2$.

59. $\frac{\partial^2 u}{\partial x^2}-3xy\frac{\partial^2 u}{\partial x\partial y}+2y^2\frac{\partial^2 u}{\partial y^2}+3y\frac{\partial u}{\partial y}=0$; $u\big|_{y=1}=2x$, $\frac{\partial u}{\partial y}\Big|_{y=1}=x$.

60. $3x^2\frac{\partial^2 u}{\partial x^2}-16xy\frac{\partial^2 u}{\partial x\partial y}+16y^2\frac{\partial^2 u}{\partial y^2}+15x\frac{\partial u}{\partial x}=0$; $u\big|_{x=1}=2y^2$, $\frac{\partial u}{\partial x}\Big|_{x=1}=\frac{20}{3}y^2$.

61. $\frac{\partial^2 u}{\partial x^2}+2\sin x\frac{\partial^2 u}{\partial x\partial y}-\cos^2 x\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+(\sin x+\cos x+1)\frac{\partial u}{\partial y}=0$;

$u\big|_{y=-\cos x}=1+2\sin x$, $\frac{\partial u}{\partial y}\Big|_{y=-\cos x}=\sin x$.

62. $\frac{\partial^2 u}{\partial x^2}+2\sin x\frac{\partial^2 u}{\partial x\partial y}-\cos^2 x\frac{\partial^2 u}{\partial y^2}+\cos x\frac{\partial u}{\partial y}=0$; $u\big|_{y=-\cos x}=1+\cos x$,

$\frac{\partial u}{\partial y}\Big|_{y=-\cos x}=0$.

63. $\frac{\partial^2 u}{\partial x^2}-2\sin x\frac{\partial^2 u}{\partial x\partial y}-(3+\cos^2 x)\frac{\partial^2 u}{\partial y^2}-\cos x\frac{\partial u}{\partial y}=0$; $u\big|_{y=\cos x}=\sin x$,

$\frac{\partial u}{\partial y}\Big|_{y=\cos x}=\frac{1}{2}e^x$.

64. $\frac{\partial^2 u}{\partial x^2}-2\sin x\frac{\partial^2 u}{\partial x\partial y}-(3+\cos^2 x)\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+(2-\sin x-\cos x)\frac{\partial u}{\partial y}=0$;

$u\big|_{y=\cos x}=0$, $\frac{\partial u}{\partial y}\Big|_{y=\cos x}=e^{\frac{x}{2}}\cos x$.

Xususiy hosilali differensial tenglamalar almashtirish yordamida kanonik ko'rinishga keltirilgan, dastlabki tenglamaning berilgan boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini toping:

65. $\frac{\partial^2 u}{\partial\xi\partial\eta}-2\frac{\partial u}{\partial\xi}=0$; $\xi=2x+3y,\ \eta=4x-5y,\ u|_{x=0}=1,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=2.$

66. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$; $\xi=3x+8y,\ \eta=4x-5y,\ u|_{x=0}=5,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=7.$

67. $\frac{\partial^2 u}{\partial\xi\partial\eta}-4\frac{\partial u}{\partial\xi}=0$; $\xi=3x+7y,\ \eta=4x-5y,\ u|_{x=0}=1,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=2.$

68. $\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial u}{\partial\xi}=0$; $\xi=3x-4y,\ \eta=5x+6y,\ u|_{x=0}=2,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=3.$

69. $\frac{\partial^2 u}{\partial\xi\partial\eta}-3\frac{\partial u}{\partial\xi}=0$; $\xi=2x+3y,\ \eta=5x-4y,\ u|_{x=0}=1,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=1.$

70. $\frac{\partial^2 u}{\partial\xi\partial\eta}-2\frac{\partial u}{\partial\eta}=0$; $\xi=5x-6y,\ \eta=x+2y,\ u|_{x=0}=4,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=1.$

71. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{4}\frac{\partial u}{\partial\eta}=0$; $\xi=2x-3y,\ \eta=3x+4y,\ u|_{x=0}=2,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=1.$

72. $\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial u}{\partial\eta}=0$; $\xi=4x-3y,\ \eta=5x+2y,\ u|_{x=0}=3,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=5.$

73. $\frac{\partial^2 u}{\partial\xi\partial\eta}-3\frac{\partial u}{\partial\eta}=0$; $\xi=3x-4y,\ \eta=3x+5y,\ u|_{x=0}=y,\ \left.\frac{\partial u}{\partial x}\right|_{x=0}=1.$

74. $\frac{\partial^2 u}{\partial\xi\partial\eta}+2\frac{\partial u}{\partial\eta}=0$; $\xi=2x+3y,\ \eta=3x+5y,\ u|_{y=0}=2x,\ \left.\frac{\partial u}{\partial y}\right|_{y=0}=3.$

75. $\frac{\partial^2 u}{\partial\xi\partial\eta}-4\frac{\partial u}{\partial\xi}=0$; $\xi=3x+y,\ \eta=2y-5x,\ u|_{y=0}=3x+5,\ \left.\frac{\partial u}{\partial y}\right|_{y=0}=4.$

76. $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$; $\xi=x^2y^3,\ \eta=y,\ u|_{x=1}=3y^3+5,\ \left.\frac{\partial u}{\partial x}\right|_{x=1}=3y+1.$

77. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{3\eta}\frac{\partial u}{\partial\xi}=0$; $\xi=x^2y^3,\ \eta=x,\ u|_{y=1}=2x,\ \left.\frac{\partial u}{\partial x}\right|_{y=1}=3x^2+1.$

78. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{2}{\eta}\frac{\partial u}{\partial\xi}=0$; $\xi=y^2x^3,\ \eta=x,\ u|_{y=1}=2x^2,\ \left.\frac{\partial u}{\partial y}\right|_{y=1}=3x+1.$

79. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{4}{\eta}\frac{\partial u}{\partial\xi}=0$; $\xi=xy^3,\ \eta=y,\ u|_{x=1}=3y,\ \left.\frac{\partial u}{\partial x}\right|_{x=1}=2+3y.$

80. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$; $\xi=x^3y^2,\ \eta=x,\ u|_{y=1}=2x^3,\ \left.\frac{\partial u}{\partial y}\right|_{y=1}=3x\cdot$

81. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{4}{\eta}\frac{\partial u}{\partial\xi}=0$; $\xi=xy^3,\ \eta=y,\ u|_{x=1}=1+2y,\ \left.\frac{\partial u}{\partial x}\right|_{x=1}=3y^2.$

82. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{5}{3\eta}\dfrac{\partial u}{\partial\xi}=0;\quad \xi=x^3y^4,\ \eta=y,\ u\big|_{x=1}=3y^5,\ \dfrac{\partial u}{\partial x}\Big|_{x=1}=3y^4.$

83. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{3\eta}\dfrac{\partial u}{\partial\xi}=0;\quad \xi=x^2y^4,\ \eta=y,\ u\big|_{x=1}=y,\ \dfrac{\partial u}{\partial x}\Big|_{x=1}=3y+2.$

84. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{3}{\eta}\dfrac{\partial u}{\partial\xi}=0;\quad \xi=x^3y^2,\ \eta=y,\ u\big|_{x=1}=3y^2,\ \dfrac{\partial u}{\partial x}\Big|_{x=1}=3y+2.$

85. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{9}{2\eta}\dfrac{\partial u}{\partial\xi}=0;\quad \xi=x^3y^2,\ \eta=x,\ u\big|_{y=1}=x^3,\ \dfrac{\partial u}{\partial x}\Big|_{y=1}=x^2-2.$

86. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}-\dfrac{2}{\eta}\dfrac{\partial u}{\partial\xi}=0;\quad \xi=x^2y^3,\ \eta=x,\ u\big|_{y=1}=x^2+1,\ \dfrac{\partial u}{\partial x}\Big|_{y=1}=x.$

87. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{3\eta}\dfrac{\partial u}{\partial\xi}=0;\quad \xi=x^3y^4,\ \eta=x,\ u\big|_{y=1}=4x^2,\ \dfrac{\partial u}{\partial x}\Big|_{y=1}=6x.$

Xarakteristikada berilgan masalalarni yeching:

88. $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial y^2};$ $y+x=0$ da $u(x,y)=\varphi(x)$, $y-x=0$ da $u(x,y)=\psi(x)$,

$\varphi(0)=\psi(0)$.

89. $\dfrac{\partial^2 u}{\partial x^2}+6\dfrac{\partial^2 u}{\partial x\partial y}+5\dfrac{\partial^2 u}{\partial y^2}=0;$ $y-x=0$ da $u(x,y)=\varphi(x)$,

$5x-y=0$ da $u(x,y)=\psi(x)$, $\varphi(0)=\psi(0)$.

90. $\dfrac{\partial^2 u}{\partial x^2}+6\dfrac{\partial^2 u}{\partial x\partial y}+5\dfrac{\partial^2 u}{\partial y^2}=0;$ $y=5x+3$ da $u(x,y)=\varphi(x)$,

$y=x-1$ da $u(x,y)=\psi(x)$, $\varphi(-1)=\psi(-1)$.

91. $\dfrac{\partial^2 u}{\partial x^2}-6\dfrac{\partial^2 u}{\partial x\partial y}+8\dfrac{\partial^2 u}{\partial y^2}=0;$ $y+4x=0$ da $u(x,y)=\varphi(x)$,

$y+2x+2=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

92. $3\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\partial^2 u}{\partial y^2}=0;$ $x-y-1=0$ da $u(x,y)=\varphi(x)$,

$x+3y+1=0$ da $u(x,y)=\psi(x)$, $\varphi(\tfrac{1}{2})=\psi(\tfrac{1}{2})$.

93. $4\dfrac{\partial^2 u}{\partial x^2}-8\dfrac{\partial^2 u}{\partial x\partial y}+3\dfrac{\partial^2 u}{\partial y^2}=0;$ $x+2y+1=0$ da $u(x,y)=\varphi(x)$,

$3x+2y+2=0$ da $u(x,y)=\psi(x)$, $\varphi(-\tfrac{1}{2})=\psi(-\tfrac{1}{2})$.

94. $3\dfrac{\partial^2 u}{\partial x^2}+5\dfrac{\partial^2 u}{\partial x\partial y}-2\dfrac{\partial^2 u}{\partial y^2}=0;$ $x+3y+2=0$ da $u(x,y)=\varphi(x)$,

$2x-y-1=0$ da $u(x,y)=\psi(x)$, $\varphi(\tfrac{1}{7})=\psi(\tfrac{1}{7})$.

95. $25\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}=0$; $2x-5y-4=0$ da $u(x,y)=\varphi(x)$,

$x+5y+3=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{3})=\psi(\frac{1}{3})$.

96. $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-8\frac{\partial^2 u}{\partial y^2}=0$; $4x-y+3=0$ da $u(x,y)=\varphi(x)$,

$2x+y-4=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{6})=\psi(\frac{1}{6})$.

97. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}-6\frac{\partial^2 u}{\partial y^2}=0$; $2x+y+1=0$ da $u(x,y)=\varphi(x)$,

$3x-y-2=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{5})=\psi(\frac{1}{5})$.

98. $2\frac{\partial^2 u}{\partial x^2}-7\frac{\partial^2 u}{\partial x\partial y}-4\frac{\partial^2 u}{\partial y^2}=0$; $4x+y+1=0$ da $u(x,y)=\varphi(x)$,

$x-2y+4=0$ da $u(x,y)=\psi(x)$, $\varphi(-\frac{2}{3})=\psi(-\frac{2}{3})$.

99. $\frac{\partial^2 u}{\partial x^2}+2x\frac{\partial^2 u}{\partial x\partial y}-\frac{1}{x}\frac{\partial^2 u}{\partial y^2}=0,\quad (x>0)$; $y-1=0$ da $u(x,y)=\varphi(x)$,

$x^2-y=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

100. $\frac{\partial^2 u}{\partial x^2}+2x\frac{\partial^2 u}{\partial x\partial y}=0,\quad (y>0)$; $y-x^2=0$ da $u(x,y)=\varphi(x)$,

$x-2=0$ da $u(x,y)=\psi(x)$, $\varphi(2)=\psi(4)$.

101. $2y\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}=0,\quad (x>0)$; $y-\sqrt{x}=0$ da $u(x,y)=\varphi(x)$,

$y-2=0$ da $u(x,y)=\psi(x)$, $\varphi(4)=\psi(4)$.

102. $\frac{\partial^2 u}{\partial x^2}-4x^2\frac{\partial^2 u}{\partial y^2}-\frac{1}{x}\frac{\partial u}{\partial x}=0,\quad (x>0)$, $y-x^2=0$ da $u(x,y)=\varphi(x)$,

$y+x^2+2=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

103. $\frac{\partial^2 u}{\partial x^2}+2shx\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{1}{chx}\frac{\partial u}{\partial y}-thx\frac{\partial u}{\partial x}=0$; $y-e^x=0$ da $u(x,y)=\varphi(x)$,

$y-e^{-x}=0$ da $u(x,y)=\psi(x)$, $\varphi(0)=\psi(0)$.

4.2. To'lqin tenglamasi uchun Koshining klassik masalasi

$C^2(t>0)\cap C^1(t\geq 0)$ sinfdan shunday $u(x,t)$ funksiya topilsinki, bu funksiya $t>0$ da

$$u_{tt}=a^2\Delta u+f(x,t)$$

tenglamani va quyidagi boshlang'ich shartlarni qanoatlantirsin:

$$u|_{t\to+0}=u_0(x),\ u_t|_{t\to+0}=u_1(x),$$

Bu yerda f,u_0,u_1 – berilgan funksiyalar.

Bu masalaga **Koshining klassik masalasi** deyiladi.

Agar quyidagi shartlar bajarilsa:

$f\in C^1(t\geq 0), u_0\in C^2(R^1),\ u_1\in C^1(R^1)$, n=1;

$f\in C^2(t\geq 0), u_0\in C^3(R^n),\ u_1\in C^2(R^n)$, n=2,3,

Koshining klassik masalasining yechimi mavjud, yagona va quyidagi formulalar orqali topiladi:

$n=1$ bo'lganda, Dalamber formulasi bilan

$$u(x,t)=\frac{1}{2}[u_0(x+at)+u_0(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}u_1(\xi)d\xi+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau. \quad (1)$$

$n=2$ bo'lganda, Puasson formulasi bilan, agar n=2 bo'lsa:

$$u(x,t)=\frac{1}{2\pi a}\int_0^t\int_{|\xi-x|<a(t-\tau)}\frac{f(\xi,\tau)d\xi d\tau}{\sqrt{a^2(t-\tau)^2-|\xi-x|^2}}+\frac{1}{2\pi a}\int_{|\xi-x|<at}\frac{u_1(\xi)d\xi}{\sqrt{a^2t^2-|\xi-x|^2}}+$$

$$+\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_{|\xi-x|<at}\frac{u_0(\xi)d\xi}{\sqrt{a^2t^2-|\xi-x|^2}}.$$

(2)

$n=3$ bo'lganda, Kirxgoff formulasi bilan, agar n=3 bo'lsa:

$$u(x,t)=\frac{1}{4\pi a^2}\int_{|\xi-x|<at}\frac{1}{|\xi-x|}f\left(\xi,t-\frac{|\xi-x|}{a}\right)d\xi+\frac{1}{4\pi a^2t}\int_{|\xi-x|=at}u_1(\xi)dS+\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{|\xi-x|=at}u_0(\xi)dS\right].$$

(3)

Ba'zida berilgan f,u_0,u_1 funksiyalarga qarab, $n\geq 2$ uchun quyidagi formuladan ham foydalanish mumkin:

$$u(x,t)=\sum_{k=0}^{\infty}\left[\frac{t^{2k}}{(2k)!}a^{2k}\Delta^k u_0(x_1,...,x_n)+\frac{t^{2k+1}}{(2k+1)!}a^{2k}\Delta^k u_1(x_1,...,x_n)+\frac{a^{2k}}{(2k+1)!}\int_0^t(t-\tau)^{2k+1}\Delta^k f(x_1,...,x_n,\tau)d\tau\right]$$

(4)

bu yerda, Δ – Laplas operatori bo'lib, $k=0,1,2,...$ marta mos ravishda u_0,u_1,f - funksiyalarga qo'llanilgan. (4) formuladan foydalanish, berilgan funksiyalar, ayniqsa, ko'phad bo'lganda qulaydir.

Masala: $\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} + ax + bt \\ u(x,y,z,0) = xyz \\ u_t(x,y,z,0) = xy + z \end{cases}$

masalani (4) formula bilan yeching.

Yechish: $u_0 = xyz$ funksiyaga keraklicha marta Δ operatorni qo'llaymiz: $\Delta^0 u_0 = u_0 = xyz$; $\Delta^1 u_0 = \Delta u_0(x,y,z) = u_{0xx} + u_{0yy} + u_{0zz} = 0 + 0 + 0 = 0$. Laplas operatorini keyingi qo'llashlarda ham nol hosil bo'ladi, demak, hisoblashni shu yerda to'xtatamiz.

Xuddi shu hisoblashlarni u_1, f funksiyalar uchun ham bajaramiz: $\Delta^0 u_1 = u_1 = xy + z$;

$\Delta^1 u_1 = \Delta^2 u_1 = ... = 0$; $\Delta^0 f = f = ax + bt$; $\Delta^1 f = \Delta^2 f = ... = 0$.

Hisoblashlarni (4) formulaga qo'yamiz, natijada:

$$u(x,y,z,t) = xyz + t(xy+z) + \int_0^t (t-\tau)(ax+b\tau)d\tau = xyz + t(xy+z) + \frac{axt^2}{2} + \frac{bt^3}{6}$$ yechimni olamiz.

Masala: $u_{tt} = u_{xx} + e^x$; $u|_{t=0} = \sin x$, $u_t|_{t=0} = x + \cos x$.

Koshi masalasini (1) formula bilan yeching.

Yechish: $u_0 = \sin x$, $u_1 = x + \cos x$, $f(x,t) = e^x$ berilgan funksiyalar. Masalani yechish uchun Dalamber formulasidan foydalanamiz:

$$u(x,t) = \frac{1}{2}[\sin(x+t) + \sin(x-t)] + \frac{1}{2}\int_{x-t}^{x+t}(\xi + \cos\xi)d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} e^\xi d\xi d\tau =$$

$$= \frac{1}{2}[\sin(x+t) + \sin(x-t)] + \frac{1}{2}\left(\frac{\xi^2}{2} + \sin\xi\right)\Bigg|_{x-t}^{x+t} + \frac{1}{2}\int_0^t e^\xi \Big|_{x-(t-\tau)}^{x+(t-\tau)} = \frac{1}{2}[\sin(x+t) + \sin(x-t)] +$$

$$+ \frac{1}{2}\left(\frac{(x+t)^2}{2} - \frac{(x+t)^2}{2}\right) + \frac{1}{2}[\sin(x+t) - \sin(x-t)] + \int_0^t e^x sh(t-\tau)d\tau = \sin(x+t) +$$

$$+ xt - e^x ch(t-\tau)\Big|_0^t = \sin(x+t) + xt + e^x(ch t - 1)$$

$n = 2$ va $n = 3$ bo'lgan masalalarni mos ravishda Puasson va Kirxgoff formulalari bilan yechganda, ba'zan Dekart koordinatalar sistemasidan qutb va sferik koordinatalar sistemasiga o'tib yechish ma'qul. Quyida mos ravishda Puasson va Kirxgoff formulalarining qutb va sferik koordinatalar sistemasidagi ifodalanishini keltiramiz:

Puasson formulasi:

$$u(x,t)=\frac{1}{2\pi a}\int_0^t\int_{|\xi-x|<a(t-\tau)}\frac{f(\xi,\tau)d\xi d\tau}{\sqrt{a^2(t-\tau)^2-|\xi-x|^2}}+\frac{1}{2\pi a}\int_{|\xi-x|<at}\frac{u_1(\xi)d\xi}{\sqrt{a^2t^2-|\xi-x|^2}}+$$

$$+\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_{|\xi-x|<at}\frac{u_0(\xi)d\xi}{\sqrt{a^2t^2-|\xi-x|^2}}=\frac{1}{2\pi a}\int_0^t\int_0^{a(t-\tau)}\int_0^{2\pi}\frac{f(x+\rho\cos\varphi,y+\rho\sin\varphi,\tau)}{\sqrt{a^2(t-\tau)^2-\rho^2}}\rho d\varphi d\rho d\tau+$$

$$+\frac{1}{2\pi a}\int_0^{at}\int_0^{2\pi}\frac{u_1(x+\rho\cos\varphi,y+\rho\sin\varphi)}{\sqrt{a^2t^2-\rho^2}}\rho d\varphi d\rho+\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{u_0(x+\rho\cos\varphi,y+\rho\sin\varphi)}{\sqrt{a^2t^2-\rho^2}}\rho d\varphi d\rho.$$

Kirxgoff formulasi:

$$u(x,t)=\frac{1}{4\pi a^2}\int_{|\xi-x|<at}\frac{1}{|\xi-x|}f\left(\xi,t-\frac{|\xi-x|}{a}\right)d\xi+\frac{1}{4\pi a^2t}\int_{|\xi-x|=at}u_1(\xi)dS+\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{|\xi-x|=at}u_0(\xi)dS\right]=$$

$$=\frac{1}{4\pi a^2}\int_0^{at}\int_0^{2\pi}\int_0^{\pi}f\left(x+\rho\cos\varphi\sin\theta,y+\rho\sin\varphi\sin\theta,z+\rho\cos\theta,t-\frac{\rho}{a}\right)\rho\sin\theta d\theta d\varphi d\rho+$$

$$+\frac{1}{4\pi a^2}\int_0^{at}\int_0^{2\pi}\int_0^{\pi}u_1(x+\rho\cos\varphi\sin\theta,y+\rho\sin\varphi\sin\theta,z+\rho\cos\theta)\rho^2\sin\theta d\theta d\varphi d\rho+$$

$$+\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left[\frac{1}{t}\int_0^{at}\int_0^{2\pi}\int_0^{\pi}u_0(x+\rho\cos\varphi\sin\theta,y+\rho\sin\varphi\sin\theta,z+\rho\cos\theta)\rho^2\sin\theta d\theta d\varphi d\rho\right]$$

Mustaqil bajarish uchun mashqlar

Quyidagi Koshi masalalarini yeching:

a) (n=1)

104. $u_{tt}=u_{xx}+6;\ \ u|_{t=0}=x^2,\ u_t|_{t=0}=4x.$

105. $u_{tt}=4u_{xx}+xt;\ \ u|_{t=0}=x^2,\ u_t|_{t=0}=x.$

106. $u_{tt}=u_{xx}+\sin x;\ \ u|_{t=0}=\sin x,\ u_t|_{t=0}=0.$

107. $u_{tt}=u_{xx}+e^x;\ \ u|_{t=0}=\sin x,\ u_t|_{t=0}=x+\cos x.$

108. $u_{tt}=9u_{xx}+\sin x;\ \ u|_{t=0}=1,\ u_t|_{t=0}=1.$

109. $u_{tt}=a^2u_{xx}+\sin ax;\ \ u|_{t=0}=0,\ u_t|_{t=0}=0.$

110. $u_{tt}=a^2u_{xx}+\sin at;\ \ u|_{t=0}=0,\ u_t|_{t=0}=0.$

b) (n=2)

111. $u_{tt}=\Delta u+2;\ \ u|_{t=0}=x,\ u_t|_{t=0}=y.$

112. $u_{tt}=\Delta u+6xyt;\ \ u|_{t=0}=x^2-y^2,\ u_t|_{t=0}=xy.$

113. $u_{tt}=\Delta u+x^3-3xy^2;\ \ u|_{t=0}=e^x\cos y,\ u_t|_{t=0}=e^y\sin x.$

114. $u_{tt}=\Delta u+t\sin y;\ \ u|_{t=0}=x^2,\ u_t|_{t=0}=\sin y.$

115. $u_{tt} = 2\Delta u;\ \ u|_{t=0} = 2x^2 - y^2,\ u_t|_{t=0} = 2x^2 + y^2.$

116. $u_{tt} = 3\Delta u + x^3 + y^3;\ \ u|_{t=0} = x^2,\ u_t|_{t=0} = y^2.$

117. $u_{tt} = \Delta u + e^{3x+4y};\ \ u|_{t=0} = e^{3x+4y},\ u_t|_{t=0} = e^{3x+4y}.$

118. $u_{tt} = a^2\Delta u;\ \ u|_{t=0} = \cos(bx+cy),\ u_t|_{t=0} = \sin(bx+cy).$

119. $u_{tt} = a^2\Delta u;\ \ u|_{t=0} = r^4,\ u_t|_{t=0} = r^4$, bu yerda $r = \sqrt{x^2+y^2}$.

120. $u_{tt} = a^2\Delta u + r^2 e^t;\ \ u|_{t=0} = 0,\ u_t|_{t=0} = 0$, bu yerda $r = \sqrt{x^2+y^2}$.

c) (n=3)

121. $u_{tt} = \Delta u + 2xyz;\ \ u|_{t=0} = x^2 + y^2 - 2z^2,\ u_t|_{t=0} = 1.$

122. $u_{tt} = 8\Delta u + t^2 x^2;\ \ u|_{t=0} = y^2,\ u_t|_{t=0} = z^2.$

123. $u_{tt} = 3\Delta u + 6r^2;\ \ u|_{t=0} = x^2 y^2 z^2,\ u_t|_{t=0} = xyz$, bu yerda $r = \sqrt{x^2+y^2+z^2}$.

124. $u_{tt} = \Delta u + 6te^{x\sqrt{2}} \sin y \cos z,\ \ u|_{t=0} = e^{x+y} \cos z\sqrt{2},\ \ u_t|_{t=0} = e^{3y+4z} \sin 5x.$

125. $u_{tt} = a^2\Delta u\ u|_{t=0} = r^4,\ u_t|_{t=0} = r^4$, bu yerda $r = \sqrt{x^2+y^2+z^2}$.

126. $u_{tt} = a^2\Delta u + r^2 e^t,\ \ u|_{t=0} = 0,\ u_t|_{t=0} = 0$, bu yerda $r = \sqrt{x^2+y^2+z^2}$.

127. $u_{tt} = a^2\Delta u + \cos x \sin y e^z,\ \ u|_{t=0} = x^2 e^{y+z},\ u_t|_{t=0} = \sin x e^{y+z}.$

128. $u_{tt} = a^2\Delta u + xe^t \cos(3y+4z),\ \ u|_{t=0} = xy\cos z,\ u_t|_{t=0} = yze^x.$

129. $u_{tt} = a^2\Delta u,\ \ u|_{t=0} = \cos r,\ u_t|_{t=0} = \cos r$, bu yerda $r = \sqrt{x^2+y^2+z^2}$.

4.3. Issiqlik o'tkazuvchanlik tenglamasi uchun Koshi masalasi

$C^2(t>0)\cap C(t\geq 0)$ sinfdan shunday $u(x,t)$ funksiya topilsinki, bu funksiya $x \in R^n$, $t>0$ da

$$u_t = a^2\Delta u + f(x,t)$$

tenglamani va quyidagi boshlang'ich shartni qanoatlantirsin:

$$u|_{t\to+0} = u_0(x),$$

bu yerda f, u_0 - berilgan funksiyalar va $|u_0| \leq M$, $M>0$ - biror son.

Bu masalaga issiqlik o'tkazuvchanlik tenglamasi uchun Koshining klassik masalasi deyiladi.

Agar $f\in C^2(t\geq 0)$ funksiya va uning barcha ikkinchi tartibigacha hosilalari har bir $0\leq t\leq T$ sohada chegaralangan, $u_0\in C(R^n)$ funksiya chegaralangan bo‘lsa, u vaqtda Koshining klassik masalasining yechimi mavjud, yagona va quyidagi Puasson formulasi orqali topiladi:

$$u(x,t)=\frac{1}{\left(2a\sqrt{\pi t}\right)^n}\int_{R^n}u_0(\xi)e^{-\frac{|x-\xi|^2}{4a^2t}}d\xi+\int_0^t\int_{R^n}\frac{f(\xi,\tau)}{\left[2a\sqrt{\pi(t-\tau)}\right]^n}e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}}d\xi d\tau.$$

(5)

Quyidagi formuladan ham foydalansa bo‘ladi:

$$u(x,t)=\sum_{k=0}^{\infty}\left[\frac{t^k}{k!}a^{2k}\Delta^k u_0(x_1,...,x_n)+\frac{a^{2k}}{(2k+1)!}\int_0^t(t-\tau)^{2k+1}\Delta^k f(x_1,...,x_n,\tau)d\tau\right].$$

(6)

<u>**Masala.**</u> $\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2}+t+e^t$, $u|_{t=0}=2$. Koshi masalasini yeching.

<u>**Yechish:**</u> Bu masalani yechish uchun (5) formuladan foydalanamiz. Bu holda berilganlar quyidagilardan iborat: $a=2$, $u_0(x)=2$, $f(x,t)=t+e^t$. Ularni (5) formulaga etib qo‘yamiz:

$$u(x,t)=\frac{1}{2\cdot 2\sqrt{\pi t}}\int_{-\infty}^{\infty}2e^{-\frac{(x-\xi)^2}{16t}}d\xi+\int_0^t\int_{-\infty}^{\infty}\frac{\tau+e^\tau}{4\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{16(t-\tau)}}d\xi d\tau=I_1+I_2,$$

(7)

bu yerda $I_1=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2}{16t}}d\xi$ va $I_2=\int_0^t\int_{-\infty}^{\infty}\frac{\tau+e^\tau}{4\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{16(t-\tau)}}d\xi d\tau$. Integralarni alohida-alohida hisoblaymiz.

$$I_1=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2}{16t}}d\xi=\left|\begin{array}{l}\frac{x-\xi}{4\sqrt{t}}=\eta \text{ belgilash kiritamiz,}\\ \xi=x-4\sqrt{t}\eta\\ d\xi=-4\sqrt{t}d\eta\\ \xi=-\infty\rightarrow\eta=\infty\\ \xi=\infty\rightarrow\eta=-\infty\end{array}\right|=\frac{1}{2\sqrt{\pi t}}\int_{\infty}^{-\infty}\left(-4\sqrt{t}e^{-\eta^2}\right)d\eta=$$

$$=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\eta^2}d\eta=\left|\int_{-\infty}^{\infty}e^{-\eta^2}d\eta=\sqrt{\pi}\text{ - Puasson integrali}\right|=\frac{2}{\sqrt{\pi}}\cdot\sqrt{\pi}=2,$$

demak, $I_1 = 2$.

$I_2 = \int_0^t \int_{-\infty}^{\infty} \frac{\tau + e^\tau}{4\sqrt{\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{16(t-\tau)}} d\xi d\tau$ - integralni hisoblashda ham yuqoridagi kabi fikr yuritib, hisoblashlarni bajaramiz va quyidagi natijani olamiz: $I_2 = \frac{t^2}{2} + e^t - 1$. Ikkala integralni (7) ga qo‘yamiz, natijada quyidagi yechimni hosil qilamiz: $u(x,t) = \frac{t^2}{2} + e^t + 1$.

Mustaqil bajarish uchun mashqlar

(5) yoki (6) formulalar yordamida quyidagi Koshi masalalarini yeching.

a) (n=1)

1. $u_t = 4u_{xx} + t + e^t$, $u|_{t=0} = 2$.
2. $u_t = u_{xx} + 3t^2$, $u|_{t=0} = \sin x$.
3. $u_t = u_{xx} + e^{-t}\cos x$, $u|_{t=0} = \cos x$.
4. $u_t = u_{xx} + e^t \sin x$, $u|_{t=0} = \sin x$.
5. $u_t = u_{xx} + \sin t$, $u|_{t=0} = e^{-x^2}$.
6. $4u_t = u_{xx}$, $u|_{t=0} = e^{2x-x^2}$.
7. $u_t = u_{xx}$ $u|_{t=0} = xe^{-x^2}$.
8. $4u_t = u_{xx}$, $u|_{t=0} = \sin xe^{-x^2}$.

b) (n=2)

9. $u_t = \Delta u + e^t$, $u|_{t=0} = \cos x \sin y$.
10. $u_t = \Delta u + \sin t \sin x \sin y$, $u|_{t=0} = 1$.
11. $u_t = \Delta u + \cos t$, $u|_{t=0} = xye^{-x^2-y^2}$.
12. $8u_t = \Delta u + 1$, $u|_{t=0} = e^{-(x-y)^2}$.
13. $2u_t = \Delta u$, $u|_{t=0} = \cos xy$.

c) (n=3)

14. $u_t = 2\Delta u + t\cos x$, $u|_{t=0} = \cos y \sin z$.

15. $u_t = 3\Delta u + e^t$, $u|_{t=0} = \sin(x - y - z)$.

16. $4u_t = \Delta u + \sin 2z$, $u|_{t=0} = \frac{1}{4}\sin 2z + e^{-x^2}\cos y$.

17. $u_t = \Delta u + \cos(x - y + z)$, $u|_{t=0} = e^{-(x+y-z)^2}$.

18. $u_t = \Delta u$, $u|_{t=0} = \cos(xy)\sin z$.

d) Quyidagi Koshi masalalarini yeching

$$u_t = \Delta u, \qquad u|_{t=0} = u_0(x), \quad x \in R^n$$

bu yerda u_0 quyidagicha aniqlanadi:

19. $u_0 = \cos\sum_{k=1}^{n} x_k$.   20. $u_0 = e^{-|x|^2}$.

21. $u_0 = \left(\sum_{k=1}^{n} x_k\right) e^{-|x|^2}$.   22. $u_0 = \left(\sin\sum_{k=1}^{n} x_k\right) e^{-|x|^2}$.

23. $u_0 = e^{-\left(\sum_{k=1}^{n} x_k\right)^2}$.

5-BOB. O'ZGARUVCHILARNI AJRATISH (FURYE) USULI

Ushbu bobda tor tebranish va issiqlik o'tkazuvchanlik tenglamalariga qo'yilgan aralash masalalarni yechishning Furye usuli o'rganilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

5.1. Giperbolik turdagi tenglama

Uchlari $x=0$ va $x=l$ nuqtalarda mahkamlangan tor tebranishi tenglamasi masalasi uchun **Furye** yoki **o'zgaruvchilarni ajratish** usulini bayon qilamiz.

Erkin tor tebranish tenglamasining:

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

boshlang'ich:

$$u|_{t=0}=u_0(x),\ u_t|_{t=0}=u_1(x) \qquad (2)$$

va chegaraviy:

$$u|_{x=0}=0,\ u|_{x=l}=0 \qquad (3)$$

shartlarni qanoatlantiruvchi $u(x,t)$ yechimini $D=\{(x,t):0<x<l;t>0\}$ sohada aniqlaylik.

Dastlab, (1) tenglamaning xususiy yechimlarini quyidagi ko'rinishda qidiramiz:

$$u(x,t)=X(x)T(t), \qquad (4)$$

bu funksiyalar aynan nolga teng emas va (3) chegaraviy shartlarni qanoatlantirsin.

(4) funksiyani (1) tenglamaga qo'yib quyidagi oddiy differensial tenglamalarga kelamiz:

$$T''(t)+a^2\lambda T(t)=0, \quad (5)$$

$$X''(x)+\lambda X(x)=0, \quad (6)$$

bu yerda $\lambda = const$.

Chegaraviy shartlar quyidagicha bo'ladi:

$$X(0)=0, \quad X(l)=0. \quad (7)$$

Natijada biz (6)-(7) Shturm-Liuvill masalasi deb ataluvchi masalaga kelamiz.

Bu masalaning xos sonlari:

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2 \quad k=1,2,...$$

va bu xos sonlarga quyidagi xos funksiyalar mos keladi:

$$X_k(x)=\sqrt{\frac{2}{l}}\sin\frac{\pi kx}{l}.$$

$\lambda=\lambda_k$ bo'lganda (5) tenglama quyidagi umumiy yechimga ega:

$$T_k(t)=a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}.$$

Shuning uchun

$$u_k(x,t)=X_k(x)T_k(t)=\left(a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}$$

funksiyalar har qanday a_k va b_k uchun (1) masalani va (3) chegaraviy shartlarni qanoatlantiradi.

(2)-(3) shartlarni qanoatlantiruvchi (1) masalaning yechimini qator ko'rinishida qidiramiz:

$$u(x,t)=\sum_{k=1}^{\infty}X_k(x)T_k(t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l} \quad (8)$$

Agar bu qator tekis yaqinlashuvchi bo'lib, uni hadma-had ikki marta differensiallash mumkin bo'lsa, u vaqtda qator yig'indisi (1) tenglamani va (3) chegaraviy shartlarni qanoatlantiradi.

a_k va b_k doimiy koeffitsiyentlarni shunday aniqlaymizki, (8) qator yig'indisi (2) boshlang'ich shartlarni qanoatlantirsin, u holda quyidagi tengliklarga kelamiz:

$$u_0(x)=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi x}{l}, \qquad (9)$$

$$u_1(x)=\sum_{k=1}^{\infty}\frac{k\pi a}{l}b_k\sin\frac{k\pi x}{l}. \qquad (10)$$

(9) va (10) formulalar $u_0(x)$ va $u_1(x)$ funksiyalarning $(0,\ l)$ intervalda sinuslar bo'yicha Furye qatoriga yoyilmasini beradi. Bu yoyilmalarning koeffitsiyentlari quyidagi formulalar bilan topiladi:

$$a_k=\frac{2}{l}\int_0^l u_0(x)\sin\frac{k\pi x}{l}dx,$$

$$b_k=\frac{2}{k\pi a}\int_0^l u_1(x)\sin\frac{k\pi x}{l}dx.$$

<u>Masala:</u> Quyidagi masalani yeching:

$$u_{tt}=u_{xx}+u,\ 0<x<l,\ u|_{x=0}=0,\ u|_{x=l}=t,\ u|_{t=0}=0,\ u_t|_{t=0}=\frac{x}{l}.$$

Chegaraviy shartlar noldan farqli bo'lgani uchun, yechimni $u=v+w$ ko'rinishda qidiramiz, bu yerda $w=\mu_1(t)+\frac{x}{l}(\mu_2(t)-\mu_1(t))$, $\mu_1(t)=0,\ \mu_2(t)=t$. U holda $w(x,t)=\frac{xt}{l}$, yechim esa

$$u(x,t)=v(x,t)+\frac{xt}{l} \qquad (*)$$

ko'rinishda bo'ladi. Yechimdagi $v(x,t)$ funksiya quyidagi masalani qanoatlantiradi:

$$v_{tt}=v_{xx}+v+\frac{xt}{l},\ \ 0<x<l,\ v|_{x=0}=0,\ v|_{x=l}=0,\ v|_{t=0}=0,\ v_t|_{t=0}=0. \qquad (11)$$

Berilgan tenglamaning $\lambda_n=\left(\frac{\pi n}{l}\right)^2$ xos sonlarini va $\sin\frac{\pi n}{l}x$ xos funksiyalarini aniqlaymiz. Shunga, asosan, yechimni quyidagi ko'rinishda qidiramiz:

$$v(x,t)=\sum_{n=1}^{\infty}g_n(t)\sin\frac{\pi n}{l}x. \qquad (12)$$

Tenglamaning ozod hadi $f(x,t)=\frac{xt}{l}$ funksiyani Furye qatoriga yoyamiz:

$$f(x,t)=\sum_{n=1}^{\infty} f_n(t)\sin\frac{\pi n}{l}x. \qquad (13)$$

$f_n(t)$ - Furye koeffitsiyentlarini quyidagi formula yordamida aniqlaymiz: $f_n(t)=\frac{2}{l}\int_0^l f(\xi,t)\sin\frac{\pi n}{l}\xi d\xi=\frac{2}{l}\int_0^l \frac{\xi t}{l}\sin\frac{\pi n}{l}\xi d\xi$. Integralni bo‘laklab integrallab, natijada

$$f_n(t)=(-1)^{n+1}\frac{2t}{\pi n} \qquad (14)$$

tenglikni hosil qilamiz.

(12) va (13) funksiyalarni (14)ni hisobga olgan holda (11) masaladagi tengliklarga qo‘yamiz, natijada noma’lum $g_n(t)$ funksiya uchun quyidagi Koshi masalasini olamiz:

$$\begin{cases} g''_n(t)+\left(\left(\frac{\pi n}{l}\right)^2-1\right)g_n(t)=(-1)^{n+1}\frac{2t}{\pi n} \\ g'_n(t)=0, \quad g_n(t)=0. \end{cases} \qquad (15)$$

(15) masalani yechishda, dastlab, tenglamaning yechimini quyidagi ko‘rinishda qidiring: $g_n(t)=\bar{g}_n(t)+g^*_n(t)$, bu yerda $\bar{g}_n(t)$ - berilgan tenglamaga mos bir jinsli tenglamaning umumiy yechimi, $g^*_n(t)$ - berilgan tenglamaning xususiy yechimi bo‘lib, o‘ng tomonga qarab tanlanadi, bizning holda, $g^*_n(t)=at$ ko‘rinishda qidirish mumkin.

(15) masalani yechib, natijada (11) masalaning yechimini aniqlaymiz:

$$v(x,t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 2}{\pi n\left(\left(\frac{\pi n}{l}\right)^2-1\right)}\left(t-\frac{\sin\left(\left(\sqrt{\left(\frac{\pi n}{l}\right)^2-1}\right)\cdot t\right)}{\sqrt{\left(\frac{\pi n}{l}\right)^2-1}}\right)\sin\frac{\pi n}{l}x. \qquad (16)$$

(16) funksiyani (*) ga qo‘yib, berilgan masalaning yechimini olamiz, ya’ni:

$$u(x,t)=\frac{xt}{l}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 2}{\pi n\left(\left(\frac{\pi n}{l}\right)^2-1\right)}\left(t-\frac{\sin\left(\left(\sqrt{\left(\frac{\pi n}{l}\right)^2-1}\right)\cdot t\right)}{\sqrt{\left(\frac{\pi n}{l}\right)^2-1}}\right)\sin\frac{\pi n}{l}x.$$

Mustaqil bajarish uchun masalalar

Quyidagi aralash masalalarni yeching:

1. $u_{tt}=u_{xx}-4u$, $(0<x<1)$ $u|_{x=0}=0$, $u|_{x=1}=0$, $u|_{t=0}=x^2-x$, $u_t|_{t=0}=0$.
2. $u_{tt}+2u_t=u_{xx}-u$, $(0<x<\pi)$ $u|_{x=0}=0$, $u|_{x=\pi}=0$, $u|_{t=0}=\pi x-x^2$, $u_t|_{t=0}=0$.
3. $u_{tt}+2u_t=u_{xx}-u$ $(0<x<\pi)$; $u_x|_{x=0}=0$, $u|_{x=\pi}=0$, $u|_{t=0}=0$, $u_t|_{t=0}=x$.
4. $u_{tt}+u_t=u_{xx}$, $(0<x<1)$ $u|_{x=0}=t$, $u|_{x=1}=0$, $u|_{t=0}=0$, $u_t|_{t=0}=1-x$.
5. $u_{tt}=u_{xx}+u$, $(0<x<2)$ $u|_{x=0}=2t$, $u|_{x=2}=0$, $u|_{t=0}=0$, $u_t|_{t=0}=0$.
6. $u_{tt}=u_{xx}+u$, $(0<x<l)$ $u|_{x=0}=0$, $u|_{x=l}=t$, $u|_{t=0}=0$, $u_t|_{t=0}=\frac{x}{l}$.
7. $u_{tt}=u_{xx}+x$ $(0<x<\pi)$; $u|_{x=0}=0$, $u|_{x=\pi}=0$, $u|_{t=0}=\sin 2x$, $u_t|_{t=0}=0$.
8. $u_{tt}+u_t=u_{xx}+1$ $(0<x<1)$; $u|_{x=0}=0$, $u|_{x=1}=0$, $u|_{t=0}=0$, $u_t|_{t=0}=0$.
9. $u_{tt}-u_{xx}+2u_t=4x+8e^t\cos x\left(0<x<\frac{\pi}{2}\right)$; $u_x|_{x=0}=2t$, $u|_{x=\frac{\pi}{2}}=\pi t$, $u|_{t=0}=\cos x$,

   $u_t|_{t=0}=2x$.
10. $u_{tt}-u_{xx}-2u_t=4t(\sin x-x)\left(0<x<\frac{\pi}{2}\right)$; $u|_{x=0}=3$, $u_x|_{x=\frac{\pi}{2}}=t^2+t$, $u|_{t=0}=3$,

    $u_t|_{t=0}=x+\sin x$.
11. $u_{tt}-3u_t=u_{xx}+u-x(4+t)+\cos\frac{3x}{2}$ $(0<x<\pi)$; $u_x|_{x=0}=t+1$, $u|_{x=\pi}=\pi(t+1)$,

    $u|_{t=0}=x$, $u_t|_{t=0}=x$.
12. $u_{tt}-7u_t-=u_{xx}+2u_x-2t-7x+e^{-x}\sin 3x$ $(0<x<\pi)$; $u|_{x=0}=0$, $u|_{x=\pi}=\pi t$,

    $u|_{t=0}=0$, $u_t|_{t=0}=x$.
13. $u_{tt}+2u_t=u_{xx}+8u+2x(1-4t)+\cos 3x\left(0<x<\frac{\pi}{2}\right)$; $u_x|_{x=0}=t$, $u|_{x=\frac{\pi}{2}}=\frac{\pi t}{2}$,

    $u|_{t=0}=0$, $u_t|_{t=0}=x$.
14. $u_{tt}=u_{xx}+4u+2\sin^2 x$ $(0<x<\pi)$; $u_x|_{x=0}=0$, $u_x|_{x=\pi}=0$, $u|_{t=0}=0$, $u_t|_{t=0}=0$

.

15. $u_{tt}=u_{xx}+10u+2\sin 2x\cos x\left(0<x<\frac{\pi}{2}\right)$; $u_x\big|_{x=0}=0$, $u_x\big|_{x=\frac{\pi}{2}}=0$, $u\big|_{t=0}=0$,

$u_t\big|_{t=0}=0$.

16. $u_{tt}-3u_t=u_{xx}+2u_x-3x-2t$ $(0<x<\pi)$; $u_{x=0}=0$, $u\big|_{x=\pi}=\pi t$, $u\big|_{t=0}=e^{-x}simx$

, $u_t\big|_{t=0}=x$.

17. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=f(x)$,

$\frac{\partial u}{\partial t}(x,0)=F(x)$;

18. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=5\sin\frac{3\pi x}{l}-\frac{1}{2}\sin\frac{8\pi x}{l}$,

$\frac{\partial u}{\partial t}(x,0)=0$;

19. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=0$,

$\frac{\partial u}{\partial t}(x,0)=6\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}+\sin\frac{7\pi x}{l}$;

20. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$,

$u(x,0)=\frac{1}{3}\sin\frac{2\pi x}{l}+4\sin\frac{5\pi x}{l}-\frac{1}{4}\sin\frac{8\pi x}{l}$,

$\frac{\partial u}{\partial t}(x,0)=A\sin\frac{s\pi x}{l}+B\sin\frac{p\pi x}{l}$; $A,B=const$ $s,p\in N$.

21. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=Ax$,

$\frac{\partial u}{\partial t}(x,0)=0$;

22. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=0$,

$$\frac{\partial u}{\partial t}(x,0)=\begin{cases}0, & 0\le x\le\alpha,\\ \upsilon_0, & \alpha<x<\beta;\\ 0, & \beta\le x\le\ell\end{cases}$$

23. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=\frac{4hx(l-x)}{l^2}$,

$\frac{\partial u}{\partial t}(x,0)=0$;

24. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$,

$$u(x,0)=\begin{cases}\dfrac{h}{c}x, & 0\le x\le c,\\ \dfrac{h(x-l)}{(c-l)}, & c<x\le l,\end{cases},\quad \frac{\partial u}{\partial t}(x,0)=0;$$

25. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $u(l,t)=0$,

$u(x,0)=\frac{16h}{5}[(\frac{x}{l})^4-2(\frac{x}{l})^3+(\frac{x}{l})], h>0$, $\dfrac{\partial u}{\partial t}(x,0)=0$;

26. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=f(x)$,

$\dfrac{\partial u}{\partial t}(x,0)=F(x)$;

27. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=A\sin\frac{3\pi x}{2l}+B\sin\frac{11\pi x}{2l}$,

$\dfrac{\partial u}{\partial t}(x,0)=0$;

28. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=0$,

$\dfrac{\partial u}{\partial t}(x,0)=\frac{1}{2}\sin\frac{7\pi x}{2l}-\frac{1}{3}\sin\frac{9\pi x}{2l}$;

29. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\sin\frac{5\pi x}{2l}$,

$\dfrac{\partial u}{\partial t}(x,0)=\sin\frac{3\pi x}{2l}$;

30. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=0$, $\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

31. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\dfrac{hx}{l}$, $\dfrac{\partial u}{\partial t}(x,0)=0$;

32. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\frac{1}{4}\sin\frac{3\pi x}{2l}-\frac{1}{5}\sin\frac{5\pi x}{2l}$,

$\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

33. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=x$, $\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

34. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(0,t)=0,\ \dfrac{\partial u}{\partial x}(l,t)=0,\ u(x,0)=Ax$,

$\dfrac{\partial u}{\partial t}(x,0)=\sin\frac{\pi x}{2l}-2\sin\frac{3\pi x}{2l}$;

35. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=f(x),\ \dfrac{\partial u}{\partial t}(x,0)=F(x)$;

36. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=A\cos\frac{5\pi x}{2l}+B\sin\frac{7\pi x}{2l}$,

$\dfrac{\partial u}{\partial t}(x,0)=0$;

37. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=0$,

$\dfrac{\partial u}{\partial t}(x,0)=2\cos\frac{3\pi x}{2l}-\frac{2}{7}\sin\frac{7\pi x}{2l}$;

38. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=\cos\frac{\pi x}{2l}$,

$\dfrac{\partial u}{\partial t}(x,0)=\cos\frac{3\pi x}{2l}-\frac{1}{2}\cos\frac{3\pi x}{2l}$;

39. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=0$, $\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

40. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=\dfrac{h(l-x)}{l},\ \dfrac{\partial u}{\partial t}(x,0)=0$;

41. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=\frac{1}{5}\cos\frac{5\pi x}{2l}-\frac{1}{4}\cos\frac{3\pi x}{2l}$,

$\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

42. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=l-x$,

$\dfrac{\partial u}{\partial t}(x,0)=\cos\frac{\pi x}{2l}-3\cos\frac{3\pi x}{2l}$;

43. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $u(l,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=A(l-x)$,

$\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

44. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0,\ u(x,0)=f(x)$,

$\dfrac{\partial u}{\partial t}(x,0)=F(x)$;

45. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=0$,

$\dfrac{\partial u}{\partial t}(x,0)=\cos^2\frac{2\pi x}{l}$;

46. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\sin^2\frac{5\pi x}{l}$,

$\dfrac{\partial u}{\partial t}(x,0)=0$

47. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$,

$u(x,0)=1+\cos\frac{2\pi x}{l}-\frac{1}{3}\cos\frac{3\pi x}{l}$, $\dfrac{\partial u}{\partial t}(x,0)=2\cos\frac{6\pi x}{l}-\frac{2}{3}\cos\frac{2\pi x}{l}$;

48. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\dfrac{hx}{l}$,

$\dfrac{\partial u}{\partial t}(x,0)=0$;

49. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\sin^2\frac{\pi x}{l}$,

$\dfrac{\partial u}{\partial t}(x,0)=x$;

50. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=x$,

$\dfrac{\partial u}{\partial t}(x,0)=\upsilon_0$;

51. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=l-x$,

$\dfrac{\partial u}{\partial t}(x,0)=\cos^2\frac{\pi x}{l}$;

52. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $\dfrac{\partial u}{\partial x}(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)=0$, $u(x,0)=\cos\dfrac{3\pi x}{l}$,

$\dfrac{\partial u}{\partial t}(x,0)=l-x$.

53. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)+hu(l,t)=0$, $h>0$,

$u(x,0)=f(x)$, $\dfrac{\partial u}{\partial t}(x,0)=F(x)$;

54. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(l,t)+hu(l,t)=0$,

$h>0$, $\dfrac{\partial u}{\partial t}(x,0)=0$;

55. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(0,t)=0$, $\frac{\partial u}{\partial x}(l,t)+hu(l,t)=0$, $h>0$,

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

56. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(0,t)=0$, $\frac{\partial u}{\partial x}(l,t)+hu(l,t)=0$, $h>0$,

$u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=1$;

57. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(0,t)=0$, $\frac{\partial u}{\partial x}(l,t)+hu(l,t)=0$, $h>0$,

$u(x,0)=Ax$, $\frac{\partial u}{\partial t}(x,0)=0$;

58. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $u(l,t)=0$, $\frac{\partial u}{\partial x}(0,t)-hu(0,t)=0$, $h>0$,

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

59. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(l,t)=0$, $\frac{\partial u}{\partial x}(0,t)-hu(0,t)=0$, $h>0$,

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

60. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(0,t)-hu(0,t)=0$, $\frac{\partial u}{\partial x}(l,t)+hu(l,t)=0$,

$h>0$ $u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

61. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(0,t)-h_1u(0,t)=0$, $\frac{\partial u}{\partial x}(l,t)+h_2u(l,t)=0$,

$h_1>0, h_2>0$, $u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

62. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $u(0,t)=0$, $\frac{\partial^2 u}{\partial t^2}(l,t)=-h\frac{\partial u}{\partial x}(l,t)$,

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

63. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $u(0,t)=0$, $\frac{\partial^2 u}{\partial t^2}(l,t)=-h\frac{\partial u}{\partial x}(l,t)$,

$u(x,0)=Ax$, $\frac{\partial u}{\partial t}(x,0)=0$;

64. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial u}{\partial x}(0,t)=0$, $\frac{\partial^2 u}{\partial t^2}(l,t)=-h\frac{\partial u}{\partial x}(l,t)$,

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

65. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, $(t>0)$, $0<x<l$, $\frac{\partial^2 u}{\partial t^2}(0,t)=h\frac{\partial u}{\partial x}(0,t)$, $\frac{\partial^2 u}{\partial t^2}(l,t)=-h\frac{\partial u}{\partial x}(l,t)$,

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=F(x)$;

66. $\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial u}{\partial x}(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=f(x)$,

$\frac{\partial u}{\partial t}(x,0)=F(x)$;

67. $\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial u}{\partial x}(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=\cos x$, $\frac{\partial u}{\partial t}(x,0)=0$;

68. $\frac{\partial^2 u}{\partial t^2}-2u=\frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=f(x)$,

$\frac{\partial u}{\partial t}(x,0)=F(x)$;

69. $\frac{\partial^2 u}{\partial t^2}-5u=\frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=f(x)$,

$\frac{\partial u}{\partial t}(x,0)=F(x)$;

70. $\frac{\partial^2 u}{\partial t^2}-10u=\frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=0$;

71. $\frac{\partial^2 u}{\partial t^2}-10u=\frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=\frac{1}{9}\sin x+\sin 3x$,

$\frac{\partial u}{\partial t}(x,0)=F(x)$;

72. $\frac{\partial^2 u}{\partial t^2}-17u=\frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$; $\frac{\partial u}{\partial x}\left(\frac{\pi}{2},t\right)=0$; $u(x,0)=f(x)$,

$\frac{\partial u}{\partial t}(x,0)=F(x)$;

5.2. Parabolik turdagi tenglama

Bir jinsli ingichka sterjenda issiqlik tarqalish masalasini ko'rib chiqamiz, uning yon sirti issiqlik o'tkazmaydi, $x=0$ va $x=l$ chegaralarida esa nol temperatura saqlanadi deb faraz qilamiz. Ushbu masala uchun Furye yoki o'zgaruvchilarni ajratish usulini bayon qilamiz.

Quyidagi masalani qaraylik:

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad (17)$$

tenglamaning boshlang'ich:

$$u|_{t=0}=u_0(x), \qquad (18)$$

va chegaraviy:

$$u|_{x=0}=0,\ u|_{x=l}=0. \qquad (19)$$

shartlarni qanoatlantiruvchi $u(x,t)$ yechimini $D=\{(x,t):0<x<l;t>0\}$ sohada topish talab etilsin. Dastlab, (17) tenglamaning xususiy yechimlarini quyidagi ko'rinishda qidiramiz:

$$u(x,t)=X(x)T(t), \qquad (20)$$

bu funksiyalar aynan nolga teng emas va $X(x)$ funksiya (19) chegaraviy shartlarni qanoatlantiradi.

(20) funksiyani (17) tenglamaga qo'yib quyidagi oddiy differensial tenglamalarga kelamiz:

$$T'(t)+a^2\lambda T(t)=0, \qquad (21)$$

$$X''(x)+\lambda X(x)=0, \qquad (22)$$

bu yerda $\lambda=const$.

$X(x)$ funksiya uchun chegaraviy shartlar quyidagidan iborat:

$$X(0)=0,\quad X(l)=0. \qquad (23)$$

Natijada biz Shturm-Liuvill (22)-(23) masalasiga kelamiz.

Bu masalaning xos sonlari

$$\lambda_k=\left(\frac{\pi k}{l}\right)^2 \quad k=1,2,...$$

bo'lib, ularga quyidagi xos funksiyalar mos keladi:

$$X_k(x)=\sin\frac{\pi kx}{l}.$$

$\lambda=\lambda_k$ bo'lganda (21) tenglama quyidagi umumiy yechimga ega:

$$T_k(t)=a_k e^{-\left(\frac{k\pi a}{l}\right)^2 t}.$$

Shuning uchun

$$u_k(x,t)=X_k(x)T_k(t)=a_k e^{-\left(\frac{\pi ka}{l}\right)^2 t}\sin\frac{k\pi x}{l}$$

funksiya har qanday a_k uchun (17) masalani va (19) chegaraviy shartlarni qanoatlantiradi.

(18)-(19) shartlarni qanoatlantiruvchi (17) masalaning yechimini qator ko'rinishida qidiramiz:

$$u(x,t)=\sum_{k=1}^{\infty}X_k(x)T_k(t)=\sum_{k=1}^{\infty}a_k e^{-\left(\frac{k\pi a}{l}\right)^2 t}\sin\frac{k\pi x}{l} \qquad (24)$$

Agar bu qator tekis yaqinlashuvchi bo'lib, uni t o'zgaruvchi bo'yicha bir marta x o'zgaruvchi bo'yicha ikki marta differensiallash mumkin bo'lsa, u vaqtda qator yig'indisi (17) tenglamani va (19) chegaraviy shartlarni qanoatlantiradi.

a_k doimiy koeffitsiyentlarni shunday aniqlaymizki, bunda (24) qator yig'indisi (18) boshlang'ich shartlarni qanoatlantirsin. U holda quyidagi tengliklarga kelamiz:

$$u_0(x)=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi x}{l}, \qquad (25)$$

(25) formula $u_0(x)$ funksiyaning $(0,l)$ intervalda sinuslar bo'yicha Furye yoyilmasini beradi. Bu yoyilmaning koeffitsiyentlari quyidagi formula bilan topiladi:

$$a_k=\frac{2}{l}\int_0^l u_0(x)\sin\frac{k\pi x}{l}dx\ .$$

<u>Masala:</u> Quyidagi masalani Furye usulida yeching:

$$u_t=u_{xx}+u,\ (0<x<l)\ u|_{x=0}=0,\ u|_{x=l}=0,\ u|_{t=0}=13x. \qquad (26)$$

Dastlab, (26) tenglamaning xususiy yechimlarini (20) ko'rinishda qidiramiz.

$X(x)$ va $T(t)$ funksiyalar aynan nolga teng emas va $X(x)$ masaladagi chegaraviy shartlarni qanoatlantirsin.

(20) funksiyani (26) masaladagi tenglamaga qo'yib quyidagi oddiy differensial tenglamalarga kelamiz:

$$T'(t)+\lambda T(t)=0, \qquad (27)$$

$$X''(x)+(\lambda+1)X(x)=0, \qquad (28)$$

bu yerda $\lambda=const$.

Chegaraviy shartlar quyidagicha bo‘ladi:

$$X(0) = 0, \quad X(l) = 0 . \qquad (29)$$

Natijada biz Shturm-Liuvill (28)-(29) masalasiga kelamiz.

Bu masalaning xos sonlari:

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2 - 1$$

bo‘lib va bu xos sonlarga quyidagi xos funksiyalar mos keladi:

$$X_n(x) = \sin\frac{\pi n x}{l} .$$

$\lambda = \lambda_n$ bo‘lganda (27) tenglama quyidagi umumiy yechimga ega:

$$T_n(t) = a_n e^{-\left(\left(\frac{\pi n}{l}\right)^2 - 1\right)t} .$$

Shuning uchun

$$u_n(x,t) = X_n(x)T_n(t) = a_n e^{-\left(\left(\frac{\pi n}{l}\right)^2 - 1\right)t} \sin\frac{n\pi x}{l}$$

funksiya har qanday a_n uchun berilgan masalani qanoatlantiradi.

Berilgan masalaning yechimini qator ko‘rinishida qidiramiz:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} a_n e^{-\left(\left(\frac{\pi n}{l}\right)^2 - 1\right)t} \sin\frac{n\pi x}{l} .$$

a_n doimiy koeffitsiyentlarni shunday aniqlaymizki, bunda bu qator yig‘indisi boshlang‘ich shartlarni qanoatlantirsin. U holda quyidagi tenglikni hosil qilamiz:

$$13 \cdot x = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} ,$$

bu tenglik $u_0(x) = 13x$ funksiyaning $(0,l)$ intervalda sinuslar bo‘yicha Furye qatoriga yoyilmasini beradi. Bu yoyilmaning koeffitsiyentlari quyidagi formula bilan topiladi:

$$a_n = \frac{2}{l}\int_0^l 13 \cdot x \cdot \sin\frac{n\pi x}{l}dx .$$

Bu yerda integralni bo‘laklab integrallab, $a_n = \frac{26 \cdot l}{\pi n} \cdot (-1)^{n+1}$ larga ega bo‘lamiz. U vaqtda izlanayotgan yechim quyidagi ko‘rinishda bo‘ladi:

$$u(x,t)=\frac{26\cdot l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\left(\left(\frac{n\pi}{l}\right)^2-1\right)t}\sin\frac{n\pi x}{l}.$$

Mustaqil bajarish uchun masalalar

Quyidagi aralash masalalarni yeching:

73. $u_t=u_{xx},\ (0<x<l)\ u|_{x=0}=0,\ u|_{x=l}=0,\ u|_{t=0}=A=const.$

74. $u_t=u_{xx},\ (0<x<l)\ u|_{x=0}=0,\ u|_{x=l}=0,\ u|_{t=0}=Ax(l-x),\ \mathrm{A}=const.$

75. $u_t=u_{xx},\ (0<x<l)\ u|_{x=0}=0,\ (u_x+hu)|_{x=l}=0,\ u|_{t=0}=u_0(x).$

76. $u_t=u_{xx},\ (0<x<l)\ (u_x-hu)|_{x=0}=0,\ (u_x+hu)|_{x=l}=0,\ u|_{t=0}=u_0(x).$

77. $u_t=u_{xx},\ (0<x<l)\ u_x|_{x=0}=0,\ u_x|_{x=l}=0,\ u|_{t=0}=u_0=const.$

78. $u_t=u_{xx},\ (0<x<l)\ u_x|_{x=0}=0,\ u_x|_{x=l}=0,\ u|_{t=0}=\begin{cases}u_0=const, & \text{agar } 0<x<\frac{l}{2}\\ 0, & \text{agar } \frac{l}{2}<x<l\end{cases}.$

$\lim\limits_{t\to\infty}u(x,t)-?$

79. $u_t=u_{xx},\ (0<x<l)\ u_x|_{x=0}=0,\ u_x|_{x=l}=0,$

$$u|_{t=0}=\begin{cases}\frac{2u_0}{l}x, & \text{agar } 0<x<\frac{l}{2}\\ \frac{2u_0}{l}(l-x), & \text{agar } \frac{l}{2}\le x<l\end{cases},$$

bu yerda $u_0=cons$. $\lim\limits_{t\to\infty}u(x,t)-?$

80. $u_t=u_{xx},\ (0<x<1)\ u_x|_{x=0}=0,\ u|_{x=1}=0,\ u|_{t=0}=x^2-1.$

81. $u_t=u_{xx}+u,\ (0<x<l)\ u|_{x=0}=0,\ u|_{x=l}=0,\ u|_{t=0}=1.$

82. $u_t=u_{xx}-4u,\ (0<x<\pi)\ u|_{x=0}=0,\ u|_{x=\pi}=0,\ u|_{t=0}=x^2-\pi x.$

83. $u_t=u_{xx},\ (0<x<l)\ u_x|_{x=0}=1,\ u|_{x=l}=0,\ u|_{t=0}=0.$

84. $u_t=u_{xx}+u+2\sin 2x\sin x,\ \left(0<x<\frac{\pi}{2}\right)\ u_x|_{x=0}=0,\ u|_{x=\frac{\pi}{2}}=0,\ u|_{t=0}=0.$

85. $u_t=u_{xx}-2u_x+x+2t,\ (0<x<1),\ u|_{x=0}=0,\ u|_{x=1}=0,\ u|_{t=0}=e^x\sin\pi x.$

86. $u_t=u_{xx}+u+2\sin 2x\cos x,\ \left(0<x<\frac{\pi}{2}\right)\ u|_{x=0}=0,\ u_x|_{x=\frac{\pi}{2}}=1,\ u|_{t=0}=x.$

87. $u_t=u_{xx}+u+2\sin 2x\sin x,\ (0<x<\pi)\ u_x|_{x=0}=0,\ u_x|_{x=\pi}=2\pi t,\ u|_{t=0}=0.$

88. $u_t-u_{xx}+2u_x-u=e^x\sin x-t$, $(0<x<\pi)$ $u|_{x=0}=1+t$, $u|_{x=\pi}=1+t$,

$u|_{t=0}=1+e^x\sin 2x$.

89. $u_t-u_{xx}-u=xt(2-t)+2\cos t$, $(0<x<\pi)$ $u_x|_{x=0}=t^2$, $u_x|_{x=\pi}=t^2$,

$u|_{t=0}=\cos 2x$.

90. $u_t-u_{xx}-9u=4\sin^2 t\cos 3x-9x^2-2$, $(0<x<\pi)$ $u_x|_{x=0}=0$, $u_x|_{x=\pi}=2\pi$,

$u|_{t=0}=x^2+2$.

91. $u_t=u_{xx}+6u+2t(1-3t)-6x+2\cos x\cos 2x$, $\left(0<x<\frac{\pi}{2}\right)$ $u_x|_{x=0}=1$,

$u_x|_{x=\pi}=t^2+\frac{\pi}{2}$, $u|_{t=0}=x$.

92. $u_t=u_{xx}+6u+x^2(1-6t)-2(t+3x)+\sin 2x$, $(0<x<\pi)$ $u_x|_{x=0}=1$,

$u_x|_{x=\pi}=2\pi t+1$, $u|_{t=0}=x$.

93. $u_t=u_{xx}+4u_x+x-4t+1+e^{-2x}\cos^2\pi x$, $(0<x<1)$, $u|_{x=0}=t$, $u|_{x=1}=2t$,

$u|_{t=0}=0$.

6-BOB. INTEGRAL TENGLAMALAR

Integral tenglamalar nazariyasi hozirgi zamon matematikasining muhim va murakkab yo'nalishlaridan biriga aylanib bormoqda. Integral tenglamalarning turlari shu qadar ko`payib ketdiki, ularga umumiy ta`rif berishning iloji bo`lmay qoldi. Shunday bo`lsa-da, integral tenglamaning mavjud ilmiy adabiyotlarda qabul qilingan ta`rifini eslatib o`tamiz.

Ta'rif. Agar tenglamadagi noma'lum funksiya shu funksiyaning argumenti bo`yicha olinadigan integral ishorasi ostida bo`lsa, bunday tenglama integral tenglama deb ataladi.

Integral tenglamalarning ba'zilari va ularni yechish usullari bilan quyida tanishamiz.

6.1. Fredgol'm tenglamalari. Ketma-ket yaqinlashish usuli

Matematik fizikaning ko'pgina masalalari $u(t)$ noma`lum funksiyaga nisbatan

$$\int_a^b K(x,t)u(t)dt = f(x), \qquad (1)$$

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt \qquad (2)$$

ko'rinishdagi integral tenglamalarga keltiriladi. Bu tenglamalarda $f(x)$ - ozod had va $K(x,t)$ tenglamaning yadrosi berilgan funksiyalar, λ - (2) tenglamaning parametri, integrallash chegaralari a va b berilgan haqiqiy o`zgarmas sonlardir. (1) va (2) tenglamalar mos ravishda Fredgol'mning birinchi va ikkinchi turdagi integral tenglamalari deyiladi. (2) tenglamadagi noma'lum funksiya $u(x)$ integral ishorasidan tashqarida ham ishtirok etmoqda. Bu tenglamalardagi $f(x)$ funksiya $I(a \le x \le b)$ kesmada, $K(x,t)$ yadro esa

$Q(a \le x \le b,\ a \le t \le b)$ yopiq sohada berilgan va uzluksiz funksiyalar deb hisoblanadi.

Agar (2) integral tenglamada $f = 0$ bo'lsa, unda u

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \qquad (3)$$

ko'rinishda bo'lib, bu tenglama (2) tenglamaga mos bir jinsli ikkinchi turdagi Fredgol'm integral tenglamasi deyiladi.

Nihoyat, ushbu

$$\varphi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (4)$$

tenglamaga uchinchi tur integral tenglama deb ataladi. Agar I kesmada $\varphi(x) = 0$ bo`lsa, undan (1) tenglama; $\varphi(x) = 1$ bo`lsa, undan (2) tenglama kelib chiqadi. Yuqorida biz tanishgan integral tenglamalarning barchasida noma`lum $u(x)$ funksiya bir argumentlidir, ya`ni birgina x erkli o`zgaruvchining funksiyasidir. Misol uchun quyidagi integral tenglamani olaylik:

$$u(x) = 3x - 2 + 3\int_0^1 xtu(t)dt,$$

Bunda

$f(x) = 3x - 2,\qquad K(x,t) = xt,\qquad a = 0\qquad b = 1$

$\lambda = 3$

Demak, bu tenglama Fredgol'mning ikkinchi tur tenglamalaridan ekan.

Ta'rif. Agar $u(x),\ x \in [a,b]$ funksiyani (1) yoki (2) integral tenglamaga olib qo'yganda bu tenglama ayniyatga aylansa, u holda bu funksiya shu mos tenglamaning yechimi deb aytiladi.

Misol: $u(x) = \sin\frac{\pi x}{2}$ funksiya quyidagi integral tenglamaning yechimi ekanligini ko'rsating:

$$u(x) - \frac{\pi^2}{4}\int_0^1 K(x,t)u(t)dt = \frac{x}{2}, \quad \text{bunda}$$

$$K(x,t)=\begin{cases}\frac{x(2-t)}{2}, & 0\le x\le t,\\ \frac{t(2-x)}{2}, & t<x\le 1.\end{cases}$$

Yechish: Tenglamaning chap tomonini yadro ko‘rinishining hisobiga, o‘zgartiramiz:

$$u(x)-\frac{\pi^2}{4}\left(\int_0^x K(x,t)u(t)dt+\int_x^1 K(x,t)u(t)dt\right)=$$
$$=u(x)-\frac{\pi^2}{4}\left(\int_0^x \frac{t(2-x)}{2}u(t)dt+\int_x^1 \frac{x(2-t)}{2}u(t)dt\right)=$$
$$=u(x)-\frac{\pi^2}{4}\left(\frac{2-x}{2}\int_0^x tu(t)dt+\frac{x}{2}\int_x^1(2-t)u(t)dt\right).$$

Hosil bo‘lgan tenglamaga $u(x)=\sin\frac{\pi}{2}x$ ni qo‘yib,

$$\sin\frac{\pi}{2}x-\frac{\pi^2}{4}(2-x)\int_0^x\frac{t\sin\frac{\pi}{2}t}{2}dt+x\int_x^1(2-t)\frac{\sin\frac{\pi}{2}t}{2}dt=\sin\frac{\pi}{2}x-$$
$$-\frac{\pi^2}{4}\left((2-x)\left(-\frac{t}{\pi}\cos\frac{\pi t}{2}+\frac{2}{\pi^2}\sin\frac{\pi t}{2}\right)\Big|_{t=0}^{t=x}+x\left(-\frac{2-t}{\pi}\cos\frac{\pi t}{2}-\frac{2}{\pi^2}\sin\frac{\pi t}{2}\right)\Big|_{t=x}^{t=1}\right)=\frac{x}{2}$$

ekanligiga ishonch hosil qilamiz. Demak, $u(x)=\sin\frac{\pi}{2}x$ funksiya berilgan integral tenglamaga qo‘yganda ayniyat hosil bo‘ldi. Bu esa $u(x)=\sin\frac{\pi}{2}x$ funksiya tenglamaning yechimi ekanligini ko‘rsatadi.

Endi ikkinchi turdagi Fredgol’m integral tenglamasini ketma – ket yaqinlashish usuli bilan yechamiz. (2) tenglamada $K(x,y)$ va $f(x)$ funksiyalar o‘zlari aniqlangan sohalarda uzluksiz bo‘lgani uchun

$$\int_a^b|K(x,y)|dy\le M,\quad a\le x\le b,\quad \max_{a\le x\le b}|f(x)|=m, \tag{5}$$

bo‘ladi.

Agar (2) tenglama λ parametri

$$|\lambda|<\frac{1}{M(b-a)} \tag{6}$$

shartni qanoatlantirsa, u holda bu tenglamaning yagona $u(x)$ yechimi mavjud bo‘lib, uni ketma-ket yaqinlashish usuli bilan topish mumkin.

Nolinchi yaqinlashish sifatida (2) tenglamaning ozod hadini qabul qilamiz:

$$u_0(x)=f(x).$$

Birinchi yaqinlashishni

$$u_1(x)=f(x)+\lambda\int_a^b K(x,y)f(y)dy$$

munosabat bilan aniqlaymiz. Bu jarayonni davom ettirib *n*-yaqinlashishni

$$u_n(x)=f(x)+\lambda\int_a^b K(x,y)u_{n-1}(y)dy, \quad n=1,2,... \tag{7}$$

formula bilan aniqlaymiz.

Shunday qilib, (7) rekkurent munosabatlarni qanoatlantiruvchi

$$u_0(x), u_1(x), ..., u_n(x), ... \tag{8}$$

funksiyalar ketma-ketligiga ega bo'lamiz.

Matematik analizdan ma'lumki, (9) ketma-ketlikning yaqinlashishi

$$u_0(x)+\sum_{n=1}^{\infty}[u_n(x)-u_{n-1}(x)] \tag{9}$$

qatorning yaqinlashishiga teng kuchlidir. (7) formulani

$$\begin{aligned}u_n(x)&=f(x)+\lambda\int_a^b K(x,y)[u_{n-1}(y)-u_{n-2}(y)+u_{n-2}(y)]dy=\\&=f(x)+\lambda\int_a^b K(x,y)u_{n-2}(y)dy+\lambda\int_a^b K(x,y)[u_{n-1}(y)-u_{n-2}(y)]dy=\\&=u_{n-1}(x)+\lambda\int_a^b K(x,y)[u_{n-1}(y)-u_{n-2}(y)]dy, \quad n=2,3,4,...\end{aligned} \tag{10}$$

ko'rinishida yozib olamiz.

(6) ga asosan, (10) dan darhol quyidagi tengsizliklar kelib chiqadi:

$$|u_0(x)|\le m,$$
$$|u_1(x)-u_0(x)|\le m|\lambda|M(b-a),$$
$$|u_2(x)-u_1(x)|\le m|\lambda|^2M^2(b-a)^2,$$
....
$$|u_n(x)-u_{n-1}(x)|\le m|\lambda|^nM^n(b-a)^n.$$

Shunday qilib, (9) qatorning har bir hadi musbat sonli

$$\sum_{n=0}^{\infty} m|\lambda|^n M^n (b-a)^n \qquad (11)$$

qatorning mos hadidan katta emas. (11) qator esa, (6) ga asosan yaqinlashuvchidir. Demak, (9) qator, natijada uzluksiz funksiyalarning (8) ketma-ketligi uzluksiz $u(x)$ funksiyaga absolyut va tekis yaqinlashadi. (7) tenglikda $n=\infty$ limitga o'tib,

$$u(x) = f(x) + \lambda\int_a^b K(x,y)u(y)dy$$

tenglikni hosil qilamiz, bu esa $u(x)$ funksiya (2) tenglamaning yechimi ekanligini ko'rsatadi. Endi (2) tenglamaning $u(x)$ dan boshqa yechimi yo`qligini ko'rsatish qiyin emas. Buning uchun aksincha, ya'ni (2) tenglamaning $u(x)$ dan boshqa yana bitta $v(x)$ yechimi bor deb faraz qilamiz. U holda bu yechimlarning ayirmasi $w(x) = u(x) - v(x)$ (3) bir jinsli tenglamaning yechimidan iborat bo`ladi, ya`ni:

$$w(x) = \lambda\int_a^b K(x,y)w(y)dy,$$

$$w_0 = \max_{a\le x\le b}|w(x)|$$

deb belgilab olsak, oxirgi tenglikdan

$$w_0 \le |\lambda| M w_0$$

tengsizlikka ega bo'lamiz. Agar $w_0 \neq 0$ bo'lsa, oxirgi tengsizlik (7) tengsizlikka ziddir. Demak, $w_0 = 0$, bundan $w(x) = 0$, ya'ni $u(x) = v(x)$ ekanligi kelib chiqadi.

6.2. Volterra tenglamalari. Ketma-ket yaqinlashish usuli

Ta'rif. Ushbu

$$\lambda\int_a^x K(x,y)\varphi(y)dy = f(x) \qquad (12)$$

$$\varphi(x) = f(x) + \lambda\int_a^x K(x,y)\varphi(y)dy \qquad (13)$$

integral tenglamalarga mos ravishda Volterraning birinchi va ikkinchi tur integral tenglamalari deyiladi. Bunda $\varphi(x)$ – noma'lum funksiya, λ tenglamaning parametri, $f(x)$ – ozod had $I(a \le x \le b)$ kesmada va $K(x,y)$ tenglamaning yadrosi – $R(a \le x \le b,$ $a \le y \le x)$ yopiq sohada berilgan deb hisoblanadi.

Volterra ikkinchi tur (13) integral tenglamasini ketma-ket yaqinlashish usuli bilan yechamiz. 6.1 paragrafdagi mulohazalarni qaytarib,

$$\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots \qquad (14)$$

funksiyalar ketma-ketligini hosil qilamiz, bunda

$$\varphi_0(x) = f(x), \quad \varphi_n(x) = f(x) + \lambda \int_a^x K(x,y)\varphi_{n-1}(y)dy,$$

$$m = \max|f(x)|, \quad N = \max|K(x,y)|$$

Belgilashlar kiritildi. Bu holda

$$|\varphi_0(x)| \le m,$$

$$|\varphi_1(x) - \varphi_0(x)| = \left|\lambda \int_a^x K(x,y)\varphi_0(y)dy\right| \le |\lambda| mN(x-a), \ldots,$$

$$|\varphi_n(x) - \varphi_{n-1}(x)| \le m\frac{|\lambda|^n N^n (x-a)^n}{n!}, \quad n = 1,2,\ldots \qquad (15)$$

tengsizliklarga ega bo'lamiz.

Musbat hadli $m\sum_{n=0}^{\infty}\frac{|\lambda|^n N^n (x-a)^n}{n!} = me^{|\lambda|N(x-a)}$ funksional qator λ parametrning ixtiyoriy chekli qiymatida tekis yaqinlashuvchi bo'lgani uchun (15) tengsizliklarga asosan (14) funksiyalar ketma-ketligi absolyut va tekis yaqinlashuvchi bo'lib, uning limiti bo'lgan $\varphi(x) = \lim_{n\to\infty}\varphi_n(x)$ funksiya (13) tenglamaning yechimidan iborat bo'ladi.

Endi (13) tenglama yechimining yagona ekanligini ko'rsatamiz.

Faraz qilaylik, (13) tenglama ikkita $\varphi(x)$ va $\psi(x)$ uzluksiz yechimlarga ega bo'lsin. Bularning ayirmasi $\omega(x) = \varphi(x) - \psi(x)$ bir jinsli

$$\omega(x) = \lambda \int_a^x K(x,y)\omega(y)dy \qquad (16)$$

tenglamani qanoatlantiradi.

$m^* = \max|\omega(x)|$ deb belgilab olsak, (16) dan

$$|\omega(x)| \le |\lambda| \int_a^x |K(x,y)||\omega(y)| dy \le |\lambda| N m^* (x-a)$$

tengsizlik kelib chiqadi. Bundan foydalanib (16) tenglikdan

$$|\omega(x)| \le |\lambda| \int_a^x |K(x,y)||\omega(y)| dy \le |\lambda|^2 N^2 m^* \frac{(x-a)^2}{2}$$

tengsizlikni hosil qilamiz. Bu jarayonni davom ettirib, ixtiyoriy natural n uchun

$$|\omega(x)| \le |\lambda|^n N^n m^* \frac{(x-a)^n}{n!}$$

tengsizlikni hosil qilamiz. Bu tengsizlikdan $n \to \infty$ da $\omega(x) = 0$ yoki $\varphi(x) = \psi(x)$ ekanligi kelib chiqadi.

Shunday qilib quyidagi xulosaga keldik. Volterraning ikkinchi tur (13) integral tenglamasi, uning yadrosi $K(x,y)$ va ozod hadi $f(x)$ uzluksiz funksiyalar bo'lganda λ parametrning har bir chekli qiymati uchun yagona yechimga ega bo'ladi.

Bu esa Volterraning ikkinchi tur integral tenglamasi har bir λ uchun ham yechimga ega bo'lavermaydigan Fredgol'mning ikkinchi tur integral tenglamasidan tubdan farq qilishini ko'rsatadi.

Misol. Ushbu

$$u(x) = x + \int_0^x (t-x) u(t) dt$$

Tenglamani ketma-ket yaqinlashish usulidan foydalanib yeching.

Ko'rinib turibdiki,

$$f(x) = x \quad \text{va} \quad \lambda = 1.$$

Endi quyidagi munosabatlardagi ifodalarni hisoblab chiqamiz:

$$u_0(x) = f(x) = x;$$

$$u_1(x) = \int_0^x (t-x) t dt = \left[\frac{t^3}{3} - x\frac{t^2}{2}\right]_{t=0}^{t=x} = \frac{x^3}{3} - \frac{x^3}{2} = -\frac{x^3}{3!};$$

$$u_2(x)=\int_0^x (t-x)\left(-\frac{t^3}{3!}\right)dt=\frac{x^5}{5!};$$

$$u_3(x)=\int_0^x (t-x)\left(-\frac{t^5}{5!}\right)dt=\frac{x^5}{7!};$$

va hokazo. Bu ifodalarning hosil bo'lishidagi qonuniyat ko'rinib turibdi. Ularning yig'indisini hisoblasak, izlanayotgan yechimni hosil qilamiz:

$$u(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...=\sin x$$

6.3. Iteratsiyalangan yadro. Rezolventa.

(2) ko'rinishdagi

$$\varphi(x)=f(x)+\lambda\int_a^b K(x,t)\varphi(t)dt \qquad (2)$$

Fredgol'm ikkinchi turdagi integral tenglama berilgan bo'lsin. (7) tengsizlik bajarilganda (8) funksiyalar ketma-ketligi (2) tenglamaning $u(x)$ yechimiga yaqinlashishi isbotlangan edi. Endi shu ketma-ketlikning har bir hadini batafsilroq o'rganamiz. Ma'lumki,

$$\varphi_1(x)=f(x)+\lambda\int_a^b K(x,y)f(y)dy,$$

so'ngra

$$\varphi_2(x)=f(x)+\lambda\int_a^b K(x,t)\varphi_1(t)dt=$$

$$=f(x)+\lambda\int_a^b K(x,t)f(t)dt+\lambda^2\int_a^b K(x,t)dt\int_a^b K(t,y)f(y)dy.$$

Ikkilangan integralda interallash tartibini o'zgartirib,

$$K_2(x,y)=\int_a^b K(x,t)K(t,y)dt$$

kabi belgilab olib,

$$\varphi_2(x)=f(x)+\lambda\int_a^b K(x,y)f(y)dy+\lambda^2\int_a^b K_2(x,y)f(y)dy$$

tenglikni hosil qilamiz.

Bu jarayonni davom ettirib,

$$\varphi_n(x) = f(x) + \lambda \int_a^b \sum_{i=1}^{n} \lambda^{i-1} K_i(x,y) f(y) dy \tag{17}$$

tenglikka ega bo'lamiz, bunda $K_i(x,y)$lar

$$K_1(x,y) = K(x,y),$$

$$K_i(x,y) = \int_a^b K(x,t) K_{i-1}(t,y) dt, \quad i = 2,3,\dots \tag{18}$$

rekurent munosabat bilan aniqlandi. $K_i(x,y)$funksiyalar iteratsiyalangan (takroriy) yadrolar deb ataladi.

Integratsiyalangan yadrolarni (18) ga nisbatan umumiyroq

$$K_i(x,y) = \int_a^b K_r(x,t) K_{i-r}(t,y) dt \tag{19}$$

formula bilan ifodalash mumkin. Haqiqatan ham, (17) da $K_{i-1}(t,y)$ yadroni yana shu (18) formula yordamida K_{i-2} bilan ifodalab,

$$K_i(x,y) = \int_a^b K(x,t_1) \left[\int_a^b K(t_1,t_2) K_{i-2}(t_2,y) dt_2 \right] dt_1 = \int_a^b \int_a^b K(x,t_1) K(t_1,t_2) K_{i-2}(t_2,y) dt_1 dt_2$$

tenglikni hosil qilamiz. $K_{i-2}(t_2,y)$ yadroni K_{i-3} orqali ifodalash mumkin va hokazo. Bu jarayonni davom ettirib, oxirida

$$K_i(x,y) = \int_a^b \dots \int_a^b K(x,t_1) K(t_1,t_2) \dots K(t_{i-1},y) dt_1 \dots dt_{i-1}$$

formulaga kelamiz. t_r o'zgaruvchi bo'yicha integralni ajratib, oxirgi formulani

$$K_i(x,y) = \int_a^b dt_r \left\{ \int_a^b \dots \int_a^b K(x,t_1) K(t_1,t_2) \dots K(t_{i-1},y) dt_1 \dots dt_{r-1} \times \right.$$

$$\left. \times \int_a^b \dots \int_a^b K(t_r,t_{r+1}) K(t_{r+1},t_{r+2}) \dots K(t_{i-1},y) dt_{r+1} \dots dt_{i-1} \right\}$$

ko'rinishda yozib olamiz. (20) formulaga asosan figurali qavs ichidagi birinchi integral $K_r(x,t_r)$ ga, ikkinchi integral esa $K_{i-r}(t_r,y)$ ga teng.

Shunday qilib,

$$K_l(x,y)=\int_a^b K_r(x,t_r)K_{l-r}(t_r,y)dt_r$$

bunda t_r ni t ga almashtirib (19) formulaga kelamiz.

(8) ketma-ketlikning yaqinlashishini isbotlangandagi mulohazalarni qaytarib, $a \le x \le b,\ a \le y \le b$ kvadratda

$$\sum_{l=1}^{\infty} \lambda^{l-1} K_l(x,y)$$

qatorning tekis yaqinlashishiga ishonch hosil qilish mumkin.

Bu qatorning yig'indisi $R(x,y,\lambda)$ ni $K(x,y)$ yadroning yoki (2) integral tenglamaning rezolventasi yoki hal qiluvchi yadrosi deyiladi.

(17) da $n \to \infty$ deb limitda o'tib, (2) tenglamaning yechimini rezolventa yordamida

$$\varphi(x)=f(x)+\lambda\int_a^b R(x,y,\lambda)f(y)dy$$

ko'rinishida yozib olishimiz mumkin.

$R(x,y,\lambda)$ rezolventa $Q(a \le x \le b,\ a \le t \le b)$ yopiq sohada uzluksiz bo'ladi. Shu sababli, avvalgi formuladan $f(x)$ bilan bir qatorda (1) tenglamaning $\varphi(x)$ yechimining uzluksizligi kelib chiqadi.

Shunga o'xshash, Volterra (13) integral tenglamasining yechimini rezolventa orqali yozish qiyin emas. Shu maqsadda matematik analiz kursidan ma'lum bo'lgan Dirixle formulasini eslatib o'tamiz.

Faraz qilaylik, $f(x,y)$ funksiya $x=y,\ x=a,\ y=b$ to'g'ri chiziqlardan tashkil topgan teng yonli uchburchakda uzluksiz bo'lsin. U holda Δ bo'yicha olingan

$$J=\iint_{\Delta} f(x,y)dxdy$$

integralni ikki usul bilan hisoblash mumkin. Avval x o'zgaruvchi bo'yicha a dan y gacha, keyin y bo'yicha a dan b gacha integrallash mumkin, ya'ni

$$J=\int_a^b dy\int_a^y f(x,y)dx.$$

So'ngra y bo'yicha x dan b gacha, x o'zgaruvchi bo'yicha a dan b gacha integrallash mumkin, ya'ni

$$J=\int_a^b dx\int_x^b f(x,y)dy.$$

Oxirgi ikki tengliklardan

$$\int_a^b dy\int_a^y f(x,y)dx=\int_a^b dx\int_x^b f(x,y)dy$$

tenglik kelib chiqadi. Bu tenglik Dirixle formulasi deyiladi.

(13) tenglama uchun birinchi yaqinlashishni

$$\varphi_1(x)=f(x)+\lambda\int_a^x K(x,y)f(y)dy$$

formula bilan aniqlagan edik.

Ikkinchi yaqinlashish

$$\varphi_2(x)=f(x)+\lambda\int_a^x K(x,t)\varphi_1(t)dt=$$

$$=f(x)+\lambda\int_a^x K(x,t)\left[f(t)+\lambda\int_a^t K(t,y)f(y)dy\right]dt=$$

$$=f(x)+\lambda\int_a^x K(x,t)f(t)dt+\lambda^2\int_a^x K(x,t)dt\int_a^t K(t,y)f(y)dy$$

tenglik bilan aniqlanadi. Oxirgi ikkilangan integralga Dirixle formulasini qo'llaymiz:

$$\int_a^x K(x,t)dt\int_a^t K(t,y)f(y)dy.=\int_a^x f(y)dy\int_x^y K(x,t)K(t,y)dt$$

Agar

$$K_2(x,y)=\int_x^y K(x,t)K(t,y)dt$$

deb belgilasak,

$$\varphi_2(x)=f(x)+\lambda\int_a^x K(x,y)f(y)dy+\lambda^2\int_a^x K_2(x,y)f(y)dy$$

bo'ladi.

Bu jarayonni davom ettirib, xuddi Fredgol'm tenglamasidek,

$$\varphi_n(x) = f(x) + \lambda \int_a^x \sum_{i=1}^{n} \lambda^{n-1} K_i(x,y) f(y) dy \qquad (20)$$

tenglikka ega bo'lamiz, bunda

$$K_1(x,y) = K(x,y)$$

$$K_i(x,y) = \int_y^x K(x,t) K_{i-1}(t,y) dt, \qquad i = 2,3,\ldots$$

6.2 paragrafdagi mulohazalardan λ parametrning ixtiyoriy chekli qiymatida

$$\sum_{i=1}^{\infty} \lambda^{i-1} K_i(x,y)$$

qatorning absolyut va tekis yaqinlashishi kelib chiqadi. Bu qatorning yig'indisini $R(x,y,\lambda)$ orqali belgilab olamiz. Bu holda ham $R(x,y,\lambda)$ ga (13) Volterra tenglamasining rezolventasi deyiladi.

(20) tenglikda $n \to \infty$ deb limitda o'tib, (13) tenglamaning yechimini rezolventa orqali yozib olamiz:

$$\varphi(x) = f(x) + \lambda \int_a^x R(x,y,\lambda) f(y) dy.$$

Misol. Ushbu

$$u(x) = x + \int_0^x (t-x) u(t) dt$$

tenglama rezolventa usuli bilan yechilsin.

Quyidagilarni hisoblaymiz:

$$K_1 = K(x,t) = t - x = -(x-t),$$

$$K_2(x,t)=\int_t^x (x-s)(s-t)ds=\int_t^x (x-s)(x+s-t-x)ds=\int_t^x (x-s)[(x-t)-(x-s)]ds$$

$$=\int_t^x (x-s)[(x-t)-(x-s)]ds=(x-t)\int_t^x (x-s)ds-\int_t^x (x-s)^2 ds=-(x-t)\left[\frac{1}{2}(x-s)^2\right]_{s=t}^{x}+\frac{1}{3}\left[(x-3)^3\right]_{s=t}^{x}$$

$$=\frac{1}{2}(x-t)^3-\frac{1}{3}(x-t)^3=\frac{(x-t)^3}{3!}.$$

Xuddi shu kabi $K_3(x,t)$ ni topamiz:

$$K_3(x,t)=-\int_t^x (x-s)\frac{(s-t)^3}{3!}dt=-\frac{1}{3!}\int_t^x (x-t-s+t)(s-t)^3 ds=-\frac{(x-t)^5}{5!}$$

va hokazo. Bularni $\Gamma(x,t,\lambda)=K_1(x,t)+\lambda K_2(x,t)+\lambda^2 K_3(x,t)+\ldots$ formulaga qo‘yib, rezolventani hosil qilamiz:

$$\Gamma(x,t,\lambda)=-(x-t)+\frac{(x-t)^3}{3!}-\frac{(x-t)^5}{5!}+\ldots=-\sin(x-t).$$

U holda berilgan tenglamaning yechimi

$$u(x)=x-\int_0^x \sin(x-t)t\,dt$$

bo‘ladi. O‘ng tomondagi integralni hisoblab quyidagi natijani olamiz:

$$u(x)=\sin x.$$

Misol. Quyidagi tenglamaning iteratsiyalangan (takrorlangan) yadro yordamida rezolventasi va yechimini toping:

$$\varphi(x)-\lambda\int_0^1 xt\varphi(t)dt=f(x).$$

Yechish: Birin – ketin quyidagilarga ega bo‘lamiz:

$$K_1(x,t)=xt,$$

$$K_2(x,t)=\int_0^1 xs\cdot st\,ds=xt\frac{s^3}{3}\Big|_0^1=\frac{xt}{3},$$

$$K_3(x,t)=\frac{1}{3}\int_0^1 xs\cdot st\,ds=\frac{xt}{3^2},$$

....

$$K_n(x,t)=\frac{xt}{3^{n-1}}.$$

Agarda $|\lambda|<3$ bo'lsa, u holda rezolventa

$R(x,t,\lambda)=\sum_{n=1}^{\infty}K_n(x,t)\lambda^{n-1}=xt\sum_{n=1}^{\infty}\left(\frac{\lambda}{3}\right)^{n-1}=\frac{3xt}{3-\lambda}$ ga teng bo'ladi. Bundan

foydalanib yechimni ushbu $\varphi(x)=f(x)+\lambda\int_0^1\frac{3xt}{3-\lambda}f(t)dt$ ko'rinishda topamiz.

Misol. $\varphi(x)-\lambda\int_0^{2\pi}\sin(x-2t)\varphi(t)dt=f(x)$ tenglama ixtiyoriy chekli λ uchun yechimga ega ekanligini ko'rsating.

Yechish: $K(x,t)=\sin(x-2t)$ ekanligidan ikkinchi takroriy yadroni

topamiz
$$K_2(x,t)=\int_0^{2\pi}\sin(x-2s)\sin(s-2t)ds=\frac{1}{2}\int_0^{2\pi}[\cos(x+2t-3s)-\cos(x-2t-s)]ds=$$
$$=\frac{1}{2}\left(-\frac{1}{3}\sin(x+2t-3s)+\sin(x-2t-s)\right)\Bigg|_{s=0}^{s=2\pi}=0.$$

Bu yerdan barcha takroriy yadrolar uchun $K_n(x,t)=0$ bo'ladi. Shunday qilib, rezolventa $R(x,t)=\sin(x-2t)$ ko'rinishga ega va yechim ixtiyoriy chekli λ uchun $\varphi(x)=f(x)+\lambda\int_0^{2\pi}\sin(x-2t)f(t)dt$ bo'ladi.

Bu misoldagi yadro $x,t\in[0,2\pi]$ kesmada o'z – o'ziga ortogonaldir. O'z – o'ziga ortogonal yadrolar uchun ikkinchi takroriy yadro $K_2(x,t)=0$ bo'ladi va rezolventa integral tenglamaning yadrosi bilan ustma – ust tushadi.

6.4. Ajralgan yadroli Fredgol'm tenglamalari

Ta'rif. Agar (2) Fredgol'm ikkinchi tur tenglamasida ishtirok etayotgan yadro ushbu

$$K(x,t)=\sum_{i=1}^{n}a_i(x)b_i(t) \qquad (21)$$

ko'rinishga ega bo'lsa, bunday yadroga ajralgan (o'zgaruvchilari ajralgan) yadro deyiladi, $a_i(x)$ va $b_i(t)$ lar $[a,b]$ kesmada uzluksiz funksiyalar.

Ajralgan yadro uchun (2) integral tenglamani chiziqli algebraik tenglamalar sistemasiga keltirib yechish mumkin. Haqiqatan ham,

$$u(x)=f(x)+\lambda\int_a^b K(x,t)u(t)dt$$

tenglamaga (21) yadroni qo'yib, quyidagi ko'rinishdagi tenglamaga kelamiz:

$$u(x)=f(x)+\lambda\sum_{i=1}^{n}C_i a_i(x), \qquad (22)$$

bu yerda $C_i=\int_a^b b_i(t)u(t)dt$ – noma'lum sonlar.

Shunday qilib, ajralgan yadroli (2) tenglamaning yechimini (22) ko'rinishda qidirish kerak. Bu funksiyani (2) tenglamaga qo'yib, hosil bo'lgan tenglikning o'ng va chap tomonlaridagi $a_i(x)$ funksiyalar oldidagi ifodalarni har bir $i=1,2,...,n$ lar uchun tenglab, C_i larga nisbatan algebraik tenglamalar sistemasini hosil qilamiz:

$$C_i=\lambda\sum_{j=1}^{n}C_j\alpha_{ij}+\beta_i,\ i=1,2,...,n,$$

bu $\alpha_{ij}=\int_a^b a_i(t)b_j(t)dt$, $\qquad \beta_i=\int_a^b f(t)b_i(t)dt$.

Bu sistemani yechib, C_i larni va demak, (2) tenglamaning yechimi $u(x)$ funksiyani hosil qilamiz.

Bu usulni $n=3$ uchun batafsil bayon qilamiz. Bu holda $C_i, i=1,2,3$ lar quyidagicha aniqlanadi:

$$\int_a^b b_1(t)u(t)dt=C_1,\qquad \int_a^b b_2(t)u(t)dt=C_2,\qquad \int_a^b b_3(t)u(t)dt=C_3. \qquad (23)$$

Bu integrallardagi $u(t)$ funksiya noma'lum bo'lgani sababli, C_1, C_2 va C_3 lar ham noma'lum sonlar bo'lib, ularni topish talab qilinadi. Shu maqsadda (23) ni (22) ga $n=3$ uchun qo'yamiz:

$$u(x)=f(x)+\lambda a_1(x)C_1+\lambda a_2(x)C_2+\lambda a_3(x)C_3. \qquad (24)$$

(24) ifoda yordamida (23) tengliklarning birinchisini o'zgartiramiz:

$$C_1=\int_a^b b_1(t)u(t)dt=\int_a^b b_1(t)[f(t)+\lambda a_1(t)C_1+\lambda a_2(t)C_2+\lambda a_3(t)C_3]dt=$$

$$=\int_a^b b_1(t)f(t)dt+\lambda C_1\int_a^b b_1(t)a_1(t)dt+\lambda C_2\int_a^b b_1(t)a_2(t)dt+\lambda C_3\int_a^b b_1(t)a_3(t)dt. \quad (25)$$

O‘ng tomondagi aniq integrallar o‘zgarmas sonlar bo‘ladi va ularni quyidagicha belgilab olamiz:

$$\int_a^b b_1(t)f(t)dt=A_1, \qquad \int_a^b b_1(t)a_1(t)dt=a_{11},$$

$$\int_a^b b_1(t)a_2(t)dt=a_{12}, \qquad \int_a^b b_1(t)a_3(t)dt=a_{13}.$$

U holda (25) tenglik

$$C_1=A_1+\lambda C_1a_{11}+\lambda C_2a_{12}+\lambda C_3a_{13}$$

ko‘rinishiga keladi. Bundagi C_1, C_2, C_3 noma’lum sonlarni o‘z ichiga oluvchi hadlarni tenglik ishorasining bir tomoniga o‘tkazsak,

$$(1-\lambda a_{11})C_1-\lambda a_{12}C_2-\lambda a_{13}C_3=A_1$$

uch noma’lumli chiziqli algebraik tenglama hosil bo‘ladi.

Shunga o‘xshash yana ikkita algebraik tenglamani keltirib chiqarish uchun (23) tenglamalarning ikkinchi va uchinchisiga murojaat qilamiz:

$$C_2=\int_a^b b_2(t)u(t)dt=\int_a^b b_2(t)[f(t)+\lambda a_1(t)C_1+\lambda a_2(t)C_2+\lambda a_3(t)C_3]dt=$$

$$=\int_a^b b_2(t)f(t)dt+\lambda C_1\int_a^b b_2(t)a_1(t)dt+\lambda C_2\int_a^b b_2(t)a_2(t)dt+\lambda C_3\int_a^b b_2(t)a_3(t)dt.$$

Bundagi integrallarni quyidagicha belgilaymiz:

$$\int_a^b b_2(t)f(t)dt=A_2, \qquad \int_a^b b_2(t)a_1(t)dt=a_{21},$$

$$\int_a^b b_2(t)a_2(t)dt=a_{22}, \qquad \int_a^b b_2(t)a_3(t)dt=a_{23}.$$

U holda

$$C_2=A_2+\lambda C_1a_{21}+\lambda C_2a_{22}+\lambda C_3a_{23}$$

yoki

$$-\lambda a_{21}C_1+(1-\lambda a_{22})C_2-\lambda a_{23}C_3=A_2$$

hosil bo'ladi.

Xuddi shuningdek, (23) dan:

$$C_3=\int_a^b b_3(t)u(t)dt=\int_a^b b_3(t)[f(t)+\lambda a_1(t)C_1+\lambda a_2(t)C_2+\lambda a_3(t)C_3]dt.$$

Buni ham yuqoridagilar kabi o'zgartirsak, ushbu

$$-\lambda a_{31}C_1-\lambda a_{32}C_2+(1-\lambda a_{33})C_3=A_3$$

natija hosil bo'ladi; bunda

$$\int_a^b b_3(t)f(t)dt=A_3,\qquad \int_a^b b_3(t)a_1(t)dt=a_{31},$$

$$\int_a^b b_3(t)a_2(t)dt=a_{32},\qquad \int_a^b b_3(t)a_3(t)dt=a_{33}.$$

Shunday qilib, biz C_i larga nisbatan quyidagi chiziqli algebraik tenglamalar sistemasini hosil qildik:

$$\left.\begin{aligned}(1-\lambda a_{11})C_1-\lambda a_{12}C_2-\lambda a_{13}C_3=A_1\\ -\lambda a_{21}C_1+(1-\lambda a_{22})C_2-\lambda a_{23}C_3=A_2\\ -\lambda a_{31}C_1-\lambda a_{32}C_2+(1-\lambda a_{33})C_3=A_3\end{aligned}\right\}\qquad(26)$$

Bu sistemadagi A_i lar va a_{ij} lar ma'lum sonlardir, chunki ularga mos integrallar ishorasi ostidagi funksiyalar masalada berilgan bo'ladi.

Endi (26) sistemani oliy algebradagi Kramer formulalari yordamida yechamiz:

$$C_1=\frac{D_1}{D},\quad C_2=\frac{D_2}{D},\quad C_3=\frac{D_3}{D}.\qquad(27)$$

Bu formulalarda

$$D=\begin{vmatrix}1-\lambda a_{11} & -\lambda a_{12} & -\lambda a_{13}\\ -\lambda a_{21} & 1-\lambda a_{22} & -\lambda a_{23}\\ -\lambda a_{31} & -\lambda a_{32} & 1-\lambda a_{33}\end{vmatrix}\qquad(28)$$

Ma'lumki, D_1 ni topish uchun (28) determinantda birinchi ustun elementlari o'rniga (26) dagi A_1,A_2,A_3 ozod hadlarni qo'yish kerak, D_2 va D_3 lar ham shu usulda topiladi. Shuni ham ta'kidlab o'tishimiz zarurki, (26) sistemadagi A_1,A_2,A_3 larning kamida bittasi noldan farqli bo'lganda, (28) determinantning noldan farqli bo'lishi shart.

Demak, λ parametrning D determinantni nolga aylantirmaydigan hamma qiymatlari uchun (24) ko'rinishdagi yadroli (2) Fredgol'm tenglamalarini shu usulda yechish mumkin ekan. Shubhasiz, bu masalada ishtirok etayotgan barcha integrallar mavjud deb faraz qilinadi.

Misol. Ushbu tenglama yechilsin:

$$u(x) = x^2 + \lambda\int_0^1 (1+xt)u(t)dt.$$

Bu misoldagi λ parametr umumiy holda berilgan bo'lib, $K(x,t) = 1 + xt$ yadro yuqoridagi (21) ko'rinishda ifodalangan. Tenglamaning o'ng tomonidagi integralni ikkiga ajratib,

$$\int_0^1 (1+xt)u(t)dt = \int_0^1 u(t)dt + x\int_0^1 tu(t)dt$$

tenglikni hosil qilamiz.

So'ngra quyidagicha

$$C_1 = \int_0^1 u(t)dt, \quad C_2 = \int_0^1 tu(t)dt$$

kabi belgilashlar kiritamiz. U holda berilgan integral tenglama

$$u(x) = x^2 + \lambda C_1 + \lambda C_2 x$$

ko'rinishida yoziladi. Noma'lum funksiyaning bu ifodasidan foydalanib, C_1 bilan C_2 ni hisoblaymiz:

$$C_1 = \int_0^1 u(t)dt = \int_0^1 (t^2 + \lambda C_1 + \lambda C_2 t)dt =$$

$$= \left[\frac{1}{3}t^3 + \lambda C_1 t + \frac{1}{2}\lambda C_2 t^2\right]_0^1 = \frac{1}{3} + \lambda C_1 + \frac{1}{2}\lambda C_2$$

yoki

$$(1-\lambda)C_1 - \frac{1}{2}\lambda C_2 = \frac{1}{3}.$$

Xuddi shuningdek,

$$C_2 = \int_0^1 tu(t)dt = \int_0^1 t(t^2 + \lambda C_1 + \lambda C_2 t)dt =$$

$$= \left[\frac{1}{4}t^4 + \frac{1}{2}\lambda C_1 t^2 + \frac{1}{3}\lambda C_2 t^3\right]_0^1 = \frac{1}{4} + \frac{1}{2}\lambda C_1 + \frac{1}{3}\lambda C_2$$

yoki

$$-\frac{1}{2}\lambda C_1+(1-\frac{1}{3}\lambda)C_2=\frac{1}{4}.$$

Shunday qilib, quyidagi chiziqli algebraik tenglamalar sistemasi hosil bo'ldi:

$$\left.\begin{aligned}(1-\lambda)C_1-\frac{1}{2}\lambda C_2=\frac{1}{3},\\ -\frac{1}{2}\lambda C_1+(1-\frac{1}{3}\lambda)C_2=\frac{1}{4}.\end{aligned}\right\}$$

Bu sistemaning yechimini Kramer formulalariga asosan yozamiz:

$$C_1=\frac{D_1}{D},\quad C_2=\frac{D_2}{D};$$

Bu yerda

$$D=\begin{vmatrix}1-\lambda & -\frac{1}{2}\lambda\\ -\frac{1}{2}\lambda & 1-\frac{1}{3}\lambda\end{vmatrix}=\frac{1}{12}(\lambda^2-16\lambda+12)\neq 0,$$

$$D_1=\begin{vmatrix}\frac{1}{3} & -\frac{1}{2}\lambda\\ \frac{1}{4} & 1-\frac{1}{3}\lambda\end{vmatrix}=\frac{1}{72}(\lambda+24),$$

$$D_2=\begin{vmatrix}1-\lambda & \frac{1}{3}\\ -\frac{1}{2}\lambda & \frac{1}{4}\end{vmatrix}=\frac{1}{12}(3-\lambda).$$

Demak,

$$C_1=\frac{D_1}{D}=\frac{1}{6}\cdot\frac{\lambda+24}{\lambda^2-16\lambda+12},\qquad C_2=\frac{D_2}{D}=\frac{3-\lambda}{\lambda^2-16\lambda+12};$$

Bularni izlanayotgan noma'lum funksiyaning yuqoridagi ifodasiga qo'yib, uni quyidagi ko'rinishda yozamiz:

$$u(x)=x^2+\frac{\lambda(3-\lambda)}{\lambda^2-16\lambda+12}x+\frac{\lambda(\lambda+24)}{6(\lambda^2-16\lambda+12)}.$$

Bu esa berilgan masalaning yechimidir. Yechim ifodasidagi kasrlarning maxraji nolga teng bo'lmasligi uchun λ parametr

$$\lambda^2-16\lambda+12=0$$

Kvadrat tenglamaning ildizi bo'lmasligi shart, ya'ni $\lambda\neq 8\pm 2\sqrt{3}$. Xususiy holda $\lambda=2$ deb faraz qilsak, yechim quyidagicha yoziladi:

$$u(x)=x^2-\frac{x}{8}-\frac{13}{24}.$$

Misol. Ushbu tenglama yechilsin:

$$u(x)=f(x)+\lambda\int_0^{\pi}\cos(x+t)u(t)dt.$$

Ma'lumki,

$$\cos(x+t)=\cos x\cos t-\sin x\sin t$$

va demak, tenglamani

$$u(x)=f(x)+\lambda\cos x\int_0^{\pi}\cos tu(t)dt-\lambda\sin x\int_0^{\pi}\sin tu(t)dt=$$

$$=f(x)+\lambda\cos x\cdot C_1-\lambda\sin x\cdot C_2$$

ko'rinishida yozish mumkin; bunda

$$C_1=\int_0^{\pi}\cos tu(t)dt,\quad C_2=\int_0^{\pi}\sin tu(t)dt.$$

Bu integrallarda $u(t)$ o'rniga uning yuqorida olingan ifodasini qo'yamiz:

$$C_1=\int_0^{\pi}\cos t[f(t)+\lambda\cos tC_1-\lambda\sin tC_2]dt=$$

$$=\int_0^{\pi}\cos tf(t)dt+\lambda C_1\int_0^{\pi}\cos^2 tdt-\lambda C_2\int_0^{\pi}\cos t\cdot\sin tdt.$$

Integrallarning qiymatlari

$$\int_0^{\pi}\cos^2 tdt=\frac{\pi}{2};\quad \int_0^{\pi}\cos t\cdot\sin tdt=0.$$

bo'lgani uchun birinchi tenglama

$$(1-\frac{\lambda\pi}{2})C_1=A$$

ko'rinishda yoziladi. Bu yerda

$$A=\int_0^{\pi}\cos tf(t)dt.$$

Xuddi shu usulda C_2ni izlaymiz:

$$C_2=\int_0^{\pi}\sin t[f(t)+\lambda\cos tC_1-\lambda\sin tC_2]dt=$$

$$=\int_0^{\pi}\sin tf(t)dt-\lambda C_2\int_0^{\pi}\sin^2 tdt+\lambda C_1\int_0^{\pi}\cos t\cdot\sin tdt;$$

$$\int_0^{\pi}\sin^2 t dt=\frac{\pi}{2};$$

bo‘lgani uchun

$$(1+\frac{\lambda\pi}{2})C_2=B,$$

bu yerda

$$B=\int_0^{\pi}\sin t f(t)dt$$

va demak,

$$C_1=\frac{2}{2-\lambda\pi}A,\quad C_2=\frac{2}{2+\lambda\pi}B.$$

Izlanayotgan yechim quyidagidan iborat:

$$u(x)=f(x)+\frac{2\lambda\cos x}{2-\lambda\pi}A-\frac{2\lambda\sin x}{2+\lambda\pi}B.$$

Bu ifodadagi kasrlarning maxrajlari nolga aylanmasligi uchun $\lambda\neq\pm\frac{2}{\pi}$ bo‘lishi kerak. Xususiy holda, agar $\lambda=1,\ f(x)=x$ deb olsak,

$$A=\int_0^{\pi}t\cos t dt=-2,\quad B=\int_0^{\pi}t\sin t dt=\pi$$

bo‘lib, yechim uchun quyidagi ifoda hosil bo‘ladi:

$$u(x)=x-\frac{4}{2-\pi}\cos x-\frac{2\pi}{2+\pi}\sin x.$$

Mustaqil bajarish uchun misollar

a) $\varphi(x)=\lambda\int_0^1 K(x,y)\varphi(y)dy+f(x)$ integral tenglamani quyidagi hollar uchun yeching:

1. $K(x,y)=x-1,\ f(x)=x.$
2. $K(x,y)=2e^{x+y},\ f(x)=e^x.$
3. $K(x,y)=x+y-2xy,\ f(x)=x+x^2.$

b) $\varphi(x)=\lambda\int_{-1}^1 K(x,y)\varphi(y)dy+f(x)$ integral tenglamani quyidagi hollar uchun yeching:

4. $K(x,y)=xy+x^2y^2,\ f(x)=x^2+x^4.$
5. $K(x,y)=x^{\frac{1}{3}}+y^{\frac{1}{3}},\ f(x)=1-6x^2.$

6. $K(x,y)=x^4+5x^3y$, $f(x)=x^2-x^4$.

7. $K(x,y)=2xy^3+5x^2y^2$, $f(x)=7x^4+3$.

8. $K(x,y)=x^2-xy$, $f(x)=x^2+x$.

9. $K(x,y)=5+4xy-3x^2-3y^2+9x^2y^2$, $f(x)=x$.

c) $\varphi(x)=\lambda\int_0^{\pi}K(x,y)\varphi(y)dy+f(x)$ integral tenglamani quyidagi hollar uchun yeching:

10. $K(x,y)=\sin(2x+y)$, $f(x)=\pi-2x$.

11. $K(x,y)=\sin(x-2y)$, $f(x)=\cos 2x$

12. $K(x,y)=\cos(2x+y)$, $f(x)=\sin x$.

13. $K(x,y)=\sin(3x+y)$, $f(x)=\cos x$.

14. $K(x,y)=\sin y+y\cos x$, $f(x)=1-\frac{2x}{\pi}$.

15. $K(x,y)=\cos^2(x-y)$, $f(x)=1+\cos 4x$.

d) $\varphi(x)=\lambda\int_0^{2\pi}K(x,y)\varphi(y)dy+f(x)$ integral tenglamani quyidagi hollar uchun yeching:

16. $K(x,y)=\cos x\cdot\cos y+\cos 2x\cos 2y$, $f(x)=\cos 3x$

17. $K(x,y)=\cos x\cdot\cos y+2\sin 2x\cdot\sin 2y$, $f(x)=\cos x$

18. $K(x,y)=\sin x\cdot\sin y+3\cos 2x\cdot\cos 2y$, $f(x)=\sin x$

e) Quyidagi integral tenglamalarning barcha xarakteristik qiymatlarini va shu xarakteristikalarga mos funksiyalarini toping.

19. $\varphi(x)=\lambda\int_0^{2\pi}\left[\sin(x+y)+\frac{1}{2}\right]\varphi(y)dy$

20. $\varphi(x)=\lambda\int_0^{2\pi}\left[\cos^2(x+y)+\frac{1}{2}\right]\varphi(y)dy$

21. $\varphi(x)=\lambda\int_0^{1}\left[x^2y^2-\frac{2}{45}\right]\varphi(y)dy$

22. $\varphi(x)=\lambda\int_0^{2\pi}\left[\left(\frac{x}{y}\right)^{2/5}+\left(\frac{y}{x}\right)^{2/5}\right]\varphi(y)dy$

23. $\varphi(x)=\lambda\int_0^{2\pi}(\sin x\cdot\sin 4y+\sin 2x\cdot\sin 3y+\sin 3x\cdot\sin 2y+\sin 4x\cdot\sin y)\varphi(y)dy$

f)

24. a va b parametrlarning qanday qiymatlarida quyidagi tenglama yechimga ega va shu qiymatlardagi yechimini toping:

$$\varphi(x)=12\int_0^1\left(xy-\frac{x+y}{2}+\frac{1}{3}\right)\varphi(y)dy+ax^2+bx-2?$$

25. a parametrning qanday qiymatlarida quyidagi tenglama yechimga ega:

$$\varphi(x)=\sqrt{15}\int_0^1\left[y\left(4x^2-3x\right)+x(4y^2-3y)\right]\varphi(y)dy+ax+\frac{1}{x}?$$

26. λ parametrning qanday qiymatlarida

$$\varphi(x)=\lambda\int_0^{2\pi}\cos(2x-y)\varphi(y)dy+f(x)$$

integral tenglama har qanday $f(x)\in C([0,2\pi])$ uchun yechimga ega, shu yechimni toping.

g) Barcha λ va ozod hadga kiruvchi barcha *a, b, c* parametrlar uchun quyidagi integral tenglamalarning yechimini toping:

27. $$\varphi(x)=\lambda\int_{-\pi/2}^{\pi/2}(y\sin x+\cos y)\varphi(y)dy+ax+b$$

28. $$\varphi(x)=\lambda\int_0^{\pi}\cos(x+y)\varphi(y)dy+a\sin x+b$$

29. $$\varphi(x)=\lambda\int_{-1}^{1}(1+xy)\varphi(y)dy+ax^2+bx+c$$

30. $$\varphi(x)=\lambda\int_{-1}^{1}(x^2y+xy^2)\varphi(y)dy+ax+bx^3$$

31. $$\varphi(x)=\lambda\int_{-1}^{1}\frac{1}{2}(xy+x^2y^2)\varphi(y)dy+ax+b$$

32. $$\varphi(x)=\lambda\int_{-1}^{1}\left[5(xy)^{1/3}+7(xy)^{2/3}\right]\varphi(y)dy+ax+bx^{1/3}$$

33. $$\varphi(x)=\lambda\int_{-1}^{1}\frac{1+xy}{1+y^2}\varphi(y)dy+ax+bx^2$$

34. $$\varphi(x)=\lambda\int_{-1}^{1}\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\varphi(y)dy+ax^2+bx+c$$

35. $\varphi(x)=\lambda\int_{-1}^{1}(xy+x^2+y^2-3x^2y^2)\varphi(y)dy+ax+b$

36. $K(x,y)$ yadroning xos sonlarini va ularga mos keluvchi xos funksiyalarini toping va barcha λ,a,b,c lar uchun quyidagi tenglamani yeching

1. $K(x,y)=3x+xy-5x^2y^2$, $f(x)=ax$.
2. $K(x,y)=3xy+5x^2y^2$, $f(x)=ax^2+bx$.

37. $K(x,y)$ yadroning xos sonlarini va ularga mos keluvchi xos funksiyalarini toping va barcha λ,a,b,c lar uchun quyidagi tenglamani yeching:

$$\varphi(x)=\lambda\int_{-\pi}^{\pi}K(x,y)\varphi(y)dy+f(x)$$

1. $K(x,y)=x\cos y+\sin x$, $f(x)=a+b\cos x$.
2. $K(x,y)=x\sin y+\cos x$, $f(x)=ax+b$.

l) Quyidagi integral tenglamalarni yeching va $R(x,y;\lambda)$ rezolventasini toping:

38. $\varphi(x)=\lambda\int_{-1}^{1}\sin(x+y)\varphi(y)dy+f(x)$

39. $\varphi(x)=\lambda\int_{-1}^{1}(1-y+2xy)\varphi(y)dy+f(x)$

40. $\varphi(x)=\lambda\int_{-\pi}^{\pi}(x\sin y+\cos x)\varphi(y)dy+ax+b$

41. $\varphi(x)=\lambda\int_{0}^{2\pi}(\sin x\sin y+\sin 2x\sin 2yy)\varphi(y)dy+f(x)$

p) Har qanday λ parametr uchun quyidagi integral tenglamalar yechimga ega bo'ladigan a,b,c parametrlarning barcha qiymatlarini toping:

42. $\varphi(x)=\lambda\int_{-1}^{1}(xy+x^2y^2)\varphi(y)dy+ax^2+bx+c$

43. $\varphi(x)=\lambda\int_{-1}^{1}(1+xy)\varphi(y)dy+ax^2+bx+c$, bu yerda $a^2+b^2+c^2=1$.

44. $\varphi(x)=\lambda\int_{-1}^{1}\frac{1+xy}{\sqrt{1-y^2}}\varphi(y)dy+x^2+b$

45. $\varphi(x)=\lambda\int_{-1}^{1}(xy-\frac{1}{3})\varphi(y)dy+ax^2-bx+1$

46. $\varphi(x)=\lambda\int_{-1}^{1}(x+y)\varphi(y)dy+ax+b+1$

47. $\varphi(x)=\lambda\int_{0}^{2\pi}\cos(2x+4y)\varphi(y)dy+e^{ax+b}$

48. $\varphi(x)=\lambda\int_{-1}^{1}(\sin x\sin 2y+\sin 2x\sin 4y)\varphi(y)dy+ax^2+bx+c$

49. $\varphi(x)=\lambda\int_{-1}^{1}(1+x^2+y^3)\varphi(y)dy+ax+bx^3$

q) Quyidagi integral tenglamalarning xos sonlarini va ularga mos keluvchi xos funksiyalarini toping:

50. $\varphi(x_1,x_2)=\lambda\int_{-1}^{1}\int_{-1}^{1}\left[x_1+x_2+\frac{3}{32}(y_1+y_2)\right]\varphi(y_1,y_2)dy_1dy_2$

51. $\varphi(x)=\lambda\int_{|y|<1}(|x|^2+|y|^2)\varphi(y)dy, x=(x_1,x_2)$

52. $\varphi(x)=\lambda\int_{|y|<1}\frac{1+|y|}{1+|x|}\varphi(y)dy, x=(x_1,x_2,x_3)$

7-BOB. ELLIPTIK TURDAGI TENGLAMALAR

Ushbu bobda elliptik turdagi tenglamalar haqida umumiy ma'lumot berilgan bo'lib, ularga qo'yilgan korrekt masalalarni yechish o'rganilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

7.1. Umumiy tushunchalar va fundamental yechim

Issiqlik maydonlari (sterjen) qaralganda, bu maydonlarda issiqlik tarqalish masalalari ko'rilgan edi. U maydonlar statsionar bo'lmagan maydonlar bo'lib, issiqlik tarqalish jarayoni vaqtga bog'liq edi.

Endi issiqlik tarqalish jarayonini statsionar deb qaraymiz, ya'ni vaqt o'tishi bilan maydondagi temperatura o'zgarmaydi. Bunday maydonlar statsionar temperaturali maydonlar deyiladi.

a) Bir jinsli sterjenda issiqlik tarqalish jarayoni statsionar bo'lsin, u holda issiqlik tarqalish tenglamasida $\frac{\partial u}{\partial t}=0$ bo'lib tenglama

$$\frac{\partial^2 u}{\partial x^2}=0 \qquad (1)$$

ko'rinishga keladi.

Agar sterjenga doim issiqlik manbalari ta'sir etsa, tenglama

$$\frac{\partial^2 u}{\partial x^2}=-g \qquad (2)$$

ko'rinishda bo'ladi.

b) Bir jinsli membranada issiqlik tarqalish jarayoni statsionar bo'lsa, issiqlik tarqalish tenglamasi

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (3)$$

ko'rinishda yoziladi. Agar membranaga doimiy issiqlik manbalari ta'sir etsa, tenglama

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-g \qquad (4)$$

ko'rinishni oladi.

c) Bir jinsli qattiq jism uch o'lchovli fazoda qaralayotgan bo'lsa va unda issiqlik tarqalish jarayoni statsionar bo'lsa, u holda issiqlik tarqalish tenglamasi

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad (5)$$

bo'lib, agar unga issiqlik manbalari ta'sir etsa, tenglama ko'rinishi

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-g \qquad (6)$$

kabi bo'ladi.

Yuqorida yozilgan (1), (3), (5) tenglamalar mos ravishda bir, ikki, uch o'lchovli Laplas tenglamalari deyiladi. (2), (4), (6) tenglamalar esa bir, ikki, uch o'lchovli Puasson tenglamalari deyiladi.

S sirt bilan chegaralangan qandaydir D sohani qaraylik. D soha ichida u(x,y,z) temperaturaning statsionar tarqalish masalasi quyidagicha qo'yiladi:

D soha ichida $\Delta u=-f(x,y,z)$ tenglamani va quyidagi chegaraviy shartlardan bittasini:

I. $u=f_1$, S da (birinchi chegaraviy masala)

II. $\frac{\partial u}{\partial n}=f_2$, S da (ikkinchi chegaraviy masala)

III. $\frac{\partial u}{\partial n}+h(u-f_3)=0$, S da (uchinchi chegaraviy masala)

qanoatlantiruvchi u(x,y,z) funksiya topilsin, bunda keyingi tengliklarda n - S sirt o'tkazilgan normal, h - berilgan doimiy son, f_1, f_2, f_3 - berilgan funksiyalar.

Laplas yoki Puasson tenglamasiga qo‘yilgan 1-chegaraviy masalaga Dirixle masalasi, 2-chegaraviy masalaga esa Neyman masalasi deyiladi.

Δ orqali 2-tartibli xususiy hosilalarning quyidagi differensial operatorini belgilaymiz:

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Ushbu Δ differensial operator Laplas operatori,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \qquad (7)$$

tenglama – n o‘lchovli Laplas tenglamasi deyiladi.

(7) tenglamaga mos kvadratik xarakteristik forma quyidagicha aniqlanadi:

$$Q = \sum_{i=1}^{n} \lambda_i^2,$$

va bu forma E_n fazoning hamma nuqtalarida musbat aniqlangan. Bundan esa (7) tenglama E_n fazoda elliptik ekanligi kelib chiqadi.

<u>Ta’rif.</u> 2-tartibli uzluksiz xususiy hosilalarga ega bo‘lgan va Laplas tenglamasini qanoatlantiruvchi (ya’ni uning yechimi bo‘lgan) $u(x)$ funksiya garmonik funksiya deyiladi.

x va ξ o‘zgaruvchilarning funksiyasi bo‘lgan quyidagi funksiya ham x, ham ξ bo‘yicha Laplas tenglamasini qanoatlantirishini to‘g‘ridan-to‘g‘ri tekshirish mumkin:

$$E(x,\xi) = \begin{cases} \frac{1}{n-2}|\xi - x|^{2-n}, & n > 2, \\ -\log|\xi - x|, & n = 2, \end{cases} \qquad (8)$$

bu yerda, $|\xi - x| = \sqrt{(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + ... + (\xi_n - x_n)^2}$. Haqiqatan, $x \neq \xi$ bo‘lganda (8) dan quyidagini olamiz:

$$\frac{\partial^2 E}{\partial x_i^2} = -|\xi - x|^{-n} + n|\xi - x|^{-n-2}(\xi_i - x_i)^2. \qquad i = 1,2,...,n \qquad (9)$$

(9) ni etib (8) ga qo‘ysak quyidagiga ega bo‘lamiz:

$$\Delta E = -n|\xi - x|^{-n} + n|\xi - x|^{-n-2} \sum_{i=1}^{n} (\xi_i - x_i)^2 = 0.$$

$E(x,\xi)$ funksiya x va ξ o'zgaruvchilarga nisbatan simmetrik bo'lganligi uchun, bu funksiya ξ, $\xi \neq x$ o'zgaruvchi bo'yicha ham Laplas tenglamasini qanoatlantiradi.

(8) formula orqali aniqlangan $E(x,\xi)$ funksiya Laplas tenglamasining elementar yoki fundamental yechimi deyiladi.

$n=3$ bo'lgan holda fundamental yechim x (yoki ξ) nuqtada joylashgan birlik zaryadning potensialini bildiradi.

<u>**Masala.**</u> $u=u(x_1,...,x_n)$ garmonik funksiya berilgan $\frac{\partial u}{\partial x_1}\cdot\frac{\partial u}{\partial x_2}$, $(n=2)$ funksiya garmonik funksiya bo'lish yoki bo'lmasligini tushuntiring.

<u>**Yechish.**</u> $v(x_1,x_2)=\frac{\partial u}{\partial x_1}\cdot\frac{\partial u}{\partial x_2}$ belgilash kiritamiz. Garmonik funksiya ta'rifidan foydalanamiz. Funksiyadan o'zgaruvchilar bo'yicha ikkinchi tartibli xususiy hosilalarni olamiz:

$$\frac{\partial^2 v(x_1,x_2)}{\partial x_1^2}=\frac{\partial^3 u}{\partial x_1^3}\cdot\frac{\partial u}{\partial x_2}+2\frac{\partial^2 u}{\partial x_1^2}\cdot\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial u}{\partial x_1}\cdot\frac{\partial^3 u}{\partial x_2\partial x_1^2},$$

$$\frac{\partial^2 v(x_1,x_2)}{\partial x_2^2}=\frac{\partial^3 u}{\partial x_2^3}\cdot\frac{\partial u}{\partial x_1}+2\frac{\partial^2 u}{\partial x_2^2}\cdot\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial u}{\partial x_1}\cdot\frac{\partial^3 u}{\partial x_1\partial x_2^2}.$$

Laplas tenglamasini qanoatlantirishini ko'rsatamiz,

$$\Delta v=\frac{\partial^2 v(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 v(x_1,x_2)}{\partial x_1^2}=\frac{\partial^3 u}{\partial x_1^3}\cdot\frac{\partial u}{\partial x_2}+2\frac{\partial^2 u}{\partial x_1^2}\cdot\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial u}{\partial x_1}\cdot\frac{\partial^3 u}{\partial x_2\partial x_1^2}+$$

$$+\frac{\partial^3 u}{\partial x_2^3}\cdot\frac{\partial u}{\partial x_1}+2\frac{\partial^2 u}{\partial x_2^2}\cdot\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial u}{\partial x_2}\cdot\frac{\partial^3 u}{\partial x_1\partial x_2^2}=\frac{\partial u}{\partial x_2}\left(\frac{\partial^3 u}{\partial x_1^3}+\frac{\partial^3 u}{\partial x_1\partial^2 x_2}\right)+\frac{\partial u}{\partial x_1}\left(\frac{\partial^3 u}{\partial x_2\partial^2 x_1}+\frac{\partial^3 u}{\partial x_2^3}\cdot\frac{\partial u}{\partial x_1}\right)+$$

$$+2\frac{\partial^2 u}{\partial x_1\partial x_2}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)=\frac{\partial u}{\partial x_2}\cdot\frac{\partial}{\partial x_1}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+\frac{\partial u}{\partial x_1}\cdot\frac{\partial}{\partial x_2}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+2\frac{\partial^2 u}{\partial x_1\partial x_2}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right),$$

masala shartiga asosan

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=0,$$

bundan $\frac{\partial^2 v(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 v(x_1,x_2)}{\partial x_1^2}=0$, ya'ni $\Delta v=0$. Demak, berilgan funksiya garmonik.

<u>**Masala.**</u> $x_1^3+kx_1x_2^2$ berilgan funksiya garmonik bo'ladigan k doimiyning qiymatini toping.

Yechish. $u(x_1,x_2)=x_1^3+kx_1x_2^2$ belgilash kiritamiz. Garmonik funksiya ta'rifidan foydalanamiz. Funksiyadan o'zgaruvchilar bo'yicha ikkinchi tartibli xususiy hosilalarni olamiz:

$$\frac{\partial^2 u(x_1,x_2)}{\partial x_1^2}=6x_1,\quad \frac{\partial^2 u(x_1,x_2)}{\partial x_2^2}=2kx_1$$

Laplas tenglamasini qanoatlantiradi, ya'ni

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=0,$$

bundan

$$6x_1+2kx_1=0,$$

$$k=-3.$$

Demak, $k=-3$ bo'lganda berilgan funksiya garmonik funksiya bo'ladi.

Mustaqil bajarish uchun mashqlar

1. Laplas operatorining quyidagi koordinatalar sistemasidagi ifodasini toping.

a) egri chiziqli koordinatalarda

$$x=\varphi(\xi,\eta),\ y=\psi(\xi,\eta).$$

b) qutb koordinatalarida

$$x=r\cos\varphi,\ y=r\sin\varphi$$

c) silindrik koordinatalarida

$$x=r\cos\varphi,\ y=r\sin\varphi,\ z=z$$

d) sferik koordinatalarida

$$x=r\sin\upsilon\cos\varphi,\ y=r\sin\upsilon\sin\varphi,\ z=r\cos\upsilon$$

e) yassi sferoidal koordinatalarida

$$x=\xi\eta\sin\varphi,\ y=\sqrt{(\xi^2-1)(1-\eta^2)},\ z=\xi\eta\cos\varphi.$$

2. $u=u(x_1,\dots,x_n)$ garmonik funksiya berilgan quyida yozilgan funksiyalardan qaysi biri garmonik funksiya bo'lish yoki bo'lmasligini tushuntiring.

a) $u(x+h), h=(h_1,\dots,h_n)$-doimiy vektor;

b) $u(\lambda x), \lambda$ – skalyar doimiy;

c) $u(Cx), C$ – doimiy ortogonal matrisa;

d) $\frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2}, n = 2;$

e) $\frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2}, n > 2;$

f) $x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + x_3 \frac{\partial u}{\partial x_3}, n = 3;$

j) $x_1 \frac{\partial u}{\partial x_1} - x_2 \frac{\partial u}{\partial x_2}, n = 2;$

h) $x_2 \frac{\partial u}{\partial x_2} - x_1 \frac{\partial u}{\partial x_1}, n = 2;$

k) $\dfrac{\frac{\partial u}{\partial x_1}}{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2}, n = 2;$

l) $\left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2, n = 2;$

m) $\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2, n = 2.$

3. Quyida garmonik bo‘lgan funksiyalar berilgan. k doimiyning qiymatini toping.

a) $x_1^3 + kx_1x_2^2;$

b) $x_1^2 + x_2^2 + kx_3^2;$

c) $e^{2x_1} ch kx_2;$

d) $\sin 3x_1 ch kx_2;$

e) $\frac{1}{|x|^k}, |x|^2 = \sum_{i=1}^{n} x_i^2, |x| \neq 0.$

4. $u(x, y)$ funksiyani garmonik deb faraz qilsak, $\varphi(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ funksiyaning analitik bo‘lishini ko‘rsating.

5. $u(x,y)=\operatorname{Re} f(z)$, ya`ni $u(x,y)$ funksiya $f(z)$ funksiyaning haqiqiy qismiga teng bo'lsa, D sohada egri chiziq integralidan foydalanib, $f(z)$ ning analitik bo'lishini keltirib chiqaring, agar:

5.1. $u=x^3-3xy^2$.

5.2. $u=e^x \sin y$.

5.3. $u=\sin x ch y$.

6. $u_x-v_y=0, \quad u_y+v_x=0$ Koshi-Riman tenglamalar sistemasidan foydalanib, u(x,y) funksiya bilan qo'shma garmonik bog`langan v(x,y) funksiyani toping:

a) $u(x,y)=xy^3-yx^3$;

b) $u(x,y)=e^y \sin x$;

c) $u(x,y)=sh x \sin y$;

d) $u(x,y)=ch x \cos my$;

e) $u(x,y)=sh x \cos y$;

f) $u(x,y)=ch x \sin y$.

7. Koshi-Riman tenglamalar sistemasidan foydalanib, u(x,y) garmonik funksiyani toping:

a) $u_x(x,y)=3x^2y-y^3$;

b) $u_y(x,y)=e^x \cos y$;

c) $u_x(x,y)=e^x \sin y$;

d) $u_y(x,y)=x^2-y^2+x+y$;

e) $u_x(x,y)=xy+x^2-y^2$.

8. Agar:

a) $u_y=e^x \cos z-2y$;

b) $u_x=sh x \cos z+2xy$;

c) $u_z=xy^2-xz^2+6xz+x$;

d) $u_z=e^x(x\cos y-y\sin y)+2z$.

bo'lsa, $u=u(x,y,z)$ garmonik funksiyani toping.

9. Agar:

a) $u_x(x,y)=y^3-3x^2y$;

b) $u_y(x,y)=e^y\cos x$;

c) $u_y(x,y)=shx\sin y$;

d) $u_x(x,y)=chx\sin y$;

e) $u_x(x,y)=xy$.

bo'lsa, u(x,y) garmonik funksiyaga bog`liq bo'lgan v(x,y) funksiyani toping.

7.2. Chegaraviy masalalarni doirada va doira tashqarisida Furye usuli bilan yechish

Doira uchun Dirixle masalasi:

$D=\{\rho^2=x^2+y^2<a^2\}$doirada

$$\Delta u=0 \tag{10}$$

ikki o'lchovli Laplas tenglamasining

$$u|_{\rho=a}=f \tag{11}$$

birinchi chegaraviy shartni qanoatlantiruvchi yechimini topish masalasini ko'raylik, bu yerda f berilgan funksiya.

Dastlab $S=\{x^2+y^2=a^2\}$ aylanada $f\in C^1$ deb faraz qilaylik (keyinchalik bu shartni olib tashlaymiz).

Markazi doira markazida bo'lgan (ρ,φ) qutb koordinatalar sistemasini kiritamiz. Unda (10) tenglama

$$\Delta u=\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\cdot\frac{\partial^2 u}{\partial\varphi^2}=0 \tag{12}$$

ko'rinishini oladi.

(12), (11) masalani o'zgaruvchilarni ajratish usuli bilan yechamiz ya`ni (12) tenglama yechimini $u(\rho,\varphi)=R(\rho)\cdot\Phi(\varphi)\neq 0$ ko'rinishda izlaymiz. Bundan esa

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+\lambda\Phi(\varphi)=0,\ \ \Phi(\varphi)\neq 0 \tag{13}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right)-\lambda R(\rho)=0,\ R\ \ (\rho)\neq 0 \qquad (14)$$

tenglamani hosil qilamiz. (13) tenglamaning yechimi $\Phi(\varphi)=A\cos n\sqrt{\lambda}\varphi+B\sin n\sqrt{\lambda}\varphi$ bo‘lib, bu yerda A,B ixtiyoriy o‘zgarmaslar. Ko‘rinib turibdiki, φ burchak 0 dan 2π gacha o‘zgarganda bir qiymatli $u(\rho,\varphi)$funksiya yana o‘z qiymatiga qaytishi kerak, ya`ni $u(\rho,\varphi+2\pi)=u(\rho,\varphi)$ yechim davriy bo‘ladi. Bundan esa $\Phi(\varphi+2\pi)=\Phi(\varphi)$ ham davri 2π bo‘lgan davriy funksiya bo‘ladi. Bu esa faqat $\sqrt{\lambda}=n$ -butun son bo‘lsagina mumkin va

$$\Phi_n(\varphi)=A_n\cos n\varphi+B_n\sin n\varphi.$$

Endi $R(\rho)$funksiyaga nisbatan Eyler tenglamasi hosil bo‘ladi, uning yechimini $R(\rho)=\rho^{\mu}$ko‘rinishda izlaymiz. Buni (14) tenglamaga qo‘yamiz va ρ^{μ} ga qisqartirib, $n^2=\mu^2$ yoki $\mu=\pm n\ (p>0)$ tenglikni olamiz. Demak $R(\rho)=C\rho^{n}+D\rho^{-n}$ bunda C,D - ixtiyoriy o‘zgarmaslar.

Agar $D\neq 0$ bo‘lsa $\rho\to 0$ da $u=R(\rho)\Phi(\varphi)\to\infty$ va $u(\rho,\varphi)$ funksiya sohaning ichida garmonik bo‘lmaydi. Shu sababli, agar ichki masalani qarayotgan bo‘lsak $R(\rho)=C\rho^{n}$ ya’ni $\mu=n$ deb olish maqsadga muvofiq bo‘ladi. Shuningdek, tashqi masala uchun $R(\rho)=D\rho^{-n},\ (\mu=-n)$ deb olish kerak, chunki tashqi masalada yechim $\rho\to\infty$ chegaralangan bo‘lishi kerak.

Shunday qilib, ichki va tashqi Dirixle masalalarining xususiy yechimlari mos ravishda $\rho\leq a$ bo‘lganda: $u_n(\rho,\varphi)=\rho^{n}(A_n\cos n\varphi+B_n\sin n\varphi)$ va $\rho\geq a$ bo‘lganda: $u_n(\rho,\varphi)=\rho^{n}(A_n\cos n\varphi+B_n\sin n\varphi)$ bo‘ladi.

Shuni ham ta`kidlash lozimki, $\rho=0$ nuqtada (12) Laplas operatori ma`nosini yo‘qotadi. Biz $\Delta u_n=0$ tenglik, $\rho=0$ da ham bajarili ishni ko‘rsatish uchun $\rho^n\cos n\varphi$ va $\rho^n\sin n\varphi$ xususiy yechimlar $\rho^n e^{in\varphi}=(\rho e^{i\varphi})^n=(x+iy)^n$ funksiyaning haqiqiy va mavhum qismlari ekanligidan foydalanamiz. Bu x va y ga nisbatan ko‘p had bo‘lib, $\rho>0$ da $\Delta u=0$ hamda uzluksiz ikki marta differensiallanuvchi bo‘lgani uchun $\rho=0$ da ham $\Delta u=0$ tenglama chiziqli bo'lgani uchun bu xususiy yechimlar **yig'indisi** ham mos masalalar yechimi bo'ladi: $u(\rho,\varphi)=\sum_{n=0}^{\infty}\rho^{n}(A_n\cos n\varphi+B_n\sin n\varphi)$ ichki masala uchun;

$u(\rho,\varphi)=\sum_{n=0}^{\infty}\rho^{-n}(A_n\cos n\varphi+B_n\sin n\varphi)$ tashqi masala uchun.

A_n va B_n koeffisiyentlarni aniqlash uchun (11) chegaraviy shartdan foydalanamiz:

$$u(a,\varphi)=\sum_{n=0}^{\infty}a^{n}(A_n\cos n\varphi+B_n\sin n\varphi)=f(\varphi) \qquad (15)$$

va $f(\varphi)$ funksiyaning Furye qatoriga yoyilmasini yozamiz (uni mavjud degan faraz bilan)

$$f(\varphi)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}(\alpha_n\cos n\varphi+\beta_n\sin n\varphi) \qquad (16)$$

bu yerda $\alpha_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\psi)\cos n\psi\, d\psi$ $(n=0,1,2,\ldots)$,

(17)

$$\beta_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\psi)\sin n\psi\, d\psi\ (n=0,1,2,\ldots) \qquad (18)$$

(15) va (16) qatorlarni tenglashtirib, ichki masala uchun:

$$A_0=\frac{\alpha_0}{2},\ A_n=\frac{\alpha_n}{a^n},\ B_n=\frac{\beta_n}{a^n},$$

tashqi masala uchun esa

$$A_0=\frac{\alpha_0}{2},\ A_n=a^n\alpha_n,\ B_n=a^n\beta_n$$

qiymatlarni topamiz.

Shunday qilib, doirada Dirixlening ichki masalasi uchun

$$u(\rho,\varphi)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^n(\alpha_n\cos n\varphi+\beta_n\sin n\varphi) \qquad (19)$$

yechimni, tashqi masala uchun esa

$$u(\rho,\varphi)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\left(\frac{a}{\rho}\right)^n(\alpha_n\cos n\varphi+\beta_n\sin n\varphi) \qquad (20)$$

yechimni hosil qilamiz.

Bu funksiyalar haqiqatan ham izlanayotgan yechim bo'lishini ko'rsatish uchun, ularni hadma-had differensiallab, hosil bo'lgan qatorlar ham yaqinlashuvchi bo'lishini hamda chegarada uzluksiz bo'lib, chegaraviy qiymatni qabul qilishini isbotlash lozim bo'ladi. (19), (20) qatorlarni bitta ko'rinishda yozib olamiz:

$$u(\rho,\varphi)=\sum_{n=1}^{\infty} t^n(\alpha_n \cos n\varphi+\beta_n \sin n\varphi)+\frac{\alpha_0}{2}, \quad (21)$$

bu yerda $t=\begin{cases}\frac{\rho}{a}\le 1 & (\rho\le a - ichki),\\ \frac{a}{\rho}\le 1 & (\rho\ge a - tashqi),\end{cases}$

α_n, β_n lar esa $f(\varphi)$ funksiyaning Furye koeffisiyentlari.

(19), (20) qatorlarni $t<1$ bo'lganda istagancha differensiallash mumkin. Qatorning umumiy hadi $u_n=t^n(\alpha_n \cos n\varphi+\beta_n \sin n\varphi)$ ni qaraylik hamda uni φ bo'yicha k marta differensiallaymiz:

$$\frac{\partial^k u_n}{\partial\varphi^k}=t^n n^k\left[\alpha_n \cos\left(n\varphi+\frac{\pi k}{2}\right)+\beta_n \sin\left(n\varphi+\frac{\pi k}{2}\right)\right]$$

Agar $|\alpha_n|<M$, $|\beta_n|<M$ desak, quyidagi bahoga ega bo'lamiz

$$\left|\frac{\partial^k u_n}{\partial\varphi^k}\right|<t^n\cdot n^k\cdot 2M.$$

Birorta $\rho_0<a$ (ichki masala uchun) va $\rho_1=\frac{a^2}{\rho_0}>a$ (tashqi masala uchun) qiymatlarni fiksirlaymiz, bunda $t_0=\frac{\rho_0}{a}<1$ bo'ladi va ushbu qatorni qaraymiz $\sum_{n=1}^{\infty} t^n n^k(|\alpha_n|+|\beta_n|)\le 2M\sum_{k=1}^{\infty} t_0^n\cdot n^k \quad (t<t_0)$

Ko'ramizki, bu qator ixtiyoriy chekli k uchun $t<t_0<1$ bo'lganda yaqinlashadi. Shuning uchun (19), (20) qatorlarni mos ravishda ichki, tashqi nuqtasida k marta differensiallash mumkin.

Xuddi shunga o'xshash ko'rsatish mumkinki, (19) va (20) qatorlarni $\rho_0<a$ va $\rho_1>a$ (doiraning ichi va tashqarisida) mos ravishda ρ o'zgaruvchi bo'yicha ham istagancha differensiallash mumkin. Fiksirlangan ρ_0 ning ixtiyoriyligidan esa (19) va (20) qatorlarni doiraning mos ravishda ichki va tashqi nuqtasida differensiallash mumkin bo'ladi, hadma-had hosila olish mumkinligidan esa, superpozitsiya prinsipini qo'llash o'rinli ekanligi kelib chiqadi. Demak, koeffisiyentlari (17) va (18) formulalar bilan aniqlanadigan (19) va (20) funksiyalar (10) tenglamani va (11) chegaraviy shartni mos ravishda doiraning ichida va tashqi sohasida qanoatlantiradi.

__Masala.__ $x^2+y^2=r^2<R^2$ doirada Dirixle masalasini yeching:

$$\Delta u(x,y)=0,\ 0\le r<R,$$

$$u(x,y)=g(x,y),\ r=R$$

bu yerda, $g(x,y)=2x^2-x-y$.

__Yechish.__ Yechim (19) qator ko'rinishida bo'ladi, koeffisiyentlari (17) va (18) formulalar yordamida aniqlanadi. $g(x,y)$ funksiyani qutb koordinatalar sistemasida yozib olamiz: $g(r,\varphi)=2r^2\cos^2\varphi-r\cos\varphi-r\sin\varphi$ va $r=R$ da $g=2R^2\cos^2\varphi-R\cos\varphi-R\sin\varphi$ bo'ladi.

$$\alpha_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(\psi)\cos n\psi d\psi=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2R^2\cos^2\psi-R\cos\psi-R\sin\psi\right)\cos n\psi d\psi$$

$$\beta_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(\psi)\sin n\psi d\psi=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2R^2\cos^2\psi-R\cos\psi-R\sin\psi\right)\sin n\psi d\psi$$

$$\alpha_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2R^2\frac{1+\cos 2\psi}{2}-R\cos\psi-R\sin\psi\right)d\psi=2R^2$$

$$\alpha_1=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2R^2\cos^2\psi-R\cos\psi-R\sin\psi\right)\cos\psi d\psi=-R$$

$$\alpha_2=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2R^2\cos^2\psi-R\cos\psi-R\sin\psi\right)\cos 2\psi d\psi=R^2$$

$$\beta_1=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2R^2\cos^2\psi-R\cos\psi-R\sin\psi\right)\sin\psi d\psi=-R$$

Qolgan barcha koeffisiyentlarning qiymatlari nolga teng. Topilgan natijalarni (19) qatorga qo'yib, berilgan masalaning yechimini olamiz:

$$u(r,\varphi)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(\alpha_n\cos n\varphi+\beta_n\sin n\varphi)=R^2-r\cos\varphi+r^2\cos 2\varphi-r\sin\varphi.$$

Oxirgi tenglikda dekart koordinatalar sistemasiga o'tamiz va masalaning yechimini olamiz:

$$u(x,y)=R^2-x+x^2-y^2-y.$$

<u>**Masala.**</u> $a\le r<b, 0\le\varphi\le 2\pi$ halqa ichida quyidagi $u(r)$ chegaraviy masalalarning yechimlarini toping: $\Delta u(r)=0, u(a)=T, u(b)=U.$

<u>**Yechish.**</u> Bir o'lchovli holda Laplas tenglamasi quyidagicha:

$$\Delta u(r)=\frac{1}{r}\left(\frac{d}{dr}\left(r\frac{du}{dr}\right)\right)=0.$$

Tenglamaning yechimi: $u(r)=C_1\ln r+C_2$. C_1,C_2 larni chegaraviy shartlardan topamiz:

$$u(a)=C_1\ln a+C_2=T$$
$$u(b)=C_1\ln b+C_2=U$$
$$C_1=\frac{T-U}{\ln\frac{a}{b}}$$
$$C_2=T-\frac{T-U}{\ln\frac{a}{b}}\ln a$$

Demak, yechim quyidagicha:

$$u(r)=T+\frac{T-U}{\ln\frac{a}{b}}\ln\frac{r}{a}$$

bo'ladi.

10. $x^2+y^2=r^2<R^2$ doirada Dirixle masalasini yeching:

$\Delta u(x,y)=0,\ 0\le r<R,$

$u(x,y)=g(x,y),\ r=R.$

Bu yerda:

a) $g(x,y)=x+xy;$

b) $g(x,y)=2(x^2+y);$

c) $g(x,y)=4y^3;$

d) $g(x,y)=x^2-2y^2;$

e) $g(x,y)=4xy^2;$

f) $g(x,y)=\frac{1}{R}y^2+Rxy;$

g) $g(x,y)=2x^2-x-y$.

11. $x^2+y^2=r^2<R^2$ doiradan tashqarida Drixle masalasini yeching:

$$\Delta u(x,y)=0,\quad R<r<\infty,$$
$$u(x,y)=g(x,y),\quad r=R.\ |u(x,y)|<\infty$$

Bu yerda:

a) $g(x,y)=y+2xy$;

b) $g(x,y)=ax+by+c$;

c) $g(x,y)=x^2-y^2$;

d) $g(x,y)=x^2+1$;

e) $g(x,y)=y^2-xy$;

f) $g(x,y)=y^2+x+y$;

g) $g(x,y)=2x^2-x+y$.

12. $x^2+y^2=r^2<R^2$ doirada Puasson tenglamasiga qo'yilgan Drixle masalasini yeching:

$$\Delta u(x,y)=f(x,y), 0\le r<R,$$
$$u(x,y)=g(x,y), r=R.$$

Agar:

a) $f(x,y)=1, g(x,y)=0$;

b) $f(x,y)=x, g(x,y)=0$;

c) $f(x,y)=-1, g(x,y)=\frac{y^2}{2}$;

d) $f(x,y)=y, g(x,y)=1$;

e) $f(x,y)=4, g(x,y)=1$.

13. $x^2+y^2=r^2<R^2$ doirada to'g'ri qo'yilgan Neyman masalasining

$$\Delta u(x,y)=0,\quad 0\le r<R,$$
$$\frac{\partial u(x,y)}{\partial y}=g(x,y),\quad r=R.$$

bajarilish shartini toping.

Agar:

a) $g(x,y)=A$;

b) $g(x,y)=2x^2+A$;

c) $g(x,y)=2xy;$

d) $g(x,y)=Ay^2-B;$

e) $g(x,y)=Ax^2+By^2+y.$

bo‘lsa, to‘g`ri qo‘yilgan masalaning yechimini toping, bu yerda A,B – doimiy.

14. $x^2+y^2=r^2<R^2$ doira tashqarisida g(x,y) funksiya uchun to‘g`ri qo‘yilgan Neyman masalasining yechimini toping

$\Delta u(x,y)=0,\quad R<r<\infty,$

$\frac{\partial u(x,y)}{\partial y}=g(x,y),\ \ r=R,\ \ |u(x,y)|<\infty$

Agar:

a) $g(x,y)=y^2-A;$

b) $g(x,y)=x^2+Ay-B;$

c) $g(x,y)=2xy-Ay+B,$

d) $g(x,y)=x^2-Ay^2+B;$

bo‘lsa, masalaning yechimini toping, bu yerda A,B – doimiy.

15. Agar quyidagilar berilgan bo‘lsa, $K:0\le r<R, 0\le\varphi\le\pi$ doirada $u(R,\varphi)-u(R_1,\varphi)=f(\varphi)$ shartni qanoatlantiruvchi $u(r,\varphi)\in C^1(K)$ garmonik funksiyani toping, bu yerda $0<R_1<R, \int_0^{2\pi} f(\varphi)d\varphi=0;$

a) $f(\varphi)=\sin\varphi;$

b) $f(\varphi)=\cos\varphi;$

c) $f(\varphi)=\cos^2\varphi+C;$

d) $f(\varphi)=\sin 2\varphi+\cos 3\varphi;$

e) $f(\varphi)=A\cos^2\varphi+B\sin\varphi,$

f) $f(\varphi)=\sin\varphi-3\cos^2\varphi+C;$

bu yerda A,B,C – doimiy.

16. $K: 0 \le r \le R, 0 \le \varphi \le \pi$ doira tashqarisida $u(R,\varphi) - u(R_1,\varphi) = f(\varphi)$ shartni qanoatlantiruvchi $u(r,\varphi) \in C^1(C\bar{K})$ garmonik funksiyani toping, bu yerda $0 < R_1 < R, \int_0^{2\pi} f(\varphi) d\varphi = 0;$

a) $f(\varphi) = 3\sin 2\varphi;$

b) $f(\varphi) = 5\sin^2\varphi - A;$

c) $f(\varphi) = \sin^3\varphi + 2;$

d) $f(\varphi) = \sin\varphi + 3\cos^2\varphi - A;$

e) $f(\varphi) = \sin\varphi + \cos 5\varphi;$

A – doimiy

17. $\Delta u(x,y,z) = 0,$
$u(x,y,0) = g(x,y), \quad u_z(x,y,0) = h(x,y)$

Laplas tenglamasiga qo‘yilgan Koshi masalasini yeching.

(Ko‘rsatma: $u(x) = \sum_{k=0}^{\infty} (-1)^k \left[\frac{x_n^{2k}}{(2k)!} \Delta^k \tau(x_1,...,x_{n-1}) + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^k \nu(x_1,...,x_{n-1}) \right]$

formuladan foydalaning, bu yerda $\tau(x_1,...,x_{n-1}), \nu(x_1,...,x_{n-1})$ – boshlang‘ich shartda berilgan funksiyalar.)

Agar:

a) $g(x,y) = x + 2y, h(x,y) = 2x - y^2;$

b) $g(x,y) = xe^y, h(x,y) = 0;$

c) $g(x,y) = xy + x^2, h(x,y) = e^x + y;$

d) $g(x,y) = x\sin y, h(x,y) = \cos y;$

e) $g(x,y) = x^3 + 2, h(x,y) = 2x^2 - y;$

f) $g(x,y) = \cos 2x, h(x,y) = x - 2\sin 2y;$

bo‘lsa.

$a \le r < b, 0 \le \varphi \le 2\pi$ **halqa ichida quyidagi u(r) chegaraviy masalalarning yechimlarini toping.**

18. $\Delta u(r) = 0, u(a) = T, u(b) = U.$

19. $\Delta u(r) = 0, u(a) = T, u_r(b) = U.$

20. $\Delta u(r) = 0, u_r(a) = T, u(b) = U.$

21. $\Delta u(r) = 0, u_r(a) = T, u_r(b) = U.$

22. $\Delta u(r)=0, u(a)=T, u_r(b)+hu(b)=U.$

23. $\Delta u(r)=0, u_r(a)-hu(a)=T, u(b)=U.$

24. $\Delta u(r)=0, u_r(a)=T, u_r(b)+hu(b)=U.$

25. $\Delta u(r)=0, u_r(a)-hu(a)=T, u_r(b)=U.$

26. $\Delta u(r)=0, u_r(a)-hu(a)=T, u_r(b)+hu(b)=U.$

27. $\Delta u(r)=0, u(a)=T, u(c)=hu(b), a<c<b, h\neq 0.$

28. $K: x^2+y^2+2x<0$ aylanada

$\Delta u(x,y)=f(x,y), (x,y)\in K,$

$u(x,y)=g(x,y), (x,y)=\partial K,$

masalani yeching, agar:

a) $f(x,y)=0, g(x,y)=4x^3+6x-1;$

b) $f(x,y)=0, g(x,y)=x^2+2y;$

c) $f(x,y)=0, g(x,y)=2y^2-x;$

d) $f(x,y)=4, g(x,y)=2xy+1;$

e) $f(x,y)=24y, g(x,y)=y.$

7.3. Chegaraviy masalalarni to'rtburchak sohada Furye usuli bilan yechish

Elliptik turdagi tenglamalarga to'rtburchak sohada qo'yilgan chegaraviy masalalarni, tor tebranish va issiqlik o'tkazuvchanlik tenglamalariga qo'yilgan aralash masalalarni Furye usulida yechish algoritmi asosida yechiladi.

Masala. Laplas tenglamasiga $0<x<p, 0<y<s$ to'g'ri to'rtburchak sohada qo'yilgan chegaraviy masalani yeching:

$u(0,y)=u_x(p,y)=0,\ u(x,0)=0,\ u(x,s)=f(x);$

Yechish. Ikki o'lchovli Laplas tenglamasi quyidagicha:

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Berilgan masalaning yechimini quyidagi ko'rinishda qidiramiz:

$$u(x,y)=X(x)\cdot Y(y).$$

bu yerda $X(x)$ - x o'zgaruvchining funksiyasi, $Y(y)$ - y o'zgaruvchining funksiyasi. Ular uchun quyidagi tenglamalar hosil bo'ladi:

$$X''(x)+\lambda X(x)=0$$
$$Y''(y)-\lambda Y(y)=0$$

chegaraviy shartlardan foydalansak, $X(x)$ funksiya uchun quyidagi ko'rinishni oladi: $X(0)=0, X'(p)=0$.

Natijada masalani yechsak: $X_n(x)=\sin\frac{2n+1}{p}\pi x$,

$$Y_n(y)=a_n e^{\frac{2n+1}{p}\pi y}+b_n e^{-\frac{2n+1}{p}\pi y}.$$

Masalaning yechimi: $u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)=\sum_{n=0}^{\infty}\left(a_n e^{\frac{2n+1}{p}\pi y}+b_n e^{-\frac{2n+1}{p}\pi y}\right)\sin\frac{2n+1}{p}\pi x.$

Qolgan chegaraviy shartlardan foydalanib, yig'indidagi noma'lum koeffisiyentlar uchun quyidagi tenglamalar sistemasini olamiz:

$$\sum_{n=0}^{\infty}(a_n+b_n)=0,$$
$$\sum_{n=0}^{\infty}\left(a_n e^{\frac{2n+1}{p}\pi s}+b_n e^{-\frac{2n+1}{p}\pi s}\right)\sin\frac{2n+1}{p}\pi x=f(x),$$

bundan,

$$b_n=-a_n,$$
$$\sum_{n=0}^{\infty}a_n\left(e^{\frac{2n+1}{p}\pi s}-e^{-\frac{2n+1}{p}\pi s}\right)\sin\frac{2n+1}{p}\pi x=f(x).$$

Oxirgi tenglik $f(x)$ funksiyaning Furye qatoriga yoyilmasini beradi.

Demak, berilgan masalaning yechimi:

$$u(x,y)=\sum_{n=0}^{\infty}\left(a_n sh\left(\frac{2n+1}{p}\pi y\right)\sin\left(\frac{2n+1}{p}\pi x\right)\right).$$

bu yerda, $a_n=\frac{1}{p\cdot sh\left(\frac{2n+1}{p}\pi s\right)}\int_0^p f(x)\sin\left(\frac{2n+1}{p}\pi x\right)dx.$

29. Laplas tenglamasiga $0<x<p, 0<y<s$ to'g'ri to'rtburchak sohada qo'yilgan chegaraviy masalani yeching:

a) $u(0,y)=u_x(p,y)=0, u(x,0)=0, u(x,s)=f(x);$

b) $u_x(0,y)=u_x(p,y)=0, u(x,0)=A, u(x,s)=Bx;$

c) $u_x(0,y)=u(p,y)=0, u(x,0)=0, u_y(x,s)=Bx;$

d) $u(0,y)=U, u_x(p,y)=0, u_y(x,0)=T\sin\frac{\pi x}{2p}, u(x,s)=0;$

e) $u(0,y)=0, u_x(p,y)=q, u(x,0)=0, u_y(x,s)=U;$

f) $u(0,y)=0, u(p,y)=Ty, u(x,0)=0, u_y(x,s)=\frac{sTx}{p}.$

30. $0<x<\infty, 0<y<l$ yarim tekislikda chegaraviy shartlarni qanoatlantiruvchi Laplas tenglamasining yechimi:

a) $u(x,0)=u_y(x,l)=0, u(0,y)=f(y), u(\infty,y)=0;$

b) $u(x,0)=u_y(x,l)+hu(x,l)=0,$
$u(0,y)=f(y),\ u(\infty,y)=0, h>0;$

c) $u(x,0)=u(x,l)=0,\ u(0,y)=y(l-y),\ u(\infty,y)=0;$

d) $u_y(x,0)-hu(x,0)=0,\ u(x,l)=0,$
$u(0,y)=l-y, u(\infty,y)=0,\ h>0.$

31. $0<r<R$ doirada quyidagi chegaraviy qiymatlarni qanoatlantiruvchi garmonik funksiyani toping:

a) $u(R,\varphi)=\varphi(2\pi-\varphi);$

b) $u(R,\varphi)=\varphi\sin\varphi;$

c) $u_r(R,\varphi)+hu(R,\varphi)=T+Q\sin\varphi+U\cos 3\varphi;$

d) $u_r(R,\varphi)=f(\varphi).$

32. $0<r<R$ doira tashqarisida quyidagi Laplas tenglamasiga qo'yilgan $u=u(r,\varphi)$ chegaraviy masalani yeching:

a) $u(R,\varphi)=T\sin\frac{\varphi}{2};$

b) $u(R,\varphi)=\frac{1}{2}+\varphi\sin 2\varphi;$

c) $u_r(R,\varphi)+hu(R,\varphi)=f(\varphi);$

d) $u_r(R,\varphi)=U(\varphi+\cos\varphi).$

33. $a<r<b$ halqa ichida chegaraviy qiymatlarni qanoatlantiruvchi $u=u(r,\varphi)$ garmonik funksiyani toping:

a) $u(a,\varphi)=0, u(b,\varphi)=A\cos\varphi;$

b) $u(a,\varphi)=A, u(b,\varphi)=B\sin 2\varphi;$

c) $u(a,\varphi)=q\cos\varphi, u(b,\varphi)=Q+T\sin 2\varphi;$

d) $u(a,\varphi)=T+U\cos\varphi, u_r(b,\varphi)=hu(b,\varphi)=0;$

34. $a<r<b, 0<\varphi<\alpha$ doira sektorida chegaraviy shartlarni qiymatlarni qanoatlantiruvchi garmonik funksiyani toping:

a) $u(r,0)=u(r,\alpha)=0, u(R,\varphi)=A\varphi;$

b) $u_\varphi(r,0)=u(r,\alpha)=0, u(R,\varphi)=f(\varphi);$

c) $u_\varphi(r,0)=u_\varphi(r,\alpha)=0, u(R,\varphi)=U\varphi,$

d) $u(r,0)=u(r,\alpha)=0, u_r(R,\varphi)=Q;$

e) $u(r,0)=u_\varphi(r,\alpha)+hu(r,\alpha), u_r(R,\varphi)+\gamma u(R,\varphi)=0.$

8-BOB. GIPERBOLIK SISTEMALAR

Ushbu bobda xususiy hosilali differensial tenglamalar sistemasi haqida umumiy ma'lumotlar berilib, birinchi tartibli xususiy hosilali differensial tenglamalar sistemasining klassifikatsiyasi, kanonik ko'rinishga keltirish, umumiy yechimini topish, shuningdek, giperbolik sistemalarga qo'yilgan Koshi va aralash masalalarni yechish o'rganilgan. Mavzuga doir mustaqil yechish uchun misol va masalalar keltirilgan.

8.1. Umumiy tushunchalar. Giperbolik sistemalarni kanonik ko'rinishga keltirish va umumiy yechimini topish

Quyidagi tenglamalar sistemasi berilgan bo'lsin

$$\begin{cases} A_{11}(x,t)\dfrac{\partial u_1}{\partial t}+A_{12}(x,t)\dfrac{\partial u_2}{\partial t}+B_{11}(x,t)\dfrac{\partial u_1}{\partial x}+B_{12}(x,t)\dfrac{\partial u_2}{\partial x}=f_1(x,t), \\ A_{21}(x,t)\dfrac{\partial u_1}{\partial t}+A_{22}(x,t)\dfrac{\partial u_2}{\partial t}+B_{21}(x,t)\dfrac{\partial u_1}{\partial x}+B_{22}(x,t)\dfrac{\partial u_2}{\partial x}=f_2(x,t) \end{cases} \quad (1)$$

Bu yerda, $A_{11}(x,t), A_{12}(x,t), B_{11}(x,t), B_{12}(x,t), A_{21}(x,t), A_{22}(x,t), B_{21}(x,t), B_{22}(x,t)$ – sistema koeffisiyentlari, $f_1(x,t), f_2(x,t)$ – ozod hadlar bo'lib, berilgan funksiyalar. $u_1(x,t), u_2(x,t)$ – noma'lum funksiyalar. (1) sistemani matritsaviy shaklda yozib olamiz, bu uchun quyidagi belgilashlar kiritamiz:

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}; \quad B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}; \quad F=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}; \quad u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Natijada, berilgan (1) sistema quyidagi ko'rinishni oladi:

$$A\frac{\partial u}{\partial t}+B\frac{\partial u}{\partial x}=F \quad (1)$$

$u_1(x,t), u_2(x,t)$ funksiyalarning to'la differensiallarini yozamiz:

$$\begin{cases} du_1 = \dfrac{\partial u_1}{\partial t}dt + \dfrac{\partial u_1}{\partial x}dx, \\ du_2 = \dfrac{\partial u_2}{\partial t}dt + \dfrac{\partial u_2}{\partial x}dx. \end{cases} \qquad (2)$$

(1) va (2) sistemani birgalikda qaraymiz:

$$\begin{cases} A_{11}\dfrac{\partial u_1}{\partial t} + A_{12}\dfrac{\partial u_2}{\partial t} + B_{11}\dfrac{\partial u_1}{\partial x} + B_{12}\dfrac{\partial u_2}{\partial x} = f_1, \\ A_{21}\dfrac{\partial u_1}{\partial t} + A_{22}\dfrac{\partial u_2}{\partial t} + B_{21}\dfrac{\partial u_1}{\partial x} + B_{22}\dfrac{\partial u_2}{\partial x} = f_2, \\ dt\dfrac{\partial u_1}{\partial t} + dx\dfrac{\partial u_1}{\partial x} = du_1, \\ dt\dfrac{\partial u_2}{\partial t} + dx\dfrac{\partial u_2}{\partial x} = du_2. \end{cases} \qquad (*)$$

Hosil bo'lgan (*) sistema $\frac{\partial u_1}{\partial t}, \frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial t}, \frac{\partial u_2}{\partial x}$ noma'lumlarga nisbatan chiziqli tenglamalar sistemasini tashkil qiladi. (*) tenglamalar sistemasining matrisaviy shakli quyidagicha:

$$\begin{cases} A\dfrac{\partial u}{\partial t} + B\dfrac{\partial u}{\partial x} = f \\ dtE\dfrac{\partial u}{\partial t} + dxE\dfrac{\partial u}{\partial x} = du \end{cases}$$

(*) tenglamalar sistemasi noldan farqli yechimga ega bo'lishi uchun

$$\begin{vmatrix} A & B \\ dtE & dxE \end{vmatrix} \neq 0$$

bo'lishi kerak.

Ta'rif.

$$\begin{vmatrix} A & B \\ dtE & dxE \end{vmatrix} = 0 \qquad (3)$$

tenglikni qanoatlantiruvchi chiziqlar (1) ***sistemaning xarakteristikalari*** deyiladi.

Xarakteristikalar ustida munosabatni aniqlash uchun quyidagi kengaytirilgan matritsani qarashimiz kerak

$$\begin{pmatrix} A & B & f \\ dtE & dxE & du \end{pmatrix}.$$

Agar ushbu matritsaning rangi asosiy matritsaning rangiga teng bo'lsa, u holda ***xarakteristikalar ustida munosabat aniqlangan deyiladi***, ya'ni

$$r\begin{pmatrix} A & B & f \\ dtE & dxE & du \end{pmatrix} = r\begin{pmatrix} A & B \\ dtE & dxE \end{pmatrix}.$$

Misol.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial t} + \rho_0 c_0^2 \dfrac{\partial u}{\partial x} = 0 \end{cases}$$

Akustika tenglamalari sistemasining xarakteristikalarini aniqlang va xarakteristikalar ustida munosabatni quring. Sistemaning umumiy yechimini toping.

Yechish. Berilgan sistemani matrisaviy shaklda yozib olamiz:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} u \\ p \end{pmatrix} + \begin{pmatrix} 0 & \dfrac{1}{\rho_0} \\ \rho_0 c_0^2 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Bundan,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ B = \begin{pmatrix} 0 & \dfrac{1}{\rho_0} \\ \rho_0 c_0^2 & 0 \end{pmatrix};\ f = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\ U = \begin{pmatrix} u \\ p \end{pmatrix}$$

$$A\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} = f$$

Ta'rifdan foydalanib xarakteristikalarini aniqlaymiz:

$$\begin{vmatrix} 1 & 0 & 0 & \dfrac{1}{\rho_0} \\ 0 & 1 & \rho_0 c_0^2 & 0 \\ dt & 0 & dx & 0 \\ 0 & dt & 0 & dx \end{vmatrix} = 0$$

$$dt\begin{vmatrix} 1 & 0 & \dfrac{1}{\rho_0} \\ 0 & \rho_0 c_0^2 & 0 \\ dt & dx & 0 \end{vmatrix} + dx\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho_0 c_0^2 \\ dt & 0 & dx \end{vmatrix} = -c_0^2 dt^2 + dx^2 = dx^2 - c_0^2 dt^2 = 0.$$

$$(dx - c_0 dt)(dx + c_0 dt) = 0,$$

$$\frac{dx}{dt}=c_0,\quad x-c_0t=const,$$

$$\frac{dx}{dt}=-c_0,\quad x+c_0t=const.$$

Demak, tovush tarqalish tenglamalari sistemasi quyidagi xarakteristikalarga ega: $x-c_0t=const$, $x+c_0t=const$

Endi ushbu tenglamaning kengaytirilgan matritsasini yozib olamiz:

$$\left(\begin{array}{cccc|cc} 1 & 0 & 0 & \frac{1}{\rho_0} & 0 \\ 0 & 1 & \rho_0c_0^2 & 0 & 0 \\ dt & 0 & dx & 0 & du \\ 0 & dt & 0 & dx & dp \end{array}\right)$$

Xarakteristikalar ustida munosabat quramiz, bu uchun kengaytirilgan matrisaning 4-tartibli ixtiyoriy minorini hisoblaymiz (faqat oxirgi qator o'chirilmaydi!):

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \rho_0c_0^2 & 0 \\ dt & 0 & dx & du \\ 0 & dt & 0 & dp \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \rho_0c_0^2 & 0 \\ dt & 0 & dx & du \\ 0 & dt & 0 & dP \end{vmatrix}=\begin{vmatrix} 1 & \rho_0c_0^2 & 0 \\ 0 & dx & du \\ dt & 0 & dp \end{vmatrix}=dxdp+\rho_0c_0^2dtdu=0.$$

$$\frac{dx}{dt}dp+\rho_0c_0^2du=0.$$

$$\frac{dx}{dt}=c_0 \Rightarrow c_0dp+\rho_0c_0^2du=0,\ d(p+\rho_0c_0u)=0,$$

Berilgan sistema uchun $x-c_0t=const$ xarakteristika ustida munosabat quyidagicha: $c_0dp+\rho_0c_0^2du=0$.

$$\frac{dx}{dt}=-c_0 \Rightarrow -c_0dp+\rho_0c_0^2du=0,\ d(p-\rho_0c_0u)=0$$

Berilgan sistema uchun $x+c_0t=const$ xarakteristika ustida munosabat quyidagicha:

$$-c_0dp+\rho_0c_0^2du=0.$$

Xarakteristika munosabatlardan foydalanib berilgan sistemaning umumiy yechimini topish mumkin:

$$p+\rho_0 c_0 u=f_1(x-c_0 t); \quad p-\rho_0 c_0 u=f_2(x+c_0 t).$$

Yuqoridagi tenglamar sistemasidan noma'lum $U=\begin{pmatrix} u \\ p \end{pmatrix}$ -vertor funksiyani topamiz:

Natijada berilgan akustika tenglamalari sistemasining umumiy yechimi quyidagicha:

$$u=\frac{1}{2\rho_0 c_0}\left(f_1(x+c_0 t)-f_2(x+c_0 t)\right),$$

$$p=\frac{1}{2}\left(f_1(x-c_0 t)+f_2(x+c_0 t)\right).$$

Barcha xarakteristikalari haqiqiy va turlicha bo'lgan ***sistemalar giperbolik sistemalar*** deyiladi.

Giperbolik sistemalarga Koshi masalasi xususiy hosilalari differensial tenglamalarga qo'yilgan kabi $t=0$ da Ox o'qining biror bir intervalida qo'yiladi.

Quyidagi

$$A\frac{\partial U}{\partial t}+B\frac{\partial U}{\partial x}=0 \tag{4}$$

sistemani qaraylik. Agar $t=0$ chiziq xarakteristika bo'lmasa $\frac{\partial U}{\partial t}$ hosila sohaning barcha nuqtalarida mavjud bo'ladi. Faraz qilaylik

$$\det\|A\|\neq 0.$$

A matritsaga teskari A^{-1} mavjud (4) sistemaning chap tomonini A^{-1} ga ko'paytiramiz

$$A^{-1}A\frac{\partial U}{\partial t}+A^{-1}B\frac{\partial U}{\partial x}=0.$$

$A^{-1}A=E, A^{-1}B=C$ deb belgilasak,

$$\frac{\partial U}{\partial t}+C\frac{\partial U}{\partial x}=0 \tag{5}$$

Agar (4) sistemamiz bir jinsli bo'lmasa, ya'ni

$$A\frac{\partial U}{\partial t}+B\frac{\partial U}{\partial x}=f \tag{4'}$$

$A^{-1}f=g$ bilan belgilasak,

$$\frac{\partial U}{\partial t}+B\frac{\partial U}{\partial x}=g \tag{5'}$$

sistemaga kelamiz.

Biz asosan (5) ko'rinishidagi sistemalarni qaraymiz.

$$\begin{vmatrix} E & C \\ dtE & dxE \end{vmatrix} = 0$$

ko'rinishida bo'ladi.

Faraz qilaylik, tenglamalar sistemasi n o'zgaruvchili bo'lsin. dt ni determinantdan tashqariga chiqaramiz:

$$dt^n \begin{vmatrix} E & C \\ E & \frac{dx}{dt}E \end{vmatrix} = 0,$$

$$dt^n \begin{vmatrix} 0 & C - \frac{dx}{dt}E \\ E & \frac{dx}{dt}E \end{vmatrix} = 0.$$

$$dt^n (-1)^n \left| C - \frac{dx}{dt}E \right| = 0$$

ga kelamiz.

$$\frac{dx}{dt} = k_i(x,t) \qquad (6)$$

(6) tenglik bilan aniqlanadigan chiziqlar ***xarakteristikalar*** deyiladi.

Xarakteristikani aniqlayotganda k_i qiymatlar

$$|C - kE| = 0 \qquad (7)$$

tenglikdan topiladi.

(7) tenglamaning ildizlari k_i lar haqiqiy va turlicha bo'lsa, u holda qarayotgan sistemamiz giperbolik sistema deyiladi.

Quyidagi sistemani qaraylik:

$$A(x,t,U)\frac{\partial U}{\partial t} + B(x,t,U)\frac{\partial U}{\partial x} = f(x,t,U) \qquad (8)$$

(8) ko'rinishidagi sistemaga ***Kvazichiziqli sistema*** deyiladi.

Chiziqli tenglamalar sistemasi uchun aytilgan mulohazalar Kvazichiziqli sistemalar uchun ham o'rinli.

Misol. Berilgan giperbolik sistemani kanonik ko'rinishga keltiring va umumiy yechimini toping:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0. \end{cases}$$

<u>Yechish.</u> Berilgan sistemani matrisaviy shaklda yozib olamiz:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Bunda,

$$A=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\ B=\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix},\ \mathrm{U}=\begin{pmatrix}u\\v\end{pmatrix},\ f=\begin{pmatrix}0\\0\end{pmatrix}.$$

Xarakteristik tenglamani yechamiz:

$$|B-kE|=\left|\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}-\begin{pmatrix}k & 0\\0 & k\end{pmatrix}\right|=0.$$

Ildizlari: $k_1=1,\ k_2=-1$.

B matrisaning xos vektorlarini topamiz:

$$(B-k_i E)z=0,$$

$$z_1=\begin{pmatrix}1\\-1\end{pmatrix},\quad z_2=\begin{pmatrix}1\\1\end{pmatrix}.$$

Xos vektorlardan quyidagi matrisani tuzamiz:

$$Z=\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$$

Ushbu matrisaga teskari matrisa:

$$Z^{-1}=\frac{1}{2}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}$$

$U=ZV$ almashtirish yordamida tenglama kanonik ko‘rinishga keladi:

$$\frac{\partial V}{\partial t}+K\frac{\partial V}{\partial x}=0,$$

bu yerda, $K=Z^{-1}BZ=\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$, $V=\begin{pmatrix}v_1\\v_2\end{pmatrix}$.

Berilgan tenglamaning kanonik ko‘rinishi quyidagicha:

$$\begin{cases}\dfrac{\partial v_1}{\partial t}-\dfrac{\partial v_1}{\partial x}=0\\[2mm]\dfrac{\partial v_2}{\partial t}+\dfrac{\partial v_2}{\partial x}=0\end{cases}$$

Ushbu sistemaning yechimi: $\begin{array}{l}v_1(x,t)=f_1(x+t),\\ v_2(x,t)=f_2(x-t).\end{array}$

Dastlabki sistemaning yechimini quyidagi tenglikdan aniqlaymiz:

$$U = ZV$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Bundan, berilgan sistemaning umumiy yechimi:

$$u(x,t) = f_1(x+t) - f_2(x-t),$$
$$v(x,t) = f_1(x+t) + f_2(x-t).$$

Mustaqil bajarish uchun mashqlar

Berilgan sistemalarning xarakteristikalarini aniqlang:

1. $\begin{cases} u_x - b\upsilon_x - c\upsilon_y = 0, \\ u_y - a\upsilon_x + b\upsilon_y = 0 \end{cases}$

2. $yu_{xx} + u_{yy} = 0$

3. $yu_{yy} - u_{xx} = 0$

4. $\begin{cases} xu_{xx} + 2x\upsilon_{xx} - u_{yy} - 2\upsilon_{yy} = (x+y)^2 u, \\ u_{xx} - \upsilon_{xx} - 2u_{xy} + u_{yy} - v_{yy} = 0; \end{cases}$

5. $\begin{cases} u_{xx} - 2\upsilon_{xy} - u_{yy} = 0, \\ \upsilon_{xx} + 2u_{xy} - \upsilon_{yy} = 0 \end{cases}$

$\omega = u + i\upsilon,\ z = x + iy,\ \bar{z} = x - iy\ \ \omega_{\bar{z}\bar{z}} = 0$

6. $x^2 u_{xx} - y^2 u_{yy} - 2yu_y = 0;$

7. $\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = x^2 \frac{\partial^2 u}{\partial y^2};$

8. $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

9. $(1+x^2)u_{xx} - (1+y^2)u_{yy} + xu_x + yu_y = 0;$

10. $\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right) = 0$

11. $\frac{\partial}{\partial x}\left(u_x e^{-u_x^2-u_y^2}\right) + \frac{\partial}{\partial y}\left(u_y e^{-u_x^2-u_y^2}\right) = 0.$

12. $\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2}}\right) = 0.$

13. $$\begin{cases}\dfrac{\partial u}{\partial t}+\alpha\dfrac{\partial u}{\partial x}+\beta\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}+\gamma\dfrac{\partial u}{\partial x}+\delta\dfrac{\partial \upsilon}{\partial x}=0.\end{cases};$$

14. $$\frac{\partial}{\partial x}(\rho\cdot\varphi_x)+\frac{\partial}{\partial y}(\rho\cdot\varphi_y)=0,\ \rho=\rho\left(\sqrt{\varphi_x^2+\varphi_y^2}\right)$$

15. $$\begin{cases}\dfrac{\partial u}{\partial y}-\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial}{\partial x}(\rho\cdot u)+\dfrac{\partial}{\partial y}(\rho\cdot\upsilon)=0,\end{cases}$$

$$\rho=\left(1-\upsilon^2-u^2\right)^{\sigma},\ \sigma=const.$$

16. $$\begin{cases}\dfrac{\partial p}{\partial t}+\rho_0c_0\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial \upsilon}{\partial y}\right)=0,\\ \dfrac{\partial u}{\partial t}+\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}+\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial y}=0\end{cases}$$

17. $$\begin{cases}\dfrac{\mu}{c_0}\cdot\dfrac{\partial H_1}{\partial t}+\dfrac{\partial E_3}{\partial y}-\dfrac{\partial E_2}{\partial z}=0,\\ \dfrac{\mu}{c_0}\cdot\dfrac{\partial H_2}{\partial t}+\dfrac{\partial E_1}{\partial z}-\dfrac{\partial E_3}{\partial x}=0,\\ \dfrac{\mu}{c_0}\cdot\dfrac{\partial H_3}{\partial t}+\dfrac{\partial E_2}{\partial x}-\dfrac{\partial E_1}{\partial y}=0,\\ \dfrac{\varepsilon}{c_0}\cdot\dfrac{\partial E_1}{\partial t}-\dfrac{\partial H_3}{\partial y}+\dfrac{\partial H_2}{\partial z}=0,\\ \dfrac{\varepsilon}{c_0}\cdot\dfrac{\partial E_2}{\partial t}-\dfrac{\partial H_1}{\partial z}+\dfrac{\partial H_3}{\partial x}=0,\\ \dfrac{\varepsilon}{c_0}\cdot\dfrac{\partial E_3}{\partial t}-\dfrac{\partial H_2}{\partial x}+\dfrac{\partial H_1}{\partial y}=0\end{cases}$$

18. $$\frac{\partial\psi}{\partial t}=A_1\frac{\partial\psi}{\partial x}+A_2\frac{\partial\psi}{\partial y}+A_3\frac{\partial\psi}{\partial z}+mA_4\psi$$

$$A_1=\begin{Vmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{Vmatrix};\ A_2=\begin{Vmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{Vmatrix};\ A_3=\begin{Vmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{Vmatrix};\ A_4=\begin{Vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix}$$

19. $$\begin{cases}\dfrac{\partial u}{\partial t}+2\dfrac{\partial u}{\partial x}+\upsilon=0,\\ \dfrac{\partial \upsilon}{\partial t}-\dfrac{\partial \upsilon}{\partial x}+u=0;\end{cases}$$

20. $$\begin{cases} \frac{1}{\upsilon}\cdot\frac{\partial\varphi_0}{\partial t}+\frac{\partial\varphi_1}{\partial r}+\frac{\varphi_1}{r}+\alpha_0\varphi_0=q_0, \\ \frac{3}{\upsilon}\cdot\frac{\partial\varphi_1}{\partial t}+\frac{\partial\varphi_0}{\partial r}+3\alpha_1\varphi_1=0, \end{cases} \quad v=const$$

21. $$\begin{cases} (1+x^2)\frac{\partial u_1}{\partial t}+\frac{\partial u_2}{\partial t}+\frac{x}{t}(1+x^2)\frac{\partial u_1}{\partial x}+\frac{x}{t}\frac{\partial u_2}{\partial x}+\frac{2x^2}{t}u_1=0, \\ \frac{\partial u_2}{\partial t}-\frac{t}{x}\frac{\partial u_2}{\partial x}=0; \end{cases}$$

22. $$\begin{cases} \frac{\partial(P+\tau\cos 2\psi)}{\partial x}+\frac{\partial(\tau\sin 2\psi)}{\partial y}=0, \\ \frac{\partial(\tau\sin 2\psi)}{\partial x}-\frac{\partial(P-\tau\cos 2\psi)}{\partial y}=0, \end{cases} \quad \tau=\tau(P)$$

23. $$\begin{cases} u_y-\upsilon_x=0, \\ (c^2-u^2)u_x-u\upsilon(u_y+\upsilon_x)+(c^2-\upsilon^2)\upsilon_y=0 \end{cases}$$

24. $u_t+Au_x=0,$

$$A=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3t^2 \end{Vmatrix};$$

25. $$\begin{cases} \frac{\partial u_0}{\partial t}+\frac{\partial u_1}{\partial x}=0, \\ \frac{\partial u_1}{\partial t}+\frac{2}{3}\cdot\frac{\partial u_2}{\partial x}+\frac{1}{3}\cdot\frac{\partial u_0}{\partial x}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial u_k}{\partial t}+\frac{k+1}{2k+1}\cdot\frac{\partial u_{k+1}}{\partial x}+\frac{k}{2k+1}\cdot\frac{\partial u_{k-1}}{\partial x}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial u_{N-1}}{\partial t}+\frac{N}{2N-1}\cdot\frac{\partial u_N}{\partial x}+\frac{N-1}{2N-1}\cdot\frac{\partial u_{N-2}}{\partial x}=0, \\ \frac{\partial u_N}{\partial t}+\frac{N}{2N+1}\cdot\frac{\partial u_{N-1}}{\partial x}=0 \end{cases}$$

Giperbolik sistemalarni kanonik ko'rinishga keltiring:

26. $$\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial \upsilon}{\partial x}=0, \\ \frac{\partial \upsilon}{\partial t}-\frac{\partial u}{\partial x}=0; \end{cases}$$

27. $$\begin{cases} 2u_t+(2t-1)u_x-(2t+1)\upsilon_x=0, \\ 2\upsilon_t-(2t+1)u_x+(2t-1)\upsilon_x=0; \end{cases}$$

28. $$\begin{cases}\dfrac{\partial u}{\partial t}+(1+x)\dfrac{\partial \upsilon}{\partial x}+u=0,\\ \dfrac{\partial \upsilon}{\partial t}+(1+x)\dfrac{\partial u}{\partial x}-\upsilon=0;\end{cases}$$

29. $$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial \upsilon}{\partial x}+xu=0,\\ (1+x^2)\dfrac{\partial \upsilon}{\partial t}+\dfrac{\partial u}{\partial x}-\upsilon=0;\end{cases}$$

30. $$\begin{cases}2\dfrac{\partial u}{\partial t}+4\dfrac{\partial \upsilon}{\partial x}+2\dfrac{\partial \omega}{\partial x}=2\omega-2u-\upsilon,\\ \dfrac{\partial \upsilon}{\partial t}+8\dfrac{\partial u}{\partial x}=2\omega-2u-\upsilon,\\ \dfrac{\partial \omega}{\partial t}+3\dfrac{\partial \omega}{\partial x}=2u+\upsilon+2\omega;\end{cases}$$

31. $$\begin{cases}\dfrac{\partial u}{\partial t}+6\dfrac{\partial u}{\partial x}+5\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}+5\dfrac{\partial u}{\partial x}+6\dfrac{\partial \upsilon}{\partial x}=2u,\\ 3\dfrac{\partial \omega}{\partial t}+6\dfrac{\partial \omega}{\partial x}-3\dfrac{\partial u}{\partial x}=2\upsilon+3\omega-3u.\end{cases}$$

32. $$\frac{\partial u}{\partial t}+C\frac{\partial u}{\partial x}=f,$$

$$C=\begin{Vmatrix}1 & 0 & 2\\ 0 & 3 & 4\\ 0 & -\dfrac{1}{4} & 1\end{Vmatrix}$$

33. $$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{1}{\rho_0}\dfrac{\partial \rho}{\partial x}=0,\\ \dfrac{\partial \rho}{\partial t}+\rho_0 c_0^2\dfrac{\partial u}{\partial x}=0\end{cases}$$

Giperbolik sistemalarning umumiy yechimini toping:

34. $$\begin{cases}(x-1)u_t-(x+1)\upsilon_t+u_x=0,\\ (x+1)u_t-(x-1)\upsilon_t-\upsilon_x=0;\end{cases}$$

35. $$\begin{cases}u_x+\upsilon_y=2(u_x-\upsilon_y)-3(\upsilon_x-u_y),\\ \upsilon_x+u_y=3(u_x-\upsilon_y)+2(\upsilon_x-u_y),\end{cases}$$

36. $$\begin{cases} \frac{\partial u}{\partial t}+4\frac{\partial u}{\partial x}+5\frac{\partial \upsilon}{\partial x}=0, \\ \frac{\partial \upsilon}{\partial t}+5\frac{\partial u}{\partial x}+4\frac{\partial \upsilon}{\partial x}=0, \\ \frac{\partial \omega}{\partial t}+3\frac{\partial u}{\partial x}-2\frac{\partial \omega}{\partial x}=0; \end{cases}$$

37. $$\begin{cases} \frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial x}=0, \\ \frac{\partial u_2}{\partial t}-3\frac{\partial u_2}{\partial x}+\frac{\partial u_3}{\partial x}=0, \\ \frac{\partial u_3}{\partial t}-6\frac{\partial u_2}{\partial x}+4\frac{\partial u_3}{\partial x}=0. \end{cases}$$

8.2. Giperbolik sistemalarga qo'yilgan Koshi masalasi va aralash masalani yechish

Giperbolik sistemalarga qo'yilgan Koshi masalasi va aralash masalani yechish xususiy hosilali differensial tenglamalarga qo'yilgan Koshi masalasi va aralash masalani yechish kabi. Quyidagi misollarda sistema uchun qo'yilgan masalalarni yechish ko'rsatilgan.

Masala. Giperbolik sistemaga qo'yilgan Koshi masalasini yeching:

$$\begin{cases} 2u_t-u_x-v_x=0, \\ 2v_t-u_x-v_x=0, \end{cases} \quad u(x,0)=0,\ v(x,0)=2x,\ -\infty<x<\infty$$

Yechish. Berilgan sistemani matrisaviy shaklda yozib olamiz:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}u\\ v\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Bunda,

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ B=\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix},\ \mathrm{U}=\begin{pmatrix}u\\ v\end{pmatrix},\ f=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Xarakteristik tenglamani yechamiz:

$$|B-kE|=\left|\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}-\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\right|=0.$$

Ildizlari: $k_1=0,\ k_2=-2$.

B matrisaning xos vektorlarini topamiz:

$$(B-k_1E)z=0,$$

$$z_1=\begin{pmatrix}1\\-1\end{pmatrix},\quad z_2=\begin{pmatrix}1\\1\end{pmatrix}.$$

Xos vektorlardan quyidagi matrisani tuzamiz:

$$Z=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

Ushbu matrisaga teskari matrisa:

$$Z^{-1}=\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

$U=ZV$ almashtirish yordamida tenglama kanonik ko'rinishga keladi:

$$\frac{\partial V}{\partial t}+K\frac{\partial V}{\partial x}=0,$$

bu yerda, $K=Z^{-1}BZ=\begin{pmatrix}-2&0\\0&0\end{pmatrix}$, $V=\begin{pmatrix}v_1\\v_2\end{pmatrix}$.

Berilgan tenglamaning kanonik ko'rinishi quyidagicha:

$$\begin{cases}\dfrac{\partial v_1}{\partial t}-2\dfrac{\partial v_1}{\partial x}=0\\ \dfrac{\partial v_2}{\partial t}=0\end{cases}$$

Ushbu sistemaning yechimi: $v_1(x,t)=f_1(x+2t)$, $v_2(x,t)=f_2(x)$.

Dastlabki sistemaning yechimini quyidagi tenglikdan aniqlaymiz:

$$U=ZV$$

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

Bundan, berilgan sistemaning umumiy yechimi:

$$u(x,t)=f_1(x+t)-f_2(x),$$
$$v(x,t)=f_1(x+t)+f_2(x).$$

Endi berilgan sistema uchun Koshi masalasini yechamiz:

$u(x,0)=0,\ v(x,0)=2x,\ -\infty<x<\infty$

$$\begin{cases}u(x,0)=f_1(x)-f_2(x)=0,\\ v(x,t)=f_1(x)+f_2(x)=2x.\end{cases}$$

$$\begin{cases}f_1(x)=x,\\ f_2(x)=x.\end{cases}$$

Bundan

$$\begin{cases} f_1(x+t) = x+t, \\ f_2(x) = x. \end{cases}$$

Demak, berilgan masalaning yechimi:

$$\begin{cases} u(x,t) = t, \\ v(x,t) = 2x+t. \end{cases}$$

Masala. Akustika tenglamalar sistemasi

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} = 0, \\ \dfrac{\partial p}{\partial t} + \rho_0 C_0^2 \dfrac{\partial u}{\partial x} = 0, \end{cases}$$

berilgan bo‘lib, $u = 0, x = 0, x = l$ chegaraviy shartlarni qanoatlantiradi.

Yechish. Xususiy yechimni quyidagi ko‘rinishda qidiramiz:

$$u = T(t)U(x)$$
$$p = T(t)P(x)$$

Agar yechim mavjud bo‘lsa, u holda T, P, U lar o‘zaro quyidagicha bog‘langan:

$$\frac{T'(t)}{T(t)} = -\frac{1}{\rho_0}\frac{P'(x)}{U(x)} = \lambda = const$$

$$\frac{T'(t)}{T(t)} = -\rho_0 C_0^2 \frac{U'(x)}{P(x)} = \lambda = const$$

Bundan $T(t) = const e^{\lambda t}$ va shuning uchun xususiy yechimlar:

$$U = e^{\lambda t} \cdot U(x),$$
$$P = e^{\lambda t} \cdot P(x),$$

ko‘rinishida bo‘ladi.

Ma’lumki, $u(x)$ uchun, $u(0) = u(l) = 0$ chegaraviy shartlar bajarilishi kerak. U, P ga bog‘liq oddiy differensial tenglamaga kelamiz:

$$\begin{cases} \lambda U + \dfrac{P}{\rho_0}\dfrac{dP}{dx} = 0 \\ \lambda P + \rho_0 C_0 \dfrac{dU}{dx} = 0 \end{cases}$$

Bu tenglamalarning umumiy yechimi quyidagicha:

$$U = Ae^{\frac{\lambda x}{C_0}} + Be^{-\frac{\lambda x}{C_0}}$$

$$P = -\rho_0 C_0 A e^{\frac{\lambda x}{C_0}} + \rho_0 C_0 B e^{-\frac{\lambda x}{C_0}}$$

A, B doimiylarni $U(0)=U(l)=0$ chegaraviy shartlardan aniqlaymiz. Bu shartlar bir jinsli chiziqli tenglamalar sistemasiga kelamiz:

$$\begin{cases} A+B=0 \\ Ae^{\frac{\lambda l}{C_0}}+Be^{-\frac{\lambda l}{C_0}}=0 \end{cases}$$

Agar

$$D(\lambda)=\begin{vmatrix} 1 & 1 \\ e^{\frac{\lambda l}{C_0}} & e^{-\frac{\lambda l}{C_0}} \end{vmatrix}=e^{-\frac{\lambda l}{C_0}}-e^{\frac{\lambda l}{C_0}}=-2sh\frac{\lambda l}{C_0}=0$$

bo‘lsa, u holda yuqoridagi sistema nol bo‘lmagan yechimga ega ya’ni:

$$\lambda=\frac{ik\pi C_0}{l}\ (k\text{-butun son})$$

$A=\frac{1}{2};\ B=-\frac{1}{2}$ deb olamiz.

$$U=i\frac{e^{i\frac{k\pi}{l}x}-e^{-i\frac{k\pi}{l}x}}{2i}=i\sin\frac{k\pi}{l}x,$$

$$P=-\rho_0 C_0\frac{e^{i\frac{k\pi}{l}x}+e^{-i\frac{k\pi}{l}x}}{2}=-\rho_0 C_0\cos\frac{k\pi}{l}x$$

$$\lambda U+\frac{P}{\rho_0}\frac{dP}{dx}=0$$

$$\lambda P+\rho_0 C_0\frac{dU}{dx}=0$$

sistema noldan farqli yechimga ega bo‘ladigan qiymatlari λ parametrning xos qiymati, shu xos sonlarga mos yechimlar xos funksiyani tashkil etadi.

Xos qiymat va xos funksiya quyidagi formula bilan aniqlanadi:

$$\lambda_k=i\frac{k\pi C_0}{l},\quad u_k=i\sin\frac{k\pi}{l}x,\quad p_k=-\rho_0 C_0\cos\frac{k\pi}{l}x.$$

Xususiy yechimlar cheksiz ko‘p:

$$u_k=e^{\lambda_k t}U_k(x)$$

$$p_k=e^{\lambda_k t}P_k(x)$$

Ma’lumki, ushbu tenglamalar ixtiyoriy chekli chiziqli kombinatsiyasi ham, ya’ni ushbu tenglamalar:

$$u = \sum_k a_k u_k \quad \begin{pmatrix} u \\ p \end{pmatrix} = \sum_k a_k \begin{pmatrix} u_k \\ p_k \end{pmatrix}.$$
$$p = \sum_k a_k p_k$$

Quyidagi sistemani

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0$$
$$\frac{\partial p}{\partial t} + \rho_0 C_0^2 \frac{\partial u}{\partial x} = 0$$

va

$$u(0,t) = u(P,t) = 0$$

chegaraviy shartlarni qanoatlantiradi. Bu masala yechimi odatda $u(x,0) = \varphi(x)$, $P(x,0) = \varphi(x)$, $\{\varphi(x); \psi(x)\}$ vektor funksiyani chekli chiziqli kombinatsiyada apraksimatsiyalaymiz:

$$\begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} \approx \sum a_k \begin{pmatrix} U_k(x) \\ P_k(x) \end{pmatrix}$$

tabiiyki

$$\tilde{u}(x,t) = \sum a_k e^{\lambda_k t} U_k(x)$$
$$\tilde{p}(x,t) = \sum a_k e^{\lambda_k t} P_k(x)$$

yechimlar $u(x,t)$ va $p(x,t)$ yechimlarni aproksimatsiyalaydi.

Kompleks xususiy yechimi:

$$u_k = ie^{i\frac{k\pi C_0}{l}t}\sin\frac{k\pi}{l} = i\cos\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x\sin\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x$$

$$p_k = -\rho_0 C_0 e^{i\frac{k\pi C_0}{l}t}\cos\frac{k\pi}{l}x = -\rho_0 C_0\left(\cos\frac{k\pi C_0}{l}t\cos\frac{k\pi}{l}x + i\sin\frac{k\pi C_0}{l}t\cos\frac{k\pi}{l}x\right)$$

Chiziqli kombinatsiyalar

$$\sum a_k \begin{pmatrix} u_k \\ p_k \end{pmatrix} = \sum a_k \begin{pmatrix} \frac{u_k + u_{-k}}{2} \\ \frac{p_k + p_{-k}}{2} \end{pmatrix} + \sum i a_k \begin{pmatrix} \frac{u_k - u_{-k}}{2i} \\ \frac{p_k - p_{-k}}{2i} \end{pmatrix}$$

Shunday qilib, chiziqli kombinatsiyadan quyidagi xususiy yechimga ega bo'lamiz:

$$\frac{u_k + u_{-k}}{2} = -\sin\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x, \quad \frac{p_k + p_{-k}}{2} = -\rho_0 C_0 \sin\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x$$
$$\frac{u_k - u_{-k}}{2} = \cos\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x, \quad \frac{p_k - p_{-k}}{2} = -\rho_0 C_0 \sin\frac{k\pi C_0}{l}t\cos\frac{k\pi}{l}x.$$

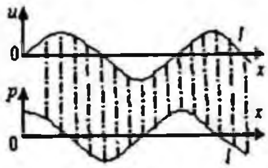
Yechim

$$\begin{pmatrix} u \\ p \end{pmatrix} = a\begin{pmatrix} -\sin\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x \\ -\rho_0 C_0\cos\frac{k\pi C_0}{l}t\cos\frac{k\pi}{l}x \end{pmatrix} + b\begin{pmatrix} \cos\frac{k\pi C_0}{l}t\sin\frac{k\pi}{l}x \\ -\rho_0 C_0\sin\frac{k\pi C_0}{l}t\cos\frac{k\pi}{l}x \end{pmatrix} =$$

$$= \sqrt{a^2+b^2}\begin{pmatrix} \cos\frac{k\pi C_0(t+\tau)}{l}\sin\frac{k\pi}{l}x \\ -\rho_0 C_0\sin\frac{k\pi C_0(t+\tau)}{l}\cos\frac{k\pi}{l}x \end{pmatrix}$$

ko'rinishida tasvirlanadi.

$x=0,\ x=l$ qo'zg'almas tekislik orasida gaz qatlamining tebranishi tik to'lqin deb ataladi.



1-chizmada qandaydir vaqtdagi to'lqin tezligi va bosimi taqsimoti grafigi keltirilgan.

"Tik to'lqinlar" nomi shuni ifodalaydiki nuqtaning tebranishi uchun tezlik amplitudasi (yoki 1-chizma p bosim) nolga teng bo'lsa, yoki hamma vaqt ekstremal bo'ladi.

Tugun nuqtada bosim amplitudasi maksimal bo'ladi. Bundan tashqari izoh berish kerakki, u siljish fazosi bo'yicha bosim siljishi o'shanga qarab siljigan bo'ladi.

Akustika tenglamalar sistemasi uchun yaratilgan Furye usulini qarab chiqdik.

$u(0,t)=u(l,t)$ chegaraviy shartlarni cheklashdagi xususiy yechimlar yig'indisi tik to'lqinni ifodalaydi.

<u>**Masala.**</u> $\begin{cases} \frac{\partial u}{\partial t} + 2\frac{\partial \vartheta}{\partial x} = 0 \\ \frac{\partial \vartheta}{\partial t} + 5\frac{\partial u}{\partial x} = 0 \end{cases}$, $u(0,t)=\vartheta(1,t)=0$, $u(x,0)=x$, $\vartheta(x,0)=0$.

<u>**Yechish:**</u> Yechimni quyidagi ko'rinishda qidiramiz:

$$u = T(t)U(x)$$

$$\vartheta = T(t)V(x)$$

$$T'(t)\cdot U(x) + 2V'(x)\cdot T(t) = 0 \qquad :T(t)U(x)$$

$$V(x)T'(t) + 5U'(x)T(t) = 0 \qquad :T(t)V(x)$$

$$\begin{cases} \dfrac{T'(t)}{T(t)} + 2\dfrac{V'(x)}{U(x)} = 0 \\ \dfrac{T'(t)}{T(t)} + 5\dfrac{U'(x)}{V(x)} = 0 \end{cases}$$

$$\frac{T'(t)}{T(t)} = -2\frac{V'(x)}{U(x)} = -5\frac{U'(x)}{V(x)} = \lambda, \quad T'(t) - \lambda T(t) = 0,$$

$$-2\frac{V'(x)}{U(x)} = -5\frac{U'(x)}{V(x)} = \lambda, \quad U(x) = \frac{-2V'(x)}{\lambda}, \quad U'(x) = \frac{-2V''(x)}{\lambda}$$

$$\frac{10}{\lambda}\frac{V''(x)}{V(x)} = \lambda, \; 10V''(x) - \lambda^2 V(x) = 0, \; V(x) = e^{kx}, \; V'(x) = ke^{kx},$$

$$V''(x) = k^2 e^{kx}, \; 10k^2 e^{kx} - \lambda^2 e^{kx} = 0, \; e^{kx}(10k^2 - \lambda^2) = 0, \; k = \pm\frac{\lambda}{\sqrt{10}}.$$

$$V(x) = C_1 e^{\frac{\lambda}{\sqrt{10}}x} + C_2 e^{-\frac{\lambda}{\sqrt{10}}x}, \; V'(x) = \frac{C_1}{\sqrt{10}}\lambda e^{\frac{\lambda}{\sqrt{10}}x} - \frac{C_2}{\sqrt{10}}\lambda e^{-\frac{\lambda}{\sqrt{10}}x}.$$

$$U(x) = \frac{2\left(\frac{C_1}{\sqrt{10}}\lambda e^{\frac{\lambda}{\sqrt{10}}x} - \frac{C_2}{\sqrt{10}}\lambda e^{-\frac{\lambda}{\sqrt{10}}x}\right)}{\lambda} = \frac{2\left(C_1 e^{\frac{\lambda}{\sqrt{10}}x} - C_2 \lambda e^{-\frac{\lambda}{\sqrt{10}}x}\right)}{\sqrt{10}}.$$

$$U = \frac{e^{\lambda t}2\left(\frac{C_1}{\sqrt{10}}\lambda e^{\frac{\lambda}{\sqrt{10}}x} - \frac{C_2}{\sqrt{10}}\lambda e^{-\frac{\lambda}{\sqrt{10}}x}\right)}{\lambda}, \quad U\big|_{x=0} = -e^{\lambda t}\frac{2(C_1 - C_2)}{\sqrt{10}} = 0,$$

$$C_1 - C_2 = 0, \; C_1 = C_2.$$

$$V = e^{\lambda t}\left(C_1 e^{\frac{\lambda}{\sqrt{10}}} + C_2 e^{-\frac{\lambda}{\sqrt{10}}}\right), \; V\big|_{x=1} = e^{\lambda t}\left(C_1 e^{\frac{\lambda}{\sqrt{10}}} + C_2 e^{-\frac{\lambda}{\sqrt{10}}}\right) = 0,$$

$$2\left(\frac{e^{\frac{\lambda}{\sqrt{10}}} + e^{-\frac{\lambda}{\sqrt{10}}}}{2}\right) = 0, \; 2ch\frac{\lambda}{\sqrt{10}} = 0. \quad \frac{\lambda}{\sqrt{10}} = \left(\frac{\pi}{2} + \pi n\right)i, \; x = \sqrt{10}i\left(\frac{\pi}{2} + \pi n\right)$$

$$U(x) = -\frac{2\left(C_1 e^{i\left(\frac{\pi}{2}+\pi n\right)x} - C_1 \lambda e^{-i\left(\frac{\pi}{2}+\pi n\right)x}\right)}{\sqrt{10}} = \frac{2C_1\left(\cos\left(\frac{\pi}{2} + \pi n\right)x - i\sin\left(\frac{\pi}{2} + \pi n\right)x\right)}{\sqrt{10}} -$$

$$-\frac{\cos\left(\frac{\pi}{2} + \pi n\right)x - i\sin\left(\frac{\pi}{2} + \pi n\right)x}{\sqrt{10}} = \frac{4C_1 i\sin\left(\frac{\pi}{2} + \pi n\right)x}{\sqrt{10}}.$$

$$V(x) = C_1 e^{i\left(\frac{\pi}{2}+\pi n\right)x} + C_1 \lambda e^{-i\left(\frac{\pi}{2}+\pi n\right)x} = 2C_1\left(\frac{\pi}{2} + \pi n\right)x = \left|C_1 = \frac{1}{2}\sqrt{10}\right| = \sqrt{10}\cos\left(\frac{\pi}{2} + \pi n\right)x.$$

$$U(x)=\frac{-4i\sin\left(\frac{\pi}{2}+\pi n\right)x}{\sqrt{10}}=\left|C_1=\frac{1}{2}\sqrt{10}\right|=-2i\sin\left(\frac{\pi}{2}+\pi n\right)x.$$

$$U(x,t)=e^{\sqrt{10}i\left(\frac{\pi}{2}+\pi n\right)t}\left(2i\sin\left(\frac{\pi}{2}+\pi n\right)x\right),\quad V(x,t)=e^{\sqrt{10}i\left(\frac{\pi}{2}+\pi n\right)t}\left(\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\right)=$$

$$=-\left[\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right]\cdot 2i\sin\left(\frac{\pi}{2}+\pi n\right)x.$$

$$U(x,t)=\left[\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right]\cdot\left(\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\right).$$

$$\frac{U_n+U_{-n}}{2}=\frac{\left(\left(-2i\sin\left(\frac{\pi}{2}+\pi n\right)x\right)\left(\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right)\right)}{2}+$$

$$+\frac{\left(\left(-2i\sin\left(\frac{\pi}{2}+\pi n\right)x\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)-i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right)\right)}{2}=$$

$$=2\sin\left(\frac{\pi}{2}+\pi n\right)x\left(\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right).$$

$$\frac{V_n+V_{-n}}{2}=\frac{\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\left(\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\right)}{2}+$$

$$+\frac{\left(\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right)\left(\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\right)}{2}=$$

$$=\sqrt{10}\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\cos\left(\frac{\pi}{2}+\pi n\right)x.$$

$$\frac{U_n-U_{-n}}{2}=\frac{\left(\left(-2i\sin\left(\frac{\pi}{2}+\pi n\right)x\right)\left(\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right)\right)}{2}-$$

$$-\frac{\left(\left(-2i\sin\left(\frac{\pi}{2}+\pi n\right)x\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)-i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right)\right)}{2}=$$

$$=2i\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\sin\left(\frac{\pi}{2}+\pi n\right)x$$

$$\frac{V_n-V_{-n}}{2}=\frac{\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\left(\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\right)}{2}-$$

$$-\frac{\left(\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)+i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\right)\left(\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\right)}{2}=$$

$$=i\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x.$$

Demak, berilgan masalaning yechimi quyidagicha:

$$\begin{pmatrix}U\\V\end{pmatrix}=a\sum\begin{pmatrix}2\sin\left(\frac{\pi}{2}+\pi n\right)x\sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\\ \sqrt{10}\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\cos\left(\frac{\pi}{2}+\pi n\right)x\end{pmatrix}+$$

$$+b\sum\begin{pmatrix}2\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\sin\left(\frac{\pi}{2}+\pi n\right)x\\ \sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\end{pmatrix}.$$

Boshlang‘ich shartlardan foydalanib, noma’lum koeffisiyentlarni Furye usulidan foydalanib topamiz:

$$u(x,0)=b\sum 2\sin\left(\frac{\pi}{2}+\pi n\right)x=x,$$

$$2b=2\int_0^1 x\sin\left(\frac{\pi}{2}+\pi n\right)xdx+2\int_0^1\cos\left(\frac{\pi}{2}+\pi n\right)xdx=\frac{2}{\left(\frac{\pi}{2}+\pi n\right)^2}\sin\left(\frac{\pi}{2}+\pi n\right)x\Big|_0^1=$$

$$=\frac{2}{\left(\frac{\pi}{2}+\pi n\right)^2}\sin\left(\frac{\pi}{2}+\pi n\right)=2(-1)^n\left(\frac{\pi}{2}+\pi n\right)^{-2}.$$

$$\vartheta(x,0)=0,\quad a=0,\quad b=\frac{(-1)^n}{\left(\frac{\pi}{2}+\pi n\right)^2}.$$

Demak, berilgan masalaning yechimi:

$$\begin{pmatrix}u\\ \vartheta\end{pmatrix}=\sum 2(-1)^n\left(\frac{\pi}{2}+\pi n\right)^{-2}\begin{pmatrix}2\cos\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\sin\left(\frac{\pi}{2}+\pi n\right)x\\ \sin\sqrt{10}t\left(\frac{\pi}{2}+\pi n\right)\sqrt{10}\cos\left(\frac{\pi}{2}+\pi n\right)x\end{pmatrix}.$$

<u>Masala.</u> Endi esa Furye almashtirishni qo`llab giperbolik sistemaga qo`yilgan aralash masala qanday yechilishini

ko'rsatamiz. Giperbolik sistemalar tebranma jarayonlarini, tovush tarqalish hodisalarini ifodalaydi.

$D = \{(x,t)/0 < x < 1, t > 0\}$ sohada quyidagi masalani qaraylik:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial x} = 0. \end{cases} \quad 0 < x < 1, \quad t > 0,$$

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = 0, \quad v(x,0) = \cos \pi x.$$

<u>**Yechish.**</u> Masalani yechishda Furye almashtirishidan foydalanamiz.

Bir o`zgaruvchili funksiya uchun to`g'ri va teskari Furye almashtirishi mos ravishda quyidagicha bo`ladi:

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} dx;$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s) e^{isx} ds.$$

Ikki o`zgaruvchili bo`lgan holda to`g'ri va teskari Furye almashtirishi mos ravishda quyidagicha bo`ladi:

$$U(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-isx} dx;$$

$$V(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x,t) e^{-isx} dx;$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(s,t) e^{isx} ds;$$

$$v(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(s,t) e^{isx} ds.$$

sistemadagi tenglamalarni $\dfrac{1}{\sqrt{2\pi}} e^{-isx}$ ga ko`paytirib, R sohada integrallaymiz.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

$$\begin{cases} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{-isx}dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial v}{\partial x}e^{-isx}dx=0 \\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial v}{\partial t}e^{-isx}dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{-isx}dx=0 \end{cases}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{-isx}dx=\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-isx}dx\right]=\frac{dU}{dt}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial v}{\partial t}e^{-isx}dx=\frac{dV}{dt}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{-isx}dx=\left|\begin{matrix} e^{-isx}=u & du=-ise^{-isx}dx \\ \frac{\partial u}{\partial x}dx=dv & v=u \end{matrix}\right|=\frac{1}{\sqrt{2\pi}}u(x,t)e^{-isx}\Big|_{-\infty}^{\infty}+\frac{is}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-isx}dx=isU(s,t)$$

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$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial v}{\partial x}e^{-isx}dx=isV(s,t)$$

Bizning masalamiz uchun quyidagilar o`rinli:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{-isx}dx=\frac{dU}{dt}; \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{-isx}dx=isU(s,t),$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial v}{\partial t}e^{-isx}dx=\frac{dV}{dt}; \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial v}{\partial x}e^{-isx}dx=isV(s,t) \ ,$$

U holda berilgan sistema quyidagi ko`rinishga keladi:

$$\begin{cases} \frac{dU(s,t)}{dt}+isV(s,t)=0 \\ \frac{dV(s,t)}{dt}+isU(s,t)=0 \end{cases}$$

$$U(0,t)=U(1,t)=0, \quad U(x,0)=0, \qquad V(s,0)=\Phi(s),$$

$$\frac{d^2V}{dt^2}-s^2V=0$$

$$V_t(s,0)=0$$

Ya`ni Koshi masalani hosil qilamiz. Bu yerda $\cos\pi x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(s)e^{isx}ds$.

Ushbu masala ikkinchi tartibli differensial tenglamaga qo`yilgan Koshi masalasi.

Hosil bo‘lgan masalaning umumiy yechimini aniqlaymiz:

$$V(s,t)=\Phi(s)\frac{e^{-ist}+e^{ist}}{2}$$

Ushbu funksiyaga teskari Furye almashtirishini qo`llaymiz.

$$v(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}V(t,s)e^{isx}ds=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}(e^{-ist}+e^{ist})\Phi(s)ds=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}(e^{i(x-t)s}+e^{i(x+t)s})\Phi(s)ds=$$

$$=\left|\cos\pi x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(s)e^{isx}ds\right|=\frac{1}{2}(\cos\pi(x-t)+\cos\pi(x+t))=\cos\pi x\cos\pi t$$

Demak, $v(x,t)=\cos\pi x\cos\pi t$

Endi $U(s,t)=\frac{1}{is}\frac{dV}{dt}-s^2V$ tenglikdan foydalanamiz:

$$U(s,t)=\frac{1}{2}\Phi(s)(e^{ist}-e^{-ist})$$

Ushbu funksiyaga Furye almashtirishini qo`llab, natijada $u(x,t)=\sin\pi x\sin\pi t$ yechimni olamiz.

U holda berilgan masalaning yechimi quyidagicha bo`ladi:

$$\begin{cases}u(x,t)=\sin\pi x\sin\pi t\\ v(x,t)=\cos\pi x\cos\pi t\end{cases}$$

Shuni ta`kidlash joizki, giperbolik sistemalar, Furye almashtirishi matematikada keng tatbiqqa ega sohalaridan hisoblanadi. Xususiy hosilali differensial tenglamalarga qo`yilgan masalalar giperbolik sistemalarga qo`yilgan masalalarga keladi.

Giperbolik sistemalarga qo'yilgan Koshi masalasini yeching:

38. $\begin{cases}2u_t-u_x-\upsilon_x=0,\\ 2\upsilon_t-u_x-\upsilon_x=0,\end{cases}$ $u(x,0)=0,\ \upsilon(x,0)=2x,\ -\infty<x<\infty$

39. $\begin{cases}2u_t-(2t-1)u_x+(2t+1)\upsilon_x=0,\\ 2\upsilon_t+(2t+1)u_x-(2t-1)\upsilon_x=0,\end{cases}$ $u(x,0)=0,\ \upsilon(x,0)=2x,\ -\infty<x<\infty$

40. $\begin{cases}3u_t+2\upsilon_t-u_x-\upsilon_x=0,\\ u_t+u_x+\upsilon_x=0,\end{cases}$ $u(x,0)=0,\ \upsilon(x,0)=x,\ -\infty<x<\infty$

41. Gursa masalasini yeching: $t\ge|x|\ \frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}=0,\ u(x,x)=\varphi(x),\ x>0,$

$u(x,-x)=\psi(x),\ x<0,\ \varphi(0)=\psi(0)$

42. $\begin{cases}\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}+5\frac{\partial\upsilon}{\partial y}=0,\\ \frac{\partial u}{\partial y}-\frac{\partial\upsilon}{\partial x}=0,\end{cases}$

$u(x,x)=x,\quad x>0,$

$\upsilon(x,5x)=x^2,\ x<0.$

Giperbolik sistemalarga qo'yilgan aralash masalalarni o'zgaruvchilarni ajratish usuli bilan yeching:

43. $\begin{cases} \dfrac{\partial u}{\partial t} + 9\dfrac{\partial \upsilon}{\partial x} = 0, \\ \dfrac{\partial \upsilon}{\partial t} + \dfrac{\partial u}{\partial x} = 0. \end{cases}$

$u(0,t) = \upsilon(\pi,t) = 0,$

$u(x,0) = x^2,\ \upsilon(x,0) = 0,\ 0 \le x \le \pi;$

44.

$\begin{cases} \dfrac{\partial u}{\partial t} + 9\dfrac{\partial \upsilon}{\partial x} + \dfrac{\partial \omega}{\partial x} = 0, \\ \dfrac{\partial \upsilon}{\partial t} + \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial \omega}{\partial t} + \dfrac{\partial \omega}{\partial x} = 0; \end{cases}$

$u(x,0) = 0,\ \upsilon(x,0) = 0,\ \omega(x,0) = x^2,\ u(0,t) - 3\upsilon(0,t) = 0,\ \omega(0,t) = 0,\ \upsilon(1,t) = 0,\ 0 \le x \le 1;$

45.

$\begin{cases} \dfrac{\partial u}{\partial t} + 4\dfrac{\partial \upsilon}{\partial x} + u = 0, \\ \dfrac{\partial \upsilon}{\partial t} + \dfrac{\partial u}{\partial x} + \upsilon = 0, \end{cases}$

$u(0,t) = \upsilon(\pi,t) = 0,\ u(x,0) = 0,\ \upsilon(x,0) = \sin^2 x,\ 0 \le x \le \pi.$

46.

$\begin{cases} \dfrac{\partial u}{\partial t} + 27\dfrac{\partial \upsilon}{\partial x} + u = 0, \\ \dfrac{\partial \upsilon}{\partial t} + 3\dfrac{\partial u}{\partial x} - \upsilon = 0, \end{cases}$

$u(0,t) + \upsilon(0,t) = 0,\ u(1,t) - \upsilon(1,t) = 0,\ 0 \le x \le 1.$

47.

$\begin{cases} \dfrac{\partial u}{\partial t} + 27\dfrac{\partial \upsilon}{\partial x} - u = 0, \\ \dfrac{\partial \upsilon}{\partial t} + 3\dfrac{\partial u}{\partial x} - \upsilon = 0, \end{cases}$

$u(0,t) = \upsilon(1,t) = 0,\ 0 \le x \le 1;$

48.

$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial \upsilon}{\partial x} + x^2 u + (1 - x^2)\upsilon = 0, \\ \dfrac{\partial \upsilon}{\partial t} + 4\dfrac{\partial u}{\partial x} + (1 - x^2)u + x^2 \upsilon = 0, \end{cases}$

$u(1,t) - \upsilon(1,t) = 0,\ 0 \le x \le 1.$

$u(0,t) = 0,$

49.

$$\begin{cases}\dfrac{\partial u}{\partial t}+2\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}+5\dfrac{\partial u}{\partial x}=0,\end{cases}$$

$u(0,t)=0,\ \upsilon(1,t)=0,\ u(x,0)=x,$

$\upsilon(x,0)=0,\ 0\le x\le 1;$

50.

$$\begin{cases}\dfrac{\partial u}{\partial t}-2\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}-3\dfrac{\partial u}{\partial x}=0,\end{cases}$$

$u(0,t)=0,\ \upsilon(1,t)=0,$

$u(x,0)=0,\ \upsilon(x,0)=x,\ 0\le x\le 1;$

51.

$$\begin{cases}\dfrac{\partial u}{\partial t}+2\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}+3\dfrac{\partial u}{\partial x}=0,\end{cases}$$

$\upsilon(0,t)=0,\ u(1,t)=0,$

$\upsilon(x,0)=x,\ u(x,0)=0,\ 0\le x\le 1;$

52.

$$\begin{cases}\dfrac{\partial u}{\partial t}-3\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}-5\dfrac{\partial u}{\partial x}=0,\end{cases}$$

$u(0,t)=0,\ \upsilon(1,t)=0,$

$u(x,0)=x,$

$\upsilon(x,0)=0,\ 0\le x\le 1;$

53. $$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial \upsilon}{\partial x}=0,\\ \dfrac{\partial \upsilon}{\partial t}+2\dfrac{\partial u}{\partial x}=0,\end{cases}$$

$\upsilon(0,t)=0,\ (1,t)=0,$

$u(x,0)=1,\ \upsilon(x,0)=0,\ 0\le x\le 1;$

Javoblar

1-bob.

1.2.3.4.5.6.7. $z=f(x^2+y^2)$. **8.** $z=f(xy+y^2)$. **9.** $z=f(\frac{y}{x}+\frac{z}{x})$. **10.** $z=f(\frac{x-y}{z}+\frac{(x+y+2z)^2}{z})$. **11.** $F(x^2-y^2,x-y+z)=0$. **12.** $F(e^{-x}-y^{-1},\frac{x-\ln|y|}{e^{-x}-y^{-1}})=0$. **13.** $F(x^2-4z,\frac{(x+y)^2}{x})=0$. **14.** $F(x^2+y^2,\frac{z}{x})=0$. **15.** $F(\frac{x^2}{y},xy-\frac{3z}{z})=0$. **16.** $F(\frac{1}{x+y}+\frac{1}{z},\frac{1}{x-y}+\frac{1}{z})=0$. **17.** $F(x^2+y^4,y(z+\sqrt{z^2+1}))=0$. **18.** $F(\frac{1}{x}-\frac{1}{y},\ln|xy|-\frac{z^2}{2})=0$.**19.** $F(x^2+y^2,arctg(\frac{x}{y})+(z+1)e^{-z})=0$. **20. 20.33.** $z=2xy$. **34.** $z=ye^x-e^{2x}+1$. **35.** $z=y^2e^{2\sqrt{x}-2}$.**36.** $u=(1-x+y)(2-2x+z)$. **37.** $u=(xy-2z)\left(\frac{x}{y}+\frac{y}{x}\right)$. **38.** $y^2-x^2-\ln\sqrt{y^2-x^2}=z-\ln|y|$. **39.** $2x^2(y+1)=y^2+4z-1$. **40.** $(x+2y)^2=2x(z+xy)$.**41.** $\sqrt{\frac{z}{y^3}}\sin x=\sin\sqrt{\frac{z}{y}}$.**42.** $2xy+1=x+3y+\frac{1}{z}$. **43.** $x-2y=x^2+y^2+z$. **54.** $z=xy+f\left(\frac{y}{x}\right)$, bu yerda f ixtiyoriy funksiya bo'lib, u uchun $f(1)=0$ shart bajariladi.

2-bob.

1. Elliptik. **2.** Giperbolik. **3.** Parabolik. **4.** Elliptik. **5.** Giperbolik. **6.** Giperbolik.**7.** Parabolik. **8.** Elliptik. **9.** Giperbolik.**10.** Elliptik.**11.** Elliptik.**12.** $u_{\xi\xi}+u_{\eta\eta}+u_\xi=0,\ \xi=x,\ \eta=3x+y$. **13.** $u_{\eta\eta}+u_\xi=0,\ \xi=x-2y,\ \eta=x$. **14.** $u_{\xi\eta}+\frac{1}{6(\xi+\eta)}\left(u_\xi+u_\eta\right)=0,\ \xi=\frac{2}{3}x^{\frac{3}{2}}+y,\ \eta=\frac{2}{3}x^{\frac{3}{2}}-y$, $x>0$; $u_{\xi\xi}+u_{\eta\eta}-\frac{1}{3\xi}u_\xi=0$, $\xi=\frac{2}{3}(-x)^{\frac{3}{2}}$, $\eta=y$, $x<0$. **15.** $u_{\xi\eta}+\frac{1}{2(\xi-\eta)}\left(u_\xi-u_\eta\right)=0,\ \xi=x+2\sqrt{y},\ \eta=x-2\sqrt{y}$, $y>0$; $u_{\xi\xi}+u_{\eta\eta}-\frac{1}{\eta}u_\eta=0,\ \xi=x,\ \eta=2\sqrt{-y}$, $y<0$. **16.** $u_{\xi\xi}-u_{\eta\eta}-\frac{1}{\xi}\left(u_\xi-u_\eta\right)=0,\ \xi=\sqrt{|x|},\ \eta=\sqrt{|y|}$, ($x>0$, $y>0$ yoki $x<0$, $y<0$) ; $u_{\xi\xi}+u_{\eta\eta}-\frac{1}{\xi}\left(u_\xi+u_\eta\right)=0,\ \xi=\sqrt{|x|},\ \eta=\sqrt{|y|}$ ($x>0$, $y<0$ yoki $x<0$, $y>0$). **17.**

$u_{\xi\xi}-u_{\eta\eta}+\frac{1}{3\xi}u_{\xi}-\frac{1}{3\eta}u_{\eta}=0,\ \xi=|x|^{\frac{3}{2}},\ \eta=|y|^{\frac{3}{2}}$, ($x>0,\ y>0$ yoki $x<0,\ y<0$) ;

$u_{\xi\xi}+u_{\eta\eta}+\frac{1}{3\xi}u_{\xi}+\frac{1}{3\eta}u_{\eta}=0,\ \xi=|x|^{\frac{3}{2}},\ \eta=|y|^{\frac{3}{2}}$, ($x>0,\ y<0$ yoki $x<0,\ y>0$).

18. $u_{\xi\xi}+u_{\eta\eta}-u_{\xi}-u_{\eta}=0,\ \xi=\ln|x|,\ \eta=\ln|y|$ (har bir kvadrantda).

19. $u_{\xi\xi}+u_{\eta\eta}+\frac{1}{2\xi}u_{\xi}+\frac{1}{2\eta}u_{\eta}=0,\ \xi=y^2,\ \eta=x^2$ (har bir kvadrantda).

20. $u_{\xi\eta}+\frac{1}{4(\eta^2-\xi^2)}(\eta u_{\xi}+\xi u_{\eta})=0,\ \xi=y^2-x^2,\ \eta=y^2+x^2$ (har bir kvadrantda).

21. $u_{\xi\xi}+u_{\eta\eta}-th\xi u_{\xi}=0,\ \xi=\ln\left(x+\sqrt{1+x^2}\right)\ \eta=\ln\left(y+\sqrt{1+y^2}\right)$

22. $u_{\xi\eta}-\frac{1}{2(\xi-\eta)}(u_{\xi}-u_{\eta})+\frac{1}{4(\xi+\eta)}(u_{\xi}+u_{\eta})=0,\ \xi=y^2+e^x,\ \eta=y^2-e^x$ (y>0 yoki y<0).

23. $u_{\xi\xi}+u_{\eta\eta}+\cos\xi u_{\eta}=0,\ \xi=x,\ \eta=y-\cos x$. **24.** $u_{\eta\eta}=u_{\xi}$. **25.** **26.** $\xi=2y+x$; $\eta=x$; elliptik, $\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **27.** $\xi=2e^x-y^2$; $\eta=x+y$; giperbolik, $\frac{\partial^2u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **28.** $\xi=5x+y$; $\eta=x$; parabolik, $\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **29.** $\xi=e^x$; $\eta=y$; elliptik, $\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$.

30. $\xi=x^2-2e^y$; $\eta=x$; parabolik, $\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **31.** $\xi=y-x^2$; $\eta=x^2+y^2$; giperbolik, $\frac{\partial^2u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **32.** $\xi=\cos x+y^3$; $\eta=x$; parabolik, $\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **33.** $\xi=yx$; $\eta=2x$; elliptik, $\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **34.** $\xi=2e^x-y^2$; $\eta=x+y$; giperbolik, $\frac{\partial^2u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **35.** $\xi=e^y\cos x$; $\eta=\frac{e^x}{x}$; giperbolik, $\frac{\partial^2u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **36.** $\xi=\cos x-\sin y$; $\eta=x$; parabolik, $\frac{\partial^2u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$.

37. $\xi=2x-y$; $\eta=\frac{1}{x}+\frac{1}{y}$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **38.** $\xi=tgy-x$; $\eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **39.** $\xi=\cos y$; $\eta=\sin x$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **40.** $\xi=\ln y-\frac{1}{x}$; $\eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **41.** $\xi=y+ctgx$; $\eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$.

42. $\xi=e^{-2x}+2y$; $\eta=e^{-2x}$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **43.** $\xi=ctgy$; $\eta=tgx$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **44.** $\xi=y\sin x$; $\eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **45.** $\xi=x-e^{y}$; $\eta=2x-e^{y}$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **46.** $\xi=y+\frac{2}{x}$; $\eta=\frac{1}{x}$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$.

47. $\xi=2x-\sin y$; $\eta=y$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **48.** $\xi=y-\ln\sin x$; $\eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **49.** $\xi=\ln\cos y$; $\eta=\ln\sin x$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **50.** $\xi=e^{arctg\frac{y}{x}}\sqrt{x^2+y^2}$; $\eta=x-y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **51.** $\xi=xy$; $\eta=3y$ yoki $\xi=\ln y+\frac{1}{2}\ln(x^2+9)$; $\eta=arctg\frac{x}{3}$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **52.** $\xi=\frac{y}{x^2}-\ln x$; $\eta=x-y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **53.** $\xi=xy+\ln x$; $\eta=x+y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=F(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta})$. **54.** $\xi=x-t$; $\eta=x$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$. **55.** $\xi=x+y$, $\eta=y$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=0$. **56.** $\xi=x+2y$, $\eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}-\frac{1}{2}\frac{\partial u}{\partial\eta}=0$. **57.** $\xi=4x+y$, $\eta=2x+y$; elliptik, $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{2}\frac{\partial u}{\partial\xi}=0$. **58.** $\xi=x+y$, $\eta=x$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}=0$. **59.** $\xi=x-y$, $\eta=x+3y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{4}\frac{\partial u}{\partial\eta}=0$. **60.**

$\xi=2y-x,\ \eta=y$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}=0$. **61.** $\xi=x+3y,\ \eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{3}\frac{\partial u}{\partial\eta}=0$. **62.** $\xi=x+2y,\ \eta=3x+2y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{4}\frac{\partial u}{\partial\eta}=0$. **63.** $\xi=x-3y,\ \ \eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}=0$. **64.** $\xi=x+2y,\ \eta=3x$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{6}(\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta})=0$. **65.** $\xi=2x-y,\ \ \eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}=0$. **66.** $\xi=x-5y,\ \eta=x-y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{4}\frac{\partial u}{\partial\eta}=0$. **67.** $\xi=x+y,\ \eta=x-y$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+2\frac{\partial u}{\partial\eta}=0$. **68.** $\xi=2x+3y,\ \ \eta=x$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{3}\frac{\partial u}{\partial\eta}=0$. **69.** $\xi=x+2y,\ \eta=2x+y$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+2\frac{\partial u}{\partial\eta}=0$. **70.** $\xi=x+3y,\ \ \ \eta=2x-y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial u}{\partial\xi}=0$. **71.** $\xi=x+y,\ \eta=y$; parabolik,

$$\frac{\partial^2 u}{\partial\eta^2}+(\alpha+\beta)\frac{\partial u}{\partial\xi}+\beta\frac{\partial u}{\partial\eta}+cu=0.$$

72. $\xi=x+y,\ \eta=3x-y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$. **73.** $\xi=x+3y,\ \eta=x+y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\eta}=0$. **74.** $\xi=xy,\ \eta=\frac{y}{x}$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$. **75.** $\xi=\ln(x+\sqrt{x^2+1}),\ \eta=\ln(y+\sqrt{y^2+1})$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=0$. **76.** $\xi=x^2+y,\ \eta=y-x^2$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$. **77.** $\xi=x^3y,\ \ \ \eta=y$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}+\frac{4}{3\eta}\frac{\partial u}{\partial\eta}=0$. **78.** $\xi=y^2+x,\ \eta=x-y^2$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$. **79.** $\xi=x,\ \eta=x+e^y$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial u}{\partial\eta}=0$. **80.** $\xi=x^2+y,\ \eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}=0$. **81.** $\xi=x^2-y^2,\ \eta=x^2$; elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\xi-\eta}\frac{\partial u}{\partial\xi}+\frac{1}{2\eta}\frac{\partial u}{\partial\eta}=0$. **82.** $\xi=x+\sin y,\ \eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}+\eta\frac{\partial u}{\partial\eta}=0$. **83.** $\xi=x+y+\cos x,\ \eta=x-y-\cos x$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{4}\cos\frac{\xi+\eta}{2}(\frac{\partial u}{\partial\eta}-\frac{\partial u}{\partial\xi})=0$. **84.** $\xi=x+\cos y,\ \eta=x$; giperbolik, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{\eta}\frac{\partial u}{\partial\eta}=0$. **85.** $\xi=y^3x,\ \ \ \eta=x$; parabolik, $\frac{\partial^2 u}{\partial\eta^2}-\frac{\xi}{\eta^2}\frac{\partial u}{\partial\xi}=0$. **86.** $\xi=xtg\frac{y}{2}$,

$\eta = x$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} - \frac{2\xi}{\xi^2 + \eta^2}\frac{\partial u}{\partial \xi} = 0$. **87.** $\xi = tgy,\ \eta = \ln x$; elliptik, $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{2\xi^2}{1+\xi^2}\frac{\partial u}{\partial \xi} = 0$. **88.** $\xi = x + \cos y$, $\eta = y$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} = 0$. **89.** $\xi = e^y - 2x,\ \eta = e^y - x$; giperbolik, $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$. **90.** $\xi = y^2,\ \eta = 4x$; elliptik, $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial u}{\partial \xi} = 0$. **91.** $\xi = y^2 + 2e^x,\ \eta = y$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0$. **92.** $\xi = x^2 + y,\ \eta = x^2$; elliptik, $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{4\eta}\frac{\partial u}{\partial \xi} = 0$. **93.** $\xi = 4x^3 - 3y^2$, $\eta = x$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} + \frac{6\eta^2}{4\eta^3 - \xi}\frac{\partial u}{\partial \eta} = 0$. **94.** $\xi = 2x + \sin y,\ \eta = y$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} = 0$. **95.** $\xi = x + 2e^{-y},\ \eta = 2x$; elliptik, $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$. **96.** $\xi = x + y + \sin x,\ \eta = x - y - \sin x$; giperbolik, $\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{4}\cos\frac{\xi + \eta}{2}\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi}\right) = 0$. **97.** $\xi = y\,tg\frac{x}{2},\ \eta = y$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} - \frac{2\xi}{\xi^2 + \eta^2}\frac{\partial u}{\partial \xi} = 0$. **98.** $\xi = y\,chx,\ \eta = shx$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{1+\eta^2}\left(\xi\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \xi}\right) = 0$. **99.** $\xi = y\sin x$, $\eta = y$; parabolik, $\frac{\partial^2 u}{\partial \eta^2} - \frac{2\xi}{\eta^2}\frac{\partial u}{\partial \xi} = 0$. **100.** $y > 0$ daelliptik, $\xi = x,\ \eta = \frac{2}{3}y^{\frac{3}{2}}$; $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{3\eta}\frac{\partial u}{\partial \eta} = 0$; $y < 0$ dagiperbolik; $\xi = x - \frac{2}{3}(-y)^{\frac{3}{2}}$, $\eta = x + \frac{2}{3}(-y)^{\frac{3}{2}}$, $\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{(\eta - \xi)}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 0$. **101.** $y > 0$ daelliptik, $\xi = x,\ \eta = 2\sqrt{y}$; $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{2\alpha - 1}{\eta}\frac{\partial u}{\partial \eta} = 0$; $y < 0$ da giperbolik, $\xi = x - 2\sqrt{-y}$, $\eta = x + 2\sqrt{-y}$; $\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\alpha - \frac{1}{2}}{(\eta - \xi)}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 0$. **102.** $\xi = x^{\frac{3}{2}},\ \eta = y^{\frac{3}{2}}$ $(x > 0,\quad y < 0)$, va $\xi = (-x)^{\frac{3}{2}},\ \eta = (-y)^{\frac{3}{2}}$ $(x > 0,\quad y < 0)$, elliptik, $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{3\xi}\frac{\partial u}{\partial \xi} + \frac{1}{3\eta}\frac{\partial u}{\partial \eta} = 0$; $\xi = (-x)^{\frac{3}{2}} - y^{\frac{3}{2}},\ \eta = (-x)^{\frac{3}{2}} + y^{\frac{3}{2}}$, $(x > 0,\quad y < 0)$, va $\xi = x^{\frac{3}{2}} - (-y)^{\frac{3}{2}},\ \eta = x^{\frac{3}{2}} + (-y)^{\frac{3}{2}}$, $(x > 0,\quad y < 0)$, giperbolik,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{3}\frac{\partial^2 u}{\partial \eta^2 - \partial \xi^2} + \left(\eta\frac{\partial u}{\partial \xi} - \xi\frac{\partial u}{\partial \eta}\right) = 0.$$

103. $\xi = \sqrt{x},\ \eta = \sqrt{y}$ $(x > 0,\quad y > 0)$, va

$\xi=\sqrt{-x}$, $\eta=\sqrt{-y}$ $(x>0,\quad y>0)$, elliptik, $\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}-\frac{1}{\xi}\frac{\partial u}{\partial\xi}-\frac{1}{\eta}\frac{\partial u}{\partial\eta}=0$;

$\xi=\sqrt{-x}$, $\quad\eta=\sqrt{y}$ $(x<0,\quad y>0)$, $\xi=\sqrt{x}$, $\eta=\sqrt{-y}$ $(x<0,\quad y>0)$,

giperbolik, $\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}-\frac{1}{\xi}\frac{\partial u}{\partial\xi}-\frac{1}{\eta}\frac{\partial u}{\partial\eta}=0$. **104.** $u_{\xi\xi}+u_{\eta\eta}+u_{\zeta\zeta}=0$, $\xi=x$, $\eta=y-x$, $\zeta=x-\frac{1}{2}y+\frac{1}{2}z$. **105.** $u_{\xi\xi}-u_{\eta\eta}+u_{\zeta\zeta}+u_\eta=0$, $\xi=\frac{1}{2}x$, $\eta=\frac{1}{2}x+y$, $\zeta=-\frac{1}{2}x-y+z$.

106. $u_{\xi\xi}-u_{\eta\eta}+2u_\xi=0$, $\xi=x+y$, $\eta=y-x$, $\zeta=y+z$.

107. $u_{\xi\xi}+u_{\eta\eta}=0$, $\xi=x$, $\eta=y-x$, $\zeta=2x-y+z$. **108.** $u_{\xi\xi}-u_{\eta\eta}-u_{\zeta\zeta}=0$, $\xi=x$, $\eta=y-x$, $\zeta=\frac{3}{2}x-\frac{1}{2}y+\frac{1}{2}z$. **109.** $u_{\xi\xi}+u_{\eta\eta}+u_{\zeta\zeta}+u_{\tau\tau}=0$, $\xi=x$, $\eta=y-x$, $\zeta=x-y+z$, $\tau=2x-2y+z+t$.

110. $u_{\xi\xi}-u_{\eta\eta}+u_{\zeta\zeta}+u_{\tau\tau}=0$, $\xi=x+y$, $\eta=y-x$, $\zeta=z$, $\tau=y+z+t$.

111. $u_{\xi\xi}-u_{\eta\eta}+u_{\zeta\zeta}+u_{\tau\tau}=0$, $\xi=x+y$, $\eta=y-x$, $\zeta=-2y+z+t$, $\tau=z-t$.

112. $u_{\xi\xi}-u_{\eta\eta}+u_{\zeta\zeta}=0$, $\xi=x$, $\eta=y-x$, $\zeta=2x-y+z$, $\tau=x+z+t$.

113. $u_{\xi\xi}+u_{\eta\eta}=0$, $\xi=x$, $\eta=y$, $\zeta=-x-y+z$, $\tau=x-y+t$.

114. $\sum_{k=1}^{n}u_{\xi_k\xi_k}=0$, $\xi_k=\sum_{l=1}^{k}x_l$, $k=1,2,\dots,n$. **115.** $\sum_{k=1}^{n}(-1)^{k+1}u_{\xi_k\xi_k}=0$, $\xi_k=\sum_{l=1}^{k}x_l$, $k=1,2,\dots,n$.

116. $\sum_{k=1}^{n}u_{\xi_k\xi_k}=0$, $\xi_1=x_1$, $\xi_k=x_k-x_{k-1}$, $k=2,3,\dots,n$. **117.** $\sum_{k=1}^{n}u_{\xi_k\xi_k}=0$, $\xi_k=\sqrt{\frac{2k}{k+1}}\left(x_k-\frac{1}{k}\sum_{l<k}x_l\right)$, $k=1,2,\dots,n$. **118.** $u_{\xi_1\xi_1}-\sum_{k=2}^{n}u_{\xi_k\xi_k}=0$, $\xi_1=x_1-x_2$, $\xi_k=\sqrt{\frac{2(k-2)}{k+2}}\left(x_k-\frac{1}{k-2}\sum_{l<k}x_l\right)$, $k=3,4,\dots,n$.

3-bob.

1. $f(y+ax)+g(y-ax)$. **2.** $f(x-y)+g(3x+y)$. **3.** $f(y)+g(x)e^{-ay}$.

4. $x-y+f(x-3y)+g(2x+y)e^{\frac{3y-x}{7}}$. **5.** $[f(x)+g(y)]e^{-bx-ay}$. **6.** $e^{x+y}+[f(x)+g(y)]e^{3x+2y}$.

7. $f(y-ax)+g(y-ax)e^{-x}$. **8.** $f(x+y)+(x-y)g(x^2-y^2)$ $(x>-y$ yoki $x<-y)$.

9. $f(xy)+|xy|^{\frac{3}{4}}g\left(\frac{x^3}{y}\right)$, (har bir kvadrantda). **10.** $f\left(\frac{x}{y}\right)+xg\left(\frac{x}{y}\right)$, $(x^2+y^2\neq 0)$.

11. $xf(y)-f'(y)+\int_0^x (x-\xi)g(\xi)e^{\xi y}d\xi$. **Ko'rsatma.** $u_x=v$ belgilash kiritib, $u=xv-v_y$, $v_{xy}-xv_x=0$ munosabatlarni oling. **12.** $yg(x)+\frac{1}{x}g'(x)+\int_0^y (y-\xi)f(\xi)e^{-x^2\xi}d\xi$. **Ko'rsatma.** $u_y=v$ belgilash kiritib, $u=\frac{1}{2x}v_x+yv$, $v_{xy}+2xyv_y=0$ munosabatlarni oling.

13. $e^{-y}\left[yf(x)+f'(x)+\int_0^y (y-\eta)g(\eta)e^{-x\eta}d\eta\right]$ **Ko'rsatma.** $u_y+u=v$ belgilash kiritib, $u=v_x+yv$, $v_{xy}+v_x+yv_y+yv=0$ munosabatlarni oling. **14.** $e^{-y^2}\left[yf(x)+f'(x)+\int_0^y (y-\eta)g(\eta)e^{-x\eta}d\eta\right]$ **Ko'rsatma.** $u_y+u=v$ belgilash kiritib, $u=v_x+2yv$, $(v_y+xv)_x+2y(v_y+xv)=0$ munosabatlarni oling. **15.** $u=\varphi(x-t)+\psi(x)$; **16.** $u=\varphi(x+y)+\psi(2x+y)$; **17.** $u=\varphi(x+2y)+\psi(x+2y)e^{\frac{x}{2}}$; **18.** $u=\varphi(4x+y)e^{x+\frac{y}{3}}+\psi(2x+y)$; **19.** $u=\varphi(x-y)+\psi(x+3y)e^{\frac{y-x}{4}}$; **20.** $u=\varphi(x+3y)+\psi(x+3y)e^{-\frac{x}{3}}$; **21.** $u=\varphi(x+2y)+\psi(3x+2y)e^{-\frac{x+2y}{4}}$; **22.** $u=\varphi(y-3x)+\psi(y-3x)e^{-x}$; **23.** $u=\varphi(2x-y)+\psi(2x-y)e^{-x}$; **24.** $u=\varphi(x-5y)e^{-\frac{x-y}{4}}+\psi(x-y)$; **25.** $u=\varphi(2x+3y)+\psi(2x+3y)e^{-\frac{x}{3}}$; **26.** $u=\varphi(2x-y)+\psi(x+3y)e^{y-2x}$; **27.** $u=\varphi(3x-y)+\psi(x+y)e^{-\frac{3x-y}{2}}$; **28.** $u=\varphi(x+3y)+\psi(x+y)e^{-\frac{x+3y}{2}}$; **29.** $u=\varphi(x)+\psi(x-e^y)e^{-x}$; **30.** $u=\varphi(x+\cos y)\frac{1}{x}+\psi(x)$; **31.** $\xi=x+y,\ \eta=5x-y$; $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{6}\frac{\partial u}{\partial\eta}=0$; $u=\varphi(x+y)+\psi(5x-y)e^{-\frac{x+y}{6}}$; **32.** $\xi=y$, $\eta=y-\cos x$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{\partial u}{\partial\eta}=0$; $u=\varphi(y)+\psi(y-\cos x)e^y$; **33.** $\xi=xy^4,\ \eta=y$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{3}{\eta}\frac{\partial u}{\partial\xi}=0$; $u=\varphi(xy^4)y^3+\psi(y)$; **34.** $\xi=x^2+y,\ \eta=x$; $\frac{\partial^2 u}{\partial\eta^2}-\frac{\partial u}{\partial\eta}=0$; $u=\varphi(x^2+y)+\psi(x^2+y)e^x$; **35.** $\xi=xy,\ \eta=y$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$; $u=y\varphi(xy)+\psi(y)$; **36.** $\xi=xy^2,\ \eta=x$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$; $u=\varphi(xy^2)x+\psi(x)$; **37.** $\xi=x^2+y,\ \eta=x$; $\frac{\partial^2 u}{\partial\eta^2}-\frac{1}{\eta}\frac{\partial u}{\partial\eta}=0$; $u=\varphi(x^2+y)x^2+\psi(x^2+y)$; **38.** $\xi=x^3y,\ \eta=x$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{2}{\eta}\frac{\partial u}{\partial\xi}=0$;

$u=\varphi(x^3y)x^2+\psi(x)$; **39.** $\xi=xy,\eta=y$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{3}{\eta}\frac{\partial u}{\partial\xi}=0$; $u=\varphi(xy)y^3+\psi(y)$;

$u=\varphi(xy^4)y^3+\psi(y)$; **40.** $\xi=\sin x+y,\eta=x$; $\frac{\partial^2 u}{\partial\eta^2}-2\frac{\partial u}{\partial\eta}=0$; $u=\varphi(\xi)+\psi(\xi)e^{2\eta}$; **41.**

$\xi=\frac{y}{x},\eta=y$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$; $u=\varphi(\frac{y}{x})y+\psi(y)$; **42.** $\xi=xy^4,\eta=x$; $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$;

$u=\frac{1}{x}\varphi(xy^4)+\psi(x)$; **43.** $\xi=xy,\eta=y$; $\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial\eta}=0$;V $u=\varphi(xy)\ln y+\psi(xy)$; **44.**

$\xi=xt,\eta=\frac{x}{t}$; $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{2\xi}\frac{\partial u}{\partial\eta}=0$; $u=\varphi(xt)+\sqrt{xt}\psi(\frac{x}{t})$; **45.** $\xi=xy^3,\eta=y$;

$\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$; $u=\varphi(xy^3)y+\psi(y)$; **46.** $\xi=x,\eta=xy^3$, $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{1}{3\xi}\frac{\partial u}{\partial\eta}=0$,

$u=\varphi(x)+x^{\frac{1}{3}}\psi(xy^3)$; **47.** $\xi=xy^2,\eta=y$, $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{3}{\eta}\frac{\partial u}{\partial\xi}=0$, $u=\varphi(xy^2)y^3+\psi(y)$; **48**

$\xi=x+y+\cos x$, $\eta=x-y-\cos x$, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$, $u=\varphi(\xi)e^{-\frac{\eta}{2}}+\psi(\eta)$; **49.**

$\xi=x+y+\cos x$, $\eta=x-y-\cos x$, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$, $u=\varphi(\xi)+\psi(\eta)$; **50.** $\xi=2x-y+\cos x$,

$\eta=2x+y-\cos x$, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$, $u=\varphi(\eta)+\psi(\xi)$; **51.** $\xi=2x-y+\cos x$, $\eta=2x+y-\cos x$,

$\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{4}\frac{\partial u}{\partial\xi}=0$, $u=\varphi(\eta)+\psi(\xi)e^{-\frac{\eta}{4}}$; **52.** $\xi=x^2y,\eta=xy$, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{\eta}\frac{\partial u}{\partial\xi}=0$,

$u=\frac{1}{xy}\varphi(x^2y)+\psi(xy)$; **53.** $\xi=xy^{\frac{1}{4}},\eta=xy^{\frac{1}{2}}$, $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{2}{\eta}\frac{\partial u}{\partial\xi}=0$, $u=\varphi(\xi)\eta^2+\psi(\eta)$;

53. $\xi=x^2+y^2,\eta=x$, $\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{2}{\eta}\frac{\partial u}{\partial\xi}-\eta^3=0$, $u=\varphi(\xi)+\psi(\xi)\eta^3+\frac{\eta^5}{10}$; **55.** $\xi=x$,

$\eta=x^2+y$, $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{\eta}\frac{\partial u}{\partial\xi}+1=0$, $u=\frac{1}{\eta}\varphi(\xi)+\psi(\eta)-\frac{\xi\eta}{2}$.

4-bob.

1. $\frac{4}{5}\left(y^{\frac{5}{4}}-|x|^{\frac{5}{2}}\right)$; $|x|<1,\ 0<y<1$. **2.** $\sin y-1+e^{x-y}$; $-\infty<x,y>\infty$. **3.** $x-y-\frac{1}{2}+\frac{1}{2}e^{2y}$;

$-\infty<x,y>\infty$. **4.** $\frac{1}{2}\left[1-x-3y+(x+y-1)e^{2x}\right]$; $-\infty<x,y<\infty$. **5.**

$xy+\frac{3}{2}\sin\frac{2y}{3}\cos\left(x+\frac{y}{3}\right)$; $-\infty<x,y<\infty$.

$u=y^2+(x^2-1)y^7$; **43.** $u=xy^4+1$; **44.** $u=(x-1)y^5$; **45.** $\xi=x^3y^2,\ \eta=x$;

$\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{2}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\dfrac{\varphi(\xi)}{\eta^2}+\psi(\eta)$; $u=\dfrac{1}{4}y^4x^4+2+3x^2-\dfrac{1}{4}x^4$; **46.** $\xi=x^2y^3,\ \eta=y$;

$\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{1}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\dfrac{1}{\eta}\varphi(\xi)+\psi(\eta)$;

$u=y^2x^2+3-\dfrac{3}{4}yx^{\frac{4}{3}}+y^5-y^2+\dfrac{3}{4}y$; **47.** $\xi=x^4y^3,\ \eta=x$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{5}{3\eta}\dfrac{\partial u}{\partial\xi}=0$;

$u=\dfrac{\varphi(\xi)}{\eta^{\frac{5}{3}}}+\psi(\eta)$; $u=\dfrac{4}{5}y^{\frac{5}{4}}-xy^2+3x^3+3x-\dfrac{4}{5}$; **48.** $\xi=x^5y^2,\ \eta=x$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{3}{\eta}\dfrac{\partial u}{\partial\xi}=0$;

$u=\dfrac{\varphi(\xi)}{\eta^3}+\psi(\eta)$;

$u=\dfrac{25}{8}xy^{\frac{4}{3}}+\dfrac{5}{3}y^{\frac{4}{3}}+3x^2-\dfrac{25}{8}x-\dfrac{2}{3}$; **49.** $\xi=x^3y^4,\ \eta=y$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{5}{3\eta}\dfrac{\partial u}{\partial\xi}=0$;

$u=\dfrac{\varphi(\xi)}{\eta^{\frac{2}{3}}}+\psi(\eta)$;

$u=\dfrac{6}{7}x^{\frac{2}{3}}y^3+1-\dfrac{6}{7}y^3$; **50.** $\xi=xy^3,\ \eta=y$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{2}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\dfrac{\varphi(\xi)}{\eta^2}+\psi(\eta)$;

$u=\dfrac{3}{2}x^{\frac{2}{3}}-xy+3y^2-\dfrac{3}{2}+y$; **51.** $\xi=y^3x^2,\ \eta=x$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{3\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\dfrac{\varphi(\xi)}{\eta^{\frac{1}{3}}}+\psi(\eta)$;

$u=2(y^2-1)+\dfrac{1}{5}x^2(1-y^5)+3x^2$; **52.** $\xi=xy^4,\ \eta=y$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}-\dfrac{3}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\eta^3\varphi(\xi)+\psi(\eta)$;

$u=3y+(1-x^{-\frac{1}{4}})8y^2$; **53.** $u=\dfrac{x^2}{t}+(xt)^2$; **54.** $u=5x^4y^2-3x^2y^3$; **55.** $u=5x^4y^2-3x^2y^3$;

56. $u=2\sqrt{xt}$; **57.** $\xi=xy^2,\ \eta=\dfrac{y^2}{x}$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}-\dfrac{1}{2\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\eta^{\frac{1}{2}}\varphi(xt)+\psi(\eta)$;

$u=\dfrac{2y}{\sqrt{x}}+\dfrac{y}{\sqrt{x}}\ln x$; **58.** $\xi=x^3y,\ \eta=\dfrac{x^3}{y}$; $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{5}{6\eta}\dfrac{\partial u}{\partial\xi}=0$; $u=\dfrac{1}{\eta^{\frac{5}{6}}}\varphi(\xi)+\psi(\eta)$;

$u=\dfrac{1}{3}x^{\frac{7}{3}}x^2+\dfrac{3}{7}y^2x+\dfrac{18}{7}\dfrac{x}{y^{\frac{1}{3}}}-\dfrac{1}{3}\dfrac{x^2}{y^{\frac{2}{3}}}$; **59.** $u=x(1+y)$. **60.** $u=(x^4+x^{\frac{1}{3}})y^2$; **61.**

$u=1+\sin(x-y-\cos x)+e^{y+\cos x}\sin(x+y+\cos x)$; **62.** $u=1+\cos x\cdot\cos(y+\cos x)$; **63.**

$u=\sin x\cdot\cos(\dfrac{y-\cos x}{2})+e^x sh(\dfrac{y-\cos x}{2})$; **64.** $u=2e^{-\frac{2x-y-\cos x}{4}}\cos x\cdot\sin\dfrac{y-\cos x}{2}$; **65.**

$u=e^{2\eta}\varphi(\xi)+\psi(\eta)$; $u=\dfrac{3}{22}e^{\frac{11}{3}x}+\dfrac{19}{22}$; **66.** $u=e^{-\frac{1}{3}\eta}\varphi(\xi)+\psi(\eta)$; $u=-\dfrac{112}{47}e^{-\frac{47}{16}x}+\dfrac{347}{47}$; **67.**

$u=e^{4\eta}\varphi(\xi)+\psi(\eta)$; $u=\dfrac{7}{86}e^{\frac{43}{7}x}+\dfrac{79}{86}$; **68.** $u=e^{-3\eta}\varphi(\xi)+\psi(\eta)$; $u=-\dfrac{2}{19}e^{-\frac{19}{2}x}+\dfrac{40}{19}$; **69.**

$x^2+xt+4t^2+\frac{1}{6}xt^3$. **106.** $\sin x$. **107.** $xt+\sin(x+t)-(1-ch t)e^x$. **108.** $1+t+\frac{1}{9}(1-\cos 3t)\sin x$. **109.** $\frac{1}{a^2\omega^2}(1-\cos a\omega t)\sin\omega x$. **110.** $\frac{t}{\omega}-\frac{1}{\omega^2}\sin\omega t$. **111.** $x+ty+t^2$.

112. $xy(1+t^2)+x^2-y^2$. **113.** $\frac{1}{2}t^2(x^3-3xy^2)+e^x\cos y+te^y\sin x$. **114.** $x^2+t^2+t\sin y$.

115. $2x^2-y^2+(2x^2+y^2)t+2t^2+2t^3$.

116. $x^2+ty^2+\frac{1}{2}t^2(6+x^3+y^3)+t^3+\frac{3}{4}t^4(x+y)$. **117.** $e^{3x+4y}\left[\frac{25}{26}ch5t-\frac{1}{25}+\frac{1}{5}sh5t\right]$.

118. $\cos(bx+cy)\cos\left(at\sqrt{b^2+c^2}\right)+\frac{1}{a\sqrt{b^2+c^2}}\sin(bx+cy)\sin\left(at\sqrt{b^2+c^2}\right)$.

119. $(x^2+y^2)^2(1+t)+8a^2t^2(x^2+y^2)\left(1+\frac{1}{3}t\right)+\frac{8}{3}a^4t^4\left(1+\frac{1}{5}t\right)$.

120. $(x^2+y^2+4a^2)(e^t-1-t)-2at^2\left(1+\frac{1}{3}t\right)$. **121.** $x^2+y^2-2z^2+t+t^2xyz$

122. $y^2+tz^2+8t^2+\frac{8}{3}t^3+\frac{1}{12}t^4x^2+\frac{2}{45}t^6$.

123. $x^2y^2z^2+txy+3t^2(x^2+y^2+z^2+x^2y^2+x^2z^2+y^2z^2)+\frac{3}{2}t^4(3+x^2+y^2+z^2)+\frac{9}{10}t^6$.

124. $e^{x+y}\cos(z\sqrt{2})+te^{3y+4z}\sin 5x+t^3e^{x\sqrt{x}}\sin y\cos z$.

125. $(1+t)(x^2+y^2+z^2)^2+10a^2t^2\left(1+\frac{1}{3}t\right)(x^2+y^2+z^2)+a^4t^4(5+t)$.

126. $(x^2+y^2+z^2+6a^2)(e^t-1-t)-a^2t^2(3+t)$.

127. $\frac{1}{a^2}(1-\cos at)e^z\cos x\sin y+e^{y+z}\left[\frac{1}{a}sh at\sin x+\frac{at}{\sqrt{2}}sh\left(at\sqrt{2}\right)+x^2ch\left(at\sqrt{2}\right)\right]$.

128. $xy\cos z\cos at+\frac{1}{a}yze^x sh at+\frac{x}{1+25a^2}\cos(3y+4z)\left(e^t-\cos 5at-\frac{1}{5a}\sin 5at\right)$.

129.

$$\left(\cos at+\frac{1}{a}\sin at\right)\cos\sqrt{x^2+y^2+z^2}+\frac{1}{\sqrt{x^2+y^2+z^2}}\sin\sqrt{x^2+y^2+z^2}\left(t\cos at-at\sin at-\frac{1}{a}\sin at\right)$$

4.3.

1. $1+e^t+\frac{1}{2}t^2$. **2.** $t^3+e^{-t}\sin x$. **3.** $(1+t)e^{-t}\cos x$. **4.** $ch t\sin x$. **5.** $1-\cos t+(1+4t)^{-\frac{1}{2}}e^{-\frac{x^2}{1+t}}$.

6. $(1+t)^{-\frac{1}{2}}e^{\frac{2x-x^2+t}{1+t}}$. **7.** $x(1+4t)^{-\frac{3}{2}}e^{\frac{-x^2}{1+4t}}$. **8.** $(1+t)^{-\frac{1}{2}}\sin\frac{x}{1+t}e^{-\frac{4x^2+t}{4(1+t)}}$. **9.** $e^t-1+e^{-2t}\cos x\sin y$.

10. $1+\frac{1}{5}\sin x\sin y(2\sin t-\cos t+e^{-2t})$ **11.** $\sin t+\frac{xy}{(1+4t)^3}e^{-\frac{x^2+y^2}{1+4t}}$. **12.** $\frac{t}{8}+\frac{1}{\sqrt{1+t}}e^{-\frac{(x-y)^2}{1+t}}$.

13. $\frac{1}{\sqrt{1+t^2}}\cos\frac{xy}{1+t^2}e^{\frac{t(x^2+y^2)}{2(1+t^2)}}$. **14.** $\frac{1}{4}\cos x\left(e^{-2t}-1+2t\right)\cos y\cos ze^{-4t}$. **15.** $e^t-1+\sin(x-y-z)e^{-3t}$.

16. $\frac{1}{4}\left(1-e^{-t}\right)+\frac{\cos 2y}{\sqrt{1+t}}e^{-t-\frac{x^2}{1+t}}$. **17.** $\frac{1}{3}\cos(x-y+z)\left(1-e^{-3t}\right)+\frac{1}{\sqrt{1+12t}}e^{\frac{(x+y-z)^2}{1+12t}}$.

18. $\frac{\sin z}{\sqrt{1+4t^2}}\cos\frac{xy}{1+4t^2}e^{-t-\frac{t(x^2+y^2)}{1+4t^2}}$. **19.** $e^{-nt}\cos\sum_{k=1}^{n}x_k$. **20.** $(1+4t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{1+4t}}$. **21.** $(1+4t)^{-\frac{n+2}{2}}e^{-\frac{|x|^2}{1+4t}}$.

22. $(1+4t)^{-\frac{n}{2}}\sin\frac{\sum_{k=1}^{n}x_k}{1+4t}e^{-\frac{nt+|x|^2}{1+4t}}$. **23.** $\frac{1}{\sqrt{1+4nt}}e^{-\frac{1}{1+4nt}\left(\sum_{k=1}^{n}x_k\right)^2}$.

5-bob

1. $-\frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x}{(2k+1)^3}\cos\left(\sqrt{(2k+1)^2\pi^2+4t}\right)$.

2. $-\frac{8e^{-t}}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\left[\cos(2k+1)t+\sin(2k+1)t\right]\sin(2k+1)x$.

3. $8e^{-t}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left[(-1)^k-\frac{2}{\pi(2k+1)}\right]\sin\frac{(2k+1)}{2}t\cos\frac{(2k+1)}{2}x$.

4. $t(1-x)+\sum_{k=1}^{\infty}e^{-\frac{t}{2}}\frac{1}{(k\pi)^3}\left[2\cos\lambda_k t+\frac{1}{\lambda_k}\sin\lambda_k t-2\right]\sin\pi kx,\quad \lambda_k=\sqrt{(k\pi)^2-\frac{1}{4}}$.

5. $t(2-x)+\sum_{k=1}^{\infty}\left[\frac{4t}{k\pi\lambda_k^2}-\frac{k\pi^3}{\lambda_k^3}\sin\lambda_k t\right]\sin\frac{\pi kx}{2},\quad \lambda_k=\sqrt{\left(\frac{k\pi}{2}\right)^2-1}$.

6. $\frac{xt}{l}+\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{\pi k\lambda_k^2}\left[t-\frac{1}{\lambda_k}\sin\lambda_k t\right]\sin\frac{\pi kx}{l},\quad \lambda_k=\sqrt{\left(\frac{k\pi}{l}\right)^2-1}$.

7. $\sin 2x\cos 2t+\sum_{k=1}^{\infty}(-1)^k\frac{2}{k^3}\left[1-\cos kt\right]\sin kx$

8.

$$-\sum c_k\left[-1+e^{-\frac{t}{2}}\left(\cos\mu_k t+\frac{1}{\mu_k}\sin\mu_k t\right)\right]\sin(2k+1)\pi x,\quad c_k=\frac{4}{(2k+1)^3\pi^3},\quad \mu_k=\sqrt{(2k+1)^2\pi^2-\frac{1}{4}}.$$

Ko‘rsatma. Yechimni $u(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin k\pi x$ qator ko‘rinishida qidiring.

Izoh. Yechimni $u=v+\omega$ yig‘indi ko‘rinishida qidirish mumkin, bu yerda $v=\frac{1}{2}x(1-x)$ funksiya tenglamani va chegaraviy shartlarni qanoatlantiradi. U holda

$$u(x,t)=\frac{x(1-x)}{2}-\sum_{k=0}^{\infty}\left(\cos\mu_k t+\frac{1}{2\mu_k}\sin\mu_k t\right)e^{-\frac{t}{2}}\sin(2k+1)\pi x.$$

9. $2xt+\left(2e^t-e^{-t}-3te^{-t}\right)\cos x.$ **10.** $3+x\left(t+t^2\right)+\left(5te^t-8e^t+4t+8\right)\sin x.$

11. $x(t+1)+\left(\frac{1}{5}e^{\frac{5}{2}t}-e^{\frac{t}{2}}+\frac{4}{5}\right)\cos\frac{3}{2}x.$ **12.** $xt+\left(\frac{1}{10}-\frac{1}{6}e^{2t}+\frac{1}{15}e^{5t}\right)e^{-x}\sin 3x.$

13. $xt+\left(1-e^{-t}-te^{-t}\right)\cos 3x.$ **14.** $\frac{1}{8}\left(e^{2t}+e^{-2t}\right)-\frac{1}{4}-\frac{t^2}{2}\cos 2x.$ **15.**

$\frac{1}{9}\sin x(ch3t-1)+\sin 3x(cht-1).$

16. $xt+\left(2e^t-e^{2t}\right)e^{-x}\sin x.$ **17. 18. 19. 20.**

73. $\sum_{n=1}^{\infty}a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}\sin\frac{\pi nx}{l}$, bu yerda $a_n=\frac{2}{l}\int_0^l u_0(x)\sin\frac{\pi nx}{l}dx$,

$u_0(x)=A=const$, bo'lgani uchun $u(x,t)=\frac{4A}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-\frac{(2k+1)^2\pi^2a^2}{l^2}t}\sin\frac{(2k+1)\pi x}{l}.$

74. $u(x,t)=\frac{8Al^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-\frac{(2k+1)^2\pi^2a^2}{l^2}t}\sin\frac{(2k+1)\pi x}{l}.$

75. $\frac{2}{l}\sum_{n=1}^{\infty}a_n\frac{\sigma^2+\mu_n^2}{\sigma(\sigma+1)+\mu_n^2}e^{-\frac{\mu_n^2a^2t}{l^2}}\sin\frac{\mu_n x}{l}$, bu yerda $a_n=\int_0^l u_0(x)\sin\frac{\mu_n x}{l}dx,$

μ_n, $(n=1,2,\ldots)$ - $tg\mu=-\frac{\mu}{\sigma}$, $\sigma=hl>0$ tenglamaning musbat ildizlari. **76.**

$\frac{2}{l}\sum_{n=1}^{\infty}b_n e^{-\frac{\mu_n^2a^2t}{l^2}}\frac{\mu_n\cos\frac{\mu_n x}{l}+\sigma\sin\frac{\mu_n x}{l}}{\sigma(\sigma+2)+\mu_n^2}$, bu yerda $b_n=\int_0^l u_0(x)\left(\mu_n\cos\frac{\mu_n x}{l}+\sigma\sin\frac{\mu_n x}{l}\right)dx,$

μ_n, $(n=1,2,\ldots)$- $ctg\mu=\frac{1}{2}\left(\frac{\mu}{\sigma}-\frac{\sigma}{\mu}\right)$, $\sigma=hl>0$ tenglamaning musbat ildizlari.

77. u_0. **78.** $\frac{u_0}{2}+\frac{2u_0}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}e^{-\frac{(2k+1)^2\pi^2a^2t}{l^2}}\cos\frac{(2k+1)\pi x}{l}$, $\lim_{t\to\infty}u(x,t)=\frac{u_0}{2}.$

79. $\frac{u_0}{2}-\frac{4u_0}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^2}e^{-\frac{4(2k+1)^2\pi^2a^2t}{l^2}}\cos\frac{2(2k+1)\pi x}{l}$, $\lim_{t\to\infty}u(x,t)=\frac{u_0}{2}.$

80. $\frac{32}{\pi^3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^3}e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t}\cos\frac{(2n+1)\pi x}{2}$. **81.**

$\frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}e^{-\left(\left(\frac{(2k+1)\pi}{l}\right)^2+1\right)t}\sin\frac{(2k+1)\pi x}{l}$.

82. $-\frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}e^{-(2k+1)^2 t}\sin(2k+1)x$. **83.** $x-l+\frac{8l}{\pi^2}\sum_{k=0}^{\infty}\frac{e^{-\lambda_k^2 t}}{(2k+1)^2}\cos\lambda_k x,\ \lambda_k=\frac{\pi(2k+1)}{2l}$.

84. $t\cos x+\frac{1}{8}\left(e^{-8t}-1\right)\cos 3x$. **85.** $xt+\sin\pi x\, e^{x-t-\pi^2 t}$. **86.** $x+t\sin x+\frac{1}{8}\left(1-e^{-8t}\right)\sin 3x$.

87. $tx^2+\frac{1}{4}\left(e^{4t}-1\right)+t\cos 2x$. **88.** $t+1+\left(1-e^{-t}\right)e^x\sin x+e^{x-4t}\sin 2x$.

89. $xt^2+e^t+\sin t-\cos t+e^{-3t}\cos 2x$. **90.** $x^2+2e^{9t}+(2t-\sin 2t)\cos 3x$.

91. $x+t^2+\frac{1}{5}\left(e^{5t}-1\right)\cos x+\frac{1}{3}\left(-e^{-3t}+1\right)\cos 3x$.

92. $x^2t+x+\sum_{k=1}^{\infty}\frac{C_{2k-1}}{(2k-1)^2-6}\left(1-e^{-6(2k-1)^2 t}\right)\cos(2k-1)x,\ C_{2k-1}=\frac{2}{\pi}\left(\frac{1}{2k+1}-\frac{1}{2k-3}\right)$.

93. $(x+1)t+e^{-2x}\sum_{k=1}^{\infty}\frac{C_k}{k^2\pi^2+4}\left(1-e^{-(k^2\pi^2+4)t}\right)\sin k\pi x$,

$$C_k=\begin{cases}0, & \text{agar}\quad n=2m\\ \frac{1}{\pi}\left(\frac{2}{2m-1}+\frac{1}{2m+1}+\frac{1}{2m-3}\right), & \text{agar } n=2m-1.\end{cases}$$

6-bob.

1. Agar $\lambda=-2$ bo‘lsa, yechim yo‘q. Agar $\lambda\neq-2$ bo‘lsa, u holda $\varphi(x)=\frac{2x(\lambda+1)-\lambda}{\lambda+2}$.

2. Agar $\lambda\neq\lambda_1$, bu yerda $\lambda_1=\frac{1}{e^2-1}$ bo‘lsa, u holda $\frac{e^x}{1-\lambda(e^2-1)}$ bo‘ladi. $\lambda=\lambda_1$ da yechim yo‘q. **3.** Agar $\lambda\neq2$ va $\lambda\neq-6$ bo‘lsa, u holda $\frac{12\lambda^2x-24\lambda x-\lambda^2+42\lambda}{6(\lambda+6)(2-\lambda)}$. $\lambda=2$ va $\lambda=-6$ da tenglama yechimga ega emas.

4. Agar $\lambda\neq\frac{3}{2}$ va $\lambda\neq\frac{5}{2}$ bo‘lsa, u holda $\frac{5(7+2\lambda)}{7(5-2\lambda)}x^2+x^4$. Agar $\lambda=\frac{3}{2}$ bo‘lsa, $Cx+\frac{25}{7}x^2+x^4$, bu yerda C – ixtiyoriy doimiy. $\lambda=\frac{5}{2}$ da tenglama yechimga ega emas. **5.** Agar $\lambda\neq\pm\sqrt{\frac{5}{12}}$ bo‘lsa, u holda $\frac{2\lambda}{12\lambda^2-5}\left(5\sqrt[3]{x}+6\lambda\right)+1-6x^2$. $\lambda=\pm\sqrt{\frac{5}{12}}$ da tenglama yechimga ega emas. **6.**

Agar $\lambda \neq \frac{5}{2}$ va $\lambda \neq \frac{1}{2}$ bo‘lsa, u holda $\frac{5(2\lambda-3)}{3(5-2\lambda)}x^4+x^2$. Agar $\lambda=\frac{1}{2}$ bo‘lsa, $Cx^3+x^2-\frac{5}{6}x^4$, bu yerda C – ixtiyoriy doimiy. $\lambda=\frac{5}{2}$ da tenglama yechimga ega emas. **7.** Agar $\lambda \neq \frac{5}{2}$ va $\lambda \neq \frac{1}{2}$ bo‘lsa, u holda $\frac{20\lambda}{1-2\lambda}x^2+7x^4+3$. Agar $\lambda=\frac{5}{2}$ bo‘lsa, $7x^4+3-\frac{50}{3}x^2+Cx$, bu yerda C - ixtiyoriy doimiy. $\lambda=\frac{1}{2}$ da tenglama yechimga ega emas. **8.** Agar $\lambda \neq \pm\frac{3}{2}$bo‘lsa, u holda $\frac{3(5-2\lambda)}{5(3+2\lambda)}x+x^3$. Agar $\lambda=\frac{3}{2}$ bo‘lsa, $\frac{1}{5}x+x^3+Cx^2$, bu yerda C – ixtiyoriy doimiy. $\lambda=-\frac{3}{2}$ da tenglama yechimga ega emas. **9.** Agar $\lambda=\lambda_1=\frac{1}{8}$ va $\lambda \neq \frac{1}{2}$ bo‘lsa, u holda $C_1+\frac{3}{2}x$. Agar $\lambda=\lambda_2=\frac{5}{8}$ bo‘lsa, $C_2(3x^2-1)-\frac{3}{2}x$, bu yerda C_1, C_2 - ixtiyoriy doimiylar. $\lambda=\lambda_3=\frac{3}{8}$ da tenglama yechimga ega emas. Agar $\lambda \neq \lambda_i$, $i=1,2,3$ bo‘lsa, u holda $\varphi(x)=\frac{3x}{3-8\lambda}$. **10.** Agar $\lambda \neq \frac{3}{4}$ va $\lambda \neq -\frac{3}{2}$ bo‘lsa, u holda $\frac{12\lambda}{3-4\lambda}\sin 2x+\pi-2x$. Agar $\lambda=-\frac{3}{2}$ bo‘lsa, $\pi-2x-2\sin 2x+C\cos 2x$, bu yerda C – ixtiyoriy doimiy. $\lambda=\frac{3}{4}$ da tenglama yechimga ega emas. **11.** Agar $\lambda \neq -\frac{3}{4}$ va $\lambda \neq -\frac{3}{2}$ bo‘lsa, u holda $\frac{3\pi\lambda}{2(2\lambda+3)}\sin x+\cos 2x$. Agar $\lambda=-\frac{3}{4}$ bo‘lsa, $\cos 2x-\frac{3\pi}{4}\sin x+C\cos x$, bu yerda C - ixtiyoriy doimiy. $\lambda=-\frac{3}{2}$ da tenglama yechimga ega emas. **12.** Agar $\lambda \neq \pm\frac{3}{2\sqrt{2}}$ bo‘lsa, u holda $\sin x+\frac{3\pi\lambda}{8\lambda^2-9}\left(2\lambda\cos 2x+\frac{3}{2}\sin 2x\right)$. Agar $\lambda=\pm\frac{3}{2\sqrt{2}}$ bo‘lsa, tenglama yechimga ega emas. **13.** λ ning barcha qiymatlarida $\frac{\lambda\pi}{2-\lambda\pi}\sin 3x+\cos x$. **14.** Agar $\lambda \neq \pm\frac{1}{2}$ bo‘lsa, u holda $1-\frac{2x}{\pi}-\frac{\pi^2\lambda}{6(2\lambda+1)}\cos x$. Agar $\lambda=\frac{1}{2}$ bo‘lsa, $\frac{4}{3}-\frac{2x}{\pi}+(8+\pi^2\cos x)C$, bu yerda C – ixtiyoriy doimiy. $\lambda=-\frac{1}{2}$ da tenglama yechimga ega emas. **15.** Agar $\lambda \neq \frac{2}{\pi}$ va $\lambda \neq \frac{4}{\pi}$ bo‘lsa, u holda $\cos 4x + 1 + \frac{\pi\lambda}{2-\lambda\pi}$. Agar $\lambda = \frac{4}{\pi}$ bo‘lsa, $\cos 4x - 1 + C_1 \cos 2x + C_2 \sin 2x$, bu yerda C_1, C_2 - ixtiyoriy doimiylar. $\lambda = \frac{2}{\pi}$ da tenglama yechimga ega emas. **16.** Agar $\lambda \neq \frac{1}{\pi}$ bo‘lsa, u holda $\cos 3x$. Agar $\lambda = \frac{1}{\pi}$ bo‘lsa, $\cos 3x + C_1 \cos x + C_2 \cos 2x$, bu yerda C_1, C_2 - ixtiyoriy doimiylar. **17.** Agar $\lambda \neq \frac{1}{\pi}$ va $\lambda \neq \frac{1}{2\pi}$ bo‘lsa, u holda $\frac{\cos x}{1-\lambda\pi}$. Agar $\lambda = \frac{1}{2\pi}$ bo‘lsa, $2\cos x + C\sin 2x$, bu yerda C - ixtiyoriy doimiy. $\lambda = \frac{1}{\pi}$ da tenglama yechimga ega emas. **18.** Agar $\lambda \neq \frac{1}{\pi}$ va $\lambda \neq \frac{1}{3\pi}$ bo‘lsa, u holda $\frac{\sin x}{1-\lambda\pi}$. Agar $\lambda = \frac{1}{3\pi}$ bo‘lsa, $\frac{3}{2}\sin x + C\cos 2x$, bu yerda C - ixtiyoriy doimiy. $\lambda = \frac{1}{\pi}$ da tenglama yechimga ega emas. **19.** $\lambda_1 = \frac{1}{\pi}$, $\sin x + \cos x, 1$; $\lambda_2 = -\frac{1}{\pi}$, $\cos x - \sin x$. **20.** $\lambda_1 = \frac{1}{2\pi}$, 1; $\lambda_2 = \frac{2}{\pi}$, $\cos 2x$; $\lambda_3 = -\frac{2}{\pi}$, $\sin 2x$. **21.** $\lambda_1 = -45$, $3x^2 - 2$; $\lambda_2 = \frac{45}{8}$, $15x^2 - 1$. **22.** $\lambda_1 = \frac{3}{8}$, $3x^{\frac{2}{3}} + x^{-\frac{2}{3}}$; $\lambda_2 = -\frac{3}{2}$, $3x^{\frac{2}{3}} - x^{-\frac{2}{3}}$. **23.** $\lambda_1 = -\frac{2}{\pi}$, $\sin x - \sin 3x$; $\lambda_2 = \frac{2}{\pi}$, $\sin 2x + \sin 3x$, $\sin x + \sin 4x$. **24.** $a = -12, b = 12$, $-12x^2 + C_1 x + C_2$, bu yerda C_1, C_2- ixtiyoriy doimiylar. **25.** $a = \sqrt{15} - 3$, $C\left[4\sqrt{15}x^2 + 3\left(1-\sqrt{15}\right)x\right] + \frac{1}{x} - 3x$, bu yerda C- ixtiyoriy doimiy **26.** Har qanday λ parametr uchun ushbu tenglama yechimga ega: $\varphi(x) = \lambda\int_0^{2\pi}\cos(2x-y)f(y)dy + f(x)$ **27.** Agar $\lambda \neq \frac{1}{2}$ bo‘lsa, u holda $\frac{\lambda a\pi^3}{12(1-2\lambda)}\sin x + \frac{2\lambda b}{1-2\lambda} + ax + b$. Agar $\lambda = \frac{1}{2}$ da, $a = b = 0$ bo‘lsa va faqat shu holda tenglama yechimga ega bo‘lib, yechim: $\varphi(x) = C_1 \cos x + C_2$, bu yerda C_1, C_2- ixtiyoriy doimiylar. **28.** Agar $\lambda \neq \pm\frac{2}{\pi}$ (a, b-ixtiyoriy) bo‘lsa, u holda $\frac{2(a-2\lambda b)}{2+\lambda\pi}\sin x + b$. $\lambda = \frac{2}{\pi}$ da ixtiyoriy a, b larning qiymatida tenglama yechimga ega: $\varphi(x) = \frac{a\pi - 4b}{2\pi}\sin x + b + C_1 \cos x$, bu yerda C_1- ixtiyoriy doimiy; $\lambda = -\frac{2}{\pi}$ da $a\pi + 4b = 0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, yechim: $\varphi(x) = b + C_2 \sin x$, bu yerda c_2- ixtiyoriy doimiy. **29.** Agar $\lambda\neq\frac{1}{2}$va $\lambda\neq\frac{3}{2}$ (a,b,c-ixtiyoriy) bo‘lsa, u holda $\frac{2\lambda a+3c}{3(1-2\lambda)}+\frac{3b}{3-2\lambda}x+ax^2$. $\lambda=\frac{1}{2}$ da $a+3c=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, $\varphi(x)=\frac{3}{2}bx+ax^2+C_1$, bu yerda c_1- ixtiyoriy doimiy; $\lambda=\frac{3}{2}$da $b=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, yechim: $\varphi(x)=ax^2-\frac{1}{2}(a+c)+C_2x$, bu yerda c_2- ixtiyoriy doimiy. **30.** Agar $\lambda\neq\pm\frac{\sqrt{15}}{2}$ (a,b-ixtiyoriy) bo‘lsa, u holda $\frac{2\lambda(5a+3b)}{15-4\lambda^2}x^2+\frac{4\lambda^2(5a+3b)}{5(15-4\lambda^2)}x+ax+bx^3$. $\lambda=\frac{\sqrt{15}}{2}$ da $5a+3b=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, $\varphi(x)=a\left(x-\frac{5}{3}x^3\right)+C_1\left(\sqrt{\frac{5}{3}}x^2+x\right)$, bu yerda c_1- ixtiyoriy doimiy; $\lambda=-\frac{\sqrt{15}}{2}$da $5a+3b=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, yechim: $\varphi(x)=a\left(x-\frac{5}{3}x^3\right)+C_2\left(x-\sqrt{\frac{5}{3}}x^2\right)$, bu yerda c_2- ixtiyoriy doimiy. **31.** Agar $\lambda\neq3$ va $\lambda\neq5$ (a,b-ixtiyoriy) bo‘lsa, u holda $\frac{3a}{3-\lambda}x+\frac{5\lambda b}{3(5-\lambda)}x^2+b$. $\lambda=3$ da $a=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, $\varphi(x)=b\left(\frac{5}{2}x^2+1\right)+C_1$, bu yerda c_1- ixtiyoriy doimiy; $\lambda=5$da $b=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, yechim: $\varphi(x)=C_2x^2-\frac{3}{2}ax$, bu yerda c_2- ixtiyoriy doimiy. **32.** Agar $\lambda\neq\frac{1}{6}$ (a,b-ixtiyoriy) bo‘lsa, u holda $\frac{30\lambda a+7b}{7(1-6\lambda)}x^{\frac{1}{3}}+ax$. $\lambda=\frac{1}{6}$ da $5a+7b=0$ bo‘lsa, va faqat shu holda tenglama yechimga ega bo‘lib, $\varphi(x)=-\frac{7}{5}bx+C_1x^{\frac{1}{3}}+C_2x^{\frac{2}{3}}$, bu yerda c_1 va c_2- ixtiyoriy doimiylar. **33.** Agar $\lambda\neq\frac{2}{\pi}$ va $\lambda\neq\frac{2}{4-\pi}$ (a,b-ixtiyoriy) bo‘lsa, u holda $\frac{2a+\lambda b(4-\pi)}{2-\lambda\pi}+\frac{2}{2-\lambda(4-\pi)}x+bx^2$. $\lambda=\frac{2}{\pi}$ da

$a\pi+b(4-\pi)=0$ bo'lsa, va faqat shu holda tenglama yechimga ega bo'lib, $\varphi(x)=\frac{\pi}{2(\pi-2)}x+bx^2+C$, bu yerda C - ixtiyoriy doimiy; $\lambda=\frac{2}{4-\pi}$ da tenglama yechimga ega emas. **34.** Agar $\lambda\neq\pm\frac{1}{2}\sqrt{\frac{5}{3}}$ (a,b,c-ixtiyoriy) bo'lsa, u holda $\frac{5\lambda(14a+36\lambda b+42c)}{21(5-12\lambda^2)}x^{\frac{1}{3}}+\frac{28\lambda^2a+30\lambda b+35}{7(5-12\lambda^2)}+ax^2+bx$. $\lambda=\frac{1}{2}\sqrt{\frac{5}{3}}$ da $15\sqrt{3}b+7\sqrt{5}(a+c)=0$ bo'lsa, va faqat shu holda tenglama yechimga ega bo'lib, $\varphi(x)=ax^2+bx+c+C_1\left(x^{\frac{1}{3}}+\sqrt{\frac{3}{5}}\right)$, bu yerda C_1- ixtiyoriy doimiy; $\lambda=-\frac{1}{2}\sqrt{\frac{5}{3}}$ da $15\sqrt{3}b-7\sqrt{5}(a+3c)=0$ bo'lsa, va faqat shu holda tenglama yechimga ega bo'lib, yechim: $\varphi(x)=ax^2+bx+c+C_1\left(x^{\frac{1}{3}}-\sqrt{\frac{3}{5}}\right)$, bu yerda C_2- ixtiyoriy doimiy. **35.** Agar $\lambda\neq-\frac{15}{8}$ va $\lambda\neq\frac{3}{2}$ (a,b-ixtiyoriy) bo'lsa, u holda $\frac{30(b-1)\lambda}{15+8\lambda}x^2+\frac{3a\lambda^2}{3-2\lambda}x+\frac{36\lambda^2(b-1)}{(15+8\lambda)(3-2\lambda)}$. $\lambda=-\frac{15}{8}$ da $b=1$ bo'lsa, va faqat shu holda tenglama yechimga ega bo'lib, $\varphi(x)=\frac{17}{2}ax+1-20a+C(x^2+1)$, bu yerda C - ixtiyoriy doimiy; $\lambda=\frac{3}{2}$ da $a=b=0$ bo'lsa, va faqat shu holda tenglama yechimga ega bo'lib, $\varphi(x)=C_1x+C_2$, bu yerda C_1 va C_2- ixtiyoriy doimiylar. **36.1.** $\lambda_1=\frac{3}{2}, \varphi_1=x$; $\lambda_2=-\frac{1}{2}, \varphi_2=3x-4x^2$; agar $\lambda_1\neq\frac{3}{2}$ va $\lambda_2\neq-\frac{1}{2}$ bo'lsa, $\varphi(x)=\frac{3ax}{3-2\lambda}$ (a-ixtiyoriy); $\lambda=\frac{3}{2}$ da tenglama yechimga ega, agar $a=0$ bo'lsa va $\varphi(x)=\frac{3}{4}ax+C_2(3x-4x^2)$, bu yerda C_2- ixtiyoriy doimiy. **36.2.** $\lambda_1=\frac{1}{2}$, $\varphi_1^{(1)}=x$, $\varphi_1^{(2)}=x^2$; agar $\lambda\neq\frac{1}{2}$ bo'lsa, $\varphi(x)=\frac{ax^2+bx}{1-2\lambda}$; $\lambda=\frac{1}{2}$ da tenglama yechimga ega, agar $a=b=0$ bo'lsa, va $\varphi(x)=C_1x^2+C_2x$ bu yerda C_1 va C_2- ixtiyoriy doimiylar. **37.1.** $\lambda_1=\frac{1}{\pi}$, $\varphi_1=\sin x$; agar $\lambda\neq\frac{1}{\pi}$ bo'lsa, $\varphi(x)=a+b\cos x+\lambda b\pi x+\frac{2\pi^2\lambda^2b}{1-\pi\lambda}\sin x$; $\lambda=\frac{1}{\pi}$ da tenglama yechimga ega, agar

$b=0$ bo‘lsa, va $\varphi(x)=a+C\sin x$ bu yerda C - ixtiyoriy doimiy. **37.2.** $\lambda_1=\frac{1}{\pi}$, $\varphi_1=x$; agar $\lambda\neq\frac{1}{2\pi}$ bo‘lsa, $\varphi(x)=\frac{ax}{1-2\pi\lambda}+b+2\lambda b\pi\cos x$ (bu yerda a,b – ixtiyoriy); $\lambda=\frac{1}{2\pi}$ da tenglama yechimga ega, agar $a=0$ bo‘lsa, va $\varphi(x)=b(1+\cos x)+Cx$ bu yerda C - ixtiyoriy doimiy. **38.** $\varphi(x)=\lambda\int_0^{\pi}\frac{\sin(x+y)+\lambda\frac{\pi}{2}\cos(x-y)}{\Delta(\lambda)}f(y)dy+f(x)$, agar $\Delta(\lambda)\neq 0$ bo‘lsa, bu yerda

$\Delta(\lambda)=1-\lambda^2\frac{\pi^2}{4}$; $\lambda=\frac{2}{\pi}$ da tenglama yechimga ega, agar $f_1+f_2=0$ bo‘lsa, bu yerda $f_1=\int_0^{\pi}\cos y f(y)dy$, $f_2=\int_0^{\pi}\sin y f(y)dy$, va yechim: $\varphi(x)=C_1(\sin x+\cos x)+\frac{2}{\pi}f_1\sin x+f(x)$ (C_1 - ixtiyoriy doimiy); $\lambda=-\frac{2}{\pi}$ da tenglama yechimga ega, agar $f_1-f_2=0$ bo‘lsa va yechim: $\varphi(x)=C_2(\sin x-\cos x)-\frac{2}{\pi}f_1\sin x+f(x)$ (C_2 - ixtiyoriy doimiy);

$$R(x,y;\lambda)=\frac{\sin(x+y)+\frac{\lambda\pi}{2}\cos(x-y)}{\Delta(\lambda)}$$ - rezolventa.

39. $\varphi(x)=\lambda\int_{-1}^{1}\frac{1-\frac{4}{3}\lambda+y(2x-4\lambda x-1)}{\Delta(\lambda)}f(y)dy+f(x)$, agar $\Delta(\lambda)\neq 0$ bo‘lsa, bu yerda $\Delta(\lambda)=(1-2\lambda)(1-\frac{4}{3}\lambda)$; $\lambda=\frac{1}{2}$ da tenglama yechimga ega, agar $f_1=3f_2$ bo‘lsa, bu yerda $f_1=\int_{-1}^{1}f(x)dx$, $f_2=\int_{-1}^{1}xf(x)dx$, va yechim: $\varphi(x)=\left(x-\frac{1}{2}\right)f_1+f(x)+C_1$ (C_1 - ixtiyoriy doimiy); $\lambda=\frac{3}{4}$ da tenglama yechimga ega, agar $f_2=0$ bo‘lsa va yechim: $\varphi(x)=-\frac{3}{2}f_1+f(x)+C_2(x+1)$ (C_2 - ixtiyoriy doimiy); $R(x,y;\lambda)=\frac{1-\frac{4}{3}\lambda+y(2x-4\lambda x-1)}{\Delta(\lambda)}$ - rezolventa. **40.** $\varphi(x)=\lambda\int_{-\pi}^{\pi}\left(\frac{x\sin y}{1-2\pi\lambda}+\cos x\right)(ay+b)dy+ax+b=\frac{ax}{1-2\pi\lambda}+2\pi\lambda b\cos x+b$, agar $\lambda\neq\frac{1}{2\pi}$ bo‘lsa (a,b –ixtiyoriy); $\lambda=\frac{1}{2\pi}$ da tenglama yechimga ega, agar $a=0$

bo‘lsa, yechim: $\varphi(x) = b(\cos x + 1) + Cx$ (C - ixtiyoriy doimiy);
$R(x, y; \lambda) = \frac{x \sin y}{1 - 2\pi\lambda} + \cos x$ - rezolventa.

41. $\varphi(x) = \lambda \int_0^{2\pi} \frac{\sin x \sin y + \sin 2x \sin 2y}{1 - \pi\lambda} f(y)dy + f(x)$, agar $\lambda \neq \frac{1}{\pi}$ bo‘lsa; $\lambda = \frac{1}{\pi}$ da tenglama yechimga ega, agar $\int_0^{2\pi} \sin y f(y)dy = \int_0^{2\pi} \sin 2y f(y)dy = 0$ bo‘lsa, yechim: $\varphi(x) = f(x) + C_1 \sin x + C_2 \sin 2x$ (C_1, C_2 - ixtiyoriy doimiylar);
$R(x, y; \lambda) = \frac{\sin x \sin y + \sin 2x \sin 2y}{1 - \pi\lambda}$ - rezolventa. **42.** $b = 0,\ 3a + 5c = 0$.

43. $a = \frac{3}{\sqrt{10}},\ b = 0,\ c = -\frac{1}{\sqrt{10}}$; $a = -\frac{3}{\sqrt{10}},\ b = 0,\ c = \frac{1}{\sqrt{10}}$. **44.** $a = 0,\ b = -\frac{1}{2}$. **45.** $a = 6$.

46. $a = 0,\ b = -1$. **47.** a, b - ixtiyoriy. **48.** a, b, c - ixtiyoriy. **49.** $7a + 5b = 0$. **50.** $\lambda_1 = 1$, $\varphi_1 = 4(x_1 + x_2) + 1$; $\lambda_2 = -1$, $\varphi_2 = 4(x_1 + x_2) - 1$. **51.** $\lambda_1 = \frac{4\sqrt{3} - 6}{\pi}$, $\varphi_1 = 2 + \sqrt{3}(x_1^2 + x_2^2)$; $\lambda_2 = -\frac{4\sqrt{3} + 6}{\pi}$, $\varphi_2 = \sqrt{3}(x_1^2 + x_2^2) - 1$. **52.** $\lambda_1 = \frac{3}{4\pi}$, $\varphi_1 = \frac{1}{1 + r}$, bu yerda $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

7-bob.

1. Analiz kursidan ma’lumki, $x_1, x_2, \ldots, x_n$ dekart ortogonal koordinatalari sistemasidan ixtiyoriy $y_1, y_2, \ldots, y_n$ egri chiziqli koordinalar sistemasiga o‘tishda quyidagi ifoda:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

quyidagi formula bilan ifodalanadi:

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial y_j} \left(\sqrt{g} g^{jk} \frac{\partial u}{\partial y_k}\right)$$

bu yerda $g = \det\|g_{jk}\|$, $g^{jk} = \frac{G^{jk}}{g}$, $G^{jk} = G^{kj} - g_{jk}, g_{jk}$ elementning algebraik to‘ldiruvchisi $\det\|g_{jk}\|$ da,

$$g_{jk}(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_j} \frac{\partial x_i}{\partial y_k}$$

$y_1, y_2, \ldots, y_n$ koordinatalar orthogonal bo‘lganda, $g_{jk} = 0, j \neq k$.

a)

$$\Delta u=\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi}(\sqrt{g}g^{11}\frac{\partial u}{\partial\xi})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi}(\sqrt{g}g^{12}\frac{\partial u}{\partial\eta})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}(\sqrt{g}g^{21}\frac{\partial u}{\partial\xi})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi}(\sqrt{g}g^{22}\frac{\partial u}{\partial\eta})$$

bu yerda $g=(x_\xi y_\eta-y_\xi x_\eta)^2$, $g^{11}=\frac{1}{g}(x_\eta^2+y_\eta^2)$, $g^{12}=g^{21}=-\frac{1}{g}(x_\xi x_\eta-y_\xi y_\eta)$,

$g^{22}=\frac{1}{g}(x_\xi^2+y_\xi^2)$

b) $\Delta u=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}$; **c)** $\Delta u=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}$;

d) $\Delta u=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\upsilon}\frac{\partial}{\partial\upsilon}(\sin\upsilon\frac{\partial u}{\partial\upsilon})+\frac{1}{r^2\sin^2\upsilon}\frac{\partial^2 u}{\partial\varphi^2})$

e)

$$\Delta u=\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi\eta(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[\sqrt{\frac{\xi^2-1}{1-\eta^2}}\xi\eta\frac{\partial u}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[\sqrt{\frac{1-\eta^2}{\xi^2-1}}\xi\eta\frac{\partial u}{\partial\eta}\right]+\frac{\partial}{\partial\varphi}\left[\frac{\xi^2-\eta^2}{\xi\eta}\frac{1}{\sqrt{(\xi^2-1)(1-\eta^2)}}\frac{\partial u}{\partial\varphi}\right]\right\}$$

2.a) garmonik; **b)** garmonik; **c)** garmonik; **d)** garmonik; **e)** yo‘q; **f)** garmonik; **j)** yo‘q; **h)** garmonik; **k)** garmonik. Bevosita hisob-kitoblar katta. Keyingi hisoblashlarda garmonik funksiya $u=u(x_1,x_2)$ ni $\text{Re} f(z)$, $z=x_1+ix_2$, deb olib, vektor analitik $f(z)=u+i\upsilon$ funksiyaning mavhum qismi $\upsilon(x_1,x_2)=\text{Im} f(z)$ funksiyani qurish mumkin. Koshi-Riman sharti bu holda quyidagicha: $\frac{\partial u}{\partial x_1}=\frac{\partial v}{\partial x_2}$, $\frac{\partial u}{\partial x_2}=-\frac{\partial v}{\partial x_1}$. Ko‘rinib turibdiki, $w(z)=\frac{\partial u}{\partial x_1}+i\frac{\partial v}{\partial x_1}$, funksiya analitik va Koshi-Riman shartiga asosan $w(z)=\frac{\partial u}{\partial x_1}-i\frac{\partial v}{\partial x_2}$. Shunday qilib quyidagi funksiya ham analitik

$$\frac{1}{w(z)}=\frac{1}{\frac{\partial u}{\partial x_1}-i\frac{\partial v}{\partial x_2}}=\frac{\frac{\partial u}{\partial x_1}+i\frac{\partial v}{\partial x_1}}{\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2},$$ uning haqiqiy qismi: $$\frac{\frac{\partial u}{\partial x_1}}{\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2}$$

garmonik; **l)** garmonik; **m)** yo‘q.

3.a)k=-3; **b)** k=-2; **c)** $k=\pm 2i, chkx_2=\cos 2x_2$; **d)** $k=\pm 3$; **e)** k=0, k=n-2 n>2 da.

4. $\varphi(z)$ analitik, uning $U(x,y)=u_x$ haqiqiy va $V(x,y)=-u_y$ mavhum qismlarining o'zi va birinchi tartibli hosilalari uzluksiz va Koshi-Riman shartini qanoatlantiradi $U_x-V_y=u_{xx}+u_{yy}=0$; $U_x+V_y=u_{xy}-u_{xy}=0$.

5.1. u(x,y) va v(x,y) lar f(z)=u(x,y)+iv(x,y) analitik funksiyaning haqiqiy va mavhum qismlari, Koshi-Riman sharti $u_x-v_y=0, u_y+v_x=0$ bilan bog'langan. Shuning uchun $dv=v_xdx+v_ydy=-u_ydx+u_xdy$ ifodalar funksiyalarning to'la differensiali bo'ladi, shunday qilib $u_{xx}+u_{yy}=\Delta u=0$. Bundan, $\int dv=\int -u_ydx+u_xdy$ ixtiyoriy fiksirlangan (x_0, y_0) nuqtadan to o'zgaruvchi nuqtagacha (x,y) nuqtagacha egrichiziqli integral D sohada yo'nalishga bog'liq emas.

$$f(z)=x^3-3xy^2+i\left[\int_{x_0}^{x}6xy_0dx+\int_{y_0}^{y}3(x^2-y^2)dy\right]+iC=x^3-3xy^2+i(3x^2y-y^3)+i\left(-3x_0^2y_0+y_0^3+C\right)$$

5.2. $f(z)=e^x\sin y-ie^x\cos y+i\left(e^{x_0}\cos y_0+C\right)$

5.3. $f(z)=\sin x\, ch y+i\cos x\, sh y+i\left(-\cos x_0\, sh y_0+C\right)$ **6.a)** $v(x,y)=\frac{1}{4}(x^4+y^4-6x^2y^2)+C$;

b) $v(x,y)=e^y\cos x+C$; **c)** $v(x,y)=-chx\cos y+C$; **g)** $v(x,y)=shx\sin y+C$; **d)** $v(x,y)=chx\sin y+C$ **e)** $v(x,y)=-shx\cos y+C$; **7.a)** $u(x,y)=x^3y-xy^3Cy+C_0$

d) $u(x,y)=e^x\sin y+Cx+C_0$;

c) $u(x,y)=e^x\sin y+Cy+C_0$; **e)** $u(x,y)=x^2y-\frac{y^3}{3}+xy+\frac{y^2-x^2}{2}+Cx+C_0$;

f) $u(x,y)=\frac{1}{2}x^2y-xy^2+\frac{x^3}{3}-\frac{y^3}{6}+Cy+C_0$ **8. a)** $u=ye^x\cos z-y^2+x^2+g(x,z)$, bu yerda g(x,z)-ixtiyoriy garmonik funksiya. **b)** $u=chx\cos z-y^2+yx^2+g(x,y)$, bu yerda g(x,y)-ixtiyoriy garmonik funksiya.

c) $u=xy^2z-\frac{xz^3}{3}+3xz^2-x^3+xz+g(x,y)$, bu yerda g(x,y)-ixtiyoriy garmonik funksiya.

d) $u=xze^z\cos y-yze^z\sin y+z^2-x^2)+g(x,y)$, bu yerda g(x,y)-ixtiyoriy garmonik funksiya.

9. a) $v(x,y)=\dfrac{x^4+y^4}{4}-1{,}5(xy)^2+C_0x+C_1$;

b) $v(x,y)=-e^y\sin x+C_0y+C_1$; **c)** $v(x,y)=-chx\sin y+C_0y+C_1$; **d)**

$v(x,y)=-chx\cos y+C_0x+C_1$

10. a) $u(x,y)=\left(\dfrac{R}{r}\right)^2 y+2\left(\dfrac{R}{r}\right)^4 xy$, $\sum\limits_{k=0}^{\infty}R^{-k}(a_k\cos k\varphi+b_k\sin k\varphi)=R\sin\varphi+R^2\sin 2\varphi$.

$\cos k\varphi$ va $\sin k\varphi$ oldidagi koeffisiyentlarni tenglashtirib, quyidagini olamiz:

$b_1=R^2$, $\quad a_0=a_1=a_2=...=0 \quad b_2=R^4$, $\quad b_3=b_4=...=0$. Shunday qilib,

$u(x,y)=R^2r^{-1}\sin\varphi+R^4r^{-2}\sin 2\varphi=\left(\dfrac{R}{r}\right)^2 y+2\left(\dfrac{R}{r}\right)^4 xy$

b) $u(x,y)=\left(\dfrac{R}{r}\right)^2(ax+by)+c$; **c)** $u(x,y)=\left(\dfrac{R}{r}\right)^4(x^2-y^2)$; **d)**

$u(x,y)=\dfrac{1}{2}\left(\dfrac{R}{r}\right)^4(x^2-y^2)+\left(\dfrac{R}{r}\right)^2+1$;

e) $u(x,y)=\left(\dfrac{R}{r}\right)^2-0{,}5\left(\dfrac{R}{r}\right)^4(x^2-y^2+2xy)$; **f)**

$u(x,y)=\left(\dfrac{R}{r}\right)^2-0{,}5\left(\dfrac{R}{r}\right)^4(x^2-y^2)+\left(\dfrac{R}{r}\right)^2(x+y)$;

g) $u(x,y)=R^2+\left(\dfrac{R}{r}\right)^4(x^2-y^2)-\left(\dfrac{R}{r}\right)^2(x-y)$; **11.a)** $u(x,y)=\left(\dfrac{R}{r}\right)^2 y+2\left(\dfrac{R}{r}\right)^4 xy$

$\sum\limits_{k=0}^{\infty}R^{-k}(a_k\cos k\varphi+b_k\sin k\varphi)=R\sin\varphi+R^2\sin 2\varphi$. $\cos k\varphi$ va $\sin k\varphi$ oldidagi koeffisiyentlarni tenglashtirib, quyidagini olamiz:

$b_1=R^2$, $\quad a_0=a_1=a_2=...=0 \quad b_2=R^4$, $\quad b_3=b_4=...=0$

Shunday qilib, $u(x,y)=R^2r^{-1}\sin\varphi+R^4r^{-2}\sin 2\varphi=\left(\dfrac{R}{r}\right)^2 y+2\left(\dfrac{R}{r}\right)^4 xy$.

b) $u(x,y)=\left(\dfrac{R}{r}\right)^2(ax+by)+c$; **c)** $u(x,y)=\left(\dfrac{R}{r}\right)^4(x^2-y^2)$; **d)**

$u(x,y)=\dfrac{1}{2}\left(\dfrac{R}{r}\right)^4(x^2-y^2)+\left(\dfrac{R}{r}\right)^2+1$;

e) $u(x,y)=\left(\frac{R}{r}\right)^2-0{,}5\left(\frac{R}{r}\right)^4(x^2-y^2+2xy)$; f)

$u(x,y)=\left(\frac{R}{r}\right)^2-0{,}5\left(\frac{R}{r}\right)^4(x^2-y^2)+\left(\frac{R}{r}\right)^2(x+y)$;

g) $u(x,y)=R^2+\left(\frac{R}{r}\right)^4(x^2-y^2)-\left(\frac{R}{r}\right)^2(x-y)$; **12.** a) $u(x,y)=\frac{r^2-R^2}{4}$; Tenglamaning xususiy yechimini tanlab, Laplas tenglamasiga qo'yilgan Dirixle masalasini yechish masalasiga kelamiz.

b) $u(x,y)=\frac{1}{8}(x^3+xy^2-R_2x)$; c) $u(x,y)=\frac{R^2-x^2}{2}$; d) $u(x,y)=\frac{1}{8}(y^3+yx^2-R_2y\neq 8)$;

e) $u(x,y)=r^2-R^2+1$. **13.** a) $A=0$ da $u(x,y)=const$. $A\neq 0$ da masala xato qo'yilgan. b) $A=\frac{R}{2}$ da $u(x,y)=\frac{R}{2}(x^2-y^2)+const$. $A\neq\frac{R}{2}$ da masala xato. c) u(x,y)=Rxy+const; d) $B=\frac{AR^2}{2}$ da $u(x,y)=-\frac{AR}{4}(x^2-y^2)+const$. $B\neq\frac{AR^2}{2}$ da masala xato. e) $B=A$ da $u(x,y)=\frac{AR}{2}(x^2-y^2)+Ry+const$. $B\neq A$ da masala xato. **14.** a) $A=\frac{R^2}{2}$ da $u(x,y)=\frac{R^5}{4r^4}(x^2-y^2)+const$. $A\neq\frac{R^2}{2}$ da masala xato.

b) $B=\frac{R^2}{2}$ da $u(x,y)=\frac{R^5}{4r^4}(y^2-x^2)-\frac{AR^3}{r^2}y+const$. $B\neq\frac{R^2}{2}$ da masala xato.

c) $B=\frac{AR^2}{2}$ da $u(x,y)=\frac{AR^5}{4r^4}(x^2-y^2)-\frac{R^5}{r^4}xy+const$. $B\neq\frac{AR^2}{2}$ da masala xato.

d) $B=(A-1)\frac{R^2}{2}$ da $u(x,y)=\frac{(1+A)R^5}{4r^4}(y^2-x^2)+const$ $B\neq(A-1)\frac{R^2}{2}$ da masala xato.

15. a) $u(r,\varphi)=\frac{r}{R-R_1}\sin\varphi+const$. $u(r,\varphi)=\sum_{k=0}^{\infty}r^k(a_k\cos k\varphi+b_k\sin k\varphi)$. Bundan

$u(R,\varphi)-u(R_1,\varphi)=\sum_{k=0}^{\infty}(R^k-R_1^k)(a_k\cos k\varphi+b_k\sin k\varphi)=\sin\varphi$

$a_0=a_1=a_2=...=0$ $\quad b_1=\frac{1}{R-R_1}$, $\quad b_2=b_3=...=0$. Shuning uchun

$u(r,\varphi)=\frac{r}{R-R_1}\sin\varphi+a_0,\ a_0=const$; b) $u(r,\varphi)=\frac{r}{R-R_1}\cos\varphi+const$; c) $C=-\frac{1}{2}$ da

$u(r,\varphi)=\frac{r^2\cos2\varphi}{2(R^2-R_1^2)}+const$. $C\neq-\frac{1}{2}$ da $\int_0^{2\pi}f(\varphi)d\varphi=0$ shart bajarilmaydi.

d) $u(r,\varphi)=\dfrac{r^2\sin2\varphi}{R^2-R_1^2}+\dfrac{r^3\cos3\varphi}{R^3-R_1^3}+const$; **e)** $B=-A$da $u(r,\varphi)=A\dfrac{r^2\cos2\varphi}{R^2-R_1^2}+const$. $B\neq-A$ da $\int_0^{2\pi}f(\varphi)d\varphi=0$ shart bajarilmaydi. **f)** $u(r,\varphi)=\dfrac{r\sin\varphi}{R-R_1}+\dfrac{3r^2\cos2\varphi}{2(R^2-R_1^2)}+const$, $C=\dfrac{3}{2}$. $C\neq1{,}5$ da $\int_0^{2\pi}f(\varphi)d\varphi=0$ shart bajarilmaydi. **16.a)** $u(r,\varphi)=\dfrac{3R^2R_1^2\sin2\varphi}{(R^2-R_1^2)r^2}+const$

b) $u(r,\varphi)=-\dfrac{5R^2R_1^2\cos2\varphi}{2(R^2-R_1^2)r^2}+const$, $A=\dfrac{5}{2}$. $A\neq2{,}5$da $\int_0^{2\pi}f(\varphi)d\varphi=0$ shart bajarilmaydi. **c)** masala yechimga ega emas. **d)** $A=\dfrac{3}{2}$da

$u(r,\varphi)=\dfrac{RR_1\sin\varphi}{(R-R_1)r}+\dfrac{3R^2R^2\cos2\varphi}{2(R^2-R_1^2)r^2}+const$ $A\neq1{,}5$ da $\int_0^{2\pi}f(\varphi)d\varphi=0$ shart bajarilmaydi.

e) $u(r,\varphi)=\dfrac{RR_1\sin\varphi}{(R-R_1)r}+\dfrac{R^5R_1^5\cos5\varphi}{(R^5-R_1^5)r^5}+const$ **17. a)** $u=x+2y+z(2x-y^2)+\dfrac{z^3}{3}$; **b)** $u=xe^y\cos z$;

c) $u=x(x+y)+z(y-z)+e^x\sin z$; **d)** $u=x\sin y\,\mathrm{ch}z+\mathrm{sh}z\cos y$; **e)** $u=x^3+z(2x^2-y)-3xz^2-\dfrac{2}{3}z^3+2$;

f) $u=xz+\cos2x\,\mathrm{ch}2z-\sin2x\,\mathrm{ch}2z$; **18.** $u=T+(U-T)\dfrac{\ln\frac{r}{b}}{\ln\frac{b}{a}}$; **19.** $u=T+bU\ln\dfrac{r}{a}$; **20.** $u=U+aT\ln\dfrac{r}{b}$; **21.** $aT=bU$da $u=aT\ln r+const$,aks holda masala xato qo‘yilgan bo‘ladi. **22.** $u=T+\dfrac{b(U-hT)\ln\frac{r}{a}}{1+bh\ln\frac{b}{a}}$; **23.** $u=U+\dfrac{a(T+hU)\ln\frac{r}{b}}{1+ah\ln\frac{b}{a}}$; **24.** $u=\dfrac{bU-aT}{bh}+aT\ln\dfrac{r}{b}$; **25.** $u=\dfrac{bU-aT}{ah}+bU\ln\dfrac{r}{a}$; **26.**

$u=\dfrac{abh(T\ln\frac{r}{b}+U\ln\frac{r}{a})+bU-aT}{h(a+b+abh\ln\frac{b}{a})}+aT\ln\dfrac{r}{b}$; **27.** $u=\dfrac{h\ln\frac{r}{b}-\ln\frac{r}{c}}{h\ln\frac{a}{b}-\ln\frac{a}{c}}$;

28. a) $u(x,y)=x^3-3x^2-3xy^2+3y^2+12x-1$; **b)** $u(x,y)=\dfrac{1}{2}(x^2-y^2)-x+2y$; **c)** $u(x,y)=y^2-x^2-3x$;

d) $u(x,y)=(x+y)^2+2x+1$; **e)** $u(x,y)=3y(x+1)^2+3y^2-2y$; **29.a)**

$$u(x,y)=\sum_{k=0}^{\infty}a_k\sin\frac{(2k+1)\pi}{2p}x\,sh\frac{(2k+1)\pi}{2p}y;\quad a_k=\frac{2}{p}sh^{-1}\frac{(2k+1)\pi s}{2p}\int_0^p f(x)\frac{(2k+1)\pi}{2p}x\,dx;$$

b) $$u(x,y)=\frac{(pB-2A)y}{2s}+A-\frac{4pB}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2sh\frac{(2k+1)\pi s}{p}}\cos\frac{(2k+1)\pi}{p}x\,sh\frac{(2k+1)\pi}{p}y;$$

c) $$u(x,y)=\frac{8Bp^2}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k\pi^2(2k+1)^2-2}{(2k+1)^2ch\frac{(2k+1)\pi s}{2p}}sh\frac{(2k+1)\pi y}{2p}\cos\frac{(2k+1)\pi x}{2p};$$ **d)**

$$u(x,y)=U+\frac{2p}{\pi}\left[Tsh\frac{\pi}{2p}y-\left(ch^{-1}\frac{\pi s}{2p}\right)\left(\frac{2U}{p}+Tsh\frac{\pi s}{2p}\right)ch\frac{\pi}{2p}y\right]\cdot\sin\frac{\pi}{2p}x-$$

$$-\frac{4U}{\pi}\sum_{k=1}^{\infty}\frac{ch^{-1}\frac{(2k+1)\pi s}{2p}}{(2k+1)}ch\frac{(2k+1)\pi}{2p}y\sin\frac{(2k+1)\pi}{2p}x$$

e)

$$u(x,y)=\frac{4qs}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\cos\frac{(2k+1)\pi p}{s}}sh\frac{(2k+1)\pi x}{s}\sin\frac{(2k+1)\pi y}{s}+$$

$$+\frac{4U}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)sh\frac{(2k+1)\pi s}{2p}}sh\frac{(2k+1)\pi y}{2p}\sin\frac{(2k+1)\pi x}{2p}.$$

f) $$u(x,y)=\frac{2sT}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{1}{sh\frac{k\pi s}{p}}sh\frac{k\pi y}{p}\sin\frac{k\pi x}{p}+\frac{1}{sh\frac{k\pi p}{s}}sh\frac{k\pi x}{s}\sin\frac{k\pi y}{s}\right);$$

30.a) $$u(r,\varphi)=\sum_{k=0}^{\infty}a_ke^{-\frac{(2k-1)\pi x}{2l}}\sin\frac{(2k+1)\pi y}{2l},\quad a_k=\frac{2}{l}\int_0^l f(y)\sin\frac{(2k+1)\pi y}{2l}dy;$$

b) $$u(x,y)=\sum_{k=1}^{\infty}\left\{\frac{2(h^2+\lambda_k^2)}{l(h^2+\lambda_k^2)+k}\int_0^l f(\xi)\cos\lambda_k\xi d\xi\right\}e^{-\lambda_k x}\cos\lambda_k y,\qquad \lambda tg\lambda l=h.$$

c) $$u(x,y)=\frac{8l}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-\frac{(2k+1)\pi x}{l}}\sin\frac{(2k+1)\pi y}{l};$$

d)

$$u(x,y)=2(1+hl)\sum_{k=1}^{\infty}\frac{e^{-\lambda_k x}}{\lambda_k[l(h^2+\lambda_k^2)+h]}y_k(y),\qquad y_k(y)=\lambda_k\cos\lambda_k y+h\sin\lambda_k y,\qquad h tg\lambda l=-\lambda.$$

31.a) $u(r,\varphi)=\frac{2\pi^2}{3}-4\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{r}{R}\right)^k\cos k\varphi$; **b)**

$$u(r,\varphi)=-1-\frac{r}{2R}\cos\varphi+\frac{\pi r}{R}\sin\varphi+2\sum_{k=2}^{\infty}\frac{1}{k^2-1}\left(\frac{r}{R}\right)^k\cos k\varphi;$$

c) $u(r,\varphi)=\frac{T}{h}+\frac{Qr}{1+hR}\sin\varphi+\frac{Ur^3}{R^2(3+Rh)}\cos 3\varphi$; **d)** $u(r,\varphi)=C+\sum_{k=1}^{\infty}r^k(A_k\cos k\varphi+B_k\sin k\varphi)$;

$$A_k=\frac{1}{k\pi R^{k-1}}\int_0^{2\pi}f(\varphi)\cos k\varphi d\varphi,\ B_k=\frac{1}{k\pi R^{k-1}}\int_0^{2\pi}f(\varphi)\sin k\varphi d\varphi,\ \int_0^{2\pi}f(\varphi)d\varphi=0$$

32.a) $u(r,\varphi)=\frac{2T}{\pi}+\frac{4T}{\pi}\sum_{k=1}^{\infty}\frac{1}{1-4k^2}\left(\frac{R}{r}\right)^k\cos k\varphi$;

b) $u(r,\varphi)=C+\frac{4R^2}{3r}\cos\varphi+\frac{R^3}{4r^2}\cos 2\varphi-\frac{\pi R^3}{r^2}\sin 2\varphi+4R\sum_{k=1}^{\infty}\frac{1}{1-4k^2}\left(\frac{R}{r}\right)^k\cos k\varphi$

c) $u(r,\varphi)=-\frac{A_0}{2\pi h}-\frac{R}{\pi}\sum_{k=1}^{\infty}\frac{1}{k+hR}\left(\frac{R}{r}\right)^k(A_k\cos k\varphi+B_k\sin k\varphi)$

$A_k=\int_0^{2\pi}f(\varphi)\cos k\varphi d\varphi,\ B_k=\int_0^{2\pi}f(\varphi)\sin k\varphi d\varphi$, **d)**

$$u(r,\varphi)=\pi u-\frac{RU}{r}\sin\varphi+2U\sum_{k=2}^{\infty}\frac{2k^2-1}{k(1-k^2)}\left(\frac{R}{r}\right)^k\sin k\varphi;$$

33.a) $u(r,\varphi)=\frac{A}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\cos\varphi$;**b)** $u(r,\varphi)=A\frac{\ln\frac{r}{b}}{\ln\frac{a}{b}}+\frac{ab^2}{b^4-a^4}\left(r^2-\frac{a^4}{r^2}\right)\sin 2\varphi$;

c) $u(r,\varphi)=Q+\frac{a^2q}{b^2+a^2}\left(r-\frac{b^2}{r}\right)\cos\varphi+\frac{rb^2}{b^4+a^4}\left(r^2+\frac{a^4}{r^2}\right)\sin 2\varphi$;

d) $u(r,\varphi)=T\frac{1+hb\ln\frac{b}{r}}{1+hb\ln\frac{a}{b}}+abU\frac{(1-hb)\frac{r}{b}+(1+hb)\frac{b}{r}}{b^2+a^2+hb(b^2-a^2)}\cos\varphi$;

34.a) $u(r,\varphi)=\frac{2Aa^2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{r}{R}\right)^{\frac{k\pi}{a}}\sin\frac{k\pi\varphi}{a}$;

b) $u(r,\varphi)=\sum_{k=0}^{\infty}a_k r^{\frac{(2k-1)\pi}{2a}}\cos\frac{(2k+1)\pi}{2a}\varphi$, $a_k=\frac{2}{a}R^{-\frac{(2k-1)\pi}{2a}}\int_0^{2\pi}f(\varphi)\cos\frac{(2k+1)\pi}{2a}\varphi d\varphi$,

c) $u(r,\varphi)=\frac{aU}{2}-\frac{4aU}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{r}{R}\right)^{\frac{k\pi}{a}}\cos\frac{k\pi\varphi}{a}$; **d)** $u(r,\varphi)=\frac{4aQR}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{r}{R}\right)^{\frac{k\pi}{a}}\sin\frac{k\pi\varphi}{a}$;

e) $u(r,\varphi)=2QR\sum_{k=1}^{\infty}\left\{\frac{2(h^2+\lambda_k^2)(1-\cos a\lambda_k)}{\lambda_k(yR+\lambda_k)[1+a(h^2+\lambda_k^2)]}\right\}\left(\frac{r}{R}\right)^{\lambda_k}\sin\lambda_k\varphi.$

8-bob. $\frac{\partial y}{\partial x}=\frac{-b\pm\sqrt{b^2+ac}}{a}$;

1. $2\sqrt{y}=\pm+C\ y>0;\ y<0$; **2.** $2x^{1/2}+y=C_1;\ -2x^{1/2}+y=C_2;\ x\ \ +y=C_3$. **3.** **4.** $y=C_1x$, $xy=C_2$. **5.** $x\pm y=C$; **6.** $y=C$. **7.** $\arcsin\ x\pm\arcsin\ y=C$. **8.** **9.** $\frac{\partial y}{\partial x}=-2u_xu_y\pm\sqrt{2u_x^2+2u_y^2-1}$. **10.**

$$\frac{(\partial y)^2}{\sqrt{1+u_y^2}}+u_xu_y\left[\frac{1}{\left(\sqrt{1+u_y^2}\right)^3}+\frac{1}{\left(\sqrt{1+u_x^2}\right)^3}\right]dxdy+\frac{(dx)^2}{\sqrt{1+u_x^2}}=0.$$

11. $\frac{(\alpha-\delta)^2}{4}+\gamma\beta>0$. **12.** $\frac{d}{d\omega}[\omega\rho(\omega)]>0,\ \omega=\sqrt{\varphi_x^2+\varphi_y^2}$. **13.** $\tau\left[\tau^2-c_0^2\left(\xi^2+\eta^2\right)\right]=0$. **14.** **15.**

$$\frac{\mu\varepsilon}{c_0^2}-\tau^2\left(\frac{\mu\varepsilon}{c_0^2}\tau^2-\xi^2-\eta^2-\zeta^2\right)^2=0.\ \begin{vmatrix}A & B\\ C & D\end{vmatrix}=|AD-BC|,\ CA=AC,$$

16. $\left(\tau^2-\xi^2-\eta^2-\zeta^2\right)^2=0$. **17.** **18.** **19.** $dv+udt=0\ x+t=C_1;\ du+vdt=0\ x-2t=C_2$. **20.**

$$\pm\frac{\sqrt{3}}{\upsilon}d\varphi_1+\frac{1}{\upsilon}d\varphi_0+dt\left(-q_0+\frac{q_1}{r}+\alpha_0\varphi_0\pm\sqrt{3}\alpha_1\varphi_1\right)=0,\ r=\pm\frac{\upsilon}{\sqrt{3}}t.$$

21. $x^2+t^2=C_1,\ u_2=C_2;\ x=C_3t,\ t\left(1+x^2\right)du_1+tdu_2+2u_1x^2dt=0$. **22.**

$2\tau d\psi\pm dP\sqrt{1-\tau'^2}=0,\ \frac{dy}{dx}=\frac{\sin 2\psi\pm\sqrt{1-\tau'^2}}{\cos 2\psi+\tau'}$. **23. 1)** $u^2+\upsilon^2<c^2$ **2)**

$u^2+\upsilon^2>c^2\ d\upsilon(c^2-\upsilon^2)+du\left[-u\upsilon\mp\sqrt{c^2\left(u^2+\upsilon^2-c^2\right)}\right]<0$

$(c^2-\upsilon^2)dx=\left[-u\upsilon\mp\sqrt{c^2\left(u^2+\upsilon^2-c^2\right)}\right]dy.$

24. $u_1=C_1,\ x-t=C_2;\ u_2=C_3,\ x-\frac{t^3}{3}=C_4;\ u_3=C_5,\ x+t=C_6;\ u_4=C_7,\ x+t^3=C_8;$

25.
$$I_N=\begin{vmatrix}-\lambda & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\ \frac{1}{3} & -\lambda & \frac{2}{3} & 0 & \cdots & \cdots & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 0 & \frac{k}{2k+1} & -\lambda & \frac{k+1}{2k+1} & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{N-1}{2N-1} & -\lambda & \frac{N}{2N-1}\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{N}{2N+1} & -\lambda\end{vmatrix}$$

$I_k=\alpha_kP_{k+1}(\lambda),\ \alpha_k=const$

$u_0+3P_1(\lambda_k)u_1+\cdots+(2N+1)P_N(\lambda_k)u_N=C$

$x-\lambda_kt=C_k,\ \lambda_k\ I_N(\lambda)=0.$

26. $\begin{cases} \dfrac{\partial u_1}{\partial t}+\dfrac{\partial u_1}{\partial x}=0, \\ \dfrac{\partial u_2}{\partial t}-\dfrac{\partial u_2}{\partial x}=0, \end{cases}$ $u_1=u-\upsilon,\ u_2=u+\upsilon.$

27. $\begin{cases} \dfrac{\partial u_1}{\partial t}-\dfrac{\partial u_1}{\partial x}=0, \\ \dfrac{\partial u_2}{\partial t}+2t\dfrac{\partial u_2}{\partial x}=0, \end{cases}$ $u_1=u+\upsilon,\ u_2=u+\upsilon.$

28. $\begin{cases} \dfrac{\partial u_1}{\partial t}+(1+x)\dfrac{\partial u_1}{\partial x}+u_2=0, \\ \dfrac{\partial u_2}{\partial t}-(1+x)\dfrac{\partial u_2}{\partial x}+u_1=0, \end{cases}$ $u_1=u+\upsilon,\ u_2=u+\upsilon.$

29. $\begin{cases} \dfrac{\partial u_1}{\partial t}+\dfrac{1}{\sqrt{1+x^2}}\dfrac{\partial u_1}{\partial x}+\dfrac{x^3+x-1}{2(1+x^2)}u_1+\dfrac{x^3+x+1}{2(1+x^2)}u_2-\dfrac{x(u_1-\upsilon_1)}{2\sqrt{1+x^2}(1+x^2)}=0, \\ \dfrac{\partial u_2}{\partial t}-\dfrac{1}{\sqrt{1+x^2}}\dfrac{\partial u_2}{\partial x}+\dfrac{x^3+x-1}{2(1+x^2)}u_1+\dfrac{x^3+x-1}{2(1+x^2)}u_2-\dfrac{x(u_1-\upsilon_1)}{2\sqrt{1+x^2}(1+x^2)}=0, \end{cases}$

$u_1=u+\sqrt{1+x^2}\,\upsilon,\ u_2=u-\sqrt{1+x^2}\,\upsilon.$

30. $\begin{cases} \dfrac{\partial u_1}{\partial t}+3\dfrac{\partial u_1}{\partial x}=u_2, \\ \dfrac{\partial u_2}{\partial t}+4\dfrac{\partial u_2}{\partial x}=8u_1, \\ \dfrac{\partial u_3}{\partial t}-4\dfrac{\partial u_3}{\partial x}=2u_3, \end{cases}$ $u_1=\omega\ u_2=2u+\upsilon+2\omega,\ u_3=-14u+7\upsilon+2\omega.$

31. $\begin{cases} \dfrac{\partial u_1}{\partial t}+11\dfrac{\partial u_1}{\partial x}=u_1+u_2, \\ \dfrac{\partial u_2}{\partial t}+\dfrac{\partial u_2}{\partial x}=-u_1-u_2, \\ \dfrac{\partial u_3}{\partial t}+2\dfrac{\partial u_3}{\partial x}=3u_1+2u_2+u_3, \end{cases}$ $u_1=u+\upsilon\ u_2=u-\upsilon,\ u_3=-4u+5\upsilon+9\omega.$

32. 33.

34. $\begin{cases} u=g(t+2x)+f(t-x^2) \\ \upsilon=g(t+2x)-f(t-x^2) \end{cases}$ **35.** $\begin{cases} u=(\sqrt5+1)f(3x-\sqrt5y)+(\sqrt5-1)g(3x+\sqrt5y) \\ \upsilon=(\sqrt5+3)f(3x-\sqrt5y)+(\sqrt5-3)g(3x+\sqrt5y) \end{cases}$

36. $\begin{cases} u=f(9t-x)+g(t+x), \\ \upsilon=f(9t-x)-g(t+x), \\ \omega=\dfrac{3}{11}f(9t-x)+3g(t+x)+\omega(2t+x) \end{cases}$ **37.** $\begin{cases} u_1=f(x-t)+h(x+2t)+3g(x-3t), \\ u_2=3h(x+2t)+g(x-3t), \\ u_3=3h(x+2t)+6g(x-3t) \end{cases}$

38. $u=t,\ \upsilon=2x+t.$ **39.** $u=-t(1+t),\ \upsilon=2x-t+t^2.$ **40.** $u=-t,\ \upsilon=x+2t.$

41. $u=\varphi\left(\frac{x+t}{2}\right)+\varphi\left(\frac{x-t}{2}\right)-\varphi(0)$. **42.** $u_1=\frac{5y-x}{4}-\frac{25}{16}(x-y)^2$.

$u_2=\frac{x-5y}{20}+\frac{25}{16}(x-y)^2$.

43. 44. 45.46.47. 48.

49. $$\begin{pmatrix}u\\ \upsilon\end{pmatrix}=\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{\pi}{2}+k\pi\right)^2}\times\begin{pmatrix}2\sin\left(\frac{\pi}{2}+k\pi\right)x\cos\sqrt{10}\left(\frac{\pi}{2}+k\pi\right)t\\ -\sqrt{10}\cos\left(\frac{\pi}{2}+k\pi\right)x\sin\sqrt{10}\left(\frac{\pi}{2}+k\pi\right)t\end{pmatrix}.$$

50. $$\begin{pmatrix}u\\ \upsilon\end{pmatrix}=\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{\frac{\pi}{2}+k\pi}-\frac{1}{\left(\frac{\pi}{2}+k\pi\right)^2}\right]\times\begin{pmatrix}-\frac{4}{\sqrt{6}}\sin\left(\frac{\pi}{2}+k\pi\right)x\sin\left(\frac{\pi}{2}+k\pi\right)\sqrt{6}t\\ 2\cos\left(\frac{\pi}{2}+k\pi\right)x\cos\left(\frac{\pi}{2}+k\pi\right)\sqrt{6}t\end{pmatrix}.$$

51. $$\begin{pmatrix}u\\ \upsilon\end{pmatrix}=\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{\pi}{2}+k\pi\right)^2}\times\begin{pmatrix}-\frac{2}{3\sqrt{6}}\cos\left(\frac{\pi}{2}+k\pi\right)x\sin\sqrt{6}\left(\frac{\pi}{2}+k\pi\right)t\\ 2\sin\left(\frac{\pi}{2}+k\pi\right)x\cos\sqrt{6}\left(\frac{\pi}{2}+k\pi\right)t\end{pmatrix}.$$

52. $$\begin{pmatrix}u\\ \upsilon\end{pmatrix}=2\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{\pi}{2}+k\pi\right)^2}\times\begin{pmatrix}\sin\left(\frac{\pi}{2}+k\pi\right)x\cos\sqrt{15}\left(\frac{\pi}{2}+k\pi\right)t\\ \frac{\sqrt{5}}{3}\cos\left(\frac{\pi}{2}+k\pi\right)x\sin\sqrt{15}\left(\frac{\pi}{2}+k\pi\right)t\end{pmatrix}.$$

53. $u\equiv 1,\ \upsilon\equiv 0$.

Foydalanilgan adabiyotlar ro'yxati

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