NOTES ON OPTIMAL CONTROL THEORY

with economic models and exercises

Andrea Calogero

Dipartimento di Matematica e Applicazioni – Università di Milano-Bicocca

(andrea.calogero@unimib.it)

October 30, 2018

Contents

Chapter 1

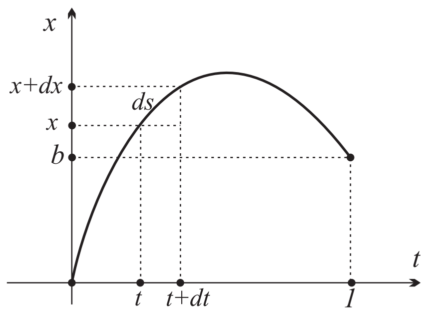
Introduction to Optimal Control

1.1 Some examples

Example 1.1.1. **The curve of minimal length and the isoperimetric problem**

Suppose we are interested to find the curve of minimal length joining two distinct points in the plane. Suppose that the two points are $(0,0)$ and (a,b). Clearly we can suppose that $a = 1$. Hence we are looking for a function $x : [0,1] \to \mathbb{R}$ such that $x(0) = 0$ and $x(1) = b$.

The length of such curve is defined by $\int_0^1 \mathrm{d}s$, i.e. as the "sum" of arcs of infinitesimal length $\mathrm{d}s$; using the picture and the Theorem of Pitagora we obtain



$$(\mathrm{d}s)^2 = (\mathrm{d}t)^2 + (\mathrm{d}x)^2$$
$$\Rightarrow \quad \mathrm{d}s = \sqrt{1+\dot{x}^2}\,\mathrm{d}t,$$

where $\dot{x} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

Hence the problem is

$$\begin{cases} \min\limits_{x} \displaystyle\int_0^1 \sqrt{1+\dot{x}^2(t)}\,\mathrm{d}t \\ x(0) = 0 \\ x(1) = b \end{cases} \tag{1.1}$$

It is well known that the solution is a line. We will solve this problem in subsection 2.5.1.

A more complicate problem is to find the closed curve in the plane of assigned length such that the area inside such curve is maximum: we call this problem the foundation of Cartagena.[1] This is the isoperimetric problem. Without loss

[1] When Cartagena was founded, it was granted for its construction as much land as a man could circumscribe in one day with his plow: what form should have the groove because it obtains the maximum possible land, being given to the length of the groove that can dig a man in a day? Or, mathematically speaking, what is the shape with the maximum area among all the figures with the same perimeter?

of generality, we consider a curve $x : [0,1] \to \mathbb{R}$ such that $x(0) = x(1) = 0$. Clearly the area delimited by the curve and the t axis is given by $\int_0^1 x(t)\,\mathrm{d}t$. Hence the problem is

$$\begin{cases} \max\limits_x \displaystyle\int_0^1 x(t)\,\mathrm{d}t \\ x(0) = 0 \\ x(1) = 0 \\ \displaystyle\int_0^1 \sqrt{1 + \dot{x}^2(t)}\,\mathrm{d}t = A > 1 \end{cases} \tag{1.2}$$

Note that the length of the interval $[0,1]$ is exactly 1 and, clearly, it is reasonable to require $A > 1$. We will present the solution in subsection 4.3.3.

Example 1.1.2. **A problem of business strategy**
A factory produces a unique good with a rate $x(t)$, at time t. At every moment, such production can either be reinvested to expand the productive capacity or sold. The initial productive capacity is $\alpha > 0$; such capacity grows as the reinvestment rate. Taking into account that the selling price is constant, what fraction $u(t)$ of the output at time t should be reinvested to maximize total sales over the fixed period $[0,T]$?
Let us introduce the function $u : [0,T] \to [0,1]$; clearly, if $u(t)$ is the fraction of the output $x(t)$ that we reinvest, $(1 - u(t))x(t)$ is the part of $x(t)$ that we sell at time t at the fixed price $P > 0$. Hence the problem is

$$\begin{cases} \max\limits_{u \in \mathcal{C}} \displaystyle\int_0^T (1 - u(t))x(t)P\,\mathrm{d}t \\ \dot{x} = ux \\ x(0) = \alpha, \\ \mathcal{C} = \{u : [0,T] \to [0,1] \subset \mathbb{R},\ u \in KC\} \end{cases} \tag{1.3}$$

where α and T are positive and fixed. We will present the solution in subsection 2.5.2 and in subsection 5.5.1.

Example 1.1.3. **The building of a mountain road**
The altitude of a mountain is given by a differentiable function y, with $y : [t_0, t_1] \to \mathbb{R}$. We have to construct a road: let us determinate the shape of the road, i.e. the altitude $x = x(t)$ of the road in $[t_0, t_1]$, such that the slope of the road never exceeds α, with $\alpha > 0$, and such that the total cost of the construction

$$\int_{t_0}^{t_1} (x(t) - y(t))^2\,\mathrm{d}t$$

is minimal. Clearly the problem is

$$\begin{cases} \min\limits_{u \in \mathcal{C}} \displaystyle\int_{t_0}^{t_1} (x(t) - y(t))^2\,\mathrm{d}t \\ \dot{x} = u \\ \mathcal{C} = \{u : [t_0, t_1] \to [-\alpha, \alpha] \subset \mathbb{R},\ u \in KC\} \end{cases} \tag{1.4}$$

where y is an assigned and continuous function. We will present the solution in subsection 2.6.1 of this problem introduced in chapter IV in [22].

Example 1.1.4. **"In boat with Pontryagin".**
Suppose we are on a boat that at time $t_0 = 0$ has distance $d_0 > 0$ from the pier of the port and has velocity v_0 in the direction of the port. The boat is equipped with a motor that provides an acceleration or a deceleration. We are looking for a strategy to arrive to the pier in the shortest time with a "soft docking", i.e. with vanishing speed in the final time T.
We denote by $x = x(t)$ the distance from the pier at time t, by $\dot{x}$ the velocity of the boat and by $\ddot{x} = u$ the acceleration ($\ddot{x} > 0$) or deceleration ($\ddot{x} < 0$). In order to obtain a "soft docking", we require $x(T) = \dot{x}(T) = 0$, where the final time T is clearly unknown. We note that our strategy depends only on our choice, at every time, on $u(t)$. Hence the problem is the following

$$\begin{cases} \min\limits_{u \in \mathcal{C}} T \\ \ddot{x} = u \\ x(0) = d_0 \\ \dot{x}(0) = v_0 \\ x(T) = \dot{x}(T) = 0 \\ \mathcal{C} = \{u : [0, \infty) \to [-1, 1] \subset \mathbb{R}\} \end{cases} \tag{1.5}$$

where d_0 and v_0 are fixed and T is free.

This is one of the possible ways to introduce a classic example due to Pontryagin; it shows the various and complex situations in the optimal control problems (see page 23 in [22]). We will solve this problem in subsection 3.5.1. △

Example 1.1.5. **A model of optimal consumption.**
Consider an investor who, at time $t = 0$, is endowed with an initial capital $x(0) = x_0 > 0$. At any time he and his heirs decide about their rate of consumption $c(t) \geq 0$. Thus the capital stock evolves according to

$$\dot{x} = rx - c$$

where $r > 0$ is a given and fixed rate to return. The investor's time utility for consuming at rate $c(t)$ is $U(c(t))$. The investor's problem is to find a consumption plain so as to maximize his discounted utility

$$\int_0^\infty e^{-\delta t} U(c(t)) dt$$

where δ, with $\delta \geq r$, is a given discount rate, subject to the solvency constraint that the capital stock $x(t)$ must be positive for all $t \geq 0$ and such that vanishes at ∞. Then the problem is

$$\begin{cases} \max\limits_{c \in \mathcal{C}} \displaystyle\int_0^\infty e^{-\delta t} U(c)\, \mathrm{d}t \\ \dot{x} = rx - c \\ x(0) = x_0 > 0 \\ x \geq 0 \\ \lim\limits_{t \to \infty} x(t) = 0 \\ \mathcal{C} = \{c : [0, \infty) \to [0, \infty)\} \end{cases} \tag{1.6}$$

with $\delta \geq r \geq 0$ fixed constants. We will solve this problem in subsections 3.7.1 and 5.7.4 for a logarithmic utility function, and in subsection 5.7.2 for a HARA utility function. △

One of the real problems that inspired and motivated the study of optimal control problems is the next and so called "moonlanding problem".

Example 1.1.6. **The moonlanding problem.**
Consider the problem of a spacecraft attempting to make a soft landing on the moon using a minimum amount of fuel. To define a simplified version of this problem, let $m = m(t) \geq 0$ denote the mass, $h = h(t) \geq 0$ and $v = v(t)$ denote the height and vertical velocity of the spacecraft above the moon, and $u = u(t)$ denote the thrust of the spacecraft's engine. Hence in the initial time $t_0 = 0$, we have initial height and vertical velocity of the spacecraft as $h(0) = h_0 > 0$ and $v(0) = v_0 < 0$; in the final time T, equal to the first time the spacecraft reaches the moon, we require $h(T) = 0$ and $v(T) = 0$. Such final time T is not fixed. Clearly

$$\dot{h} = v.$$

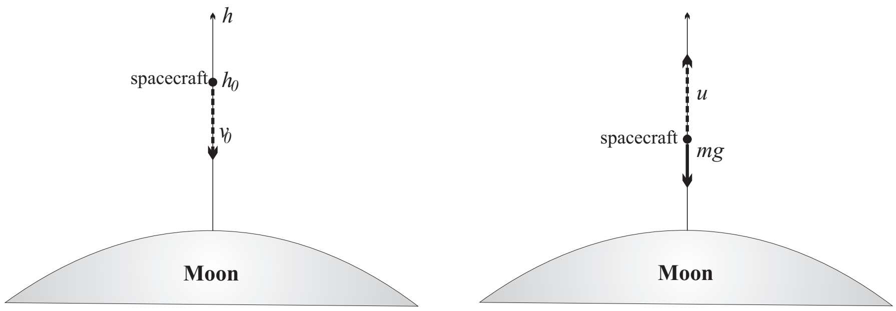
Let M denote the mass of the spacecraft without fuel, F the initial amount of fuel and g the gravitational acceleration of the moon. The equations of motion of the spacecraft is

$$m\dot{v} = u - mg$$

where $m = M + c$ and $c(t)$ is the amount of fuel at time t. Let α be the maximum thrust attainable by the spacecraft's engine ($\alpha > 0$ and fixed): the thrust u, $0 \leq u(t) \leq \alpha$, of the spacecraft's engine is the control for the problem and is in relation with the amount of fuel with

$$\dot{m} = \dot{c} = -ku,$$

with k a positive constant.



On the left, the spacecraft at time $t = 0$ and, on the right, the forces that act on it.

The problem is to land using a minimum amount of fuel:

$$\min\big(m(0) - m(T)\big) = M + F - \max m(T)$$

Let us summarize the problem

$$\begin{cases} \max_{u\in\mathcal{C}} m(T) \\ \dot{h} = v \\ m\dot{v} = u - mg \\ \dot{m} = -ku \\ h(0) = h_0, \qquad h(T) = 0 \\ v(0) = v_0, \qquad v(T) = 0 \\ m(0) = M + F \\ m(t) \geq 0, \qquad h(t) \geq 0 \\ \mathcal{C} = \{u : [0,T] \to [0,\alpha]\} \end{cases} \tag{1.7}$$

where h_0, M, F, g, $-v_0$, k and α are positive and fixed constants; the final time T is free. The solution for this problem is very hard; we will present it in subsection 3.2.4. △

1.2 Statement of problems of Optimal Control

1.2.1 Admissible control and associated trajectory

Let us consider a problem where the development of the system is given by a function

$$\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n, \qquad \text{with } \mathbf{x} = (x_1, x_2, \ldots, x_n),$$

with $n \geq 1$. At every time t, the value $\mathbf{x}(t)$ describes our system. We call $\mathbf{x}$ *state variable* (or *trajectory*): the state variable is at least a continuous function. We suppose that the system has an *initial condition*, i.e.

$$\mathbf{x}(t_0) = \boldsymbol{\alpha}, \tag{1.8}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. In many situation we require that the trajectory satisfies $\mathbf{x}$ satisfies a *final condition*; in order to do that, let us introduce a set $\mathcal{T} \subset [t_0, \infty) \times \mathbb{R}^n$ called *target set*. In this case the final condition is

$$(t_1, \mathbf{x}(t_1)) \in \mathcal{T}. \tag{1.9}$$

For example, the final condition $\mathbf{x}(t_1) = \boldsymbol{\beta}$ with $\boldsymbol{\beta}$ fixed in $\mathbb{R}^n$ has $\mathcal{T} = \{(t_1, \beta)\}$ as target set; if we have a fixed final time t_1 and no conditions on the trajectory at such final time, then the target set is $\mathcal{T} = \{t_1\} \times \mathbb{R}^n$.

Let us suppose that our system depends on some particular choice (or strategy), at every time. Essentially we suppose that the strategy of our system is given by a measurable[2] function

$$\mathbf{u} : [t_0, t_1] \to U, \qquad \text{with } \mathbf{u} = (u_1, u_2, \ldots, u_k),$$

where U is a fixed closed set in $\mathbb{R}^k$ called *control set*. We call such function $\mathbf{u}$ *control variable*. However, it is reasonable is some situations and models to

[2] In many situations that follows we will restrict our attention to the class of piecewise continuous functions (and replace "measurable" with "KC"); more precisely, we denote by $KC([t_0, t_1])$ the space of piecewise continuous function $\mathbf{u}$ on $[t_0, t_1]$, i.e. $\mathbf{u}$ is continuous in $[t_0, t_1]$ up to a finite number of points τ such that $\lim_{t\to\tau^+} \mathbf{u}(t)$ and $\lim_{t\to\tau^-} \mathbf{u}(t)$ exist and are finite.

require that the admissible controls are in KC and not only measurable (this is the point of view in the book [13]).

The fact that $\mathbf{u}$ determines the system is represented by the *dynamics*, i.e. the relation

$$\dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \tag{1.10}$$

where $g : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$. From a mathematical point of view we are interesting in solving the Ordinary Differential Equation (ODE) of the form

$$\begin{cases} \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) & \text{in } [t_0, t_1] \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ (t_1, \mathbf{x}(t_1)) \in \mathcal{T} \end{cases} \tag{1.11}$$

where $\mathbf{u}$ is an assigned function. In general, without assumption on g and $\mathbf{u}$, it is not possible to guarantee that there exists a unique solution for (1.10) defined in all the interval $[t_0, t_1]$; moreover, since the function $t \mapsto g(t, \mathbf{x}, \mathbf{u}(t))$ can be not regular, we have to be precise on the notion of "solution" of such ODE. In the next pages we will give a more precise definition of solution $\mathbf{x}$ for (1.11).

Controllability

Let us give some examples that show the difficulty to associated a trajectory to a control:

Example 1.2.1. *Let us consider*

$$\begin{cases} \dot{x} = 2u\sqrt{x} & \textit{in } [0, 1] \\ x(0) = 0 \end{cases}$$

Prove that the function $u(t) = a$, with a positive constant, is not an admissible control since the two functions $x_1(t) = 0$ and $x_2(t) = a^2t^2$ solve the previous ODE.

Example 1.2.2. *Let us consider*

$$\begin{cases} \dot{x} = ux^2 & \textit{in } [0, 1] \\ x(0) = 1 \end{cases}$$

Prove that the function $u(t) = a$, with a constant, is an admissible control if and only if $a < 1$. Prove that the trajectory associated to such control is $x(t) = \frac{1}{1-at}$.

Example 1.2.3. *Let us consider*

$$\begin{cases} \dot{x} = ux & \textit{in } [0, 2] \\ x(0) = 1 \\ x(2) = 3^6 \\ |u| \le 1 \end{cases}$$

Prove[3] that the set of admissible control is empty.

The problem to investigate the possibility to find admissible control for an optimal controls problem is called *controllability* (see section 3.4). In order to guarantee the solution of (1.11), the following well–known theorem is fundamental

Theorem 1.1. *Let us consider $G = G(t, \mathbf{x}) : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ and let G be continuous and Lipschitz continuous with respect to $\mathbf{x}$ in an open set $D \subseteq \mathbb{R}^{n+1}$ with $(t_0, \boldsymbol{\alpha}) \in D \subset \mathbb{R} \times \mathbb{R}^n$. Then, there exists a neighborhood $I \subset \mathbb{R}$ of t_0 such that the ODE*

$$\begin{cases} \dot{\mathbf{x}} = G(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \end{cases}$$

[3] Note that $0 \le \dot{x} = ux \le 3x$ and $x(0) = 1$ imply $0 \le x(t) \le e^{3t}$.

admits a unique solution $\mathbf{x} : I \to \mathbb{R}^n$, $\mathbf{x}$ *in* C^1.

Moreover, if there exist two positive constants A *and* B *such that* $\|G(t, \mathbf{x})\| \le A\|\mathbf{x}\| + B$ *for all* $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$, *then the solution of the previous ODE is defined in all the interval* $[t_0, t_1]$.

Now, let $\mathbf{u} : [t_0, t_1] \to U$ be a function in KC, i.e. continuous in $[t_0, t_1]$ up to the points $\tau_1, \tau_2, \ldots, \tau_N$, with $t_0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_N < \tau_{N+1} = t_1$, where $\mathbf{u}$ has a discontinuity of the first type. Let us suppose that there exists in $[t_0, \tau_1]$ a solution $\mathbf{x}_0$ of the ODE (1.10) with initial condition $\mathbf{x}_0(t_0) = \boldsymbol{\alpha}$. Let us suppose that there exists $\mathbf{x}_1$ solution of (1.10) in $[\tau_1, \tau_2]$ with initial condition $\mathbf{x}_0(\tau_1) = \mathbf{x}_1(\tau_1)$. In general for every i, $1 \le i \le N$, let us suppose that there exists $\mathbf{x}_i$ solution for (1.10) in $[\tau_i, \tau_{i+1}]$ with initial condition $\mathbf{x}_{i-1}(\tau_i) = \mathbf{x}_i(\tau_i)$. Finally we define the function $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n$ by

$$\mathbf{x}(t) = \mathbf{x}_i(t),$$

when $t \in [\tau_i, \tau_{i+1}]$. Such function $\mathbf{x}$ is the trajectory associated to the control $\mathbf{u}$ and initial data $\mathbf{x}(t_0) = \boldsymbol{\alpha}$.

Example 1.2.4. *Let*

$$\begin{cases} \dot{x} = ux \\ x(0) = 1 \end{cases}$$

and u *the function defined by*

$$u(t) = \begin{cases} 0 & \text{with } t \in [0, 1) \\ 1 & \text{with } t \in [1, 2] \\ t & \text{with } t \in (2, 3] \end{cases}$$

Prove that u *is admissible and that the associated trajectory* x *is*

$$x(t) = \begin{cases} 1 & \text{with } t \in [0, 1] \\ e^{t-1} & \text{with } t \in (1, 2] \\ e^{t^2/2-1} & \text{with } t \in (2, 3] \end{cases}$$

Now, if we consider a more general situation and consider a control $\mathbf{u}$ with a low regularity, the previous Theorem 1.1 doesn't give information about the solution of (1.11). To be precise, let us fix a measurable function $\mathbf{u} : [t_0, t_1] \to U$: clearly $t \mapsto g(t, \mathbf{x}, \mathbf{u}(t))$, for a fixed $\mathbf{x} \in \mathbb{R}^n$, in general is only a measurable function.

Hence we need a more general notion of solution for a ODE. Let $G : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function. We say that $\mathbf{x}$ is a *solution* for

$$\begin{cases} \dot{\mathbf{x}} = G(t, \mathbf{x}) & \text{in } [t_0, t_1] \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \end{cases} \tag{1.12}$$

if $\mathbf{x}$ is an absolutely continuous function[4] which satisfies the initial condition

[4]We recall that $\phi : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for every $\epsilon > 0$ there exists a δ such that

$$\sum_{i=1}^{n} |\phi(b_i) - \phi(a_i)| \le \epsilon$$

for any n and any collection of disjoint segments $(a_1, b_1), \ \ldots, \ (a_n, b_n)$ in $[a, b]$ with

$$\sum_{i=1}^{n} (b_i - a_i) \le \delta.$$

A fundamental property guarantees that ϕ is absolutely continuous in $[a, b]$ if and only if ϕ has a derivative ϕ' a.e., such derivative is Lebesgue integrable and

$$\phi(x) = \phi(a) + \int_a^x \phi'(s)\,\mathrm{d}s, \qquad \forall x \in [a, b].$$

and $\dot{\mathbf{x}}(t) = G(t, \mathbf{x}(t))$ for a.e. in $t \in [t_0, t_1]$.
The next result is a more precise existence theorem (see for example [14]):

Theorem 1.2. *Let us consider $G = G(t, \mathbf{x}) : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ be such that*

i. G is a measurable function;

ii. G is locally Lipschitz continuous with respect to $\mathbf{x}$ such that for every $R > 0$ there exists a function $l_R : [t_0, t_1] \to [0, \infty)$ in L^1 with

$$\|G(t, \mathbf{x}) - G(t, \mathbf{x}')\| \leq l_R(t)\|\mathbf{x} - \mathbf{x}'\|$$

for all $\mathbf{x}$, $\mathbf{x}'$ with $\|\mathbf{x}\| \leq R$, $\|\mathbf{x}'\| \leq R$ and for a.e. $t \in [t_0, t_1]$;

iii. G has at most a linear growth in the sense that there exist two positive functions $A : [t_0, t_1] \to [0, \infty)$ and $B : [t_0, t_1] \to [0, \infty)$ both in L^1 such that

$$\|G(t, \mathbf{x})\| \leq A(t)\|\mathbf{x}\| + B(t)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and for a.e. $t \in [t_0, t_1]$.

Then there exists a unique solution for (1.12).

Now we are in the position to give the precise notion of *admissible control* and *associated trajectory*:

Definition 1.1. *Let us consider a initial condition, a target set $\mathcal{T}$, a control set U and a measurable function g. We say that a measurable function $\mathbf{u} : [t_0, t_1] \to U$ is an* admissible control *(or shortly* control*) if there exists a unique solution $\mathbf{x}$ of such ODE defined on $[t_0, t_1]$, i.e. there exists an absolutely continuous function $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n$ such that is the unique solution of*

$$\begin{cases} \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) & a.e. \ in \ [t_0, t_1] \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ (t_1, \mathbf{x}(t_1)) \in \mathcal{T} \end{cases}$$

We call such solution $\mathbf{x}$ trajectory associated to $\mathbf{u}$. *We denote by $\mathcal{C}_{t_0,\boldsymbol{\alpha}}$ the set of the* admissible control for $\boldsymbol{\alpha}$ *at time t_0.*

In the next three chapters, in order to simplify the notation, we put $\mathcal{C} = \mathcal{C}_{t_0,\boldsymbol{\alpha}}$.
We say that a *dynamics is linear* if (1.10) is of the form

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t), \tag{1.13}$$

where $A(t)$ is a square matrix of order n and $B(t)$ is a matrix of order $n \times k$: moreover the elements of such matrices are continuous function in $[t_0, t_1]$. A fundamental property of controllability of the linear dynamics is the following

Proposition 1.1. *If the dynamics is linear and the trajectory has only a initial condition (i.e. $\mathcal{T} = \{t_1\} \times \mathbb{R}^n$), then every piecewise continuous function is an admissible control for (1.11), i.e. exists the associated trajectory.*

We remark that the previous proposition is false if we have a initial and a final condition on trajectory (see example 1.2.3).
The proof of the previous proposition is an easy application of the previous theorems.

For every $\tau \in [t_0, t_1]$, we define the *reachable set at time* τ as the set $R(\tau, t_0, \boldsymbol{\alpha}) \subseteq \mathbb{R}^n$ of the points $\mathbf{x}_\tau$ such that there exists an admissible control $\mathbf{u}$ and an associated trajectory $\mathbf{x}$ such that $\mathbf{x}(t_0) = \boldsymbol{\alpha}$ and $\mathbf{x}(\tau) = \mathbf{x}_\tau$.

If we consider the example 1.2.3, we have $3^6 \notin R(2, 0, 1)$.

1.2.2 Optimal Control problems

Let us introduce the functional that we would like to optimize. Let us consider the dynamics in (1.11), a function $f : [t_0, \infty) \times \mathbb{R}^{n+k} \to \mathbb{R}$, the so called *running cost* (or *running payoff*) and a function $\psi : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, the so called *pay off* .

Let U and S be control set and target respectively. Let us consider the set of admissible control $\mathcal{C}$. We define $J : \mathcal{C} \to \mathbb{R}$ by

$$J(\mathbf{u}) = \int_{t_0}^{T} f(t, \mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t + \psi(T, \mathbf{x}(T)),$$

where the function $\mathbf{x}$ is the (unique) trajectory associated to the control $\mathbf{u}$ that satisfies the initial and the final condition. This is the reason why J depends only on $\mathbf{u}$. Hence our problem is

$$\begin{cases} \max\limits_{\mathbf{u} \in \mathcal{C}} \ J(\mathbf{u}), \\ J(\mathbf{u}) = \displaystyle\int_{t_0}^{T} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(T, \mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \qquad \text{a.e. in } [t_0, T] \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ (T, \mathbf{x}(T)) \in \mathcal{T} \\ \mathcal{C} = \{\mathbf{u} : [t_0, T] \to U \subseteq \mathbb{R}^k, \ \mathbf{u} \text{ admissible}\} \end{cases} \tag{1.14}$$

We say that $\mathbf{u}^* \in \mathcal{C}$ is an *optimal control* for (1.14) if

$$J(\mathbf{u}) \leq J(\mathbf{u}^*), \qquad \forall \mathbf{u} \in \mathcal{C}.$$

The trajectory $\mathbf{x}^*$ associated to the optimal control $\mathbf{u}^*$, is called *optimal trajectory*.

In this problem and in more general problems, when f, g and ψ do not depend directly on t, we say that the problem is *autonomous*.

Chapter 2

The simplest problem of OC

In this chapter we are interested on the following problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,\mathrm{d}t \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max\limits_{\mathbf{u}\in\mathcal{C}} J(\mathbf{u}), \\ \mathcal{C} = \{\mathbf{u} : [t_0, t_1] \to U \subseteq \mathbb{R}^k,\ \mathbf{u} \text{ admissible}\} \end{cases} \tag{2.1}$$

where t_1 is fixed. The problem (2.1) is called the *simplest problem of Optimal Control* (in all that follows we shorten "Optimal Control" with OC).

2.1 The necessary condition of Pontryagin

Let us introduce the function

$$(\lambda_0, \boldsymbol{\lambda}) = (\lambda_0, \lambda_1, \ldots, \lambda_n) : [t_0, t_1] \to \mathbb{R}^{n+1}.$$

We call such function *multiplier* (or *costate variable*). We define the *Hamiltonian function* $H : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}) = \lambda_0 f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}).$$

The following result is fundamental:

Theorem 2.1 (Pontryagin)**.** *In the problem (2.1), let f and g be continuous functions with continuous derivatives with respect to $\mathbf{x}$.*
Let $\mathbf{u}^*$ be an optimal control and $\mathbf{x}^*$ be the associated trajectory.
Then there exists a multiplier $(\lambda_0^*, \boldsymbol{\lambda}^*)$, with

- ⋄ $\lambda_0^* \geq 0$ *constant,*
- ⋄ $\boldsymbol{\lambda}^* : [t_0, t_1] \to \mathbb{R}^n$ *continuous,*

such that

i) *(nontriviality of the multiplier)* $(\lambda_0^*, \boldsymbol{\lambda}^*) \neq (0, \mathbf{0})$*;*

ii) *(Pontryagin Maximum Principle, shortly PMP) for all* $t \in [t_0, t_1]$ *we have*

$$\mathbf{u}^*(t) \in \arg\max_{\mathbf{v}\in U} H(t, \mathbf{x}^*(t), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(t)), \qquad \text{i.e.}$$

$$H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) = \max_{\mathbf{v}\in U} H(t, \mathbf{x}^*(t), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(t)); \tag{2.2}$$

iii) *(adjoint equation, shortly AE) we have*

$$\dot{\boldsymbol{\lambda}}^*(t) = -\nabla_{\mathbf{x}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)), \qquad \text{a.e. } t \in [t_0, t_1]; \tag{2.3}$$

iv) *(transversality condition, shortly TC)* $\boldsymbol{\lambda}^*(t_1) = \mathbf{0}$;

v) *(normality)* $\lambda_0^* = 1$.

If in addition the functions f *and* g *are continuously differentiable in* t*, then for all* $t \in [t_0, t_1]$

$$\begin{aligned} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) &= H(t_0, \mathbf{x}^*(t_0), \mathbf{u}^*(t_0), \lambda_0^*, \boldsymbol{\lambda}^*(t_0)) + \\ &+ \int_{t_0}^t \frac{\partial H}{\partial t}(s, \mathbf{x}^*(s), \mathbf{u}^*(s), \lambda_0^*, \boldsymbol{\lambda}^*(s))\, \mathrm{d}s \end{aligned} \tag{2.4}$$

The proof of this result is very long and difficult (see [22], [13], [10], [14], [28]): in section 2.1.1 we give a proof in a particular situation. Now let us list some comments and definitions.

An admissible control $\mathbf{u}^*$ that satisfies the conclusion of the theorem of Pontryagin is called *extremal*. We define $(\lambda_0^*, \boldsymbol{\lambda}^*)$ the *associated multiplier* to the extremal $\mathbf{u}^*$ as the function that satisfies the conclusion of the mentioned theorem.

We mention that in the adjoint equation, since $\mathbf{u}^*$ is a measurable function, the multiplier $\boldsymbol{\lambda}^*$ is a solution of a ODE

$$\dot{\boldsymbol{\lambda}} = G(t, \boldsymbol{\lambda}) \qquad \text{in } [t_0, t_1]$$

(here G is the second member of (2.3)), where G is a measurable function, affine in the $\boldsymbol{\lambda}$ variable, i.e. is a linear differential equation in $\boldsymbol{\lambda}$ with measurable coefficients. This notion of solution is as in (1.12) and hence $\boldsymbol{\lambda}^*$ is an absolutely continuous function.

We remark that we can rewrite the dynamics (1.10) as

$$\dot{\mathbf{x}} = \nabla_{\boldsymbol{\lambda}} H.$$

Normal and abnormal controls

It is clearly, in the previous theorem with the simplest problem of OC (2.1), that *v.* implies $(\lambda_0^*, \boldsymbol{\lambda}^*) \neq (0, \mathbf{0})$. If we consider a more generic problem (see for example (1.14)), then it is not possible to guarantees $\lambda_0^* = 1$.

In general, there are two distinct possibilities for the constant λ_0^* :

a. if $\lambda_0^* = 0$, we say that $\mathbf{u}^*$ is *abnormal*. Then the Hamiltonian H, for such λ_0^*, does not depend on f and the Pontryagin Maximum Principle is of no use;

b. if $\lambda_0^* \neq 0$, we say that $\mathbf{u}^*$ is *normal*: in this situation we may assume that $\lambda_0^* = 1$.

Let us spend few words on this last assertion. Let $\mathbf{u}^*$ be a normal extremal control with $\mathbf{x}^*$ and $(\lambda_0^*, \boldsymbol{\lambda}^*)$ associated trajectory and associated multiplier respectively. It is an easy exercise to verify that $(\tilde{\lambda}_0, \tilde{\boldsymbol{\lambda}})$, defined by

$$\tilde{\lambda}_0 = 1, \qquad \tilde{\boldsymbol{\lambda}} = \frac{\boldsymbol{\lambda}^*}{\lambda_0^*},$$

is again a multiplier associated to the normal control $\mathbf{u}^*$. Hence, if $\mathbf{u}^*$ is *normal*, we may assume that $\lambda_0^* = 1$. These arguments give that

Remark 2.1. *In Theorem 2.1 we can replace $\lambda_0^* \geq 0$ constant, with $\lambda_0^* \in \{0, 1\}$.*

The previous theorem 2.1 guarantees that

Remark 2.2. *In the simplest optimal control problem (2.1) every extremal is normal.*

We will see in example 2.5.7 an abnormal optimal control.

On Maximum Principle with much more regularity

An important necessary condition of optimality in convex analysis[1] implies

Remark 2.3. *Let f and g be as in Theorem 2.1 with the additional assumption that they are differentiable with respect to the variable $\mathbf{u}$. Let the control set U be convex and $\mathbf{u}^*$ be optimal for (2.1). Since, for every fixed t, $\mathbf{u}^*(t)$ is a maximum for $\mathbf{v} \mapsto H(t, \mathbf{x}^*(t), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(t))$, the PMP implies*

$$\nabla_{\mathbf{u}} H(\tau, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) \cdot (\mathbf{v} - \mathbf{u}^*(t)) \leq 0, \tag{2.6}$$

for every $\mathbf{v} \in U$, $t \in [t_0, t_1]$.

When the control set coincides with $\mathbb{R}^k$ we have the following modification for the PMP:

Remark 2.4. *In the assumption of Remark 2.3, let $U = \mathbb{R}^k$ be the control set for (2.1). In theorem 2.1 we can replace the PMP with the following new formulation*

$$(\mathrm{PMP}_0) \qquad \nabla_{\mathbf{u}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) = \mathbf{0}, \qquad \forall\, t \in [t_0, t_1],$$

---

[1] *Theorem.* Let U be a convex set in $\mathbb{R}^k$ and $F : U \to \mathbb{R}$ be differentiable. If $\mathbf{v}^*$ is a point of maximum for F in U, then

$$\nabla F(\mathbf{v}^*) \cdot (\mathbf{v} - \mathbf{v}^*) \leq 0, \qquad \forall \mathbf{v} \in U. \tag{2.5}$$

Proof: If $\mathbf{v}^*$ is in the interior of U, then $\nabla F(\mathbf{v}^*) = 0$ and (2.5) is true. Let $\mathbf{v}^*$ be on the boundary of U: for all $\mathbf{v} \in U$, let us consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(s) = F((1-s)\mathbf{v}^* + s\mathbf{v})$. The formula of Mc Laurin gives $f(s) - f(0) = f'(0)s + o(s)$, where $o(s)/s \to 0$ for $s \to 0^+$. Since $\mathbf{v}^*$ is maximum we have

$$\begin{aligned} 0 &\geq F((1-s)\mathbf{v}^* + s\mathbf{v}) - F(\mathbf{v}^*) \\ &= f(s) - f(0) \\ &= f'(0)s + o(s) \\ &= \nabla F(\mathbf{v}^*) \cdot (\mathbf{v} - \mathbf{v}^*)s + o(s). \end{aligned}$$

Since $s \geq 0$, (2.5) is true. □

On the Hamiltonian along the optimal path

In (2.4) let us consider the function

$$t \mapsto h(t) := H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) \tag{2.7}$$

is a very regular function in $[t_0, t_1]$ even though the control $\mathbf{u}^*$ may be discontinuous. More precisely and the very interesting propriety of (2.4) is the following

Remark 2.5. *In the problem (2.1), let f and g be continuous functions with continuous derivatives with respect to t and $\mathbf{x}$. Let $\mathbf{u}^*$ be a piecewise continuous optimal control and $\mathbf{x}^*$ be the associated trajectory. Let the control set U be closed. Then*

- *relation (2.4) is a consequence of the relations i)–v) of Theorem 2.1;*
- *the function h in (2.7) is absolutely continuous in $[t_0, t_1]$ and*

$$\dot{h}(t) = \frac{\partial H}{\partial t}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)).$$

This result is an easy consequence of the following lemma

Lemma 2.1. *Let $\widetilde{h} = \widetilde{h}(t, \mathbf{u}) : [t_0, t_1] \times \mathbb{R}^k \to \mathbb{R}$ be a continuous function with continuous derivatives with respect to t. Let $U \subset \mathbb{R}^k$ be closed and let $\mathbf{u}^* : [t_0, t_1] \to U$ be a left piecewise continuous function such that*

$$\mathbf{u}^*(t) \in \arg\max_{\mathbf{v} \in U} \widetilde{h}(t, \mathbf{v}), \qquad \forall t \in [t_0, t_1].$$

Let $\widehat{h} : [t_0, t_1] \to \mathbb{R}$ be a function such that

$$\widehat{h}(t) = \max_{\mathbf{v} \in U} \widetilde{h}(t, \mathbf{v}) = \widetilde{h}(t, \mathbf{u}^*(t)) \tag{2.8}$$

in $[t_0, t_1]$. Then $\widehat{h}$ is absolutely continuous in $[t_0, t_1]$ and

$$\widehat{h}(t) = \widehat{h}(t_0) + \int_{t_0}^{t} \frac{\partial \widetilde{h}}{\partial t}(s, \mathbf{u}^*(s))\, \mathrm{d}s, \qquad \forall t \in [t_0, t_1]. \tag{2.9}$$

Proof. First let us prove that $\widehat{h}$ is continuous. By the assumption $\widehat{h}$ is left piecewise continuous and hence we have only to prove $\widehat{h}$ is right continuous in $[t_0, t_1]$, i.e. for every fixed $t \in [t_0, t_1]$

$$\lim_{\tau \to 0^+} \widetilde{h}(t + \tau, \mathbf{u}^*(t + \tau)) = \widetilde{h}(t, \mathbf{u}^*(t)) :$$

since $\widetilde{h}$ is continuous, the previous relation is equivalent to

$$\lim_{\tau \to 0^+} \widetilde{h}(t, \mathbf{u}^*(t + \tau)) = \widetilde{h}(t, \mathbf{u}^*(t)). \tag{2.10}$$

Relation (2.8) implies that for a fixed $t \in [t_0, t_1]$ and for every small τ we have

$$\widetilde{h}(t, \mathbf{u}^*(t + \tau)) \le \widetilde{h}(t, \mathbf{u}^*(t)), \tag{2.11}$$

$$\widetilde{h}(t + \tau, \mathbf{u}^*(t)) \le \widetilde{h}(t + \tau, \mathbf{u}^*(t + \tau)). \tag{2.12}$$

Hence considering the limits for $\tau \to 0^+$ and using the continuity of $\widetilde{h}$

$$\lim_{\tau\to 0^+} \widetilde{h}(t, \mathbf{u}^*(t+\tau)) \le \lim_{\tau\to 0^+} \widetilde{h}(t, \mathbf{u}^*(t)) = \lim_{\tau\to 0^+} \widetilde{h}(t+\tau, \mathbf{u}^*(t)) \le$$
$$\le \lim_{\tau\to 0^+} \widetilde{h}(t+\tau, \mathbf{u}^*(t+\tau)) = \lim_{\tau\to 0^+} \widetilde{h}(t, \mathbf{u}^*(t+\tau)).$$

Hence the previous inequalities are all equalities and (2.10) holds.

Now let t be a point of continuity of $\mathbf{u}^*$. Since $\widetilde{h}$ has derivative w.r.t. t, the mean theorem implies that every small τ there exist θ_1 and θ_2 in $[0, 1]$ such that, using (2.11) and (2.12),

$$\begin{aligned}
\frac{\partial \widetilde{h}}{\partial t}(t+\theta_1\tau, \mathbf{u}^*(t))\tau &= \widetilde{h}(t+\tau, \mathbf{u}^*(t)) - \widetilde{h}(t, \mathbf{u}^*(t)) \\
&\le \widetilde{h}(t+\tau, \mathbf{u}^*(t+\tau)) - \widetilde{h}(t, \mathbf{u}^*(t)) \\
&\le \widetilde{h}(t+\tau, \mathbf{u}^*(t+\tau)) - \widetilde{h}(t, \mathbf{u}^*(t+\tau)) \\
&= \frac{\partial \widetilde{h}}{\partial t}(t+\theta_2\tau, \mathbf{u}^*(t+\tau))\tau
\end{aligned}$$

Dividing by τ we obtain

$$\frac{\partial \widetilde{h}}{\partial t}(t+\theta_1\tau, \mathbf{u}^*(t)) \le \frac{\widetilde{h}(t+\tau, \mathbf{u}^*(t+\tau)) - \widetilde{h}(t, \mathbf{u}^*(t))}{\tau} \le \frac{\partial \widetilde{h}}{\partial t}(t+\theta_2\tau, \mathbf{u}^*(t+\tau)).$$

Since $\frac{\partial \widetilde{h}}{\partial t}$ is continuous and t is a point of continuity of $\mathbf{u}^*$, taking the limits as $\tau \to 0$ in the previous inequalities we have

$$\frac{\partial \widetilde{h}}{\partial t}(t, \mathbf{u}^*(t)) = \frac{\mathrm{d}\widetilde{h}}{\mathrm{d}t}(t, \mathbf{u}^*(t)) = \frac{\mathrm{d}\widehat{h}}{\mathrm{d}t}(t) = \widehat{h}'(t), \tag{2.13}$$

for every point of continuity for $\mathbf{u}^*$. Since $\widetilde{h}$ has continuous derivative with respect to t, hence $\widehat{h}$ has a piecewise continuous derivative: this implies that $\widehat{h}$ has a derivative $\widehat{h}'$ a.e., such derivative is Lebesgue integrable in $[t_0, t_1]$; finally, since $\widehat{h}'$ is piecewise continuous, clearly (2.9) holds. □

Let us notice that if $\widetilde{h}$ has only piecewise continuous derivative, the previous proof holds and we obtain the relation (2.13) for the points t where $\mathbf{u}^*$ and $\frac{\partial \widetilde{h}}{\partial t}$ are continuous functions.

Proof of Remark 2.5. We apply the previous lemma 2.1 to the function

$$\widetilde{h}(t, \mathbf{u}) = H(t, \mathbf{x}^*(t), \mathbf{u}, \lambda_0^*, \boldsymbol{\lambda}^*(t)) :$$

we have to prove that the assumptions of such lemma are satisfied. First, if $\mathbf{u}^*$ is not left piecewise continuous, then we have to modify the control in a set of null measure. Now, it is clear that the function $\widetilde{h}$ is continuous since H is continuous in all its variables. Let us check that $\frac{\partial \widetilde{h}}{\partial t}(t, u)$ is continuous for every point t of continuity for $\mathbf{u}^*$:

$$\begin{aligned}
\frac{\partial \widetilde{h}}{\partial t}(t, \mathbf{u}) = {}& \frac{\partial H}{\partial t}(t, \mathbf{x}^*(t), \mathbf{u}, \lambda_0^*, \boldsymbol{\lambda}^*(t)) + \nabla_{\mathbf{x}} H(t, \mathbf{x}^*(t), \mathbf{u}, \lambda_0^*, \boldsymbol{\lambda}^*(t)) \cdot \dot{\mathbf{x}}^*(t) + \\
&+ \nabla_{\boldsymbol{\lambda}} H(t, \mathbf{x}^*(t), \mathbf{u}, \lambda_0^*, \boldsymbol{\lambda}^*(t)) \cdot \dot{\boldsymbol{\lambda}}^*(t)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\partial H}{\partial t}(t, \mathbf{x}^*(t), \mathbf{u}, \lambda_0^*, \boldsymbol{\lambda}^*(t)) + \\
& +\nabla_{\mathbf{x}} H(t, \mathbf{x}^*(t), \mathbf{u}, \lambda_0^*, \boldsymbol{\lambda}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) + \\
& -g(t, \mathbf{x}^*(t), \mathbf{u}) \cdot \nabla_{\mathbf{x}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t));
\end{aligned}
$$

the assumptions on f and g give the claim. □

An immediate consequence of (2.4) is the following important and useful result

Remark 2.6 (autonomous problems)**.** *If the problem is autonomous, i.e. f and g does not depend directly by t, and f, g are in C^1, then (2.4) implies that*

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) = \text{constant} \qquad \text{in } [t_0, t_1] \tag{2.14}$$

2.1.1 The proof in a particular situation

In this section we consider a "simplest" optimal control problem (2.1) with two fundamental assumptions that simplify the proof of the theorem of Pontryagin:

a. we suppose that the control set is $U = \mathbb{R}^k$.

b. We suppose that the set $\mathcal{C} = \mathcal{C}_{t_0,\boldsymbol{\alpha}}$, i.e. the set of admissible controls, does not contain discontinuous function, is non empty and is open. We remark that with a linear dynamics, these assumptions on $\mathcal{C}$ are satisfied.

In order to prove the mentioned theorem, we need a technical lemma:

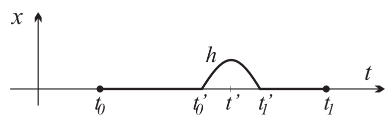
Lemma 2.2. *Let $\varphi \in C([t_0, t_1])$ and*

$$\int_{t_0}^{t_1} \varphi(t) h(t)\, \mathrm{d}t = 0 \tag{2.15}$$

for every $h \in C([t_0, t_1])$. Then φ is identically zero on $[t_0, t_1]$.

Proof. Let us suppose that $\varphi(t') \neq 0$ for some point $t' \in [t_0, t_1]$: we suppose that $\varphi(t') > 0$ (if $\varphi(t') < 0$ the proof is similar). Since φ is continuous, there exists an interval $[t'_0, t'_1] \subset [t_0, t_1]$ containing t' such that φ is positive.
Let us define the function $h : [t_0, t_1] \to \mathbb{R}$ as

$$h(t) = -(t - t'_0)(t - t'_1)\, \mathbf{1}_{[t'_0, t'_1]}(t),$$



where $\mathbf{1}_A$ is the indicator function on the set A. Hence

$$\int_{t_0}^{t_1} \varphi(t) h(t)\, \mathrm{d}t = -\int_{t'_0}^{t'_1} \varphi(t)\, (t - t'_0)(t - t'_1)\, \mathrm{d}t > 0. \tag{2.16}$$

On the other hand, (2.15) implies that $\int_{t_0}^{t_1} \varphi(t) h(t)\, \mathrm{d}t = 0$. Hence (2.16) is absurd and there does not exist such point t'. □

Theorem 2.2. *Let us consider the problem*

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max\limits_{\mathbf{u}\in\mathcal{C}} J(\mathbf{u}) \\ \mathcal{C} = \left\{\mathbf{u} : [t_0,t_1] \to \mathbb{R}^k,\ \mathbf{u} \in C([t_0,t_1])\right\} \end{cases}$$

with $f \in C^1([t_0,t_1] \times \mathbb{R}^{n+k})$ *and* $g \in C^1([t_0,t_1] \times \mathbb{R}^{n+k})$; *moreover let* $\mathcal{C}$ *be open and non empty.*
Let $\mathbf{u}^*$ *be the optimal control and* $\mathbf{x}^*$ *be the optimal trajectory. Then there exists a multiplier* $\boldsymbol{\lambda}^* : [t_0,t_1] \to \mathbb{R}^n$ *continuous such that*

$$(\mathrm{PMP}_0) \qquad \nabla_{\mathbf{u}} H(t,\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)) = \mathbf{0}, \quad \forall t \in [t_0,t_1] \tag{2.17}$$

$$(\mathrm{AE}) \qquad \nabla_{\mathbf{x}} H(t,\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)) = -\dot{\boldsymbol{\lambda}}^*(t), \quad \forall t \in [t_0,t_1] \tag{2.18}$$

$$(\mathrm{TC}) \qquad \boldsymbol{\lambda}^*(t_1) = \mathbf{0}, \tag{2.19}$$

where $H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = f(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{u})$.

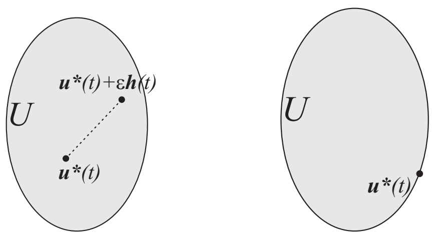
Proof. Let $\mathbf{u}^* \in \mathcal{C}$ be optimal control and $\mathbf{x}^*$ its trajectory. Let us fix a continuous function $\mathbf{h} = (h_1,\ldots,h_k) : [t_0,t_1] \to \mathbb{R}^k$. For every constant $\boldsymbol{\epsilon} \in \mathbb{R}^k$ we define the function $\mathbf{u}_{\boldsymbol{\epsilon}} : [t_0,t_1] \to \mathbb{R}^k$ by

$$\mathbf{u}_{\boldsymbol{\epsilon}} = \mathbf{u}^* + (\epsilon_1 h_1,\ldots,\epsilon_k h_k) = (u_1^* + \epsilon_1 h_1,\ldots,u_k^* + \epsilon_k h_k). \tag{2.20}$$

Since $\mathcal{C}$ is open, for every Let us show that $\mathbf{u}_{\boldsymbol{\epsilon}}$ $\boldsymbol{\epsilon}$ with $\|\boldsymbol{\epsilon}\|$ sufficiently small, $\mathbf{u}_{\boldsymbol{\epsilon}}$ is an admissible control.[2] Hence, for such $\mathbf{u}_{\boldsymbol{\epsilon}}$ there exists the associated trajectory: we denote by $\mathbf{x}_{\boldsymbol{\epsilon}} : [t_0,t_1] \to \mathbb{R}^n$ such trajectory associated[3] to the control $\mathbf{u}_{\boldsymbol{\epsilon}}$ in (2.20). Clearly

$$\mathbf{u}_{\mathbf{0}}(t) = \mathbf{u}^*(t), \qquad \mathbf{x}_{\mathbf{0}}(t) = \mathbf{x}^*(t), \qquad \mathbf{x}_{\boldsymbol{\epsilon}}(t_0) = \boldsymbol{\alpha}. \tag{2.21}$$

[2] We remark that the assumption $U = \mathbb{R}^k$ is crucial. Suppose, for example, that $U \subset \mathbb{R}^2$ and let us fix $t \in [t_0,t_1]$. If $u^*(t)$ is an interior point of U, for every function $\mathbf{h}$ and for $\boldsymbol{\epsilon}$ with modulo sufficiently small, we have that $\mathbf{u}_{\boldsymbol{\epsilon}}(t) = (u_1^*(t) + \epsilon_1 h_1(t),\ u_2^*(t) + \epsilon_2 h_2(t)) \in U$. If $u^*(t)$ lies on the boundary of U, is impossible to guarantee that, for every $\mathbf{h}$, $\mathbf{u}(t) = \mathbf{u}^*(t) + \boldsymbol{\epsilon}\cdot\mathbf{h}(t) \in U$.



The case $\mathbf{u}^*(t)$ in the interior of U; the case $\mathbf{u}^*(t)$ on the boundary of U.

[3] For example, if $n = k = 1$ and the dynamics is linear we have, for every ϵ,

$$\begin{cases} \dot{x}_\epsilon(t) = a(t)x_\epsilon(t) + b(t)[u^*(t) + \epsilon h(t)] \\ x_\epsilon(t_0) = \alpha \end{cases}$$

and hence $x_\epsilon(t) = e^{\int_{t_0}^t a(s)\,\mathrm{d}s}\left(\alpha + \displaystyle\int_{t_0}^t b(s)[u^*(s) + \epsilon h(s)]e^{\int_{t_0}^s a(w)\,\mathrm{d}w}\,\mathrm{d}s\right)$.

Now, recalling that $\mathbf{h}$ is fixed, we define the function $\mathcal{J}_{\mathbf{h}} : \mathbb{R}^k \to \mathbb{R}$ as

$$\mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) = \int_{t_0}^{t_1} f\big(t, \mathbf{x}_{\boldsymbol{\epsilon}}(t), \mathbf{u}_{\boldsymbol{\epsilon}}(t)\big)\, \mathrm{d}t.$$

Since $\mathbf{u}^*$ is optimal, $\mathcal{J}_h(\mathbf{0}) \geq \mathcal{J}_h(\boldsymbol{\epsilon}),\ \forall \boldsymbol{\epsilon}$; then $\nabla_{\boldsymbol{\epsilon}} \mathcal{J}_h(\mathbf{0}) = \mathbf{0}$. Let $\boldsymbol{\lambda} : [t_0, t_1] \to \mathbb{R}^n$ be a generic continuous function. Using the dynamics we have

$$\begin{aligned}
\mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) &= \int_{t_0}^{t_1} \Big[f\big(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}\big) + \boldsymbol{\lambda} \cdot \Big(g\big(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}\big) - \dot{\mathbf{x}}_{\boldsymbol{\epsilon}} \Big) \Big]\, \mathrm{d}t \\
&= \int_{t_0}^{t_1} \big[H\big(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}\big) - \boldsymbol{\lambda} \cdot \dot{\mathbf{x}}_{\boldsymbol{\epsilon}} \big]\, \mathrm{d}t \\
\text{(by part)} \quad &= \int_{t_0}^{t_1} \Big[H\big(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}\big) + \dot{\boldsymbol{\lambda}} \cdot \mathbf{x}_{\boldsymbol{\epsilon}} \Big]\, \mathrm{d}t - \Big(\boldsymbol{\lambda} \cdot \mathbf{x}_{\boldsymbol{\epsilon}} \Big|_{t_0}^{t_1}
\end{aligned}$$

For every i, with $1 \leq i \leq k$, we have

$$\begin{aligned}
\frac{\partial \mathcal{J}_h}{\partial \epsilon_i} &= \int_{t_0}^{t_1} \Big\{ \nabla_{\mathbf{x}} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \\
&\qquad + \nabla_{\mathbf{u}} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{u}_{\boldsymbol{\epsilon}}(t) + \\
&\qquad + \dot{\boldsymbol{\lambda}} \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) \Big\}\, \mathrm{d}t + \\
&\qquad - \boldsymbol{\lambda}(t_1) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_0).
\end{aligned}$$

Note that (2.20) implies $\nabla_{\epsilon_i} \mathbf{u}_{\boldsymbol{\epsilon}}(t) = (0, \ldots, 0, h_i, 0, \ldots, 0)$, and (2.21) implies $\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_0) = \mathbf{0}$. Hence, by (2.21), we obtain

$$\begin{aligned}
\frac{\partial \mathcal{J}_h}{\partial \epsilon_i}(\mathbf{0}) &= \int_{t_0}^{t_1} \Big\{ \Big[\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) + \dot{\boldsymbol{\lambda}} \Big] \cdot \Big(\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) \Big|_{\boldsymbol{\epsilon}=0} + \\
&\qquad + \frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) h_i(t) \Big\}\, \mathrm{d}t + \\
&\qquad - \boldsymbol{\lambda}(t_1) \cdot \Big(\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) \Big|_{\boldsymbol{\epsilon}=0} \\
&= 0.
\end{aligned} \tag{2.22}$$

Now let us choose the function $\boldsymbol{\lambda}$ as the solution of the following ODE:

$$\begin{cases} \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) & \text{for } t \in [t_0, t_1] \\ \boldsymbol{\lambda}(t_1) = 0 \end{cases} \tag{2.23}$$

Since

$$\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) = \nabla_{\mathbf{x}} f(t, \mathbf{x}^*, \mathbf{u}^*) + \boldsymbol{\lambda} \cdot \nabla_{\mathbf{x}} g(t, \mathbf{x}^*, \mathbf{u}^*),$$

this implies that the previous differential equation is linear (in $\boldsymbol{\lambda}$). Hence, the assumption of the theorem implies that there exists a unique[4] solution $\boldsymbol{\lambda}^* \in$

[4]We recall that for ODE of the first order with continuous coefficients holds the theorem 1.1.

$C([t_0,t_1])$ of (2.23). Hence conditions (2.18) and (2.19) hold. For this choice of the function $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, we have by (2.22)

$$\int_{t_0}^{t_1} \frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) h_i \, \mathrm{d}t = 0, \tag{2.24}$$

for every i, with $1 \le i \le k$, and $\mathbf{h} = (h_1, \ldots, h_k) \in C([t_0,t_1])$. Lemma 2.2 and (2.24) imply that $\frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) = 0$ in $[t_0,t_1]$ and hence (2.17). □

2.2 Sufficient conditions

In order to study the problem (2.1), one of the main result about the sufficient conditions for a control to be optimal is due to Mangasarian (see [20]). Recalling that in the simplest problem every extremal control is normal (see remark 2.2), we have:

Theorem 2.3 (Mangasarian). *Let us consider the maximum problem (2.1) with $f \in C^1$ and $g \in C^1$. Let the control set U be convex. Let $\mathbf{u}^*$ be a normal extremal control, $\mathbf{x}^*$ the associated trajectory and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$ the associated multiplier (as in theorem 2.1).*
Consider the Hamiltonian function H and let us suppose that

v) the function $(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*)$ is, for every $t \in [t_0,t_1]$, concave.

Then $\mathbf{u}^*$ is optimal.

Proof. The assumptions of regularity and concavity on H imply[5]

$$\begin{aligned} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*) \quad \le \quad & H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) + \\ & +\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) \cdot (\mathbf{x} - \mathbf{x}^*) + \\ & +\nabla_{\mathbf{u}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) \cdot (\mathbf{u} - \mathbf{u}^*), \end{aligned} \tag{2.25}$$

for every admissible control $\mathbf{u}$ with associated trajectory $\mathbf{x}$, and for every $\tau \in [t_0,t_1]$. The PMP implies, see (2.6), that

$$\nabla_{\mathbf{u}} H(\tau, \mathbf{x}^*(\tau), \mathbf{u}^*(\tau), \boldsymbol{\lambda}^*(\tau)) \cdot (\mathbf{u}(\tau) - \mathbf{u}^*(\tau)) \le 0. \tag{2.26}$$

The adjoint equation *ii)*, (2.25) and (2.26) imply

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*) \le H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) - \dot{\boldsymbol{\lambda}}^* \cdot (\mathbf{x} - \mathbf{x}^*). \tag{2.27}$$

Since $\mathbf{x}$ and $\mathbf{x}^*$ are trajectory associated to $\mathbf{u}$ and $\mathbf{u}^*$ respectively, by (2.27) we have

$$\begin{aligned} f(t, \mathbf{x}, \mathbf{u}) \quad & \le \quad f(t, \mathbf{x}^*, \mathbf{u}^*) + \boldsymbol{\lambda}^* \cdot (g(t, \mathbf{x}^*, \mathbf{u}^*) - g(t, \mathbf{x}, \mathbf{u})) + \dot{\boldsymbol{\lambda}}^* \cdot (\mathbf{x}^* - \mathbf{x}) \\ & = \quad f(t, \mathbf{x}^*, \mathbf{u}^*) + \boldsymbol{\lambda}^* \cdot (\dot{\mathbf{x}}^* - \dot{\mathbf{x}}) + \dot{\boldsymbol{\lambda}}^* \cdot (\mathbf{x}^* - \mathbf{x}) \\ & = \quad f(t, \mathbf{x}^*, \mathbf{u}^*) + \frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{\lambda}^* \cdot (\mathbf{x}^* - \mathbf{x})\big). \end{aligned} \tag{2.28}$$

[5] We recall that if F is a differentiable function on a convex set $C \subseteq \mathbb{R}^n$, then F is concave in C if and only if, for every $v, v' \in C$, we have $F(v) \le F(v') + \nabla F(v') \cdot (v - v')$.

Hence, for every admissible control $\mathbf{u}$ with associated trajectory $\mathbf{x}$, we have

$$\begin{aligned}\int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t &\leq \int_{t_0}^{t_1} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + \Big(\boldsymbol{\lambda}^* \cdot (\mathbf{x}^* - \mathbf{x})\Big|_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + \\ &\quad +\boldsymbol{\lambda}^*(t_1)\cdot\big(\mathbf{x}^*(t_1)-\mathbf{x}(t_1)\big) - \boldsymbol{\lambda}^*(t_0)\cdot\big(\mathbf{x}^*(t_0)-\mathbf{x}(t_0)\big);\end{aligned} \tag{2.29}$$

since $\mathbf{x}^*(t_0) = \mathbf{x}(t_0) = \boldsymbol{\alpha}$ and the transversality condition *iii)* are satisfied, we obtain that $\mathbf{u}^*$ is optimal. □

In order to apply such theorem, it is easy to prove the next note

Remark 2.7. *If we replace the assumption v) of theorem 2.3 with one of the following assumptions*

v') for every $t \in [t_0, t_1]$, let f and g be concave in the variables $\mathbf{x}$ and $\mathbf{u}$, and let us suppose $\boldsymbol{\lambda}^*(t) \geq \mathbf{0}$, (i.e. for every i, $1 \leq i \leq n$, $\lambda_i^*(t) \geq 0$);

v") let the dynamics of problem (2.1) be linear and, for every $t \in [t_0, t_1]$, let f be concave in the variables $\mathbf{x}$ and $\mathbf{u}$;

then $\mathbf{u}^*$ is optimal.

A further sufficient condition is due to Arrow: we are interested in a particular situation of the problem (2.1), more precisely

$$\begin{cases} \max\limits_{\mathbf{u}\in\mathcal{C}} \displaystyle\int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \mathcal{C} = \{\mathbf{u} : [t_0,t_1] \to U,\ \mathbf{u} \text{ admissible }\} \end{cases} \tag{2.30}$$

with $U \subset \mathbb{R}^k$ (note that we do not require convexity of the control set). Let us suppose that it is possible to define the *maximized Hamiltonian* function $H^0 : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$H^0(t,\mathbf{x},\boldsymbol{\lambda}) = \max_{\mathbf{u}\in U} H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}), \tag{2.31}$$

where $H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = f(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{u})$ is the Hamiltonian (note that we set $\lambda_0 = 1$). We have the following result by Arrow (see [2], [9] section 8.3, [17] part II section 15, theorem 2.5 in [26]):

Theorem 2.4 (Arrow)**.** *Let us consider the maximum problem (2.30) with $f \in C^1$ and $g \in C^1$. Let $\mathbf{u}^*$ be a normal extremal control, $\mathbf{x}^*$ be the associated trajectory and $\boldsymbol{\lambda}^*$ be the associated multiplier.*
Let us suppose that the maximized Hamiltonian function H^0 exists and, for every $t \in [t_0, t_1] \times \mathbb{R}^n$, the function

$$\mathbf{x} \mapsto H^0(t,\mathbf{x},\boldsymbol{\lambda}^*)$$

is concave. Then $\mathbf{u}^*$ is optimal.

Proof. Let us consider t fixed in $[t_0, t_1]$ (and hence we have $\mathbf{x}^* = \mathbf{x}^*(t)$, $\mathbf{u}^* = \mathbf{u}^*(t)$, $\ldots$). Our aim is to arrive to prove relation (2.27) with our new assumptions. First of all we note that the definitions of H^0 imply

$$H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) = H^0(t,\mathbf{x}^*,\boldsymbol{\lambda}^*) \quad \text{and} \quad H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}^*) \le H^0(t,\mathbf{x},\boldsymbol{\lambda}^*)$$

for every $\mathbf{x}$, $\mathbf{u}$. These relations give

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}^*) - H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) \le H^0(t,\mathbf{x},\boldsymbol{\lambda}^*) - H^0(t,\mathbf{x}^*,\boldsymbol{\lambda}^*). \tag{2.32}$$

Since the function $\mathbf{x} \mapsto H^0(t,\mathbf{x},\boldsymbol{\lambda}^*)$, is concave (we recall that t is fixed) then there exists a supergradient[6] $\mathbf{a}$ in the point $\mathbf{x}^*$, i.e.

$$H^0(t,\mathbf{x},\boldsymbol{\lambda}^*) \le H^0(t,\mathbf{x}^*,\boldsymbol{\lambda}^*) + \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}^*), \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{2.33}$$

Clearly from (2.32) and (2.33) we have

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}^*) - H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) \le \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}^*). \tag{2.34}$$

In particular, choosing $\mathbf{u} = \mathbf{u}^*$, we have

$$H(t,\mathbf{x},\mathbf{u}^*,\boldsymbol{\lambda}^*) - H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) \le \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}^*). \tag{2.35}$$

Now let us define the function $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(\mathbf{x}) = H(t,\mathbf{x},\mathbf{u}^*,\boldsymbol{\lambda}^*) - H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) - \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}^*).$$

Clearly, by (2.35), G has a maximum in the point $\mathbf{x}^*$: moreover it is easy to see that G is differentiable. We obtain

$$0 = \nabla G(\mathbf{x}^*) = \nabla_{\mathbf{x}} H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) - \mathbf{a}.$$

Now, the adjoint equation and (2.34) give

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}^*) \le H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) - \dot{\boldsymbol{\lambda}}^* \cdot (\mathbf{x}-\mathbf{x}^*).$$

Note that this last relation coincides with (2.27): at this point, using the same arguments of the second part of the proof of Theorem 2.3, we are able to conclude the proof. □

[6] We recall that (see [24]) if we consider a function $F : \mathbb{R}^n \to \mathbb{R}$, we say that $\mathbf{a} \in \mathbb{R}^n$ is a *supergradient* (respectively *subgradient*) for F in the point $\mathbf{x}_0$ if

$$F(\mathbf{x}) \le F(\mathbf{x}_0) + \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_0) \quad \forall \mathbf{x} \in \mathbb{R}^n \qquad (\text{respectivelly } F(\mathbf{x}) \ge F(\mathbf{x}_0) + \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_0)\ ).$$

For every point $\mathbf{x}_0 \in \mathbb{R}^n$, we denote by $\partial^+ F(\mathbf{x}_0)$ the set of all the supergradient for F in $\mathbf{x}_0$ ($\partial^- F(\mathbf{x}_0)$ for subgradient). Clearly, if F is differentiable in $\mathbf{x}_0$ and $\partial^+ F(\mathbf{x}_0) \neq \emptyset$, then $\partial^+ F(\mathbf{x}_0) = \{\nabla F(\mathbf{x}_0)\}$. A fundamental result in convex analysis is the following:

Theorem 2.5 (Rockafellar)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ a concave (convex) function. Then, for every $\mathbf{x}_0 \in \mathbb{R}^n$, the set of the supergradients $\partial^+ F(\mathbf{x}_0)$ (set of subgradients $\partial^- F(\mathbf{x}_0)$) in $\mathbf{x}_0$ is non empty.*

2.3 First generalizations

2.3.1 Initial/final conditions on the trajectory

What happens if we modify the initial or the final condition on the trajectory? We have found the fundamental ideas in the proof of Theorem 2.2 (see (2.21)), in the proof of Theorem 2.3 and hence in the proof of Theorem 2.4: more precisely, using the notation in (2.21), if $\tilde{t}$ is the initial or the final point of the interval $[t_0, t_1]$, we have the following two possibilities:

- if $\mathbf{x}^*(\tilde{t}) = \tilde{\boldsymbol{\alpha}}$ is fixed, then $\mathbf{x}_{\boldsymbol{\epsilon}}(\tilde{t}) = \tilde{\boldsymbol{\alpha}}\ \forall \boldsymbol{\epsilon}$; hence $\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(\tilde{t}) = \mathbf{0}$ and we have no conditions on the value $\boldsymbol{\lambda}^*(\tilde{t})$;

- if $\mathbf{x}^*(\tilde{t})$ is free, then $\mathbf{x}_{\boldsymbol{\epsilon}}(\tilde{t})$ is free $\forall \boldsymbol{\epsilon}$; hence we have no information on $\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(\tilde{t})$ and we have to require the condition $\boldsymbol{\lambda}^*(\tilde{t}) = \mathbf{0}$.

We left to the reader the details, but it is clear that slight modifications on the initial/final points of the trajectory of the problem (2.1), give us some slight differences on the transversality conditions in Theorem 2.2, in Theorem 2.3 and in Theorem 2.4.

Pay attention that if the initial and the final point of the trajectory are both fixed, it is not possible to guarantee that λ_0^* is different from zero, i.e. that the extremal control is normal: note that in the case of abnormal extremal control, the previous sufficient conditions don't work (see Example 2.5.3 and Example 2.5.7).

2.3.2 On minimum problems

Let us consider the problem (2.1) where we replace the maximum with a minimum problem. Since

$$\min \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t = -\max \int_{t_0}^{t_1} -f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t,$$

clearly it is possible to solve a min problem passing to a max problem with some minus.

Basically, a more direct approach consists in replace some "words" in all the previous pages as follows

max $\to$ min

concave function $\to$ convex function.

In particular in (2.2) we obtain the Pontryagin Minimum Principle.

2.4 The case of Calculus of Variation

A very particular situation appears when the dynamics (1.10) is of the type $\dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) = \mathbf{u}$ (and hence $k = n$) and the control set U is $\mathbb{R}^n$. Clearly it is possible to rewrite the problem[7] (2.1) as

$$\begin{cases} J(\mathbf{x}) = \int_{t_0}^{t_1} f(t, \mathbf{x}, \dot{\mathbf{x}})\, dt \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max_{\mathbf{x} \in KC^1} J(\mathbf{x}) \end{cases} \tag{2.36}$$

This problems are called *Calculus of Variation problems* (shortly CoV). Clearly in this problem the control does not appear. We say that $\mathbf{x}^* \in KC^1$ is *optimal* for (2.36) if

$$J(\mathbf{x}) \leq J(\mathbf{x}^*), \qquad \forall \mathbf{x} \in KC^1, \ \mathbf{x}(t_0) = \boldsymbol{\alpha}.$$

In this section, we will show that the theorem of Euler of Calculus of Variation is an easy consequence of the theorem of Pontryagin of Optimal Control. Hence we are interested in the problem

$$\begin{cases} \max_{\mathbf{x} \in C^1} \int_{t_0}^{t_1} f(t, \mathbf{x}, \dot{\mathbf{x}})\, dt \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \end{cases} \tag{2.37}$$

with $\boldsymbol{\alpha} \in \mathbb{R}^n$ fixed. We remark that here $\mathbf{x}$ is in C^1. We have the following fundamental result

Theorem 2.6 (Euler)**.** *Let us consider the problem (2.37) with $f \in C^1$. Let $\mathbf{x}^*$ be optimal. Then, for all $t \in [t_0, t_1]$, we have*

$$\frac{d}{dt}\Big(\nabla_{\dot{\mathbf{x}}} f(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)) \Big) = \nabla_{\mathbf{x}} f(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)). \tag{2.38}$$

In calculus of variation the equation (2.38) is called *Euler equation* (shortly EU); a function that satisfies EU is called *extremal*. Let us prove this result. If we consider a new variable $\mathbf{u} = \dot{\mathbf{x}}$, we rewrite problem (2.37) as

$$\begin{cases} \max_{\mathbf{u} \in C} \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\, dt \\ \dot{\mathbf{x}} = \mathbf{u} \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \end{cases}$$

Theorem 2.2 guarantees that, for the Hamiltonian $H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot \mathbf{u}$, we have

$$\nabla_{\mathbf{u}} H(t, \mathbf{x}^*, \mathbf{u}^*) = \mathbf{0} \quad \Rightarrow \nabla_{\mathbf{u}} f + \boldsymbol{\lambda}^* = \mathbf{0} \tag{2.39}$$

$$\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*) = -\dot{\boldsymbol{\lambda}}^* \quad \Rightarrow \nabla_{\mathbf{x}} f = -\dot{\boldsymbol{\lambda}}^* \tag{2.40}$$

If we consider a derivative with respect to the time in (2.39) and using (2.39) we have

$$\frac{d}{dt}(\nabla_{\mathbf{u}} f) = -\dot{\boldsymbol{\lambda}}^* = \nabla_{\mathbf{x}} f;$$

[7] We remark that in general in a Calculus of Variation problem one assume that $\mathbf{x} \in KC^1$; in this note we are not interested in this general situation and we will assume that $\mathbf{x} \in C^1$.

taking into account $\dot{\mathbf{x}} = \mathbf{u}$, we obtain (2.38). Moreover, we are able to find the transversality condition of Calculus of Variation: (2.19) and (2.39), imply

$$\nabla_{\dot{\mathbf{x}}} f(t_1, \mathbf{x}^*(t_1), \dot{\mathbf{x}}^*(t_1)) = \mathbf{0}.$$

As in subsection 2.3.1 we obtain

Remark 2.8. *Consider the theorem 2.6, its assumptions and let us modify slightly the conditions on the initial and the final points of* $\mathbf{x}$. *We have the following transversality conditions:*

$$\text{if } \mathbf{x}^*(t_i) \in \mathbb{R}^n, \text{ i.e. } \mathbf{x}^*(t_i) \text{ is free } \Rightarrow \nabla_{\dot{\mathbf{x}}} f(t_i, \mathbf{x}^*(t_i), \dot{\mathbf{x}}^*(t_i)) = \mathbf{0},$$

where t_i is the initial or the final point of the interval $[t_0, t_1]$.

An useful remark, in some situation, is that if f does not depend on $\mathbf{x}$, i.e. $f = f(t, \dot{\mathbf{x}})$, then the equation of Euler (2.38) is

$$\nabla_{\dot{\mathbf{x}}} f(t, \dot{\mathbf{x}}^*) = c,$$

where $c \in \mathbb{R}$ is a constant. Moreover, the following remark is not so obvious:

Remark 2.9. *If $f = f(\mathbf{x}, \dot{\mathbf{x}})$ does not depend directly on t, then the equation of Euler (2.38) is*

$$f(\mathbf{x}^*, \dot{\mathbf{x}}^*) - \dot{\mathbf{x}}^* \cdot \nabla_{\dot{\mathbf{x}}} f(\mathbf{x}^*, \dot{\mathbf{x}}^*) = c, \tag{2.41}$$

where $c \in \mathbb{R}$ is a constant.

Proof. Clearly

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} f + \ddot{\mathbf{x}} \cdot \nabla_{\dot{\mathbf{x}}} f, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{x}} \cdot \nabla_{\dot{\mathbf{x}}} f) = \ddot{\mathbf{x}} \cdot \nabla_{\dot{\mathbf{x}}} f + \dot{\mathbf{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\nabla_{\dot{\mathbf{x}}} f).$$

Now let us suppose that $\mathbf{x}^*$ satisfies condition the Euler condition (2.38): hence, using the previous two equalities we obtain

$$\begin{aligned} 0 &= \dot{\mathbf{x}}^* \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big(\nabla_{\dot{\mathbf{x}}} f(\mathbf{x}^*, \dot{\mathbf{x}}^*)\Big) - \nabla_{\mathbf{x}} f(\mathbf{x}^*, \dot{\mathbf{x}}^*) \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{\mathbf{x}}^* \cdot \nabla_{\dot{\mathbf{x}}} f(\mathbf{x}^*, \dot{\mathbf{x}}^*)\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\Big(f(\mathbf{x}^*, \dot{\mathbf{x}}^*)\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{\mathbf{x}}^* \cdot \nabla_{\dot{\mathbf{x}}} f(\mathbf{x}^*, \dot{\mathbf{x}}^*) - f(\mathbf{x}^*, \dot{\mathbf{x}}^*)\Big). \end{aligned}$$

Hence we obtain (2.41). □

If we are interested to find sufficient condition of optimality for the problem (2.37), since the dynamics is linear, remark 2.7 implies

Remark 2.10. *Let us consider an extremal $\mathbf{x}^*$ for the problem (2.37) in the assumption of theorem of Euler. Suppose that $\mathbf{x}^*$ satisfies the transversality conditions. If, for every $t \in [t_0, t_1]$, the function f is concave on the variable $\mathbf{x}$ and $\dot{\mathbf{x}}$, then $\mathbf{x}^*$ is optimal.*

2.5 Examples and applications

Example 2.5.1. Consider[8]

$$\begin{cases} \max \int_0^1 (x - u^2)\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 2 \end{cases}$$

1^{st} *method:* *Clearly the Hamiltonian is $H = x - u^2 + \lambda u$ (note that the extremal is certainly normal) and theorem 2.2 implies*

$$\frac{\partial H}{\partial u} = 0 \qquad \Rightarrow \qquad -2u^* + \lambda^* = 0 \tag{2.42}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}^* \qquad \Rightarrow \qquad 1 = -\dot{\lambda}^* \tag{2.43}$$

$$\frac{\partial H}{\partial \lambda} = \dot{x^*} \qquad \Rightarrow \qquad \dot{x^*} = u^* \tag{2.44}$$

$$\lambda^*(1) = 0 \tag{2.45}$$

Equations (2.43) and (2.45) give $\lambda^* = 1 - t$; consequently, by (2.42) we have $u^* = (1-t)/2$; since the dynamics is linear, sure the previous control u^* is admissible (see Proposition 1.1). Finally, since the Hamiltonian H is concave in x and u, the sufficient conditions of Mangasarian in theorem 2.3 guarantees that the extremal u^* is optimal.

If we are interested to find the optimal trajectory, the initial condition and (2.44) give $x^* = (2t - t^2)/4 + 2$.

2^{nd} *method:* *The problem is, clearly, of calculus of variations, i.e.*

$$\begin{cases} \max \int_0^1 (x - \dot{x}^2)\, \mathrm{d}t \\ x(0) = 2 \end{cases}$$

The necessary condition of Euler (2.38) and the transversality condition give

$$\begin{aligned} \frac{df_{\dot{x}}}{dt}(t, x^*, \dot{x}^*) = f_x(t, x^*, \dot{x}^*) \qquad &\Rightarrow\ -2\ddot{x}^* = 1 \\ &\Rightarrow\ x^*(t) = -\frac{1}{4}t^2 + at + b,\ \forall a, b \in \mathbb{R} \\ f_{\dot{x}}(1, x^*(1), \dot{x}^*(1)) = 0 \qquad &\Rightarrow\ -2\dot{x}^*(1) = 0 \end{aligned}$$

An easy calculation, using the initial condition $x(0) = 2$, implies $x^*(t) = -t^2/4 + t/2 + 2$. Since the function $(x, \dot{x}) \mapsto (x - \dot{x}^2)$ is concave, then x^* is really the maximum of the problem.

Example 2.5.2. Consider[9]

$$\begin{cases} \max \int_0^2 (2x - 4u)\, \mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 5 \\ 0 \le u \le 2 \end{cases}$$

Let us consider the Hamiltonian $H = 2x - 4u + \lambda(x + u)$ (note that the extremal is certainly normal). The theorem of Pontryagin gives

$$\begin{aligned} &H(t, x^*, u^*, \lambda^*) = \max_{v \in [0,2]} H(t, x^*, v, \lambda^*) \ \Rightarrow \\ &\qquad \Rightarrow\ 2x^* - 4u^* + \lambda^*(x^* + u^*) = \max_{v \in [0,2]} (2x^* - 4v + \lambda^*(x^* + v)) \end{aligned} \tag{2.46}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}^* \ \Rightarrow\ 2 + \lambda^* = -\dot{\lambda}^* \tag{2.47}$$

$$\frac{\partial H}{\partial \lambda} = \dot{x^*} \ \Rightarrow\ \dot{x^*} = x^* + u^* \tag{2.48}$$

$$\lambda^*(2) = 0 \tag{2.49}$$

---

[8] In the example 5.5.1 we solve the same problem with the dynamics programming.

[9] In the example 5.5.3 we solve the same problem with the dynamics programming.

From (2.46) we have, for every $t \in [0,2]$,

$$u^*(t)(\lambda^*(t)-4) = \max_{v\in[0,2]} (v(\lambda^*(t)-4))$$

and hence

$$u^*(t) = \begin{cases} 2 & \text{for } \lambda^*(t)-4>0, \\ 0 & \text{for } \lambda^*(t)-4<0, \\ ? & \text{for } \lambda^*(t)-4=0. \end{cases} \tag{2.50}$$

(2.47) implies $\lambda^*(t) = ae^{-t} - 2,\ \forall a \in \mathbb{R}:$ *using (2.49) we obtain*

$$\lambda^*(t) = 2(e^{2-t}-1). \tag{2.51}$$

Since $\lambda^*(t) > 4$ *if and only if* $t \in [0, 2-\log 3)$*, the extremal control is*

$$u^*(t) = \begin{cases} 2 & \text{for } 0 \le t \le 2-\log 3, \\ 0 & \text{for } 2-\log 3 < t \le 2. \end{cases} \tag{2.52}$$

We remark that the value of the function u^* *in* $t = 2-\log 3$ *is irrelevant. Since the dynamics is linear, the previous control* u^* *is admissible (see Proposition 1.1). Finally, the Hamiltonian function* H *is concave in* (x,u) *for every* λ *fixed, and hence* u^* *is optimal.*

If we are interested to find the optimal trajectory, the relations (2.48) and (2.52), and the initial condition give us to solve the ODE

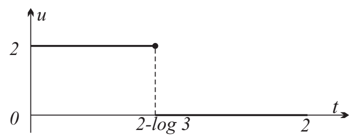
$$\begin{cases} \dot{x^*} = x^* + 2 & \text{in } [0, 2-\log 3) \\ x(0) = 5 \end{cases} \tag{2.53}$$

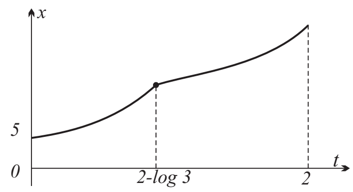
The solution is $x^*(t) = 7e^t - 2$*. Taking into account that the trajectory is a continuous function, by (2.53) we have* $x^*(2-\log 3) = 7e^{2-\log 3} - 2 = 7e^2/3 - 2$*. Hence the relations (2.48) and (2.52) give us to solve the ODE*

$$\begin{cases} \dot{x^*} = x^* & \text{in } [2-\log 3, 2] \\ x(2-\log 3) = 7e^2/3 - 2 \end{cases}$$

We obtain

$$x^*(t) = \begin{cases} 7e^t - 2 & \text{for } 0 \le t \le 2-\log 3, \\ (7e^2-6)e^{t-2} & \text{for } 2-\log 3 < t \le 2. \end{cases} \tag{2.54}$$





We note that an easy computation gives $H(t, x^*(t), u^*(t), \lambda^*(t)) = 14e^2 - 12$ *for all* $t \in [0,2]$.

△

Example 2.5.3. *Find the optimal tern for*

$$\begin{cases} \max \int_0^4 3x\,\mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 0 \\ x(4) = 3e^4/2 \\ 0 \le u \le 2 \end{cases}$$

Let us consider the Hamiltonian $H = 3x + \lambda(x+u)$*; it is not possible to guarantee that the extremal is normal, but we try to put* $\lambda_0 = 1$ *since this situation is more simple; if we will not found an extremal (more precisely a normal extremal), then we will pass to the more general Hamiltonian* $H = 3\lambda_0 x + \lambda(x+u)$ *(and in this situation certainly the extremal there exists). The theorem of Pontryagin gives*

$$H(t, x^*, u^*, \lambda^*) = \max_{v\in[0,2]} H(t, x^*, v, \lambda^*) \ \Rightarrow\ \lambda^* u^* = \max_{v\in[0,2]} \lambda^* v$$

$$\Rightarrow\ u^*(t) = \begin{cases} 2 & \text{for } \lambda^*(t) > 0, \\ 0 & \text{for } \lambda^*(t) < 0, \\ ? & \text{for } \lambda^*(t) = 0. \end{cases} \tag{2.55}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}^* \ \Rightarrow\ 3 + \lambda^* = -\dot{\lambda}^* \ \Rightarrow\ \lambda^*(t) = ae^{-t} - 3,\ \forall a \in \mathbb{R} \tag{2.56}$$

$$\frac{\partial H}{\partial \lambda} = \dot{x^*} \ \Rightarrow\ \dot{x^*} = x^* + u^* \tag{2.57}$$

Note that we have to maximize the area of the function $t \mapsto 3x(t)$ *and that* $x(t) \geq 0$ *since* $x(0) = 0$ *and* $\dot{x} = u + x \geq x \geq 0$. *In order to maximize the area, it is reasonable that the function* x *is increasing in an interval of the type* $[0, \alpha)$: *hence it is reasonable to suppose that there exists a positive constant* α *such that* $\lambda^*(t) > 0$ *for* $t \in [0, \alpha)$. *In this case, (2.55) gives* $u^* = 2$. *Hence we have to solve the ODE*

$$\begin{cases} \dot{x^*} = x^* + 2 & \text{in } [0, \alpha) \\ x(0) = 0 \end{cases} \tag{2.58}$$

The solution is $x^*(t) = 2(e^t - 1)$. *We note that for such function we have* $x^*(4) = 2(e^4 - 1) > 3e^4/2$; *hence it is not possible that* $\alpha \geq 4$: *we suppose that* $\lambda^*(t) < 0$ *for* $t \in (\alpha, 4]$. *Taking into account the final condition on the trajectory, we have to solve the ODE*

$$\begin{cases} \dot{x^*} = x^* & \text{in } (\alpha, 4] \\ x(4) = 3e^4/2 \end{cases} \tag{2.59}$$

The solution is $x^*(t) = 3e^t/2$. *We do not know the point* α, *but certainly the trajectory is continuous, i.e.*

$$\lim_{t \to \alpha^-} x^*(t) = \lim_{t \to \alpha^+} x^*(t) \ \Rightarrow\ \lim_{t \to \alpha^-} 2(e^t - 1) = \lim_{t \to \alpha^+} 3e^t/2$$

that implies $\alpha = \ln 4$. *Moreover, since the multiplier is continuous, we are in the position to find the constant* a *in (2.56): more precisely* $\lambda^*(t) = 0$ *for* $t = \ln 4$, *implies* $a = 12$, *i.e.* $\lambda^*(t) = 12e^{-t} - 3$. *Note that the previous assumptions* $\lambda^* > 0$ *in* $[0, \ln 4)$ *and* $\lambda^* < 0$ *in* $(\ln 4, 4]$ *are verified. These calculations give that* u^* *is admissible.*

Finally, the dynamics and the running cost is linear (in x *and* u*) and hence the sufficient condition are satisfied. The optimal tern is*

$$u^*(t) = \begin{cases} 2 & \text{for } t \in [0, \ln 4), \\ 0 & \text{for } t \in [\ln 4, 4] \end{cases} \tag{2.60}$$

$$x^*(t) = \begin{cases} 2(e^t - 1) & \text{for } t \in [0, \ln 4), \\ 3e^t/2 & \text{for } t \in [\ln 4, 4] \end{cases}$$

$\lambda^*(t) = 12e^{-t} - 3$. *We note that an easy computation gives* $H(t, x^*(t), u^*(t), \lambda^*(t)) = 18$ *for all* $t \in [0, 4]$. △

Example 2.5.4.

$$\begin{cases} \min \int_1^e (3\dot{x} + t\dot{x}^2)\, \mathrm{d}t \\ x(1) = 1 \\ x(e) = 1 \end{cases}$$

It is a calculus of variation problem. Since $f = 3\dot{x} + t\dot{x}^2$ *does not depend on* x, *the necessary condition of Euler implies*

$$3 + 2t\dot{x} = c,$$

where c *is a constant. Hence* $\dot{x}(t) = a/t$, $\forall a \in \mathbb{R}$, *implies the solution* $x(t) = a \ln t + b$, $\forall a, b \in \mathbb{R}$. *Using the initial and the final conditions we obtain the extremal* $x^*(t) = 1$. *Since* f *is convex in* x *and* $\dot{x}$, *the extremal is the minimum of the problem.*

△

Example 2.5.5.

$$\begin{cases} \min \int_0^{\sqrt{2}} (x^2 - x\dot{x} + 2\dot{x}^2)\, \mathrm{d}t \\ x(0) = 1 \end{cases}$$

It is a calculus of variation problem; the necessary condition of Euler (2.38) gives

$$\frac{d}{dt} f_{\dot{x}} = f_x \ \Rightarrow\ 4\ddot{x} - \dot{x} = 2x - \dot{x}$$
$$\Rightarrow\ 2\ddot{x} - x = 0 \ \Rightarrow\ x^*(t) = ae^{t/\sqrt{2}} + be^{-t/\sqrt{2}},$$

for every $a, b \in \mathbb{R}$. *Hence the initial condition* $x(0) = 1$ *gives* $b = 1 - a$. *Since there does not exist a final condition on the trajectory, we have to satisfy the transversality condition, i.e.*

$$f_{\dot{x}}(t_1, x^*(t_1), \dot{x}^*(t_1)) = 0 \ \Rightarrow\ 4\dot{x}^*(\sqrt{2}) - x^*(\sqrt{2}) - = 0$$
$$\Rightarrow\ ae + \frac{1-a}{e} - 4\left[\frac{ae}{\sqrt{2}} - \frac{1-a}{e\sqrt{2}}\right] = 0$$

Hence

$$x^*(t) = \frac{(4+\sqrt{2})e^{t/\sqrt{2}} + (4e^2 - e^2\sqrt{2})e^{-t/\sqrt{2}}}{4+\sqrt{2}+4e^2-e^2\sqrt{2}}.$$

The function $f(t, x, \dot{x}) = x^2 - x\dot{x} + 2\dot{x}^2$ *is convex in the variable* x *and* $\dot{x}$*, since its hessian matrix with respect* $(x, \dot{x})$

$$d^2 f = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix},$$

is positive definite. Hence x^* *is minimum.* △

Example 2.5.6.

$$\begin{cases} \min \int_1^2 (t^2\dot{x}^2 + 2x^2)\, \mathrm{d}t \\ x(2) = 17 \end{cases}$$

It is a calculus of variation problem; the necessary condition of Euler (2.38) gives

$$\frac{d}{dt} f_{\dot{x}} = f_x \ \Rightarrow\ t^2\ddot{x} + 2t\dot{x} - 2x = 0.$$

The homogeneity suggests to set $t = e^s$ *and* $y(s) = x(e^s)$ *: considering the derivative of this last expression with respect to* s *we obtain*

$$y'(s) = \dot{x}(e^s)e^s = t\dot{x}(t) \qquad \text{and} \qquad y''(s) = \ddot{x}(e^s)e^{2s} + \dot{x}(e^s)e^s = t^2\ddot{x}(t) + t\dot{x}(t).$$

Hence the Euler equation now is

$$y'' + y' - 2y = 0.$$

This implies $y(s) = ae^s + be^{-2s}$*, with* a*,* b *constants. The relation* $t = e^s$ *gives*

$$x^*(t) = at + \frac{b}{t^2}.$$

Note that $\dot{x^*}(s) = a - \frac{2b}{t^3}$*. The final condition* $x^*(2) = 17$ *and the transversality condition* $f_{\dot{x}}(1, x^*(1), \dot{x}^*(1)) = 0$ *give, respectively,*

$$17 = 2a + \frac{b}{4} \qquad \text{and} \qquad 2(a - 2b) = 0.$$

Hence $x^*(t) = 8t + \frac{4}{t^2}$ *is the unique extremal function which satisfies the transversality condition and the final condition. The function* $f(t, x, \dot{x}) = t^2\dot{x}^2 + 2x^2$ *is convex in the variable* x *and* $\dot{x}$ *and* x^* *is clearly the minimum.* △

The following example gives an abnormal control.

Example 2.5.7. *Let us consider*

$$\begin{cases} \max \int_0^1 u\, \mathrm{d}t \\ \dot{x} = (u - u^2)^2 \\ x(0) = 0 \\ x(1) = 0 \\ 0 \le u \le 2 \end{cases}$$

We prove that $u^* = 1$ *is an abnormal extremal and optimal. Clearly* $H = \lambda_0 u + \lambda(u - u^2)^2$*; the PMP and the adjoint equation give*

$$u^*(t) \in \arg\max_{v\in[0,2]} [\lambda_0^* v + \lambda^*(t)(v - v^2)^2], \qquad \dot{\lambda} = 0.$$

This implies that $\lambda^* = k$*, with* k *constant; hence* u^* *is constant. If we define the function* $\phi(v) = \lambda_0^* v + k(v - v^2)^2$*, the study of the sign of* ϕ' *depends on* λ_0^* *and* k *and it is not easy to obtain some information on the max.*

However, we note that the initial and the final conditions on the trajectory and the fact that $\dot{x} = (u - u^2)^2 \ge 0$*, implies that* $\dot{x} = 0$ *a.e.; hence if a control* u *is admissible, then we have* $u(t) \in \{0, 1\}$ *a.e. This implies that considering* $u^* = 1$*, for every admissible control* u*,*

$$\int_0^1 u^*(t)\, \mathrm{d}t = \int_0^1 1\, \mathrm{d}t \ge \int_0^1 u(t)\, \mathrm{d}t;$$

hence $u^* = 1$ *is maximum.*

Now, since u^* *is optimal, then it satisfies the PMP: hence we have*

$$1 = u^*(t) \in \arg\max_{v\in[0,2]} \phi(v).$$

Since we realize the previous max in an interior point of the interval $[0, 2]$*, we necessarily have* $\phi'(1) = 0$*: an easy calculation gives* $\lambda_0^* = 0$*. This proves that* u^* *is abnormal.* △

2.5.1 The curve of minimal length

We have to solve the calculus of variation problem (1.1). Since the function $f(t,x,\dot{x}) = \sqrt{1+\dot{x}^2}$ does not depend on x, the necessary condition of Euler (2.38) gives

$$\begin{aligned} f_{\dot{x}} = a \;\Rightarrow\; & \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = a \\ \Rightarrow\; & \dot{x} = c \;\Rightarrow\; x^*(t) = ct + d, \end{aligned}$$

with $a, b, c \in \mathbb{R}$ constants. The conditions $x(0) = 0$ and $x(1) = b$ imply $x^*(t) = bt$. The function f is constant and hence convex in x and it is convex in $\dot{x}$ since $\frac{\partial^2 f}{\partial \dot{x}^2} = (1+\dot{x}^2)^{-3/2} > 0$. This proves that the line x^* is the solution of the problem.

2.5.2 A problem of business strategy I

We solve[10] the model presented in the example 1.1.2, formulated with (1.3).
We consider the Hamiltonian $H = (1-u)x + \lambda x u$: the theorem of Pontryagin implies that

$$\begin{aligned} H(t,x^*,u^*,\lambda^*) &= \max_{v\in[0,1]} H(t,x^*,v,\lambda^*) \\ \Rightarrow\quad (1-u^*)x^* + \lambda^* x^* u^* &= \max_{v\in[0,1]} [(1-v)x^* + \lambda^* x^* v] \\ \Rightarrow\quad u^* x^* (\lambda^* - 1) &= \max_{v\in[0,1]} [v x^* (\lambda^* - 1)] \end{aligned} \tag{2.61}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}^* \quad\Rightarrow\quad 1 - u^* + \lambda^* u^* = -\dot{\lambda}^* \tag{2.62}$$

$$\frac{\partial H}{\partial \lambda} = \dot{x^*} \quad\Rightarrow\quad \dot{x^*} = x^* u^* \tag{2.63}$$

$$\lambda^*(T) = 0 \tag{2.64}$$

Since x^* is continuous, $x^*(0) = \alpha > 0$ and $u^* \geq 0$, from (2.63) we obtain

$$\dot{x}^* = x^* u^* \geq 0, \tag{2.65}$$

in $[0,T]$. Hence $x^*(t) \geq \alpha$ for all $t \in [0,T]$. Relation (2.61) becomes

$$u^*(\lambda^* - 1) = \max_{v\in[0,1]} v(\lambda^* - 1).$$

Hence

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda^*(t) - 1 > 0, \\ 0 & \text{if } \lambda^*(t) - 1 < 0, \\ ? & \text{if } \lambda^*(t) - 1 = 0. \end{cases} \tag{2.66}$$

Since the multiplier is a continuous function that satisfies (2.64), there exists $\tau' \in [0,T)$ such that

$$\lambda^*(t) < 1, \qquad \forall t \in [\tau', T] \tag{2.67}$$

Using (2.66) and (2.67), we have to solve the ODE

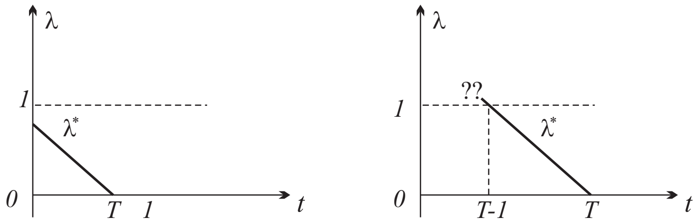
$$\begin{cases} \dot{\lambda}^* = -1 & \text{in } [\tau', T] \\ \lambda^*(T) = 0 \end{cases}$$

[10] In subsection 5.5.1 we solve the same problem with the Dynamic Programming approach.

that implies

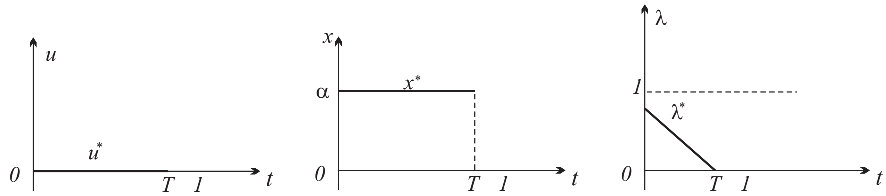
$$\lambda^*(t) = T - t, \qquad \forall t \in [\tau', T]. \tag{2.68}$$

Clearly, we have two cases: $T \leq 1$ (case A) and $T > 1$ (case B).



The case $T \leq 1$ *and the case* $T > 1$.

Case A: $T \leq 1$.
In this situation, we obtain $\tau' = 0$ and hence $u^* = 0$ and $x^* = \alpha$ in $[0, T]$.



From an economic point of view, if the time horizon is short the optimal strategy is to sell all our production without any investment. Note that the strategy u^* that we have found is an extremal: in order to guarantee the sufficient conditions for such extremal we refer the reader to the case B.
Case B: $T \geq 1$.
In this situation, taking into account (2.66), we have $\tau' = T - 1$. Hence

$$\lambda^*(T-1) = 1. \tag{2.69}$$

First of all, if there exists an interval $I \subset [0, T-1)$ such that $\lambda^*(t) < 1$, then $u^* = 0$ and the (2.62) is $\dot{\lambda}^* = -1$: this is impossible since $\lambda^*(T-1) = 1$.
Secondly, if there exists an interval $I \subset [0, T-1)$ such that $\lambda^*(t) = 1$, then $\dot{\lambda}^* = 0$ and the (2.62) is $1 = 0$: this is impossible.
Let us suppose that there exists an interval $I = [\tau'', T-1) \subset [0, T-1)$ such that $\lambda^*(t) > 1$: using (2.66), we have to solve the ODE

$$\begin{cases} \dot{\lambda}^* + \lambda^* = 0 & \text{in } [\tau'', T-1] \\ \lambda^*(T-1) = 1 \end{cases}$$

that implies

$$\lambda^*(t) = e^{T-t-1}, \qquad \text{for } t \in [0, T-1].$$

We remark the choice $\tau'' = 0$ is consistent with all the necessary conditions. Hence (2.66) gives

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T-1, \\ 0 & \text{for } T-1 < t \leq T \end{cases} \tag{2.70}$$

The continuity of the function x^*, the initial condition $x(0) = \alpha$ and the dynamics imply

$$\begin{cases} \dot{x}^* = x^* & \text{in } [0, T-1] \\ x^*(0) = \alpha \end{cases}$$

that implies $x^*(t) = \alpha e^t$; hence

$$\begin{cases} \dot{x^*} = 0 & \text{in } [T-1, T] \\ x^*(T-1) = \alpha e^{T-1} \end{cases}$$

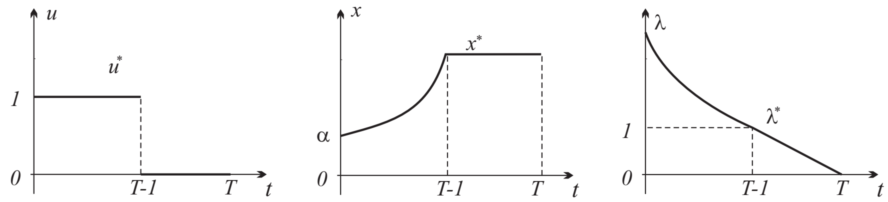
that implies $x^*(t) = \alpha e^{T-1}$. Consequently

$$x^*(t) = \begin{cases} \alpha e^t & \text{for } 0 \le t \le T-1, \\ \alpha e^{T-1} & \text{for } T-1 < t \le T \end{cases}$$

Recalling that

$$\lambda^*(t) = \begin{cases} e^{T-t-1} & \text{for } 0 \le t \le T-1, \\ T-t & \text{for } T-1 < t \le T \end{cases}$$

we have



In an economic situation where the choice of business strategy can be carried out in a medium or long term, the optimal strategy is to direct all output to increase production and then sell everything to make profit in the last period.

We remark, that we have to prove some sufficient conditions for the tern (x^*, u^*, λ^*) in order to guarantee that u^* is really the optimal strategy. An easy computation shows that the Hamiltonian is not concave. We study the maximized Hamiltonian (2.31): taking into account that $x(t) \ge \alpha > 0$ we obtain

$$H^0(t, x, \lambda) = \max_{v \in [0,1]} [(1-v)x + \lambda x v] = x + x \max_{v \in [0,1]} [(\lambda - 1)v]$$

In order to apply theorem 2.4, using the expression of λ^* we obtain

$$H^0(t, x, \lambda^*(t)) = \begin{cases} e^{T-t-1}x & \text{if } t \in [0, T-1) \\ x & \text{if } t \in [T-1, T] \end{cases}$$

Note that, for every fixed t the function $x \mapsto H^0(t, x, \lambda^*(t))$ is concave with respect to x: the sufficient condition of Arrow holds. We note that an easy computation gives $H(t, x^*(t), u^*(t), \lambda^*(t)) = \alpha e^{T-1}$ for all $t \in [0, T]$.

2.5.3 A two-sector model

This model has some similarities with the previous one and it is proposed in [26].

Consider an economy consisting of two sectors where sector no. 1 produces investment goods, sector no. 2 produces consumption goods. Let $x_i(t)$ the production in sector no. i per unit of time, $i = 1, 2$, and let $u(t)$ be the proportion of investments allocated to sector no. 1. We assume that $\dot{x}_1 = \alpha u x_1$ and $\dot{x}_2 = \alpha(1-u)x_1$ where α is a positive constant. Hence, the increase in production per unit of time in each sector is assumed to be proportional to investment allocated to the sector. By definition, $0 \le u(t) \le 1$, and if the planning period starts at $t = 0$, $x_1(0)$ and $x_2(0)$ are historically given. In this situation a number of optimal control problems could be investigated. Let us, in particular, consider the problem of maximizing the total consumption in a given planning period [0,T]. Our precise problem is as follows:

$$\begin{cases} \max_{u \in \mathcal{C}} \int_0^T x_2 \, \mathrm{d}t \\ \dot{x}_1 = \alpha u x_1 \\ \dot{x}_2 = \alpha(1-u)x_1 \\ x_1(0) = a_1 \\ x_2(0) = a_2 \\ \mathcal{C} = \{u : [0,T] \to [0,1] \subset \mathbb{R},\ u \text{ admissible}\} \end{cases}$$

where $\alpha,\ a_1,\ a_2$ and T are positive and fixed. We study the case $T > \frac{2}{\alpha}$.
We consider the Hamiltonian $H = x_2 + \lambda_1 u x_1 + \lambda_2(1-u)x_1$; the theorem of Pontryagin implies that

$$u^* \in \arg\max_{v \in [0,1]} H(t, x^*, v, \lambda^*) = \arg\max_{v \in [0,1]} [x_2^* + \lambda_1^* v x_1^* + \lambda_2^*(1-v)x_1^*]$$

$$\Rightarrow \quad u^* \in \arg\max_{v \in [0,1]} (\lambda_1^* - \lambda_2^*) v x_1^* \tag{2.71}$$

$$\frac{\partial H}{\partial x_1} = -\dot{\lambda}_1^* \quad \Rightarrow \quad -\lambda_1^* u^* \alpha - \lambda_2^* \alpha(1-u^*) = \dot{\lambda}_1^* \tag{2.72}$$

$$\frac{\partial H}{\partial x_2} = -\dot{\lambda}_2^* \quad \Rightarrow \quad -1 = \dot{\lambda}_2^* \tag{2.73}$$

$$\lambda_1^*(T) = 0 \tag{2.74}$$

$$\lambda_2^*(T) = 0 \tag{2.75}$$

Clearly (2.73) and (2.75) give us $\lambda_2^*(t) = T - t$. Moreover (2.74) and (2.75) in equation (2.72) give $\dot{\lambda}_1^*(T) = 0$. We note that

$$\lambda_1^*(T) = \lambda_2^*(T) = 0, \qquad \dot{\lambda}_1^*(T) = 0, \qquad \dot{\lambda}_2^*(T) = -1$$

and the continuity of the multiplier $(\lambda_1^*, \lambda_2^*)$ implies that there exists $\tau < T$ such that

$$T - t = \lambda_2^*(t) > \lambda_1^*(t), \qquad \forall t \in (\tau, T). \tag{2.76}$$

Since x_1^* is continuous, using the dynamics $\dot{x}_1 = \alpha u x_1$ $x_1^*(0) = a_1 > 0$ and $u^* \ge 0$, we have $\dot{x}_1(t) \ge 0$ and hence $x_1^*(t) \ge a_1 > 0$; from (2.71) we obtain

$$u^*(t) \in \arg\max_{v \in [0,1]} (\lambda_1^*(t) - T + t)v = \begin{cases} 1 & \text{if } \lambda_1^*(t) > T - t \\ ? & \text{if } \lambda_1^*(t) = T - t \\ 0 & \text{if } \lambda_1^*(t) < T - t \end{cases} \tag{2.77}$$

Hence (2.76) and (2.77) imply that, in $(\tau, T]$, we have $u^*(t) = 0$. Now (2.72) gives, taking into account (2.75),

$$\dot{\lambda}_1^* = -\lambda_2^* \alpha \quad \Rightarrow \quad \lambda_1^*(t) = \frac{\alpha}{2}(t - T)^2, \quad \forall t \in (\tau, T].$$

An easy computation shows that the relation in (2.76) holds for $\tau = T - \frac{2}{\alpha}$. Hence let us suppose that there exists $\tau' < T - \frac{2}{\alpha}$ such that

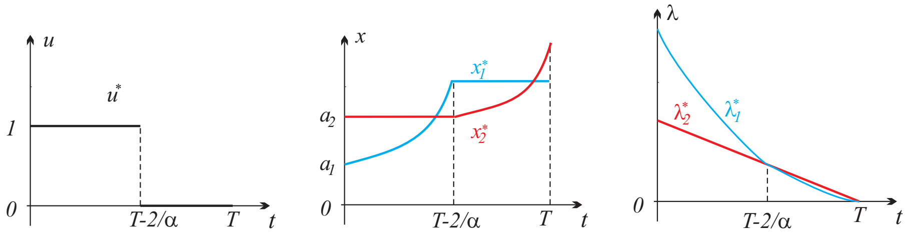
$$T - t = \lambda_2^*(t) < \lambda_1^*(t), \qquad \forall t \in (\tau', T - 2/\alpha). \tag{2.78}$$

By (2.77) we obtain, in $(\tau', T-2/\alpha)$, that $u^*(t)=1$. Now (2.72) gives, taking into account the continuity of λ_2^* in the point $T-2/\alpha$,

$$\dot\lambda_1^* = -\lambda_1^*\alpha \quad\Rightarrow\quad \lambda_1^*(t)=\frac{2}{\alpha}e^{-\alpha(t-T+2/\alpha)}, \quad \forall t\in(\tau', T-2/\alpha].$$

Since $\lambda_1^*(T-2/\alpha)=\lambda_2^*(T-2/\alpha)$, λ_2^* is a line and the function λ_1^* is convex for $t\le\tau$, we have that assumption (2.78) holds with $\tau'=0$. Using the dynamics and the initial condition on the trajectory, we obtain

$$u^*(t)=\begin{cases}1 & \text{for } 0\le t\le T-\frac{2}{\alpha},\\ 0 & \text{for } T-\frac{2}{\alpha}<t\le T\end{cases}$$
$$x_1^*(t)=\begin{cases}a_1e^{\alpha t} & \text{for } 0\le t\le T-\frac{2}{\alpha},\\ a_1e^{\alpha T-2} & \text{for } T-\frac{2}{\alpha}<t\le T\end{cases}$$
$$x_2^*(t)=\begin{cases}a_2 & \text{for } 0\le t\le T-\frac{2}{\alpha},\\ a_2e^{(\alpha t-\alpha T+2)a_1e^{\alpha T-2}} & \text{for } T-\frac{2}{\alpha}<t\le T\end{cases}$$
$$\lambda_1^*(t)=\begin{cases}\dfrac{2}{\alpha}e^{-\alpha(t-T+2/\alpha)} & \text{for } 0\le t\le T-\frac{2}{\alpha},\\ \dfrac{\alpha}{2}(t-T)^2 & \text{for } T-\frac{2}{\alpha}<t\le T\end{cases}$$
$$\lambda_2^*(t)=T-t.$$



We note that H is not convex.

• In order to guarantee some sufficient conditions we use the Arrow's sufficient condition. Taking into account that $x_1(t)\ge\alpha_1>0$ we construct the function maximized Hamiltonian $H^0=H^0(t,x_1,x_2,\lambda_1^*,\lambda_2^*)$ as follows

$$\begin{aligned}H^0 &= \max_{v\in[0,1]}[x_2^*+\lambda_1^*vx_1^*+\lambda_2^*(1-v)x_1^*]\\ &= x_2+\alpha x_1\max_{v\in[0,1]}(\lambda_1^*-\lambda_2^*)v\\ &= \begin{cases}x_2+x_1\left(2e^{-\alpha(t-T+2/\alpha)}+\alpha(t-T)\right) & \text{for } 0\le t\le T-\frac{2}{\alpha},\\ x_2 & \text{for } T-\frac{2}{\alpha}<t\le T\end{cases}\end{aligned}$$

Note that, for every fixed t the function $(x_1,x_2)\mapsto H^0(t,x_1,x_2,\lambda_1^*,\lambda_2^*)$ is concave: the sufficient condition of Arrow holds.

•• Instead of using a sufficient condition, we can prove the existence of a optimal control (see section 3.4). More precisely, taking into account Theorem 3.8 and studying its assumptions we have a compact control set $[0,1]$, a closed target set $\mathcal{T}=\{T\}\times\mathbb{R}^2$; moreover, for the dynamics we have the bounded condition

$$|\dot{\mathbf{x}}|=\left|\begin{pmatrix}\alpha ux_1\\ \alpha(1-u)x_1\end{pmatrix}\right|\le\alpha\sqrt{2}|x_1|\le\alpha\sqrt{2}|\mathbf{x}|$$

and, for every (t, x_1, x_2) with $x_1 \geq 0$ (but $x_1 < 0$ is similar) we have that

$$\begin{aligned} F_{(t,x_1,x_2)} &= \{(y_1, y_2, z) : \ y_1 = \alpha u x_1, \ y_2 = \alpha(1-u)x_1, \ z \leq x_2, \ u \in [0,1]\} \\ &= [0, \alpha x_1] \times [0, \alpha x_1] \times (-\infty, x_2], \end{aligned}$$

is a convex set. Hence Theorem 3.8 guarantees that the optimal control exists.

2.5.4 A problem of inventory and production I.

A firm has received an order for $B > 0$ units of product to be delivery by time T (fixed). We are looking for a plane of production for filling the order at the specified delivery data at minimum cost (see [17])[11]. Let $x = x(t)$ be the inventory accumulated by time t : since such inventory level, at any moment, is the cumulated past production and taking into account that $x(0) = 0$, we have that

$$x(t) = \int_0^t p(s)\, \mathrm{d}s,$$

where $p = p(t)$ is the production at time t; hence the rate of change of inventory $\dot{x}$ is the production and is reasonable to have $\dot{x} = p$.

The unit production cost c rises linearly with the production level, i.e. the total cost of production is $cp = \alpha p^2 = \alpha \dot{x}^2$; the unit cost of holding inventory per unit time is constant. Hence the total cost, at time t is $\alpha u^2 + \beta x$ with α and β positive constants, and $u = \dot{x}$. Our strategy problem is[12]

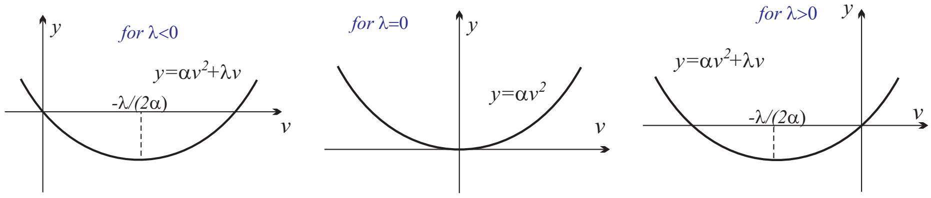
$$\begin{cases} \min\limits_u \displaystyle\int_0^T (\alpha u^2 + \beta x)\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \\ x(T) = B > 0 \\ u \geq 0 \end{cases} \tag{2.79}$$

Let us consider the Hamiltonian $H(t, x, u, \lambda) = \alpha u^2 + \beta x + \lambda u$: we are not in the situation to guarantee that the extremal is normal, but we try! The necessary conditions are

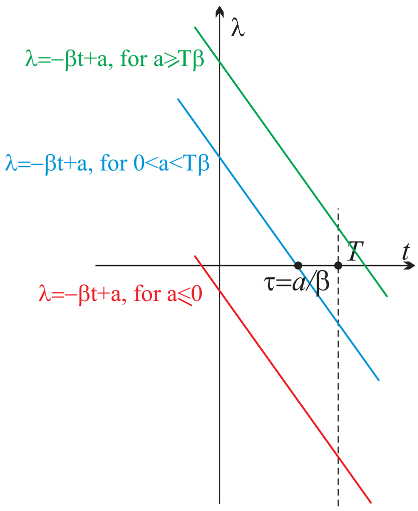
$$u^*(t) \in \arg\max_{v \geq 0} (\alpha v^2 + \beta x + \lambda v) = \arg\max_{v \geq 0} (\alpha v^2 + \lambda v) \tag{2.80}$$

$$\dot{\lambda} = -\beta \quad \Rightarrow \lambda = -\beta t + a, \tag{2.81}$$

for some constant a. Hence (2.80) gives these situations



[11] In subsection 5.5.2 we solve the same model with the Dynamic Programming.
[12] We will solve a version of this problem in subsection 5.5.2 with the Dynamic Programming.



This implies

$$u(t) = \begin{cases} 0 & \text{if } \lambda(t) \geq 0 \\ -\frac{\lambda(t)}{2\alpha} & \text{if } \lambda(t) < 0 \end{cases}$$

Taking into account (2.81), we have the three different situations as in the picture here on the right, where $\tau = \frac{a}{\beta}$.

First, $a \geq T\beta$ implies $u = 0$ in $[0, T]$ and hence, using the initial condition, $x = 0$ in $[0, T]$; this is in contradiction with $x(T) = B > 0$.

Second, $0 < a < T\beta$ implies

$$u(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau \\ -\frac{\lambda(t)}{2\alpha} = \frac{\beta t - a}{2\alpha} = \frac{\beta(t-\tau)}{2\alpha} & \text{if } \tau < t \leq T \end{cases}$$

Hence, using again the initial condition, $x(t) = 0$ in $[0, \tau]$ and, using the continuity of x in $t = \tau$,

$$x(t) = \frac{\beta}{4\alpha}(t - \tau)^2 \qquad \text{in } (\tau, T];$$

the final condition $x(T) = B$ gives $\tau = T - 2\sqrt{\frac{\alpha B}{\beta}}$. Moreover the condition $0 < a < T\beta$ gives $T > 2\sqrt{\frac{\alpha B}{\beta}}$.

Finally, the case $a \leq 0$ implies $u(t) = -\frac{\lambda(t)}{2\alpha} = \frac{\beta t - a}{2\alpha}$ in $[0, T]$ and hence

$$x(t) = \frac{\beta}{4\alpha}t^2 - \frac{a}{2\alpha}t + d \qquad \text{in } [0, T],$$

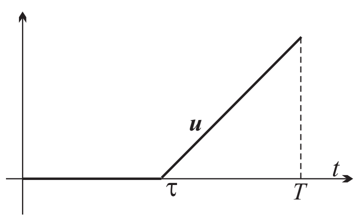
for some constant d : the conditions $x(0) = 0$ and $x(T) = B$ give

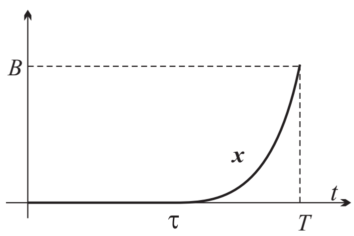
$$x(t) = \frac{\beta}{4\alpha}t^2 - \frac{4\alpha B - \beta T^2}{4\alpha T}t$$

for $T < 2\sqrt{\frac{\alpha B}{\beta}}$. Summing up, we have

- if $T > 2\sqrt{\frac{\alpha B}{\beta}}$, then with $\tau = T - 2\sqrt{\frac{\alpha B}{\beta}}$

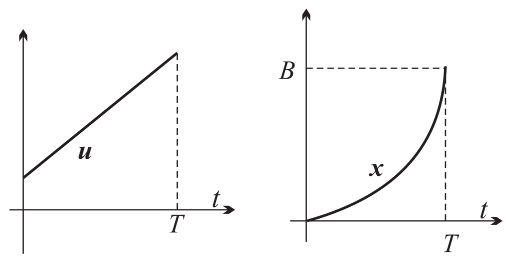
$$u^*(t) = \begin{cases} 0 & \text{if } 0 \leq t < \tau \\ \frac{\beta}{2\alpha}(t - \tau) & \text{if } \tau \leq t \leq T \end{cases} \quad \text{and} \quad x^*(t) = \begin{cases} 0 & \text{if } 0 \leq t < \tau \\ \frac{\beta}{4\alpha}(t - \tau)^2 & \text{if } \tau \leq t \leq T \end{cases}$$





- if $T \le 2\sqrt{\frac{\alpha B}{\beta}}$, then

$$u^*(t) = \frac{\beta}{2\alpha}t + \frac{4\alpha B - \beta T^2}{4\alpha T} \quad \text{and} \quad x^*(t) = \frac{\beta}{4\alpha}t^2 + \frac{4\alpha B - \beta T^2}{4\alpha T}t$$



In both the cases, we have a normal extremal and a convex Hamiltonian: hence such extremals are optimal.

2.6 Singular and bang-bang controls

The Pontryagin Maximum Principle (2.2) gives us, when it is possible, the value of the $\mathbf{u}^*$ at the point $\tau \in [t_0, t_1]$: more precisely, for every $\tau \in [t_0, t_1]$ we are looking for a unique point $\mathbf{w} = \mathbf{u}^*(\tau)$ belonging to the control set U such that

$$H(\tau, \mathbf{x}^*(\tau), \mathbf{w}, \lambda_0^*, \boldsymbol{\lambda}^*(\tau)) \ge H(\tau, \mathbf{x}^*(\tau), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(\tau)) \qquad \forall \mathbf{v} \in U. \tag{2.82}$$

In some circumstances, it is possible that using only the PMP can not be found the value to assign at $\mathbf{u}^*$ at the point $\tau \in [t_0, t_1]$: examples of this situation we have found in (2.50), (2.55) and (2.66). Now, let us consider the set $\mathcal{T}$ of the points $\tau \in [t_0, t_1]$ such that PMP gives no information about the value of the optimal control $\mathbf{u}^*$ at the point τ, i.e. a point $\tau \in \mathcal{T}$ if and only if there no exists a unique $\mathbf{w} = \mathbf{w}(\tau)$ such that it satisfies (2.82).

We say that an optimal control is *singular* if $\mathcal{T}$ contains some interval of $[t_0, t_1]$.

In optimal control problems, it is sometimes the case that a control is restricted to be between a lower and an upper bound (for example when the control set U is compact). If the optimal control $\mathbf{u}^*$ is such that

$$\mathbf{u}^*(t) \in \partial U, \qquad \forall t,$$

we say then that the control is *bang-bang*. In this case, if $\mathbf{u}^*$ switches from one extreme to the other at certain times $\tilde{\tau}$, the time $\tilde{\tau}$ is called *switching point*. For example

- in example 2.5.2, we know that the control u^* in (2.52) is optimal: the value of such control is, at all times, on the boundary $\partial U = \{0, 2\}$ of the control set $U = [0, 2]$; at time $\tilde{\tau} = 2 - \log 3$ such optimal control switches from 2 to 0. Hence $2 - \log 3$ is a switching point and u^* is bang-bang;

- in example 2.5.3, the optimal control u^* in (2.55) is bang-bang since its value belongs, at all times, to $\partial U = \{0, 2\}$ of the control set $U = [0, 2]$; the time $\log 4$ is a switching point;

- in the case B of example 1.1.2, the optimal control u^* in (2.70) is bang-bang since its value belongs, at all times, to $\partial U = \{0, 1\}$ of the control set $U = [0, 1]$; the time $T - 1$ is a switching point.

2.6.1 The building of a mountain road: a singular control

We have to solve the problem (1.4) presented in example 1.1.3 (see [22] and [17]). We note that there no exist initial or final conditions on the trajectory and hence we have to satisfy two transversality conditions for the multiplier. The Hamiltonian is $H = \lambda_0(x - y)^2 + \lambda u$, but we try to see if it possible to find a normal extremal, i.e. with $\lambda_0 = 1$:

$$(x^* - y)^2 + \lambda^* u^* = \min_{v \in [-\alpha, \alpha]} \left[(x^* - y)^2 + \lambda^* v \right]$$

$$\Rightarrow \lambda^* u^* = \min_{v \in [-\alpha, \alpha]} \lambda^* v \tag{2.83}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}^* \quad \Rightarrow \quad \dot{\lambda}^* = -2(x^* - y)$$

$$\Rightarrow \lambda^*(t) = b - 2 \int_{t_0}^{t} (x^*(s) - y(s))\, \mathrm{d}s, \ b \in \mathbb{R} \tag{2.84}$$

$$\frac{\partial H}{\partial \lambda} = \dot{x^*} \quad \Rightarrow \quad \dot{x^*} = u^* \tag{2.85}$$

$$\lambda^*(t_0) = \lambda^*(t_1) = 0 \tag{2.86}$$

We remark that (2.84) follows from the continuity of y and x. The "minimum" principle (2.83) implies

$$u^*(t) = \begin{cases} -\alpha & \text{for } \lambda^*(t) > 0, \\ \alpha & \text{for } \lambda^*(t) < 0, \\ ??? & \text{for } \lambda^*(t) = 0. \end{cases} \tag{2.87}$$

Relations (2.84) and (2.86) give

$$\lambda^*(t) = -2 \int_{t_0}^{t} (x^*(s) - y(s))\, \mathrm{d}s, \quad \forall t \in [t_0, t_1] \tag{2.88}$$

$$\int_{t_0}^{t_1} (x^*(s) - y(s))\, \mathrm{d}s = 0. \tag{2.89}$$

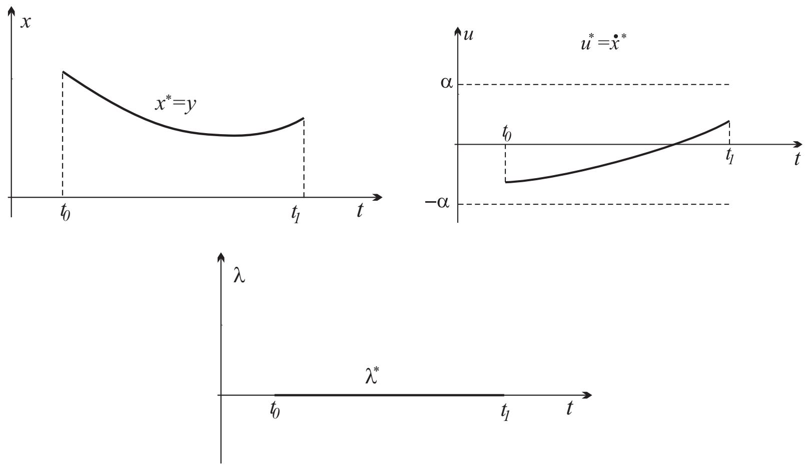
Let us suppose that there exists an interval $[c, d] \subset [t_0, t_1]$ such that $\lambda^* = 0$: clearly by (2.88) we have, for $t \in [c, d]$,

$$\begin{aligned} 0 &= \lambda^*(t) \\ &= 2 \int_{t_0}^{c} (x^*(s) - y(s))\, \mathrm{d}s - 2 \int_{c}^{t} (x^*(s) - y(s))\, \mathrm{d}s \\ &= \lambda^*(c) - 2 \int_{c}^{t} (x^*(s) - y(s))\, \mathrm{d}s \quad \forall t \in [c, d] \end{aligned}$$

and hence, since y and x^* are continuous,

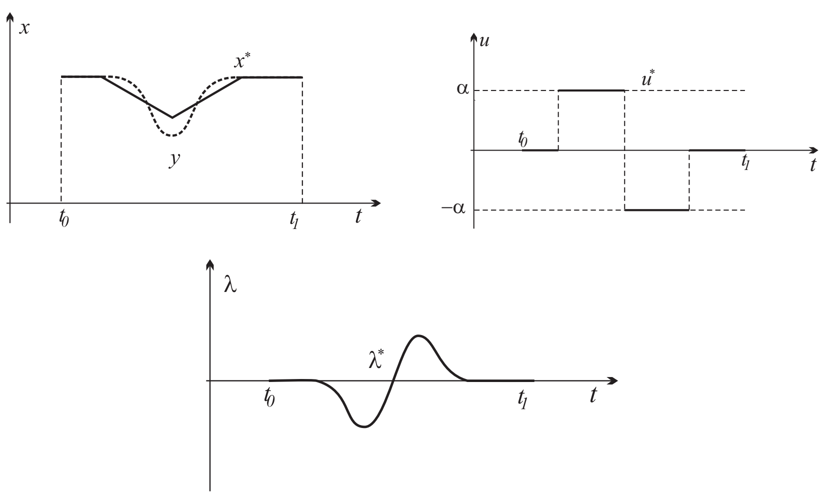
$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{c}^{t} (x^*(s) - y(s))\, \mathrm{d}s \right) = x^*(t) - y(t) = 0.$$

Hence, if $\lambda^*(t) = 0$ in $[c,d]$, then $x^*(t) = y(t)$ for all $t \in [c,d]$ and, by (2.85), $u^*(t) = \dot{y}(t)$. We remark that in the set $[c,d]$, the minimum principle has not been useful in order to determinate the value of u^*. If there exists such interval $[c,d] \subset [t_0,t_1]$ where λ^* is null, then the control is singular.
At this point, using (2.87), we are able to conclude that the trajectory x^* associated to the extremal control u^* is built with intervals where it coincides with the ground, i.e. $x^*(t) = y(t)$, and intervals where the slope of the road is maximum, i.e. $\dot{x}^*(t) \in \{\alpha, -\alpha\}$. Moreover such extremal satisfies (2.89).
Finally, we remark that the Hamiltonian is convex with respect to x and u, for every fixed t : hence the extremal is really a minimum for the problem.
Let us give three examples.
Example A: suppose that $|\dot{y}(t)| \leq \alpha,\ \forall t \in [t_0,t_1]$:



We obtain $x^* = y$ and the control is singular.

Example B: suppose that the slope $\dot{y}$ of the ground is not contained, for all $t \in [t_0,t_1]$, in $[-\alpha,\alpha]$:



In the first picture on the left, the dotted line represents the ground y, the solid represents the optimal road x^* : we remark that, by (2.89), the area of the *region between the two mentioned lines is equal to zero if we take into account the "sign" of such areas. The control is singular.*

Example 2.6.1. *Suppose that the equation of the ground is $x(t) = e^t$ for $t \in [-1,1]$ and the slope of such road must satisfy $|\dot{x}(t)| \le 1$.*

We have to solve

$$\begin{cases} \min_u \int_{-1}^{1} (x - e^t)^2 \, dt \\ \dot{x} = u \\ -1 \le u \le 1 \end{cases}$$

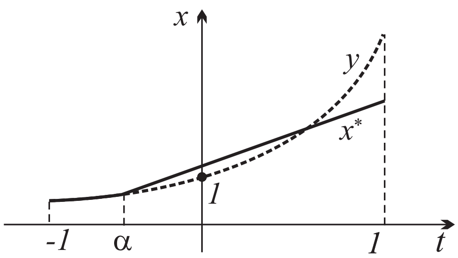
We know, for the previous consideration and calculations, that for every $t \in [-1,1]$

- *one possibility is that $x^*(t) = y(t) = e^t$ and $\lambda(t) = 0$, $|\dot{x}^*(t)| = |u^*(t)| \le 1$,*
- *the other possibility is that $\dot{x}^*(t) = u^*(t) \in \{-1,+1\}$.*

We note that for $t > 0$ the second possibility can not happen because $\dot{y}(t) > 1$. Hence let us consider the function

$$x^*(t) = \begin{cases} e^t & \text{for } t \in [-1,\alpha], \\ t + e^\alpha - \alpha & \text{for } t \in (\alpha,1], \end{cases}$$

with $\alpha \in (-1,0)$ such that (2.89) is satisfied:



$$\begin{aligned} \int_{-1}^{1} (x^*(s) - y(s)) \, ds &= \int_{\alpha}^{1} (s + e^\alpha - \alpha - e^s) \, ds \\ &= \frac{1}{2} + 2e^\alpha + \frac{1}{2}\alpha^2 - e - \alpha - \alpha e^\alpha = 0. \end{aligned} \tag{2.90}$$

For the continuous function $h : [-1,0] \to \mathbb{R}$, defined by

$$h(t) = \frac{1}{2} + 2e^t + \frac{1}{2}t^2 - e - t - te^t,$$

we have

$$\begin{aligned} h(-1) &= \frac{-e^2 + 2e + 3}{e} = -\frac{(e-3)(e+1)}{e} > 0 \\ h(0) &= \frac{5}{2} - e < 0, \end{aligned}$$

certainly there exists a point $\alpha \in (-1,0)$ such that $h(\alpha) = 0$ and hence (2.90) holds. Moreover, since $h'(t) = (e^t - 1)(1 - t) < 0$ in $(0,1)$, such point α is unique. Using (2.88), we are able to determinate the multiplier:

$$\lambda^*(t) = \begin{cases} 0 & \text{if } t \in [-1,\alpha] \\ -2\int_\alpha^t (s + e^\alpha - \alpha - e^s) ds = & \\ \quad = \frac{1}{2}(t^2 - \alpha^2) + (e^\alpha - \alpha)(t - \alpha) + e^\alpha - e^t & \text{if } t \in (\alpha,1] \end{cases}$$

Note that in the interval $[-1,\alpha]$ the PMP in (2.83) becomes

$$0 = \min_{v \in [-1,1]} 0$$

and gives us no information. Hence u^* is singular. △

2.7 The multiplier as shadow price I: an exercise

Example 2.7.1. Consider, for every $(\tau, \xi) \in [0, 2] \times [0, \infty)$ fixed, the problem

$$\begin{cases} \min \int_\tau^2 (u^2 + x^2)\, \mathrm{d}t \\ \dot{x} = x + u \\ x(\tau) = \xi \\ u \geq 0 \end{cases}$$

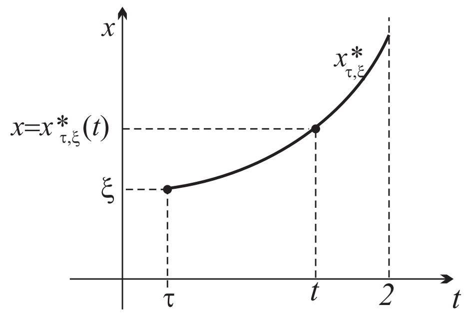
a. For every fixed (τ, ξ), find the optimal tern (x^*, u^*, λ^*). Let us denote by $({x^*}_{\tau,\xi}, u^*_{\tau,\xi}, \lambda^*_{\tau,\xi})$ such tern.

b. Calculate

$$\min_{\{u:\ \dot{x}=x+u,\ x(\tau)=\xi,\ u\geq 0\}} \int_\tau^2 (u^2 + x^2) dt = \int_\tau^2 ((u^*_{\tau,\xi})^2 + (x^*_{\tau,\xi})^2) dt$$

and denote with $V(\tau, \xi)$ such value.

c. For a given (τ, ξ), consider a point $(t, x) \in [\tau, 2] \times [0, \infty)$ on the optimal trajectory $x^*_{\tau,\xi}$, i.e. $x^*_{\tau,\xi}(t) = x$.



Consider the function $V(\tau, \xi) : [0, 2] \times [0, \infty) \to \mathbb{R}$ defined in **b.** and compute $\frac{\partial V}{\partial \xi}(t, x)$. What do you find ?

Solution **a.** Let us consider the Hamiltonian $H = u^2 + x^2 + \lambda(x + u)$; the theorem of Pontryagin gives

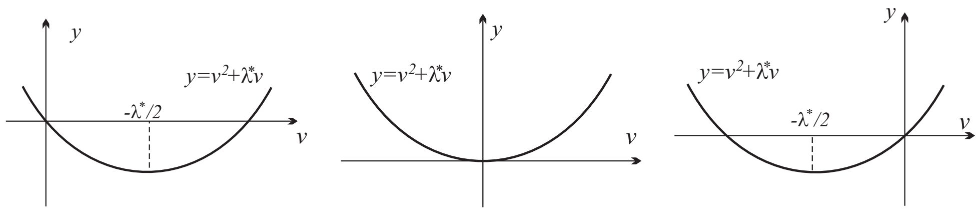
$$H(t, x^*, u^*, \lambda^*) = \min_{v\in[0,\infty)} H(t, x^*, v, \lambda^*)$$

$$\Rightarrow \ u^* \in \arg \min_{v\in[0,\infty)} (v^2 + \lambda^* v) \tag{2.91}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}^* \ \Rightarrow \ \dot{\lambda}^* = -\lambda^* - 2x^* \tag{2.92}$$

$$\lambda^*(2) = 0 \tag{2.93}$$

For every fixed t, the function $v \mapsto v^2 + \lambda^* v$ that we have to minimize represents a parabola:



The case $\lambda^*(t) < 0$; *the case* $\lambda^*(t) = 0$; *the case* $\lambda^*(t) > 0$.

Since in (2.91) we have to minimize for v in $[0, \infty)$, we obtain

$$u^*(t) = \begin{cases} 0 & \text{for } \lambda^*(t) \geq 0, \\ -\lambda^*(t)/2 & \text{for } \lambda^*(t) < 0 \end{cases} \tag{2.94}$$

Let us suppose that

$$\lambda^*(t) \geq 0, \qquad t \in [\tau, 2]. \tag{2.95}$$

Then (2.94) implies that $u^* = 0$ in $[\tau, 2]$: from the dynamics we obtain

$$\dot{x} = x + u \;\Rightarrow\; \dot{x} = x \;\Rightarrow\; x(t) = ae^t, \;\forall a \in \mathbb{R}.$$

The initial condition on the trajectory gives $x^*(t) = \xi e^{t-\tau}$. The adjoint equation (2.92) gives

$$\dot{\lambda}^* = -\lambda^* - 2\xi e^{t-\tau} \;\Rightarrow\; \lambda^*(t) = be^{-t} - \xi e^{t-\tau}.$$

By the condition (2.93) we obtain

$$\lambda^*(t) = \xi(e^{4-t-\tau} - e^{t-\tau}) \tag{2.96}$$

Recalling that $\xi \geq 0$, an easy computation shows that $\lambda^*(t) \geq 0$, for every $t \leq 2$: this is coherent with the assumption (2.95). Hence the tern

$$(u^*_{\tau,\xi}, x^*_{\tau,\xi}, \lambda^*_{\tau,\xi}) = (0, \xi e^{t-\tau}, \xi(e^{4-t-\tau} - e^{t-\tau})) \tag{2.97}$$

satisfies the necessary condition of Pontryagin. In order to guarantee some sufficient condition note that the Hamiltonian is clearly convex in (x, u): hence $u^*_{\tau,\xi}$ is optimal.

Solution **b.** Clearly, by (2.97),

$$\begin{aligned} V(\tau, \xi) &= \min_{\{u:\ \dot{x}=x+u,\ x(\tau)=\xi,\ u\geq 0\}} \int_\tau^2 (u^2 + x^2)dt \\ &= \int_\tau^2 ((u^*_{\tau,\xi})^2 + (x^*_{\tau,\xi})^2)dt \\ &= \int_\tau^2 (0^2 + \xi^2 e^{2t-2\tau})dt \\ &= \frac{\xi^2}{2}(e^{4-2\tau} - 1). \end{aligned} \tag{2.98}$$

Hence this is the optimal value for the problem, if we work with a trajectory that starts at time τ from the point ξ.

Solution **c.** Since

$$V(\tau', \xi') = \frac{(\xi')^2}{2}(e^{4-2\tau'} - 1),$$

we have

$$\frac{\partial V}{\partial \xi}(\tau', \xi') = \xi'(e^{4-2\tau'} - 1).$$

In particular, if we consider a point $(t, x) \in [\tau, 2] \times [0, \infty)$ on the optimal trajectory $x^*_{\tau,\xi}$, i.e. using (2.97) the point (t, x) is such that

$$x = x^*_{\tau,\xi}(t) = \xi e^{t-\tau},$$

we obtain

$$\frac{\partial V}{\partial \xi}(t, x) = \xi e^{t-\tau}(e^{4-2t} - 1) = \xi(e^{4-t-\tau} - e^{t-\tau}).$$

Hence we have found that

$$\frac{\partial V}{\partial \xi}(t, x^*_{\tau,\xi}(t)) = \lambda^*_{\tau,\xi}(t),$$

i.e.

Remark 2.11. *The multiplier λ^*, at time t, expresses the sensitivity, the "shadow price", of the optimal value of the problem when we modify the initial data ξ, along the optimal trajectory.*

We will see in theorem 5.9 that this is a fundamental property that holds in the general context and links the multiplier $\boldsymbol{\lambda}^*$ of the variational approach to the value function V of the dynamic programming. Two further comments on the previous exercise:

1. The function $V(\tau, \xi) : [0, 2] \times [0, \infty) \to \mathbb{R}$ is called value function and is the fundamental object of the dynamic programming. One of its property is that $V(2, \xi) = 0, \ \forall \xi$.

2. Consider the points (τ, ξ) and (τ', ξ') in $[0, 2] \times [0, \infty)$: we know that the optimal trajectories are

$$x^*_{\tau,\xi}(t) = \xi e^{t-\tau}, \qquad x^*_{\tau',\xi'}(t) = \xi' e^{t-\tau'}.$$

Now consider (τ', ξ') on the optimal trajectory $x^*_{\tau,\xi}$, i.e. the point (τ', ξ') is such that

$$\xi' = x^*_{\tau,\xi}(\tau') = \xi e^{\tau'-\tau}.$$

The previous expressions give us that, with this particular choice of (τ', ξ')

$$x^*_{\tau',\xi'}(t) = \xi' e^{t-\tau'} = \xi e^{\tau'-\tau} e^{t-\tau'} = \xi e^{t-\tau} = e^{t-\tau'} = x^*_{\tau,\xi}(t).$$

Hence the optimal trajectory associated to the initial data $(\tau'\xi')$ (with $(\tau'\xi')$ that belongs to the optimal trajectory associated to the initial data (τ, ξ)), coincides with the optimal trajectory associated to the initial data (τ, ξ). We will see that this is a fundamental property that holds in general: "the second part of an optimal trajectory is again otpimal" is the Principle of Bellman of dynamic programming (see Theorem 5.1).

Chapter 3

General problems of OC

In this chapter we will see more general problem then (2.1). In the literature there are many books devoted to this study (see for example [27], [19], [25], [5], [26]): however, the fundamental tool is the Theorem of Pontryagin that gives a necessary and useful condition of optimality.

3.1 Problems of Bolza, of Mayer and of Lagrange

Starting from the problem (2.1), let us consider t_0 fixed and T be fixed or free, with $T > t_0$. The problem

$$\begin{cases} J(\mathbf{u}) = \psi(T, \mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max\limits_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}), \end{cases} \tag{3.1}$$

with $\psi : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, is called *OC problem of Mayer.* The problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{T} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(T, \mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max\limits_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}), \end{cases} \tag{3.2}$$

is called *OC of Bolza.* The problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{T} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max\limits_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}), \end{cases} \tag{3.3}$$

is called *OC of Lagrange.* We have the following result

Remark 3.1. *The problems (3.1), (3.2) e (3.3) are equivalent.*

Clearly the problems (3.1) and (3.3) are particular cases of (3.2).

First, let us show how (3.2) can become a problem of Lagrange: since

$$\int_{t_0}^{T} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \psi(T,\mathbf{x}(T)) = \int_{t_0}^{T} \left(f(t,\mathbf{x},\mathbf{u}) + \frac{\mathrm{d}\psi(t,\mathbf{x}(t))}{\mathrm{d}t} \right) \mathrm{d}t - \psi(t_0,\mathbf{x}(t_0))$$

problem (3.2) becomes

$$\begin{cases} \tilde{J}(\mathbf{u}) = \displaystyle\int_{t_0}^{T} \left(f(t,\mathbf{x},\mathbf{u}) + \frac{\partial \psi}{\partial t}(t,\mathbf{x}) + \nabla_{\mathbf{x}} \psi(t,\mathbf{x}) \cdot g(t,\mathbf{x},\mathbf{u}) \right) \mathrm{d}t \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max\limits_{\mathbf{u} \in \mathcal{C}} \tilde{J}(\mathbf{u}) \end{cases}$$

that is clearly of Lagrange.

Secondly, let us proof how (3.3) can become a problem of Mayer: we introduce the new variable x_{n+1} defined by $\dot{x}_{n+1}(t) = f(t,\mathbf{x},\mathbf{u})$ with the condition $x_{n+1}(t_0) = 0$. Since

$$x_{n+1}(T) = x_{n+1}(t_0) + \int_{t_0}^{T} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t,$$

problem (3.3) becomes

$$\begin{cases} J(\mathbf{u}) = x_{n+1}(T) \\ (\dot{\mathbf{x}}, \dot{x}_{n+1}) = (g(t,\mathbf{x},\mathbf{u}), f(t,\mathbf{x},\mathbf{u})) \\ (\mathbf{x}(t_0), x_{n+1}(t_0)) = (\boldsymbol{\alpha}, 0) \\ \max\limits_{\mathbf{u} \in \mathcal{C}} \tilde{J}(\mathbf{u}) \end{cases}$$

that is of Mayer.

Finally, we show how the problem (3.1) becomes a problem of Lagrange: let us introduce the variable x_{n+1} as $\dot{x}_{n+1}(t) = 0$ with the condition $x_{n+1}(T) = x_{n+1}(t_0) = \frac{\psi(T,\mathbf{x}(T))}{T-t_0}$. Problem (3.1) becomes

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{T} x_{n+1}\,\mathrm{d}t \\ (\dot{\mathbf{x}}, \dot{x}_{n+1}) = (g(t,\mathbf{x},\mathbf{u}), 0) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ x_{n+1}(t_0) = \frac{\psi(T,\mathbf{x}(T))}{T-t_0} \\ \max\limits_{\mathbf{u} \in \mathcal{C}} \tilde{J}(\mathbf{u}) \end{cases}$$

that is of Lagrange.

3.2 Problems with fixed or free final time

3.2.1 Fixed final time

Let us consider $f : [t_0, t_1] \times \mathbb{R}^{n+k} \to \mathbb{R}$, $\psi : \mathbb{R}^n \to \mathbb{R}$ and let $\boldsymbol{\alpha} \in \mathbb{R}^n$ be fixed. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and let $n_1,\ n_2$ and n_3 be non negative, fixed integer such that $n_1 + n_2 + n_3 = n$. Let us consider the problem

$$\begin{cases} \max_{\mathbf{u}\in\mathcal{C}} \int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ x_i(t_1) \text{ free} \qquad 1 \le i \le n_1 \\ x_j(t_1) \ge \beta_j \quad \text{with } \beta_j \text{ fixed} \qquad n_1+1 \le j \le n_1+n_2 \\ x_l(t_1) = \beta_l \quad \text{with } \beta_l \text{ fixed} \qquad n_1+n_2+1 \le l \le n_1+n_2+n_3 \\ \mathcal{C} = \{\mathbf{u} : [t_0,t_1] \to U \subseteq \mathbb{R}^k,\ \mathbf{u} \text{ admissible}\} \end{cases} \tag{3.4}$$

where t_0 and t_1 are fixed. Since $x_i(t_1)$ is fixed for $n - n_3 < i \le n$, then the pay off function ψ depends only on $x_i(t_1)$ with $1 \le i \le n - n_3$.
We have the following necessary condition, a generalization theorem 2.1 of Pontryagin:

Theorem 3.1. *Let us consider the problem (3.4) with $f \in C^1([t_0,t_1]\times\mathbb{R}^{n+k})$, $g \in C^1([t_0,t_1] \times \mathbb{R}^{n+k})$ and $\psi \in C^1(\mathbb{R}^n)$.*
Let $\mathbf{u}^*$ be optimal control and $\mathbf{x}^*$ be the associated trajectory.
Then there exists a multiplier $(\lambda_0^*, \boldsymbol{\lambda}^*)$, with

- ⋄ *$\lambda_0^* \ge 0$ constant,*
- ⋄ *$\boldsymbol{\lambda}^* : [t_0, T^*] \to \mathbb{R}^n$ continuous,*

such that

i) the nontriviality of the multiplier holds;

ii) the Pontryagin Maximum Principle (2.2) holds,

iii) the adjoint equation (2.3) holds,

iv_{t_1}) the transversality condition is given by

- *for $1 \le i \le n_1$, we have $\lambda_i^*(t_1) = \lambda_0^* \dfrac{\partial\psi}{\partial x_i}(\mathbf{x}^*(t_1))$,*
- *for $n_1+1 \le j \le n_1+n_2$, we have $\lambda_j^*(t_1) \ge \lambda_0^* \dfrac{\partial\psi}{\partial x_j}(\mathbf{x}^*(t_1))$, $x_j^*(t_1) \ge \beta_j$ and $\left(\lambda_j^*(t_1) - \lambda_0^* \dfrac{\partial\psi}{\partial x_j}(\mathbf{x}^*(t_1))\right)\left(x_j^*(t_1) - \beta_j\right) = 0$.*

We will give an idea of the proof in Theorem 3.4.

Using Remark 3.1, it is clear that a problem with final time fixed and value of the trajectory in such final time free, it is possible to guarantee the normality of the control as in Theorem 2.1. More precisely

Remark 3.2. *Let us consider the problem (3.4) with $n_2 = n_3 = 0$, i.e. $\mathbf{x}(t_1)$ free. Then in Theorem 3.1 it is possible to guarantee that u^* is normal, i.e. $\lambda_0^* = 1$.*

A sufficient condition for the problem (3.4), with a proof similar to the one in theorem 2.3, is the following:

Theorem 3.2. *Let us consider the maximum problem (3.4) with f, g and ψ in C^1. Let the control set U be convex. Let $\mathbf{u}^*$ be a normal admissible control; let $\mathbf{x}^*$ and $(1, \boldsymbol{\lambda}^*)$ be the associated trajectory and multiplier respectively. We suppose that all the necessary conditions of theorem 3.1 hold.*
Moreover, we suppose that

v) the functions $(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*)$ and $\mathbf{x} \mapsto \psi(\mathbf{x})$ are concave functions in the variables $\mathbf{x}$ and $\mathbf{u}$, for all $t \in [t_0, t_1]$ fixed.

Then $\mathbf{u}^*$ is optimal.

We mention that the sufficient condition of Arrow works in this more general situation (see Theorem 3.4 in [26]).

Autonomous problems

It is clear that the arguments of subsection 2.1 are true. Hence

Remark 3.3. *In the assumption of Theorem 3.1, if the problem is autonomous, then we have that in $[t_0, t_1]$ (2.14) is true, i.e.*

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) = \text{constant}$$

3.2.2 Free final time

Let us consider $f : [t_0, \infty) \times \mathbb{R}^{n+k} \to \mathbb{R}$, $g : [t_0, \infty) \times \mathbb{R}^{n+k} \to \mathbb{R}$ and $\psi : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, and let $\boldsymbol{\alpha} \in \mathbb{R}^n$ be fixed. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and n_1, n_2 and n_3 be non negative, fixed integer such that $n_1 + n_2 + n_3 = n$. We consider the problem

$$\begin{cases} \max\limits_{\mathbf{u} \in \mathcal{C}} \displaystyle\int_{t_0}^{T} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(T, \mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ x_i(T) \text{ free} \qquad 1 \le i \le n_1 \\ x_j(T) \ge \beta_j \quad \text{with } \beta_j \text{ fixed} \qquad n_1 + 1 \le j \le n_1 + n_2 \\ x_l(T) = \beta_l \quad \text{with } \beta_l \text{ fixed} \qquad n_1 + n_2 + 1 \le l \le n_1 + n_2 + n_3 \\ \mathcal{C} = \{\mathbf{u} : [t_0, \infty) \to U \subseteq \mathbb{R}^k,\ \mathbf{u} \text{ admissible}\} \end{cases} \tag{3.5}$$

where t_0 is fixed and T is free with $T > t_0$. We say that $\mathbf{u}^*$ is optimal with exit time T^* if

$$\int_{t_0}^{T^*} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t + \psi(T^*, \mathbf{x}^*(T^*)) \ge \int_{t_0}^{T} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(T, \mathbf{x}(T))$$

for every admissible control $\mathbf{u}$ and for every $T \ge t_0$. Hence and optimal control has an (unique) associated exit time. We have the following result, again a generalization of the theorem of Pontryagin 2.1 (see [22]):

Theorem 3.3. *Let us consider the problem (3.5) with $f \in C^1([t_0, \infty) \times \mathbb{R}^{n+k})$, $g \in C^1([t_0, \infty) \times \mathbb{R}^{n+k})$ and $\psi \in C^1([t_0, \infty) \times \mathbb{R}^n)$.*
Let $\mathbf{u}^*$ be optimal control with exit time T^* and $\mathbf{x}^*$ be the associated trajectory.

Then there exists a multiplier $(\lambda_0^*, \boldsymbol{\lambda}^*)$, with

◇ $\lambda_0^* \geq 0$ *constant,*

◇ $\boldsymbol{\lambda}^* : [t_0, t_1] \to \mathbb{R}^n$ *continuous,*

such that

i) *the nontriviality of the multiplier holds;*

ii) *the Pontryagin Maximum Principle (2.2) holds,*

iii) *the adjoint equation (2.3) holds,*

iv_{T^*}*)* *the transversality condition is given by*

- *for* $1 \leq i \leq n_1$, *we have* $\lambda_i^*(T^*) = \lambda_0^* \dfrac{\partial \psi}{\partial x_i}(T^*, \mathbf{x}^*(T^*))$,
- *for* $n_1 + 1 \leq j \leq n_1 + n_2$, *we have* $\lambda_j^*(T^*) \geq \lambda_0^* \dfrac{\partial \psi}{\partial x_j}(T^*, \mathbf{x}^*(T^*))$, $x_j^*(T^*) \geq \beta_j$, $\left(\lambda_j^*(T^*) - \lambda_0^* \dfrac{\partial \psi}{\partial x_j}(T^*, \mathbf{x}^*(T^*))\right)\left(x_j^*(T^*) - \beta_j\right) = 0$;

moreover we have

$$H(T^*, \mathbf{x}^*(T^*), \mathbf{u}^*(T^*), \lambda_0^*, \boldsymbol{\lambda}^*(T^*)) + \lambda_0^* \frac{\partial \psi}{\partial t}(T^*, \mathbf{x}^*(T^*)) = 0. \qquad (3.6)$$

We will give an idea of the proof in Theorem 3.4.

For variable time optimal problems it is hard to find sufficient conditions of any practical value, due to an inherent lack of convexity properties in such problems.

Remark 3.4. *In the context of problem (3.5) with convex control set U, the regularity and the concavity of the Hamiltonian and the normality of the extremal are not sufficient conditions of optimality, i.e. a result similar to Theorem 3.2 does not hold in this new situation.*

In [26] appear some sufficient conditions for a large type of problems: in this note we prefer to provide some necessary conditions for the particular problems that we present and some existence results of the optimal control (see section 3.4).

Autonomous problems

In the particular case of a autonomous problem with free final time the arguments of subsection 2.1 are true. Hence, using condition (3.6), we have

Remark 3.5. *In the assumption of Theorem 3.3, if the problem is autonomous, then we have that*

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) = 0 \qquad t \in [t_0, T^*]; \qquad (3.7)$$

More details can be found in, for example, [26] (page 172) and [1] (page 189).

3.2.3 The proof of the necessary condition

In this subsection our aim is to give an idea of the proof of the generalization of Pontryagin theorem presented in theorems 3.1 and 3.3. More precisely we prove, in the spirit of the previous theorem 2.2 and in the case $n = k = 1$, the following:

Theorem 3.4. *Let us consider the problem*

$$\begin{cases} J(u) = \displaystyle\int_{t_0}^{T} f(t, x, u)\,\mathrm{d}t + \psi(T, x(T)) \\ \dot{x} = g(t, x, u) \\ x(t_0) = \alpha \quad \text{fixed} \\ \displaystyle\max_{u \in \mathcal{C}} J(u) \\ \mathcal{C} = \left\{u : [t_0, T] \to \mathbb{R},\ u \in C([t_0, T]),\ T > t_0\right\} \end{cases} \tag{3.8}$$

with f and g in C^1, and $\mathcal{C}$ open and non empty. Moreover, in the problem (3.8) we require one of the following four situations:

I. *T is fixed*[1] *and $x(T) = \beta$ is fixed;*

II. *T is fixed*[1] *and $x(T) = \beta$ is free;*

III. *T is free and $x(T) = \beta$ is fixed;*

IV. *T is free and $x(T) = \beta$ is free.*

Let u^* be the optimal control with exit time T^* and x^* be the optimal trajectory. Then there exists a multiplier (λ_0^*, λ^*), with

- λ_0^* *constant,*
- $\lambda^* : [t_0, T^*] \to \mathbb{R}$ *continuous,*

such that $(\lambda_0^*, \lambda^*) \neq (0, 0)$ and

i) the PMP_0 $\quad \dfrac{\partial H}{\partial u}(t, x^*(t), u^*(t), \lambda_0^*, \lambda^*(t)) = 0$ *holds,*

ii) the adjoint equation (2.3) holds,

iii_{T^*}) the transversality condition, depending on the previous situations, is

I. *no condition;*

II. *we have* $\lambda^*(T^*) = \lambda_0^* \dfrac{\partial \psi}{\partial x}(\mathbf{x}^*(T^*))$;

III. *we have*

$$H(T^*, \mathbf{x}^*(T^*), \mathbf{u}^*(T^*), \lambda_0^*, \boldsymbol{\lambda}^*(T^*)) + \lambda_0^* \frac{\partial \psi}{\partial t}(T^*, \mathbf{x}^*(T^*)) = 0; \tag{3.9}$$

IV. *we have* $\lambda^*(T^*) = \lambda_0^* \dfrac{\partial \psi}{\partial x}(T^*, \mathbf{x}^*(T^*))$ *and (3.9).*

[1] Note that in this case $\psi(t, x) = \psi(x)$.

iv) $\lambda_0^* \geq 0$.

Proof. First, let us proof the case IV. As in the proof of theorem 2.2, let $u^* \in \mathcal{C}$ be optimal control with exit time T^* and x^* its trajectory. Let us fix a continuous function $h : [t_0, \infty) \to \mathbb{R}$. For every constant $\epsilon \in \mathbb{R}$ we define the function $u_\epsilon = u^* + \epsilon h$. Moreover, we consider a generic C^1 exit time function $T(\epsilon) = T_\epsilon : \mathbb{R} \to [t_0, \infty)$ such that $T_0 = T^*$. Hence, for every ϵ-variation of the tern $(u^*,\ x^*,\ T^*)$ we have a new tern $(u_\epsilon,\ x_\epsilon,\ T_\epsilon)$ where $x_\epsilon : [0, T_\epsilon] \to \mathbb{R}$ is the trajectory associated to $u_\epsilon : [0, T_\epsilon] \to \mathbb{R}$. Clearly

$$\begin{array}{ll} T(0) = T^*, & x_\epsilon(t_0) = \alpha \quad \text{fixed}, \\ x_\epsilon(T_\epsilon) = \beta_\epsilon, & x_0(t) = x^*(t). \end{array}$$

Now, recalling that h is fixed and considering a constant $\lambda_0 \geq 0$, we define the function $\mathcal{J}_h : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{J}_h(\epsilon) = \lambda_0 \left(\int_{t_0}^{T_\epsilon} f\big(t, x_\epsilon(t), u_\epsilon(t)\big)\,\mathrm{d}t + \psi(T_\epsilon, \beta_\epsilon) \right).$$

Let $\lambda : [t_0, \infty) \to \mathbb{R}$ be a generic continuous function: we obtain

$$\begin{aligned} \mathcal{J}_h(\epsilon) &= \int_{t_0}^{T(\epsilon)} \Big\{ \lambda_0 f\big(t, x_\epsilon, u_\epsilon\big) + \lambda\Big(g\big(t, x_\epsilon, u_\epsilon\big) - \dot{x}_\epsilon \Big) \Big\}\,\mathrm{d}t + \lambda_0 \psi(T_\epsilon, \beta_\epsilon) \\ \text{(by part)} &= \int_{t_0}^{T_\epsilon} \Big\{ H\big(t, x_\epsilon, u_\epsilon, \lambda_0, \lambda\big) + \dot{\lambda} x_\epsilon \Big\}\,\mathrm{d}t - \Big(\lambda(T_\epsilon)\beta_\epsilon - \lambda(t_0)\alpha \Big) + \\ &\quad + \lambda_0 \psi(T_\epsilon, \beta_\epsilon) \end{aligned}$$

Since u^* is optimal and $\lambda_0 \geq 0$, we have $\frac{\mathrm{d}\mathcal{J}_h}{\mathrm{d}\epsilon}(0) = 0$. Hence[2]

$$\begin{aligned} \frac{\mathrm{d}\mathcal{J}_h}{\mathrm{d}\epsilon}(0) &= \int_{t_0}^{T^*} \left\{ \left[\frac{\partial H}{\partial x}(t, x^*, u^*, \lambda_0, \lambda) + \dot{\lambda} \right] \frac{\mathrm{d}x_\epsilon}{\mathrm{d}\epsilon}(0) + \frac{\partial H}{\partial u}(t, x^*, u^*, \lambda_0, \lambda) h \right\} \mathrm{d}t + \\ &\quad + \left[H\big(T^*, x^*(T^*), u^*(T^*), \lambda_0, \lambda(T^*)\big) + \lambda_0 \frac{\partial \psi}{\partial t}(T^*, x^*(T^*)) \right] \frac{\mathrm{d}T_\epsilon}{\mathrm{d}\epsilon}(0) + \\ &\quad - \left[\lambda(T^*) - \lambda_0 \frac{\partial \psi}{\partial x}(T^*, x^*(T^*)) \right] \frac{\mathrm{d}\beta_\epsilon}{\mathrm{d}\epsilon}(0) \\ &= 0. \end{aligned} \tag{3.10}$$

Clearly T_ϵ and $x_\epsilon(T_\epsilon) = \beta_\epsilon$ are free and we have no information on $\frac{\mathrm{d}T_\epsilon}{\mathrm{d}\epsilon}(0)$ and $\frac{\mathrm{d}\beta_\epsilon}{\mathrm{d}\epsilon}(0)$. Hence we require that λ_0 and λ solve the system

$$\begin{cases} \dot{\lambda}(t) = -\lambda(t) \dfrac{\partial g}{\partial x}(t, x^*, u^*) - \lambda_0 \dfrac{\partial f}{\partial x}(t, x^*, u^*) \quad \text{in } [t_0, T^*] \\ \lambda(T^*) = \lambda_0 \dfrac{\partial \psi}{\partial x}(T^*, x^*(T^*)) \\ \lambda_0 f(T^*, x^*(T^*), u^*(T^*)) + \lambda(T^*) g(T^*, x^*(T^*), u^*(T^*)) + \lambda_0 \dfrac{\partial \psi}{\partial t}(T^*, x^*(T^*)) = 0 \end{cases}$$

[2] We recall that

Proposition 3.1. *Let α and β in $C^1([a,b])$, with $a < b$, such that $\alpha(\epsilon) \leq \beta(\epsilon)$, for every $\epsilon \in [a,b]$. Let $A = \{(t,\epsilon) : \epsilon \in [a,b],\ \alpha(\epsilon) \leq t \leq \beta(\epsilon)\}$ and consider the function $g : A \to \mathbb{R}$. We suppose that g and $\frac{\partial g}{\partial \epsilon}$ are continuous in A. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_{\alpha(\epsilon)}^{\beta(\epsilon)} g(t,\epsilon)\,\mathrm{d}t = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{\partial g}{\partial \epsilon}(t,\epsilon)\,\mathrm{d}t + \beta'(\epsilon) g(\beta(\epsilon),\epsilon) - \alpha'(\epsilon) g(\alpha(\epsilon),\epsilon).$$

Considering the last two conditions in the point T^* we obtain

$$\lambda_0\Big(f(T^*,x^*(T^*),u^*(T^*)) + \frac{\partial\psi}{\partial x}(T^*,x^*(T^*))g(T^*,x^*(T^*),u^*(T^*)) + \\ + \frac{\partial\psi}{\partial t}(T^*,x^*(T^*))\Big) = 0 :$$

Clearly, if the big parenthesis is different from zero, then $\lambda_0^* = 0$; if the big parenthesis is zero, then we set $\lambda_0^* = 1$. Note that in both cases there exists a solution λ^* of the previous ODE and the two transversality conditions.[3] For these choices of the function $\lambda = \lambda^*$ and of the constant $\lambda_0 = \lambda_0^*$, we have by (3.10)

$$\int_{t_0}^{T^*} \frac{\partial H}{\partial u}(t,x^*,u^*,\lambda_0^*,\lambda^*)h\,\mathrm{d}t = 0, \tag{3.11}$$

for every $h \in C([t_0,\infty))$. Lemma 2.2 gives the PMP_0.

The proof of case I. is similar; here $T(\epsilon) = T^*$ and $x_\epsilon(T(\epsilon)) = \beta$ are fixed and hence

$$\frac{\mathrm{d}T}{\mathrm{d}\epsilon}(0) = 0, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}\epsilon}(0) = 0.$$

No transversality conditions appear in this case.

The other two cases are similar. □

3.2.4 The moonlanding problem

The formulation of the problem presented in (1.7) is the following:

$$\begin{cases} \max_{u\in\mathcal{C}} m(T) \\ \dot h = v \\ \dot v = \dfrac{u}{m} - g \\ \dot m = -ku \\ h(0) = h_0, \qquad h(T) = 0 \\ v(0) = v_0, \qquad v(T) = 0 \\ m(0) = M + F \\ m(t) \ge 0, \qquad h(t) \ge 0 \\ \mathcal{C} = \{u : [0,T] \to [0,\alpha],\ \text{admissible}\} \end{cases}$$

where h_0, M, F, g, $-v_0$, k and α are positive and fixed constants; the final time T is free. The target set for the problem is $\mathcal{T} = [0,\infty) \times \{(0,0)\} \times [0,\infty)$. First of all we remark that a reasonable assumption is

$$\alpha > g(M+F) \tag{3.12}$$

i.e. the possibility for the spacecraft to win the gravitation acceleration of the moon.

The Hamiltoniana is

$$H(h,v,m,u,\lambda_0,\lambda_1,\lambda_2,\lambda_3) = \lambda_1 v + \lambda_2\left(\frac{u}{m} - g\right) - \lambda_3 k u$$

[3] We mention that our proof is incomplete: we omit to prove that $(\lambda_0^*,\lambda^*) \neq (0,0)$.

with a pay off function

$$\psi(h, v, m) = m.$$

Theorem 3.1 implies

$$u(t) \in \arg \max_{w \in [0,\alpha]} \left(\lambda_1(t)v(t) + \lambda_2(t) \left(\frac{w}{m(t)} - g \right) - \lambda_3(t)kw \right)$$

$$\Rightarrow \quad u(t) = \begin{cases} \alpha & \text{if } \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k > 0 \\ ? & \text{if } \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k = 0 \\ 0 & \text{if } \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k < 0 \end{cases} \tag{3.13}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial h} \quad \Rightarrow \quad \dot{\lambda}_1 = 0 \quad \Rightarrow \quad \lambda_1(t) = a$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial v} \quad \Rightarrow \quad \dot{\lambda}_2 = -\lambda_1 \quad \Rightarrow \quad \lambda_2(t) = -at + b \tag{3.14}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial m} \quad \Rightarrow \quad \dot{\lambda}_3 = \frac{\lambda_2 u}{m^2} \tag{3.15}$$

$$\lambda_3(T) = \lambda_0 \frac{\partial \psi}{\partial m}(m(T)) \quad \Rightarrow \quad \lambda_3(T) = \lambda_0$$

where a and b are constants. Moreover, since the problem is autonomous, (3.7) implies

$$\lambda_1(t)v(t) + \lambda_2(t) \left(\frac{u(t)}{m(t)} - g \right) - \lambda_3(t)ku(t) = 0, \quad \forall t \in [0, T] \tag{3.16}$$

Let us suppose that $a = b = 0$: we easily obtain $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_0$ in $[0, T]$. The case $\lambda_0 = 0$ contradicts the nontriviality of the multiplier; then $\lambda_0 \neq 0$. The case $\lambda_0 \neq 0$ gives, using (3.16),

$$u(t) = 0 \qquad \text{in } [0, T]: \tag{3.17}$$

clear this is unreasonable: however, from a mathematical point of view, the dynamics gives $\dot{v} = -g$ which contradicts $v(0) = v_0 < 0$, $v(T) = 0$.

Now, let us suppose that a and b are not both zero: we prove that there exists at most one point $\tau \in [0, T]$ such that in (3.13) we have $\frac{\lambda_2(\tau)}{m(\tau)} - \lambda_3(\tau)k = 0$. In order to do that, we define the function $\phi : [0, T] \to \mathbb{R}$ by

$$\phi(t) = \frac{\lambda_2(t)}{m(t)} - \lambda_3(t)k.$$

Using the dynamics, (3.14) and (3.15) we obtain

$$\dot{\phi} = \frac{\dot{\lambda}_2 m - \lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 k = -\frac{a}{m};$$

this proves that ϕ is monotone. We have to show that the case $a = 0 = \dot{\phi}$ and $\phi = 0$ in $[0, T]$ does not occur; clearly $\phi = 0$ implies $\lambda_3 = \frac{b}{km}$ in all $[0, T]$. This relation and $a = 0$ in (3.16) imply $b = 0$: we know that $a = b = 0$ it's impossible. Hence, really there exists at most one point $\tau \in [0, T]$ such that $\phi(\tau)$.

Now we have four possibilities for the extremal control u

$$I^\circ \quad u(t) = \begin{cases} 0 & \text{for } t \in [0, \tau] \\ \alpha & \text{for } t \in (\tau, T] \end{cases} \qquad II^\circ \quad u(t) = \begin{cases} \alpha & \text{for } t \in [0, \tau] \\ 0 & \text{for } t \in (\tau, T] \end{cases}$$

$$III^\circ \quad u(t) = 0 \ \text{ for } t \in [0, T] \qquad IV^\circ \quad u(t) = \alpha \ \text{ for } t \in [0, T]$$

Case IV° is a.e. a particular case of case I° with $\tau = 0$. Case III° is as in (3.17). Case II° is unreasonable: however, from a mathematical point of view, the dynamics gives $\dot{v}(T) = -g$ which implies, taking into account $v(T) = 0$, that there exists t' such that $v(t) > 0$ for $t \in (t', T)$. Now the relation

$$h(t) = h(T) + \int_T^t v(s)\,\mathrm{d}s,$$

taking into account $h(T) = 0$, implies $h(t) < 0$ for $t \in (t', T)$.

Hence we focus our attention on the Case I°. In $[0, \tau]$ the dynamics and the initial conditions give

$$\begin{cases} h(t) = -\dfrac{t^2 g}{2} + v_0 t + h_0 \\ v(t) = -tg + v_0 \\ m(t) = M + F \end{cases} \tag{3.18}$$

In the (v, h)-plane, the previous relations gives the *free fall curve* γ

$$h = -\frac{v^2}{2g} + \frac{v_0^2}{2g} + h_0; \tag{3.19}$$

its interpretation is that if the spacecraft starts at the (v_0, h_0)-point (recall that $v_0 < 0,\ h_0 > 0$), the corresponding curve in (3.19) describes the free fall, i.e. with $u = 0$: we notice that v decreases.

In the interval $[\tau, T]$, putting $u(t) = \alpha$ in the dynamics we obtain

$$\begin{aligned} \dot{m} = -ku \quad &\Rightarrow \quad m(t) = m(\tau) - (t - \tau)k\alpha \\ m\dot{v}(t) = u - mg \quad &\Rightarrow \quad \dot{v} = \frac{\alpha}{m(\tau) - (t - \tau)k\alpha} - g \\ &\Rightarrow \quad v(t) = -\frac{1}{k}\log\left(\frac{m(\tau) - (t - \tau)k\alpha}{m(\tau)}\right) - (t - \tau)g + v(\tau) \\ \dot{h} = v \quad &\Rightarrow \quad h(t) = \int_\tau^t v(s)\,\mathrm{d}s + h(\tau) \end{aligned}$$

We remark that, by (3.12), we have

$$\frac{\alpha}{m(\tau) - (t - \tau)k\alpha} - g \geq \frac{\alpha}{M + F} - g > 0$$

and hence $M + F - (t - \tau)k\alpha > 0$. By using (3.18) and the continuity of the trajectory

$$\begin{cases} h(t) = \dfrac{M + F - (t - \tau)k\alpha}{k^2\alpha}\log\left(\dfrac{M + F - (t - \tau)k\alpha}{M + F}\right) + \\ \qquad\qquad + \dfrac{t - \tau}{k} - \dfrac{(t - \tau)^2 g}{2} + (t - \tau)v(\tau) + h(\tau) \\ v(t) = -\dfrac{1}{k}\log\left(\dfrac{M + F - (t - \tau)k\alpha}{M + F}\right) - (t - \tau)g + v(\tau) \\ m(t) = M + F - (t - \tau)k\alpha. \end{cases} \tag{3.20}$$

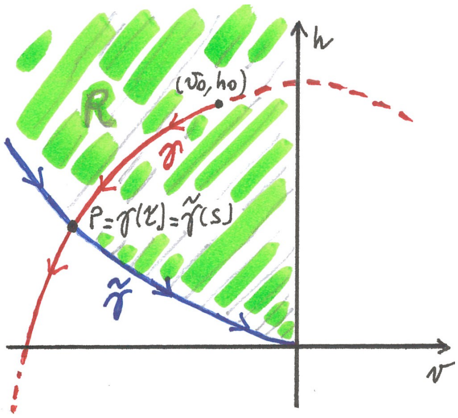
Since in the final point T we have $h(T) = v(T) = 0$ and setting $s = T - \tau$, the previous equations for h and v become

$$\begin{cases} 0 = \dfrac{M+F-sk\alpha}{k^2\alpha}\log\left(\dfrac{M+F-sk\alpha}{M+F}\right) + \dfrac{s}{k} - \dfrac{s^2 g}{2} + sv(\tau) + h(\tau) \\ 0 = -\dfrac{1}{k}\log\left(\dfrac{M+F-sk\alpha}{M+F}\right) - sg + v(\tau) \end{cases}$$

and hence

$$\begin{cases} \tilde{v}(s) := v(\tau) = \dfrac{1}{k}\log\left(\dfrac{M+F-sk\alpha}{M+F}\right) + gs \\ \tilde{h}(s) := h(\tau) = -\dfrac{M+F}{k^2\alpha}\log\left(\dfrac{M+F-sk\alpha}{M+F}\right) - \dfrac{s}{k} - \dfrac{s^2 g}{2} \end{cases} \tag{3.21}$$

The previous arguments say that τ is exactly the time necessary for a spacecraft that has a initial state $(v(\tau), h(\tau))$ to arrive on the moon with velocity zero. Hence in (3.21) we have a curve $s \mapsto \tilde{\gamma}(s) = (\tilde{v}(s), \tilde{h}(s))$: from the construction of this curve (usually called *switching curve*), its interpretation is that if the spacecraft is at a (v, h)-point on the curve corresponding to the parameter s and if it thrusts at maximum rate α, then it will arrive at $v = 0,\ h = 0$ in time s. Finally, we remark that the spacecraft burns fuel at rate $k\alpha$; since the total amount of fuel is F we have the condition $0 \le s \le \frac{F}{k\alpha}$ in our switching curve.



The initial data (v_0, h_0) *in the interior of the reachable set* R. *The* free fall curve γ *in (3.19) and the* switching curve *in (3.21). The point* $P = \tilde{\gamma} \cap \gamma$*: the switching point* τ *is such that* $P = \gamma(\tau)$*; finally, if* s *is such that* $P = \tilde{\gamma}(s)$*, then the final time* T *is* $T = \tau + s$*. In order to draw the curve* $\tilde{\gamma}$ *note that* $\tilde{\gamma}(0) = (\tilde{v}(0), \tilde{h}(0)) = (0,0)$ *and that (3.12) implies*

$$\tilde{v}'(s) = -\frac{\alpha}{M+F-sk\alpha} - g < 0$$

and

$$\begin{aligned} \tilde{h}'(s) &= \frac{M+F}{k(M+F-sk\alpha)} - \frac{1}{k} - sg \\ &= s\left(\frac{\alpha}{M+F-sk\alpha} - g\right) > 0. \end{aligned}$$

It is clear from the previous picture that for some initial data (v_0, h_0) there is no solution for the problem; moreover for every initial data (v_0, h_0) in the interior of the reachable set, there exists a unique switching point $\tau \in [0, T]$; the second part of the trajectory lies on the switching curve and spends exactly a time s to arrive in the origin. Hence $T = \tau + s$.

Now let us prove that the above construction give an extremal. The control is

$$u(t) = \begin{cases} 0 & \text{for } t \in [0, \tau] \\ \alpha & \text{for } t \in (\tau, T] \end{cases} \tag{3.22}$$

and the associated trajectory is given by (3.18) for $t \in [0, \tau]$ and by (3.20) for $t \in (\tau, T]$. Note that such trajectory is uniquely determined by the initial data; moreover we found τ and T.
The multiplier is given by

$$\lambda_1(t) = a, \quad \lambda_2(t) = -at + b, \quad \lambda_3(t) = \begin{cases} c & \text{if } t \in [0, \tau] \\ \lambda_0 - \alpha \displaystyle\int_t^T \frac{-as + b}{(m(s))^2}\, \mathrm{d}s & \text{if } t \in (\tau, T] \end{cases}$$

Since $m(s) > 0$, the last integral exists and is finite in $[\tau, T]$. By the continuity we have

$$c = \lambda_0 - \alpha \int_\tau^T \frac{-as + b}{(m(s))^2}\, \mathrm{d}s \tag{3.23}$$

We recall that τ is such that $\phi(\tau)$: hence

$$\frac{-a\tau + b}{M + F} - ck = 0 \tag{3.24}$$

Relation (3.16) implies[4]

$$0 = av_0 - bg \qquad (\text{for } \ t = 0) \tag{3.25}$$

$$0 = (-aT + b)\left(\frac{\alpha}{m(T)} - g\right) - \lambda_0 k\alpha \qquad (\text{for } \ t = T) \tag{3.26}$$

Now

$$(3.25) \quad \Rightarrow \quad a = \frac{bg}{v_0} \tag{3.27}$$

$$(3.26) \quad \Rightarrow \quad \lambda_0 = \frac{b}{k\alpha}\left(-\frac{g}{v_0}T + 1\right)\left(\frac{\alpha}{m(T)} - g\right) \tag{3.28}$$

$$(3.24) \quad \Rightarrow \quad c = \frac{b}{k(M+F)}\left(-\frac{g}{v_0}\tau + 1\right) \tag{3.29}$$

$$(3.23) \quad \Rightarrow \quad c = b\left[\frac{1}{k\alpha}\left(-\frac{g}{v_0}T + 1\right)\left(\frac{\alpha}{m(T)} - g\right) - \alpha\int_\tau^T \frac{-\frac{g}{v_0}s + 1}{(m(s))^2}\, \mathrm{d}s\right] \tag{3.30}$$

Clearly $-\frac{g}{v_0}T + 1$ is positive and (3.12) implies $\frac{\alpha}{m(T)} - g > 0$; hence if $\lambda_0 = 0$, then (3.28) implies $b = 0$; now (3.27) and (3.29) imply $a = c = 0$; finally we obtain $\lambda_3 = 0$. This is impossible by the non triviality of the multiplier.
Hence we put $\lambda_0 = 1$ and it is easy to see that there exists unique (a, b, c) such that relations (3.27)–(3.30) are true. We have only to mention that relations (3.29) and (3.30) are not in contradiction.[5] Hence we have that u in (3.22) is a normal extremal control.

[4] It is easy to see that, using (3.24), relation (3.16) for $t = \tau$ doesn't give new information.
[5] We have

$$\frac{1}{k\alpha}\left(-\frac{g}{v_0}T + 1\right)\left(\frac{\alpha}{m(T)} - g\right) - \frac{1}{k(M+F)}\left(-\frac{g}{v_0}\tau + 1\right) - \alpha\int_\tau^T \frac{-\frac{g}{v_0}s + 1}{(m(s))^2}\, \mathrm{d}s =$$
$$= \frac{1}{k}\left[\left(-\frac{g}{v_0}T + 1\right)\left(\frac{1}{m(T)} - \frac{g}{\alpha}\right) - \frac{1}{M+F}\left(-\frac{g}{v_0}\tau + 1\right) - \int_\tau^T \left(-\frac{g}{v_0}s + 1\right)\frac{-\dot m(s)}{(m(s))^2}\, \mathrm{d}s\right]$$
$$= \frac{1}{k}\left[-\frac{g}{\alpha}\left(-\frac{g}{v_0}T + 1\right) + \int_\tau^T -\frac{g}{v_0}\frac{1}{m(s)}\, \mathrm{d}s\right]$$

Now, we can apply a generic result of the existence of an optimal control (see Theorem 3.8). We know, by the previous construction, that there exists at least an admissible control u with exit time T; hence it is reasonable to restrict the original target set $\mathcal{T} = [0,\infty) \times \{(0,0)\} \times [0,\infty)$ to the new set $\mathcal{T} = [0,T] \times \{(0,0)\} \times [0,\infty)$. Moreover, we have a compact control set $[0,\alpha]$ and for the dynamics we have the bounded condition

$$
\begin{aligned}
\left| \begin{pmatrix} \dot h \\ \dot v \\ \dot m \end{pmatrix} \right| &= \left| \begin{pmatrix} v \\ u/m - g \\ -ku \end{pmatrix} \right| \\
&\le \sqrt{v^2 + \frac{\alpha^2}{M^2} + g^2 + k^2\alpha^2} \\
&\le \sqrt{\frac{\alpha^2}{M^2} + g^2 + k^2\alpha^2} + |v| \\
&\le (1+C)(1 + |(h,v,m)'|);
\end{aligned}
$$

finally, for every (t,h,v,m) we have that

$$
F_{(t,h,v,m)} = \Big\{ (y_1,y_2,y_3,z) : \ y_1 = v, \ y_2 = \frac{u}{m} - g, \ y_3 = -ku, \ z \le 0, \ u \in [0,\alpha] \Big\}
$$

is a convex set. Hence Theorem 3.8 guarantees that the optimal control exists.

3.3 The Bolza problem in Calculus of Variations.

Let us consider the problem

$$
\begin{cases}
J(\mathbf{x}) = \displaystyle\int_{t_0}^{T} f(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\,\mathrm{d}t + \psi(T,\mathbf{x}(T)) \\
\mathbf{x}(t_0) = \boldsymbol{\alpha} \\
\mathbf{x}(T) = \boldsymbol{\beta}_T \\
\displaystyle\max_{\mathbf{x}\in\mathcal{A}_B} J(\mathbf{x}) \\
\mathcal{A}_B = \{\mathbf{x} : [t_0,\infty) \to \mathbb{R}^n, \ \mathbf{x} \in \mathbf{C}^1, \ \mathbf{x}(t_0) = \boldsymbol{\alpha}\}
\end{cases}
\tag{3.31}
$$

where t_0 is fixed, and $T > t_0$, $\boldsymbol{\beta}_T \in \mathbb{R}^n$ are free. We call this problem *Bolza problem of calculus of variation*. Clearly (3.31) is a particular case of (3.5), but let us provide the necessary condition for this particular situation: hence, let us apply theorem 3.3 to our situation.
Since $\mathbf{u} = \dot{\mathbf{x}}$ and taking into account that in the case of Calculus of Variation it is possible to prove that $\lambda_0^* = 0$, we have $H = f(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda} u$: hence, as in

$$
\begin{aligned}
&= -\frac{g}{kv_0\alpha}\left[-gT + v_0 + \int_\tau^T \frac{\alpha}{m(s)}\,\mathrm{d}s\right] \\
&= -\frac{g}{kv_0\alpha}\left[-gT + v_0 + \int_\tau^T (\dot v + g)\,\mathrm{d}s\right] \\
&= 0
\end{aligned}
$$

(2.39) and (2.40), we have

$$(\mathrm{PMP}_0) \ \Rightarrow\ \nabla_{\mathbf{u}} f + \boldsymbol{\lambda}^* = \mathbf{0}$$
$$(\mathrm{AE}) \ \Rightarrow\ \nabla_{\mathbf{x}} f = -\dot{\boldsymbol{\lambda}}^*$$
$$(iv_{T^*}) \ \Rightarrow\ \boldsymbol{\lambda}^* = \nabla_{\mathbf{x}}\psi \ \Rightarrow\ f + \boldsymbol{\lambda}^*\dot{\mathbf{x}} + \frac{\partial\psi}{\partial t} = 0 \text{ in } t = T^*.$$

More precisely we have, from theorems 3.1 and 3.3, the following result:

Theorem 3.5. *Let us consider (3.31) with $f \in C^2([t_0,t_1]\times\mathbb{R}^{2n})$ and $\psi \in C^1([t_0,t_1]\times\mathbb{R}^n)$. Let $\mathbf{x}^*$ be an optimal solution with exit time T^*. Then $\mathbf{x}^*$ is extremal (i.e. satisfies EU). Moreover*

i) if T is fixed and $\boldsymbol{\beta}_T$ is free, then

$$\nabla_{\dot{\mathbf{x}}} f(T^*,\mathbf{x}^*(T^*),\dot{\mathbf{x}}^*(T^*)) + \nabla_{\mathbf{x}}\psi(T^*,\mathbf{x}^*(T^*)) = 0; \tag{3.32}$$

ii) if T is free and $\boldsymbol{\beta}_T$ is fixed, then

$$f(T^*,\mathbf{x}^*(T^*),\dot{\mathbf{x}}^*(T^*)) - \dot{\mathbf{x}}^*\cdot(T^*)\nabla_{\dot{\mathbf{x}}} f(T^*,\mathbf{x}^*(T^*),\dot{\mathbf{x}}^*(T^*)) + \frac{\partial\psi}{\partial t}(T^*,\mathbf{x}^*(T^*)) = 0; \tag{3.33}$$

iii) if T and $\boldsymbol{\beta}_T$ are free, then we have (3.32) and (3.33).

As we mention in remark 3.4, the problem to guarantees some sufficient condition is delicate when T is free. In the next example, with fixed final time, we prove directly that a extremal is really an optimal solution.

Example 3.3.1. Let us consider

$$\begin{cases} \min \displaystyle\int_0^1 (\dot{x}^2 - x)\,\mathrm{d}t + x^2(1) \\ x(0) = 0 \end{cases}$$

The solution of EU is $x(t) = -\frac{1}{4}t^2 + at + b$, with $a, b \in \mathbb{R}$. The initial condition implies $b = 0$. Since $T^* = 1$ is fixed, form (3.32) we have

$$2\dot{x}(1) + 2x(1) = 0 \quad \Rightarrow \quad a = 3/8.$$

Hence the extremal is $x^* = -t^2/4 + 3t/8$. Now, let us show that is a minimum. Let $h \in C^1([0,1])$ be such that $h(0) = 0$ and let $x = x^* + h$. Then

$$\begin{aligned} \int_0^1 (\dot{x}^2 - x)\,\mathrm{d}t + x^2(1) &= \int_0^1 (\dot{x^*}^2 + 2\dot{h}\dot{x^*} + \dot{h}^2 - x^* - h)\,\mathrm{d}t + x^{*2}(1) + 2x^*(1)h(1) + h^2(1) \\ &\geq \int_0^1 (\dot{x^*}^2 - x^*)\,\mathrm{d}t + x^{*2}(1) + \int_0^1 (2\dot{h}\dot{x^*} - h)\,\mathrm{d}t + 2x^*(1)h(1). \end{aligned}$$

Since $h(0) = 0$, $\ddot{x^*}(t) = -1/2$, $x^*(1) = 1/8$ and $\dot{x^*}(1) = -1/8$, we have

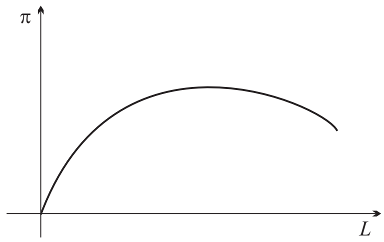
$$\begin{aligned} \int_0^1 (2\dot{h}\dot{x^*} - h)\,\mathrm{d}t + 2x^*(1)h(1) &= \Big(2h\dot{x^*}\Big]_0^1 - \int_0^1 (2h\ddot{x^*} + h)\,\mathrm{d}t + \frac{h(1)}{4} \\ &= -\frac{h(1)}{4} - \int_0^1 (-h + h)\,\mathrm{d}t + \frac{h(1)}{4} \\ &= 0. \end{aligned}$$

The previous inequality implies that x^* is a minimum. △

3.3.1 Labor adjustment model of Hamermesh.

Consider a firm that has decided to raise its labor input from L_0 to a yet undetermined level L_T after encountering a wage reduction at the initial time $t_0 = 0$. The adjustment of labor input is assumed to entail a cost C that varies, at every time, with $L'(t)$, the rate of change of L. Thus the firm has to decide on the best speed of adjustment toward L_T as well as the magnitude of L_T itself. This is the essence of the labor adjustment problem discussed in a paper by Hamermesh.

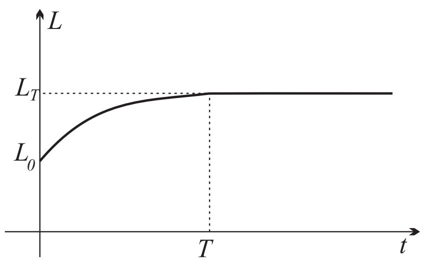
We assume that the profit of the firm be expressed by a general function $\pi(L)$, with $\pi''(L) < 0$. The labor input is taken to be the unique determinant of profit because we have subsumed all aspects of production and demand in the profit function. The cost of adjusting L is assumed to be



$$C(L') = bL'^2 + c,$$

with b and c positive constants. Thus the net profit at any time is $\pi(L) - C(L')$. The problem of the firm is to maximize the total net profit over time during the process of changing the labor input.

Inasmuch as it must choose not only the optimal L_T, but also an optimal time T^* for completing the adjustment, we have both the terminal state and terminal time free. Another feature to note about the problem is that we should include not only the net profit from $t = 0$ to $t = T^*$, but also the capitalized value of the profit in the post T^* period, which is affected by the choice of L_T and T, too.



Since the profit rate at time is $\pi(L_T)$, its present value is $\pi(L_T)e^{-\rho t}$, where $\rho > 0$ is the given discount rate. So the capitalized value of that present value is,

$$\int_T^{\infty} \pi(L)e^{-\rho t}\,\mathrm{d}t = \left(-\frac{\pi(L_T)}{\rho}e^{-\rho t}\right]_T^{\infty} = \frac{\pi(L_T)}{\rho}e^{-\rho T}.$$

Hence the problem is

$$\begin{cases} \max\limits_{L} \displaystyle\int_0^T \left(\pi(L) - bL'^2 - c\right) e^{-\rho t}\,\mathrm{d}t + \frac{\pi(L_T)}{\rho}e^{-\rho T} \\ L(0) = L_0 \\ L(T) = L_T \end{cases}$$

where T and L_T are free, $T > 0,\ L_T > L_0$.

Let us set $f(t, L, L') = \left(\pi(L) - bL'^2 - c\right) e^{-\rho t}$; EU gives us

$$L'' - \rho L' = -\frac{\pi'(L)}{2b}. \tag{3.34}$$

Conditions (3.32) and (3.33) imply

$$L'(T) - \frac{1}{2b\rho}\pi'(L_T) = 0 \tag{3.35}$$

$$bL'(T)^2 - c = 0 \tag{3.36}$$

Using (3.35), recalling that $L_T > L_0$ and hence $L' > 0$,

$$L'(T) = \sqrt{\frac{c}{b}}. \tag{3.37}$$

Now, equation (3.36) is

$$\pi'(L_T) = 2\rho\sqrt{bc}. \tag{3.38}$$

Now, in order to solve (3.34), let us specified the function π. We suppose that

$$\pi(L) = 2mL - nL^2, \quad \text{con} \quad 0 < n < m.$$

Clearly (3.34) implies

$$L'' - \rho L' - \frac{n}{b}L = -\frac{m}{b},$$

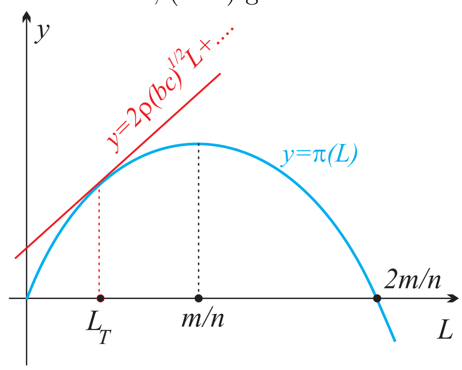
and the general solution is

$$L = \alpha e^{\left(\rho+\sqrt{\rho^2+4n/b}\right)t/2} + \beta e^{\left(\rho-\sqrt{\rho^2+4n/b}\right)t/2} + \frac{m}{n},$$

with α and β generic constants: the initial condition $L(0) = L_0$, (3.37) and (3.38) alow us to determinate α, β and T. Moreover, (3.38) gives

$$L_T = \frac{m}{n} - \frac{\rho\sqrt{bc}}{n};$$

clearly $L_T < m/n$. This tells us that the level of employment L_T to be achieved at the end of the corporate reorganization is below the level m/n, the level at which profit is maximized when we have zero cost (i.e. with $b = c = 0$).



3.4 Existence and controllability results.

In this section, we first discuss some examples

Example 3.4.1. *Let us consider the calculus of variation problem*

$$\begin{cases} J(x) = \int_0^1 t\dot{x}^2\, dt \\ x(0) = 1 \\ x(1) = 0 \\ \inf_x J(x) \end{cases}$$

Clearly $J(x) \geq 0$, for every x. Moreover let us consider the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by $x_n(t) = 1 - t^{1/n}$. It is easy to see that

$$J(x_n) = \frac{1}{n^2}\int_0^1 t^{2/n-1}\, dt = \frac{1}{2n}$$

and hence $\{x_n\}$ *is a minimizing sequence that guarantees* $\min_x J(x) = 0$. *Moreover* $x_n \to x = 0$ *and it is to see that* $J(x) = 0$ *gives* $\dot{x} = 0$, *in contradiction with the initial and the final condition on* x. *Hence the problem has no optimal solution.* △

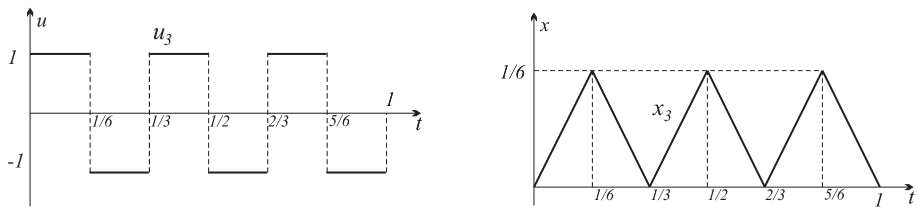
Example 3.4.2. **(Bolza)** *Let us consider this problem due to Bolza:*

$$\begin{cases} J(u) = \int_0^1 (x^2 + (1 - u^2)^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \\ x(1) = 0 \\ \inf_u J(u) \end{cases}$$

For every $n \in \mathbb{N}$, *we define the control* u_n *as*

$$u_n(t) = 1, \qquad \text{if } \frac{2i-2}{2n} < t < \frac{2i-1}{2n} \quad \text{for some } i = 1, \dots, n$$
$$u_n(t) = -1, \qquad \text{if } \frac{2i-1}{2n} < t < \frac{2i}{2n} \quad \text{for some } i = 1, \dots, n$$

We obtain



The control u_n *and its trajectory* x_n *in the case* $n = 3$.

An easy calculation gives

$$J(u_n) = 2n \int_0^{1/2n} x_n^2\,\mathrm{d}t = \frac{1}{12n^2};$$

hence $\lim_{n\to\infty} J(u_n) = 0 = \inf_u J(u)$, *since* $J(u) \ge 0$. *Again it is easy to see that the optimal control does not exist.* △

Example 3.4.3. *We consider*

$$\begin{cases} J(u) = \int_0^1 x^2\,\mathrm{d}t \\ \dot{x} = ux \\ x(0) = 1 \\ x(1) = 10 \\ 0 \le u \le 1 \\ \max_u J(u) \end{cases}$$

For every u, *with* $0 \le u \le 1$, *the dynamics gives* $\dot{x} \le x$: *the Gronwall's inequality in theorem 3.6 implies* $x(t) \le e^t \le e < 10$ *for* $t \in [0, 1]$. *Hence the class of admissible control is empty.* △

Example 3.4.4. *Let us consider a little variation of example 3.5.2:*

$$\begin{cases} \min T \\ \dot{x} = x + u \\ x(0) = 5 \\ x(T) = 0 \\ |u| \le 1 \end{cases}$$

where T *is free. For every* u, *with* $-1 \le u \le 1$, *the dynamics gives* $\dot{x} \ge x - 1$: *if we define the function* $y(t) = 1 - x(t)$, *the Gronwall's inequality*[6] *implies, for* $t \in [0, 1]$,

$$y'(t) \le y(t) \quad \Rightarrow \quad y(t) \le y(0)e^t = -4e^t \quad \Rightarrow \quad x(t) \ge 4e^t + 1.$$

Hence $x(T) = 0$ *is impossible and the class of admissible control is empty.* △

[6] We recall that

The previous examples show that we have to discuss two different questions:

A. the problem to guarantee that the set of admissible control $\mathcal{C}_{t_0,\boldsymbol{\alpha}}$ is non empty, the so called controllability problem;

B. the problem to guarantee that in $\mathcal{C}_{t_0,\boldsymbol{\alpha}}$ there exists a control such that realizes our sup (or inf);

First, let us discuss, essentially as in [13], the question **B.** (another interesting book is [26], see chapter 2). Let us consider the **minimum problem**

$$\begin{cases} \min_{\mathbf{u}\in\mathcal{C}_{t_0,\boldsymbol{\alpha}}} \int_{t_0}^T f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \psi(T,\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ (T,\mathbf{x}(T)) \in \mathcal{T} \\ \mathcal{C}_{t_0,\boldsymbol{\alpha}} = \Big\{ \mathbf{u} : [t_0,T] \to U \subset \mathbb{R}^k,\ \mathbf{u} \text{ measurable s.t.} \\ \qquad\qquad \exists!\ \mathbf{x} :\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}),\ \mathbf{x}(t_0) = \boldsymbol{\alpha},\ (T,\mathbf{x}(T)) \in \mathcal{T} \Big\} \end{cases} \tag{3.39}$$

with a control set $U \subset \mathbb{R}^k$, and with target set $\mathcal{T} \subset (t_0,\infty) \times \mathbb{R}^n$.
In all that follows, a fundamental role is played by the set $F_{(t,\mathbf{x})}$ defined, for every $(t,\mathbf{x}) \in \mathbb{R}^{n+1}$, as

$$F_{(t,\mathbf{x})} = \Big\{ (\mathbf{y},z) \in \mathbb{R}^n \times \mathbb{R} :\ \mathbf{y} = g(t,\mathbf{x},\mathbf{u}),\ z \geq f(t,\mathbf{x},\mathbf{u}),\ \mathbf{u} \in U \Big\} \tag{3.40}$$

The next results guarantee the existence of an optimal control in the set $\mathcal{C}'_{t_0,\boldsymbol{\alpha}}$ of Lebesgue-integrable function of $\mathcal{C}_{t_0,\boldsymbol{\alpha}}$. We have this first existence result

Theorem 3.7 (control set U **closed**). *Let us consider the minimum problem (3.39) with f, g and ψ continuous functions. We assume that there exists at least an admissible control, i.e. $\mathcal{C}_{t_0,\boldsymbol{\alpha}} \neq \emptyset$.*
Moreover let us suppose that

Theorem 3.6 (Gronwall's inequality). *Let y and α be a differentiable function and a continuous function respectively in $[a,b] \subset \mathbb{R}$ such that*

$$y'(t) \leq \alpha(t)y(t), \qquad \forall t \in [a,b].$$

Then

$$y(t) \leq y(a)\exp\left(\int_a^t \alpha(s)\,\mathrm{d}s\right), \qquad \forall t \in [a,b].$$

Proof. Let us define the function $v : [a,b] \to \mathbb{R}$ by

$$v(t) = \exp\left(\int_a^t \alpha(s)\,\mathrm{d}s\right).$$

Clearly we have $v' = \alpha v,\ v(a) = 1$ and $v > 0$. Hence we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y(t)}{v(t)}\right) = \frac{y'(t)v(t) - y(t)v'(t)}{v^2(t)} \leq 0$$

which implies

$$\frac{y(t)}{v(t)} \leq \frac{y(a)}{v(a)} = y(a), \qquad \forall t \in [a,b].$$

□

a. *the control set U is closed;*

b. *there exist two positive constants c_1 and c_2 such that, for every $t \in [t_0, \infty)$, $\mathbf{x}$, $\mathbf{x}' \in \mathbb{R}^n$, $\mathbf{u} \in U$, we have*

$$\|g(t, \mathbf{x}, \mathbf{u})\| \leq c_1(1 + \|\mathbf{x}\| + \|\mathbf{u}\|),$$
$$\|g(t, \mathbf{x}', \mathbf{u}) - g(t, \mathbf{x}, \mathbf{u})\| \leq c_2\|\mathbf{x}' - \mathbf{x}\|(1 + \|\mathbf{u}\|);$$

c. *the target set $\mathcal{T}$ is compact;*

d. *the set $F_{(t,\mathbf{x})}$ is convex for every $(t, \mathbf{x}) \in [t_0, \infty) \times \mathbb{R}^n$;*

f. *f is superlinear with respect the variable $\mathbf{u}$, i.e. for every $(t, \mathbf{x})$ fixed we have*

$$\lim_{\mathbf{u} \in U,\, \|\mathbf{u}\| \to \infty} \frac{f(t, \mathbf{x}, \mathbf{u})}{\|\mathbf{u}\|} = \infty. \tag{3.41}$$

Then there exists an optimal control $\mathbf{u}^* \in \mathcal{C}'_{t_0,\boldsymbol{\alpha}}$ for the minimum problem.[7]

A proof can be found in [13]. The theorem does not guarantee the existence of a piecewise continuous optimal control, it only ensure the existence of a Lebesgue-integrable optimal control. However, the risk involved in assuming that the optimal control, whose existence is ensured by theorem 3.7, is piecewise continuous are small indeed (see for example section 6 of chapter III in [13] for continuity properties of optimal control). Now, let us spend few words to comment the previous assumptions:

a. if the control set U is compact, it is possible to relax some of the other assumptions as we will see in Theorem 3.8;

b. sure assumption *b.* holds for linear dynamics, i.e. with $g(t, \mathbf{x}, \mathbf{u}) = a\mathbf{x} + b\mathbf{u}$ where a and b are continuous functions;

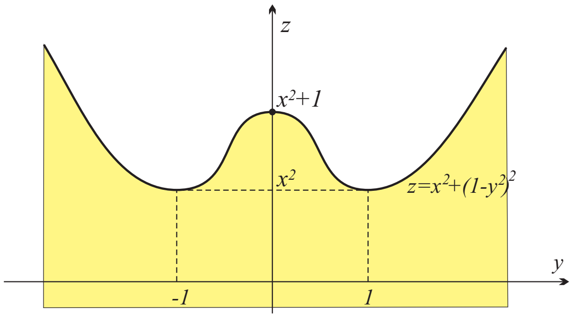
d. if the control set U is convex, the function $\mathbf{u} \mapsto f(t, \mathbf{x}, \mathbf{u})$ is convex and the dynamics is linear in $\mathbf{u}$, then assumption *d.* holds;

With the next example, we reserve a particular attention for the assumption *d.*

Example 3.4.2 [2^{nd} part]. *Consider the example 3.4.2: first we note that the dynamic is linear, the control set $U = \mathbb{R}$ is closed, the target set $\mathcal{T} = \{(1, 0)\}$ is compact, the function f is superlinear: hence all the assumptions (except* d.*) hold. For every $(t, x) \in [0, 1] \times \mathbb{R}$, we consider the set $F_{(t,x)} \subset \mathbb{R}^2$ defined as*

$$F_{(t,x)} = \left\{ (y, z) \in \mathbb{R}^2 : \ z \geq x^2 + (1 - y^2)^2, \ y = u \in U = \mathbb{R} \right\}$$

[7] If we are interested on a **maximum problem**, since $\max f = -\min(-f)$ in this theorem we have to replace "$\geq$"with "$\leq$"in the definition (3.40) of the set $F_{(t,\mathbf{x})}$ and "∞"with "$-\infty$"in the limit (3.41): in such situation exists a optimal maximum control.



The set $F_{(t,x)}$ is yellow.

Clearly $F_{(t,x)}$ *is not convex.* △

The role of the convexity of the set $F_{(t,\mathbf{x})}$ in assumption *d.* arrives form the theory of differential inclusion that allow us to construct a solution of the dynamic.

Our second result require the boundedness of the control set and weakens the assumption on target set $\mathcal{T}$:

Theorem 3.8 (control set U **compact**). *Let us consider the minimum problem (3.39) with* f, g *and* ψ *continuous functions. We assume that there exists at least an admissible control, i.e.* $\mathcal{C}_{t_0,\boldsymbol{\alpha}} \neq \emptyset$.
Moreover let us suppose that

a'. the control set U *is compact;*

b'. there exist a positive constant c_3 *such that, for every* $t \in [t_0, T_2]$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in U$, *we have*

$$\|g(t, \mathbf{x}, \mathbf{u})\| \leq c_3(1 + \|\mathbf{x}\|);$$

c'. the target set $\mathcal{T}$ *is closed and* $\mathcal{T} \subset [T_1, T_2] \times \mathbb{R}^n$ *for some* $t_0 \leq T_1 \leq T_2 < \infty$.

d. the set $F_{(t,\mathbf{x})}$ *is convex for every* $(t, \mathbf{x}) \in [t_0, T_2] \times \mathbb{R}^n$.

Then there exists an optimal control in $\mathbf{u}^* \in \mathcal{C}'_{t_0,\boldsymbol{\alpha}}$ *for the minimum problem.*[8]

A proof can be found in [26]. Note that assumptions *d.* in Theorem 3.7 and in Theorem 3.8 are the same.

3.4.1 Linear time optimal problems

Now we pass to discuss the controllability problem in **A.**. Since this is a very large ad hard problem and exhaustive discussion is not in the aim of this note, we give only some idea in the particular case of linear optimal time problem, i.e.

$$\begin{cases} \min T \\ \dot{\mathbf{x}} = M\mathbf{x} + N\mathbf{u} \\ \mathbf{x}(0) = \boldsymbol{\alpha} \\ \mathbf{x}(T) = \mathbf{0} \\ \mathbf{u} \in [-1, 1]^k \end{cases} \tag{3.42}$$

[8] If we are interested on a **maximum problem**, since $\max f = -\min(-f)$ in this theorem we have to replace "≥"with "≤"in the definition (3.40) of the set $F_{(t,\mathbf{x})}$: in such situation exists a optimal maximum control.

where T is free, M and N are $n \times n$ and $n \times k$ matrices with constant coefficients. Note that the classical example of Pontryagin since the dynamics in (1.5) is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \tag{3.43}$$

We will solve (1.5) in subsection 3.5.1.

We recall (see subsection 1.2.1) that for every $T \geq t_0$, we define the *reachable set at time* T as the set $R(T, t_0, \boldsymbol{\alpha}) \subseteq \mathbb{R}^n$ of the points $\mathbf{x}$ such that there exists an admissible control $\mathbf{u}$ and an associated trajectory $\mathbf{x}$ such that $\mathbf{x}(t_0) = \boldsymbol{\alpha}$ and $\mathbf{x}(T) = \mathbf{x}$. Moreover we define

$$R(t_0, \boldsymbol{\alpha}) = \bigcup_{T \geq t_0} R(T, t_0, \boldsymbol{\alpha})$$

as the *reachable set* from $\boldsymbol{\alpha}$.
Hence, our problem of controllability for the problem (3.42) is to guarantee that

$$\mathbf{0} \in R(0, \boldsymbol{\alpha}). \tag{3.44}$$

We say that (3.42) is *controllable* if for every $\boldsymbol{\alpha} \in \mathbb{R}^n$ we have that (3.44) holds. This problem is well exposed in [22] and in [12].

Starting from the linear ODE and the initial condition in (3.42), we have that the solution is

$$\mathbf{x}(t) = e^{tM} \left(\boldsymbol{\alpha} + \int_0^t e^{-sM} N \mathbf{u}(s) \, \mathrm{d}s \right),$$

where, as usual,

$$e^{tM} = \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}.$$

Clearly

$$\mathbf{0} \in R(T, 0, \boldsymbol{\alpha}) \quad \Leftrightarrow \quad -\boldsymbol{\alpha} = \int_0^t e^{-sM} N \mathbf{u}(s) \, \mathrm{d}s.$$

It is clear that the possibility to solve the previous problem is strictly connected with the properties of the matrices M and N; we define the *controllability matrix* $G(M, N)$ for (3.44) as the $n \times kn$-matrix

$$G(M, N) = \left[N, MN, M^2 N, \dots M^{n-1} N \right].$$

We have the following result (see for example, theorem 2.6 and 3.1 in [12]):

Theorem 3.9 (controllability for linear time optimal problem)**.** *Let us consider the problem (3.44). Let us suppose* $\operatorname{rank} G(M, N) = n$ *and* $\operatorname{Re} \theta \leq 0$ *for each eigenvalue* θ *of the matrix* M*. Then*

i. the ODE (3.44) is controllable,

ii. there exists an optimal control for the problem (3.44).

3.5 Time optimal problem

A particular case of a free final time problem in (3.5) is the following

$$\begin{cases} \min_{\mathbf{u}\in\mathcal{C}} T \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \\ \mathbf{x}(T) = \boldsymbol{\beta} \end{cases} \tag{3.45}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are fixed points in $\mathbb{R}^n$, and T is non negative and free. Hence (3.45) is the problem to transfers in the shortest possible time $\boldsymbol{\alpha}$ in $\boldsymbol{\beta}$: such problem is called *time optimal problem*: its solution has a optimal time T^*. Since $T = \int_0^T 1\,\mathrm{d}t$, we define the Hamiltonian as

$$H(t,\mathbf{x},\mathbf{u},\lambda_0,\boldsymbol{\lambda}) = \lambda_0 + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{u}) \tag{3.46}$$

and we have the following result (see for example [27], page 614):

Theorem 3.10. *Let us consider the problem (3.45) with $g \in C^1([t_0,t_1]\times\mathbb{R}^{n+k})$. Let $\mathbf{u}^*$ be optimal control with exit time T^* and $\mathbf{x}^*$ be the associated trajectory.*

Then there exists a multiplier $(\lambda_0^*, \boldsymbol{\lambda}^*)$, with

- ◇ $\lambda_0^* \geq 0$ *constant,*
- ◇ $\boldsymbol{\lambda}^* : [t_0,t_1] \to \mathbb{R}^n$ *continuous,*

such that

i) the nontriviality of the multiplier holds;

ii) the Pontryagin Minimum Principle (2.2) holds,

iii) the adjoint equation (2.3) holds,

iv_{T^*}) the transversality condition is given by

$$H(T^*,\mathbf{x}^*(T^*),\mathbf{u}^*(T^*),\lambda_0^*,\boldsymbol{\lambda}^*(T^*)) = 0, \tag{3.47}$$

In the particular case of an autonomous time optimal problem, i.e. with $g = g(\mathbf{x},\mathbf{u})$ in (3.45), condition (3.47) holds in $[0,T^*]$; more precisely by (3.7) we have

$$H(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda_0^*,\boldsymbol{\lambda}^*(t)) = 0 \qquad t\in[t_0,T^*]. \tag{3.48}$$

Since a time optimal problem is a variable time optimal problem, it is hard to give sufficient conditions of optimality (see remark 3.4). In the next examples we propose two different approach to guarantee that an extremal is really an optimal solution: we will use a Gronwall inequality or we will apply some existence result about the optimal control.

Example 3.5.1. Let us consider

$$\begin{cases} \min T \\ \dot{x} = 2x + \frac{1}{u} \\ x(0) = \frac{5}{6} \\ x(T) = 2 \\ 3 \leq u \leq 5 \end{cases}$$

where T is free. The Hamiltonian is $H = \lambda_0 + \lambda\left(2x + \frac{1}{u}\right)$. The necessary conditions give

$$\text{PMP} \quad \Rightarrow \quad u(t) \in \arg\min_{3\le v\le 5} \frac{\lambda}{v} \quad \Rightarrow \quad u(t) = \begin{cases} 5 & \text{if } \lambda > 0 \\ ? & \text{if } \lambda = 0 \\ 3 & \text{if } \lambda < 0 \end{cases} \tag{3.49}$$

$$\dot\lambda = -\frac{\partial H}{\partial x} \quad \Rightarrow \quad \dot\lambda = -2\lambda \quad \Rightarrow \quad \lambda(t) = Ae^{-2t}$$

$$(3.47) \quad \Rightarrow \quad \lambda_0 + \lambda(T)\left(2x(T) + \frac{1}{u(T)}\right) = \lambda_0 + \lambda(T)\left(4 + \frac{1}{u(T)}\right) = 0 \tag{3.50}$$

for some constant A. Note that $\lambda_0 = 0$, since $u(T) \ge 3$, implies in (3.50) that $\lambda(T) = 0$: relation $\lambda(t) = Ae^{-2t}$ gives $A = 0$ and hence $(\lambda_0, \lambda) = (0, 0)$ that is impossible.
Hence we put $\lambda_0 = 1$. Since $u(T) \ge 3$, (3.50) imply $\lambda(T) < 0$, i.e. $A < 0$. Now, (3.49) implies $u^*(t) = 3$ in $[0, T]$: the dynamics now is $\dot x = 2x + 1/3$ and with the initial condition give $x^*(t) = e^{2t} - 1/6$. The final condition $x^*(T) = 2$ implies that $T^* = \frac{1}{2}\ln\frac{13}{6}$.

In order to guarantee that T^* is really the optimal time, we have two possibilities: the first is to provide an existence result for this problem; the second is to use the Gronwall's inequality (see Example 3.5.2).

• *Using existence result: we would like to use Theorem 3.8. First, we note that there exists at least an admissible control, the control set $[1, 3]$ is compact, the dynamics is linear. We remark that we are in the position to restrict our attention to a "new" final condition of the type $(T, x(T)) \in \mathcal{T} = [T^* - \epsilon, T^* + \epsilon] \times \{2\}$, for some fixed $\epsilon > 0$: it is clear that such modification of the target set is irrelevant and the "new" optimal solution coincides with the "old" one. Now the new target set satisfies assumption* c'.
Finally, let us check condition d.*: recalling that $T = \int_0^T 1\,\mathrm{d}t$, we consider the set $F_{(t,x)} \subset \mathbb{R}^2$ defined as*

$$F_{(t,x)} = \left\{(y, z) \in \mathbb{R}^2 : \ z \ge 1, \ y = 2x + \frac{1}{u}, \ 3 \le u \le 5\right\} = \left[2x + \frac{1}{5}, 2x + \frac{1}{3}\right] \times [1, \infty).$$

Clearly $F_{(t,x)}$ is convex, for every (t, x). Hence there exists an optimal control and this proves that T^* is optimal.

•• *Using a Gronwall's inequality: For every u, the dynamics gives $\dot x \ge 2x + 1/3$: if we define the function $y(t) = 2x(t) + 1/3$, the Gronwall's inequality in theorem 3.6 implies*

$$y'(t) \le 2y(t) \quad \Rightarrow \quad y(t) \le y(0)e^{2t} \quad \Rightarrow \quad x(t) \le e^{2t} - \frac{1}{6},$$

for every trajectory x; hence $x(T) = 2$ for some $T \ge T^*$: this proves that T^* is optimal. △

Example 3.5.2. *Let us consider*

$$\begin{cases} \min T \\ \dot x = x + u \\ x(0) = 5 \\ x(T) = 11 \\ |u| \le 1 \end{cases}$$

where T is free. The Hamiltonian is $H = \lambda_0 + \lambda(x + u)$. The necessary conditions give

$$\text{PMP} \quad \Rightarrow \quad u(t) \in \arg\min_{|v|\le 1} [\lambda_0 + \lambda(x + v)] \quad \Rightarrow \quad u(t) \in \arg\min_{|v|\le 1} \lambda v \tag{3.51}$$

$$\dot\lambda = -\frac{\partial H}{\partial x} \quad \Rightarrow \quad \dot\lambda = -\lambda \quad \Rightarrow \quad \lambda(t) = Ae^{-t}$$

$$(3.47) \quad \Rightarrow \quad \lambda_0 + \lambda(T)(x(T) + u(T)) = \lambda_0 + \lambda(T)(11 + u(T)) = 0 \tag{3.52}$$

for some constant A. If $\lambda_0 = 0$, since $|u| \le 1$ we have by (3.52) that $\lambda(T) = 0$; the equation $\lambda(t) = Ae^{-t}$ implies $A = 0$ and hence $(\lambda_0, \lambda) = (0, 0)$: this is impossible.
Hence we put $\lambda_0 = 1$. It is easy to see that $|u| \le 1$ and (3.52) imply $\lambda(T) < 0$, i.e. $A < 0$. Now, (3.51) implies $u(t) = 1$ in $[0, T]$: the dynamics now is $\dot x = x + 1$ and with the initial condition $x(0) = 5$ give

$$x(t) = 6e^t - 1.$$

The final condition $x(T) = 11$ implies that $T = \ln 2$. Now condition (3.52) gives

$$1 + Ae^{-\ln 2}(11 + 1) = 0 \quad \Rightarrow \quad A = -\frac{1}{6}.$$

In order to guarantee $T^* = \ln 2$ is really the optimal time (with optimal control $u^* = 3$ and optimal trajectory $x^* = 6e^t - 1$), we have two possibilities: the first is to provide an existence result for this problem (linear time optimal problem); the second is to use the Gronwall's inequality.

• *Using existence result: first, we note that there exists at least an admissible control, the control set is $[-1, 1]$ is compact, the dynamics is linear. The matrices M, $N, G(M, N)$ are all 1×1 and equal to 1. Theorem 3.9. guarantees that there exists a optimal control.*

•• *Using a Gromwall's inequality (see Thereom 3.6): the dynamics and the control set imply that for every admissible control u its associated trajectory x satisfies*

$$\dot{x} = x + u \leq x + 1.$$

Let us define $y(t) = x(t) + 1$; using the initial condition on the trajectory and the Gronwall's inequality

$$y' \leq y \quad \Rightarrow \quad y(t) \leq y(0)e^t = 6e^t \quad \Rightarrow \quad x(t) \leq 6e^t - 1 = x^*(t)$$

for every trajectory x: hence we have $x(T) = 11$ for some $T \geq T^*$. This proves that T^* is optimal. △

3.5.1 The classical example of Pontryagin and its boat

We consider the problem (1.5) and we put $\dot{x} = x_1,\ x = x_2$: we obtain

$$\begin{cases} \min_u T \\ \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ x_1(0) = v_0, \qquad x_2(0) = d_0 \\ x_1(T) = x_2(T) = 0 \\ |u| \leq 1 \end{cases} \tag{3.53}$$

where d_0 and v_0 are generic fixed constants, T is positive and free.
The Hamiltonian is $H(t, x_1, x_2, u, \lambda_0, \lambda_1, \lambda_2) = \lambda_0 + \lambda_1 u + \lambda_2 x_1$ and the necessary conditions of Theorem 3.10 give

$$\text{PMP} \quad \Rightarrow \quad u \in \arg\min_{v \in [-1,1]} \lambda_1 v \tag{3.54}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -\lambda_2 \tag{3.55}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = 0 \tag{3.56}$$

$$(3.48) \quad \Rightarrow \quad \lambda_0 + \lambda_1(t)u(t) + \lambda_2(t)x_1(t) = 0. \tag{3.57}$$

An easy computations by (3.55) and (3.56) give

$$\lambda_2 = a, \qquad \lambda_1 = -at + b, \tag{3.58}$$

where a and b are constants. From PMP in (3.54) we have

$$u(t) = \begin{cases} -1 & \text{if } \lambda_1(t) > 0, \\ 1 & \text{if } \lambda_1(t) < 0, \\ ? & \text{if } \lambda_1(t) = 0. \end{cases}$$

First, let us prove that $a = b = 0$ is impossible. In order to do that, by (3.58) we obtain in (3.57)

$$\lambda_0 + \lambda_1(t)u(t) + \lambda_2(t)x_1(t) = \lambda_0 = 0, \forall t.$$

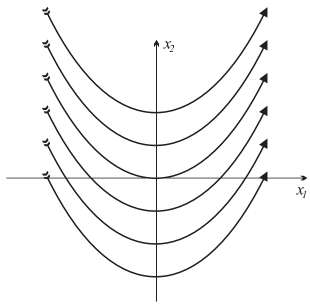
We obtain $(\lambda_0, \lambda_1, \lambda_2) = (0, 0, 0)$: this contradicts Theorem 3.10.
Hence $a = b = 0$ is impossible and λ_1 is a straight line and there exists at most a point $\tau \in [0, T]$ such that $\lambda_1(\tau) = 0$ and u has a discontinuity (a jump). And in all that follows, $\lambda_0 = 0$ or $\lambda_0 = 1$ is irrelevant.
Now we study two cases:
case A: Let us suppose $\lambda_1(t) < 0$ in $t \in (t', t'') \subset (0, \infty)$. We have, for every $t \in (t', t'')$, $u(t) = 1$ and

$$\dot{x}_1 = u \quad \Longrightarrow \quad x_1(t) = t + c, \quad \text{with } c \in \mathbb{R} \tag{3.59}$$

$$\dot{x}_2 = x_1 \quad \Longrightarrow \quad x_2(t) = t^2/2 + ct + d, \quad \text{with } d \in \mathbb{R} \tag{3.60}$$

We obtain

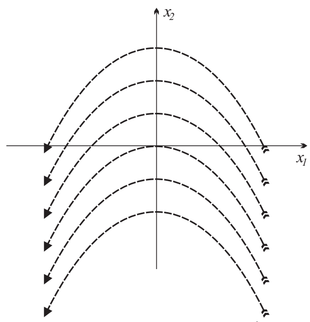
$$x_2 = \frac{1}{2}x_1^2 + d - \frac{c^2}{2}. \tag{3.61}$$



For the moment, it is not easy to find the constants c and d : however, it is clear that (3.61) represents some parabolas on the (x_1, x_2)-plane. Moreover, the dynamics $\dot{x}_2 = x_1$ gives that if $x_1 > 0$, then x_2 increases and if $x_1 < 0$ then x_2 decreases: hence there is a direction on such line when the time passes.
case B: Let $\lambda_1(t) > 0$ in $(t', t'') \subset (0, \infty)$: hence $u(t) = -1$ and, as before,

$$\dot{x}_1 = u \quad \Longrightarrow \quad x_1(t) = -t + e, \quad \text{with } e \in \mathbb{R} \tag{3.62}$$

$$\dot{x}_2 = x_1 \quad \Longrightarrow \quad x_2(t) = -t^2/2 + et + f, \quad \text{with } f \in \mathbb{R} \tag{3.63}$$

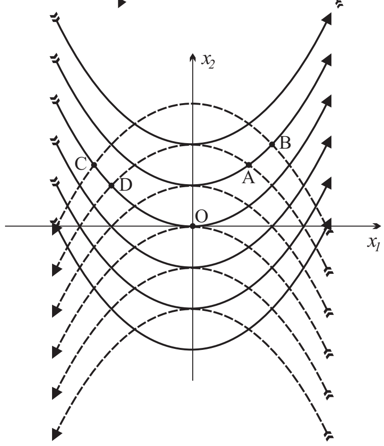


that imply

$$x_2 = -\frac{1}{2}x_1^2 + f + \frac{e^2}{2}. \tag{3.64}$$

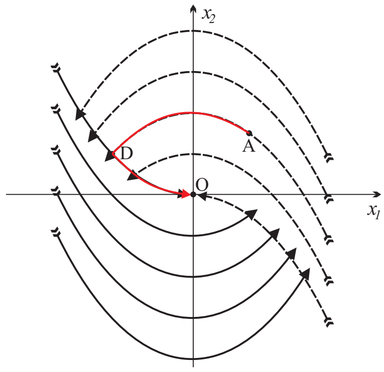
Again we have some parabolas and a precise direction for such curves.

case A+B: Now let us put together these two families of parabolas in (3.61) and (3.63). In order to start at time $t = 0$ from the point $(x_1(0), x_2(0)) = (v_0, d_0)$ and to arrive in the final and unknown time T to the point $(x_1(T), x_2(T)) = (0, 0)$, we can follow some part of such parabolas (with the right direction). It is clear that there exists infinite path: for example (see the figure) starting from A we can move on the curve as arrive in B, hence follows the dashed line and arrive in the point C and



finally to arrive in the origin along a new parabola.

We remark that every time we pass from a curve to another curve, the optimal control has a discontinuity point, i.e. u^* passes from $+1$ to -1 or vice versa. Since we know that the optimal control has at most one discontinuity point, the "red line ADO" in the figure is the unique candidate to realize the min, i.e. the minimal time T^*.



In order to guarantee that u^* is really the optimal control with exit time T^*, we suggest two possibilities:

• first we will provide an existence result for this particular type of time optimal problem called "linear"(see subsection 3.4.1). As we mentioned, the dynamics in (3.53) is exactly the system in (3.43). For such M and N matrices, we have

$$G(M,N) = [N, MN] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\det(M - \theta I) = \theta^2.$$

Since $G(M,N)$ has rank 2, and 0 is the unique eigenvalue of the matrix M, Theorem 3.9 guarantees the existence of the solution of (3.53), for every initial data (v_0, d_0).

•• Second, we can apply a generic result of the existence of an optimal control (see Theorem 3.8). We know, by the previous construction, that there exists at least an admissible control u^* with exit time T^*; hence it is reasonable to restrict the original target set $\mathcal{T} = [0,\infty) \times \{(0,0)\}$ to the new set $\mathcal{T} = [0, T^*] \times \{(0,0)\}$. Moreover, we have a compact control set $[-1,1]$ and for the dynamics we have the bounded condition

$$\|\dot{\mathbf{x}}\| = \left\| \begin{pmatrix} u \\ x_1 \end{pmatrix} \right\| \le \sqrt{1 + x_1^2} \le (1 + |x_1|) \le (1 + \|\mathbf{x}\|);$$

finally, for every (t, x_1, x_2) we have that

$$\begin{aligned} F_{(t,x_1,x_2)} &= \{(y_1, y_2, z) : \ y_1 = u, \ y_2 = x_1, \ z \ge 1, \ u \in [-1,1]\} \\ &= [-1,1] \times \{x_1\} \times [1,\infty), \end{aligned}$$

is a convex set. Hence Theorem 3.8 guarantees that the optimal control exists.

In the next example we solve a particular case of this general situation:

Example 3.5.3. *Let us consider*

$$\begin{cases} \min_u T \\ \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ x_1(0) = 2, \qquad x_2(0) = 1 \\ x_1(T) = x_2(T) = 0 \\ |u| \le 1 \end{cases}$$

Since $A = (x_1(0), x_2(0)) = (\alpha_1, \alpha_2) = (2, 1)$, by (3.62) and (3.63) we obtain $e = 2$ e $f = 1$. (3.61) gives the curve γ_1 with equation

$$x_2 = -x_1^2/2 + 3.$$

The point D is the intersection of the curve γ_2 with equation

$$x_2 = x_1^2/2,$$

and the curve γ_1 : we obtain $D = (-\sqrt{3}, 3/2)$. We note that starting from A at time $t = 0$, we arrive in D at time τ_D : such τ_D is, by (3.62), $\tau_D = 2 + \sqrt{3}$.
We restart from $D = (x_1(\tau_D), x_2(\tau_D))$ and arrive, on γ_2, to O. By (3.59) we have $x_1(\tau_D) = \tau_D + c = -\sqrt{3}$ and hence $c = -2(1 + \sqrt{3})$. By (3.61) and the equation of γ_2, we have $d = c^2/2 = 4(2 + \sqrt{3})$. We arrive in the origin at the time T that is, using (3.59) and hence $x_1(T) = T - 2(1 + \sqrt{3}) = 0$, $T = 2(1 + \sqrt{3})$. Hence

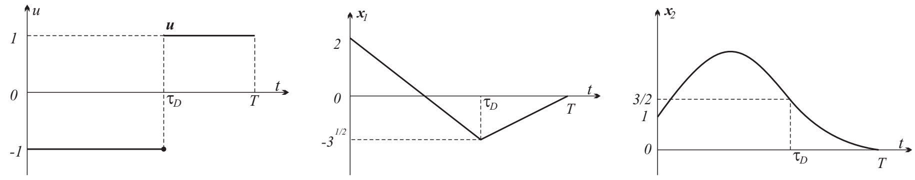
$$T^* = 2(1 + \sqrt{3}).$$

The optimal control and the optimal trajectory are (in order to guarantee that we have really the optimal control, see the discussions in the previous general case)

$$u^*(t) = \begin{cases} -1 & \text{for } t \in [0, 2 + \sqrt{3}], \\ 1 & \text{for } t \in (2 + \sqrt{3}, 2(1 + \sqrt{3})], \end{cases}$$

$$x_1^*(t) = \begin{cases} -t + 2 & \text{for } t \in [0, 2 + \sqrt{3}], \\ t - 2(1 + \sqrt{3}) & \text{for } t \in (2 + \sqrt{3}, 2(1 + \sqrt{3})], \end{cases}$$

$$x_2^*(t) = \begin{cases} -t^2/2 + 2t + 1 & \text{for } t \in [0, 2 + \sqrt{3}], \\ t^2/2 - 2(1 + \sqrt{3})t + 4(2 + \sqrt{3}) & \text{for } t \in (2 + \sqrt{3}, 2(1 + \sqrt{3})], \end{cases}$$



△

3.5.2 The Dubin car

The simple car model has three degrees of freedom, the car can be imagined as a rigid body that moves in a plane. The back wheels can only slide and that is why parallel parking is challenging. If all wheels could be rotated together, parking would be a trivial task. The position of a car can be identified, at every time t, with the triple $(x_1, x_2, \theta) \in \mathbb{R}^2 \times [0, 2\pi)$, where (x_1, x_2) are the principal directions and θ is the angle of the car with the x_1 axis.
Suppose that the car has an initial position $(x_1(0), x_2(0), \theta(0)) = (4, 0, \pi/2)$ and we are interested to park the car in the position $(0, 0)$, without constraints on the final direction of the car, in the shortest time. Clear, the fact that we have no condition on the final direction simplifies the problem.[9]

[9] If we add an assumption on the final direction of the car, as is reasonable if you are parking it, then we have the following Dubin car problem:

$$\begin{cases} \min_u T \\ \dot{x}_1 = \cos\theta \\ \dot{x}_2 = \sin\theta \\ \dot{\theta} = u \\ x_1(0) = x_{1,0}, \quad x_2(0) = x_{2,0}, \quad \theta(0) = \theta_0 \\ x_1(T) = 0, \quad x_2(T) = 0, \quad \theta(T) = \theta_1 \\ |u| \le 1 \end{cases}$$

The solution is much more complicated and is presented, for example, in [1].

From the geometry of the model (simplifying some constants to unity, i.e. we think that the velocity of the car is constant and equal to 1), the following problem can be retrieved:

$$\begin{cases} \min\limits_u T \\ \dot{x}_1 = \cos\theta \\ \dot{x}_2 = \sin\theta \\ \dot{\theta} = u \\ x_1(0) = 4, \quad x_2(0) = 0, \quad \theta(0) = \pi/2 \\ x_1(T) = 0, \quad x_2(T) = 0 \\ |u| \le 1 \end{cases} \tag{3.65}$$

The Hamiltonian is $H = \lambda_0 + \lambda_1 \cos\theta + \lambda_2 \sin\theta + \lambda_3 u$ and the necessary conditions of Theorem 3.10 give (note that $\theta(T)$ is free and hence we have a transversality condition on $\lambda_3(T)$ and note that the problem is autonomous)

$$u(t) \in \arg\min_{v\in[-1,1]} \lambda_3(t)v \tag{3.66}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \quad \Rightarrow \ \lambda_1 = c_1 \tag{3.67}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = 0 \quad \Rightarrow \ \lambda_2 = c_2 \tag{3.68}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial \theta} = \lambda_1 \sin\theta - \lambda_2 \cos\theta \tag{3.69}$$

$$\lambda_3(T) = 0 \tag{3.70}$$

$$\lambda_0 + \lambda_1(t)\cos\theta(t) + \lambda_2(t)\sin\theta(t) + \lambda_3(t)u(t) = 0 \qquad \forall t \tag{3.71}$$

where c_i are constants; changing these constants with $c_1 = \alpha\cos\beta$ and $c_2 = \alpha\sin\beta$, where $\alpha \ge 0$ and β are again constants, we obtain by (3.67)–(3.71) that

$$\lambda_1 = \alpha\cos\beta, \qquad \lambda_2 = \alpha\sin\beta \tag{3.72}$$

$$\dot{\lambda}_3 = \alpha\sin(\theta - \beta) \tag{3.73}$$

$$\lambda_0 + \alpha\cos\beta\cos\theta(T) + \alpha\sin\beta\sin\theta(T) = 0 \tag{3.74}$$

It is easy to see that if $\alpha = 0$ (i.e. $\lambda_1(t) = \lambda_2(t) = 0$), then by (3.74) we obtain $\lambda_0 = 0$; moreover (3.73) and (3.70)) give $\lambda_3(t) = 0$. This is impossible since $(\lambda_0, \lambda) \neq (0, 0)$: hence $\alpha \neq 0$. We note that if $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is a multiplier that satisfies all the conditions (3.66)–(3.71), the "new" multiplier $(\frac{\lambda_0}{\alpha}, \frac{\lambda_1}{\alpha}, \frac{\lambda_2}{\alpha}, \frac{\lambda_3}{\alpha})$ satisfies the same conditions (3.66)–(3.71). In other words, it is possible to put, without loss of generality, $\alpha = 1$.

First, let us suppose that there exists an interval (a, b) where

$$\lambda_3(t) > 0, \quad t \in (a, b);$$

(3.66) implies $u = -1$ in such interval. The dynamics gives

$$\theta(t) = -t + a + \theta(a), \qquad \forall t \in (a, b)$$

and hence

$$x_1(t) = x_1(a) - \sin(-t + a + \theta(a)) + \sin\theta(a)$$
$$x_2(t) = x_2(a) + \cos(-t + a + \theta(a)) - \cos\theta(a)$$

for every $t \in (a, b)$. Note that $t \mapsto (x_1(t), x_2(t))$ describes an arc of circumference of radius 1, in a clockwise sense.

Similar calculations give that if there exists an interval (a, b) where

$$\lambda_3(t) < 0, \quad t \in (a, b)$$

then we obtain $u(t) = +1$ and

$$\begin{aligned} \theta(t) &= t - a + \theta(a), \\ x_1(t) &= x_1(a) + \sin(t - a + \theta(a)) - \sin\theta(a) && (3.75) \\ x_2(t) &= x_2(a) - \cos(t - a + \theta(a)) + \cos\theta(a) && (3.76) \end{aligned}$$

for every $t \in (a, b)$. Again $t \mapsto (x_1(t), x_2(t))$ describes an arc of circumference of radius 1, in a counterclockwise sense.

Finally, if we have a situation where

$$\lambda_3(t) = 0, \qquad \forall t \in (a, b)$$

then, $\dot{\lambda}_3(t) = 0$ in (a, b): the continuity of θ implies

$$\sin(\theta(t) - \beta) = 0 \quad t \in (a, b),$$

i.e. θ constant in $[a, b]$. The dynamics gives

$$x_1(t) = x_1(a) + (t-a)\cos(\theta(a)), \qquad x_2(t) = x_2(a) + (t-a)\sin(\theta(a)) \tag{3.77}$$

for every $t \in (a, b)$.

Essentially, in order to construct our optimal strategy we have only three possibilities:

- we turn on the right with $\dot{\theta} = -1$, and the car describes an arc of circumference of radius 1;
- we turn on the left with $\dot{\theta} = 1$, and the car describes an arc of circumference of radius 1;
- we go straight, i.e. $\dot{\theta} = 0$.

Now, looking the position of the car, it is reasonable to think that our strategy is first to turn on the right and hence to go straight. Hence in $[0, \tau)$ we set $u(t) = 1$ and, as in (3.75)–(3.76) and using the initial conditions, we obtain

$$\begin{aligned} \theta(t) &= t + \frac{\pi}{2} \\ x_1(t) &= 3 + \sin\left(t + \frac{\pi}{2}\right) = 3 + \cos t && (3.78) \\ x_2(t) &= -\cos\left(t + \frac{\pi}{2}\right) = \sin t \end{aligned}$$

for every $t \in [0, \tau)$. In $[\tau, T]$ we go straight and hence, as in (3.77) and using the continuity of the trajectory for $t = \tau$, we have by (3.78)

$$\begin{aligned} \theta(t) &= \tau + \frac{\pi}{2} \\ x_1(t) &= 3 + \cos\tau - (t - \tau)\cos\left(\tau + \frac{\pi}{2}\right) = 3 + \cos\tau - (t - \tau)\sin\tau \\ x_2(t) &= \sin\tau + (t - \tau)\sin\left(\tau + \frac{\pi}{2}\right) = \sin\tau + (t - \tau)\cos\tau \end{aligned}$$

for every $t \in [\tau, T]$. Clearly the final condition on the position of the car gives

$$x_1(T) = 3 + \cos\tau - (T-\tau)\sin\tau = 0, \qquad x_2(T) = \sin\tau + (T-\tau)\cos\tau = 0.$$

Solving these system we obtain (using the picture below we deduce that $\sin\tau = +\sqrt{8}/3$)

$$T^* = \arccos\left(-\frac{1}{3}\right) + \sqrt{8}, \qquad \tau = \arccos\left(-\frac{1}{3}\right).$$

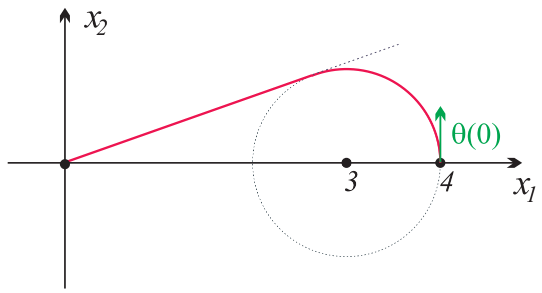
Hence

$$u^*(t) = \begin{cases} 1 & t \in [0,\tau] \\ 0 & t \in (\tau, T^*] \end{cases}$$

$$\theta^*(t) = \begin{cases} t + \pi/2 & t \in [0,\tau] \\ \tau + \pi/2 & t \in (\tau, T^*] \end{cases}$$

$$x_1^*(t) = \begin{cases} 3 + \cos t & t \in [0,\tau] \\ 8/3 - (t-\tau)\sqrt{8}/3 & t \in (\tau, T^*] \end{cases} \tag{3.79}$$

$$x_2^*(t) = \begin{cases} \sin t & t \in [0,\tau] \\ \sqrt{8}/3 - (t-\tau)/3 & t \in (\tau, T^*] \end{cases} \tag{3.80}$$



The initial data for the car is $(x_1(0), x_2(0), \theta_1(0)) = (4, 0, \pi/2)$; *the final position is* $(x_1(T^*), x_2(T^*)) = (0,0)$ *with* $\theta_1(T^*)$ *free.*

Clear, the previous arguments gives a candidate to be the optimal solution of the problem, but it is not a proof of its uniqueness. Indeed, it is possible to construct infinite paths using arcs of circumferences of radius 1 and segments to connect the point (4,0) and the origin, with continuous θ: such paths are all extremals. However, it is easy to see that in this family of extremal paths, our trajectory (x_1^*, x_2^*) in (3.79)-(3.80) is the shortest: since the modulo of the velocity of the car is constant, such trajectory is the best choice in the family of the extremals to have a minimal time. We remark that such control u^* in singular.

Finally, let us show that there exists the optimal control for our problem. We know, by the previous construction, that there exists at least an admissible control u^* with exit time T^*; hence it is reasonable to restrict the original target set $\mathcal{T} = [0,\infty) \times \{(0,0)\} \times [0, 2\pi]$ to the new set $\mathcal{T} = [0, T^*] \times \{(0,0)\} \times [0, 2\pi]$. Moreover, we have a compact control set $[-1, 1]$ and for the dynamics we have the bounded condition

$$\|\dot{\mathbf{x}}\| = \left\| \begin{pmatrix} \cos\theta \\ \sin\theta \\ u \end{pmatrix} \right\| \le \sqrt{1+u^2} \le 2;$$

finally, for every (t, x_1, x_2, θ) fixed we have that

$$F_{(t,x_1,x_2,\theta)} = \{(\cos\theta, \sin\theta)\} \times [-1, 1] \times [1, \infty),$$

is a convex set. Hence Theorem 3.8 guarantees that the optimal control exists.

3.6 Infinite horizon problems

If in the problem (3.4) we consider, with due caution, $t_1 = \infty$, then the first question is the validity of Theorem of Pontryagin 3.1. It is clear that

- with $\psi(\mathbf{x}(\infty))$ and $\mathbf{x}(\infty)$, we have to replace $\lim_{t\to\infty} \psi(\mathbf{x}(t))$ and $\lim_{t\to\infty} \mathbf{x}(t)$;
- in general, without other assumptions on f, on the control $\mathbf{u}^*$ and the associated trajectory $\mathbf{x}^*$ that are candidate to maximize (or minimize) the problem, we are not able to guarantee that the integral

$$\int_{t_0}^{\infty} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t$$

  exists and is finite.

In this context and considering the transversality condition *iii*) of Theorem of Pontryagin, one might expect that

$$\lim_{t\to\infty} \boldsymbol{\lambda}^*(t) = 0 \tag{3.81}$$

would be a natural condition for a infinite horizon problem. The next example shows that (3.81) is false:

Example 3.6.1 (The Halkin counterexample). *Let us consider the problem*

$$\begin{cases} \max J(u) \\ J(u) = \displaystyle\int_0^{\infty} (1-x)u\, \mathrm{d}t \\ \dot{x} = (1-x)u \\ x(0) = 0 \\ 0 \le u \le 1 \end{cases}$$

If we integrate the dynamics $\dot{x} + xu = u$ and taking into account the initial condition, we obtain

$$x(t) = e^{-\int_0^t u(s)\,\mathrm{d}s} \left(\int_0^t u(s) e^{\int_0^s u(v)\,\mathrm{d}v}\, \mathrm{d}s \right) = 1 - e^{-\int_0^t u(s)\,\mathrm{d}s}.$$

Hence, for every admissible control u the associated trajectory x is such that $x(t) \le 1$. Hence, using the dynamics and the initial condition,

$$J(u) = \lim_{T\to\infty} \int_0^T (1-x)u\, \mathrm{d}t = \lim_{T\to\infty} \int_0^T \dot{x}\, \mathrm{d}t = \lim_{T\to\infty} (x(T) - x(0)) \le 1.$$

Hence, every function u such that $\int_0^{\infty} u(t)\, \mathrm{d}t = \infty$ gives $J(u) = 1$ and hence is optimal. For example, consider the constant control $u_0(t) = u_0 \in (0, 1)$: it is optimal. Since the Hamiltonian is $H = (1-x)(1+\lambda)u$, the PMP implies that

$$u_0 \in \arg\max_{v\in[0,1]} (1-x)(1+\lambda)v \qquad \forall t \ge 0.$$

The previous condition gives $\lambda(t) = -1$ for every $t \ge 0$. Hence such multiplier λ is associated to the optimal control u_0 and $\lim_{t\to\infty} \lambda(t) \neq 0$. △

Hence let us consider the problem:

$$\begin{cases} \max_{\mathbf{u}\in\mathcal{C}} \int_{t_0}^{\infty} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \lim_{t\to\infty} x_i(t) = \beta_i, & \text{for } 1 \le i \le n' \\ \lim_{t\to\infty} x_i(t) \ge \beta_i, & \text{for } n' < i \le n'' \\ \lim_{t\to\infty} x_i(t) \text{ free} & \text{for } n'' < i \le n \\ \mathcal{C} = \{\mathbf{u} : [t_0,\infty) \to U \subseteq \mathbb{R}^k,\ \mathbf{u} \text{ admissible}\} \end{cases} \tag{3.82}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ are fixed in $\mathbb{R}^n$.

The problem of the transversality for this problem is treated with many details in [26] (see Theorem 3.13): here we give only a sufficient condition in the spirit of the theorem of Mangasarian:

Theorem 3.11. *Let us consider the infinite horizon maximum problem (3.82) with $f \in C^1$ and $g \in C^1$. Let the control set U be convex. Let $\mathbf{u}^*$ be a normal extremal control, $\mathbf{x}^*$ the associated trajectory and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$ the associated multiplier, i.e. the tern $(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)$ satisfies the PMP and the adjoint equation.*
Suppose that

v) the function $(\mathbf{x},\mathbf{u}) \mapsto H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}^*(t))$ is, for every $t \in [t_0,\infty)$, concave,

vi) for all admissible trajectory $\mathbf{x}$,

$$\lim_{t\to\infty} \boldsymbol{\lambda}^*(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \ge 0. \tag{3.83}$$

Then $\mathbf{u}^*$ is optimal.

Proof. The first part of the proof coincides with the proof of Theorem 2.3 of Mangasarian: hence we obtain that, for every admissible control $\mathbf{u}$ with associated trajectory $\mathbf{x}$ we have that (see (2.29))

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}^*) \le H(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}^*) - \dot{\boldsymbol{\lambda}}^* \cdot (\mathbf{x} - \mathbf{x}^*)$$

and hence, for every that $t_1 > t_0$

$$\begin{aligned} \int_{t_0}^{t_1} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t &\le \int_{t_0}^{t_1} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + \Big(\boldsymbol{\lambda}^* \cdot (\mathbf{x}^* - \mathbf{x})\Big|_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + \\ &\quad +\boldsymbol{\lambda}^*(t_1) \cdot \big(\mathbf{x}^*(t_1) - \mathbf{x}(t_1)\big) - \boldsymbol{\lambda}^*(t_0) \cdot \big(\mathbf{x}^*(t_0) - \mathbf{x}(t_0)\big). \end{aligned}$$

Now, taking into account $\mathbf{x}^*(t_0) = \mathbf{x}(t_0) = \boldsymbol{\alpha}$, the limit for $t_1 \to \infty$ of the members of the previous inequality gives

$$\int_{t_0}^{\infty} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \le \int_{t_0}^{\infty} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + \lim_{t_1\to\infty} \boldsymbol{\lambda}^*(t_1) \cdot \big(\mathbf{x}^*(t_1) - \mathbf{x}(t_1)\big). \tag{3.84}$$

Clearly the transversality condition (3.83) implies that $\mathbf{u}^*$ is optimal. □

The transversality condition in (3.83) is not so easy to guarantee, since it requires to study every admissible trajectory. Suppose that in problem (3.82) we have $n' = n'' = n$, i.e. we have a final condition on the trajectory of the type

$$\lim_{t\to\infty} \mathbf{x}(t) = \boldsymbol{\beta}, \tag{3.85}$$

for some fixed $\boldsymbol{\beta} \in \mathbb{R}^n$. It is clear by the final part the previous proof (see (3.84)), that

Remark 3.6. *Suppose that in the problem (3.82) we have only a condition of the type (3.85). Suppose that there exists a constant c such that*

$$\|\boldsymbol{\lambda}^*(t)\| \le c, \qquad \forall t \ge \tau \tag{3.86}$$

for some τ, then the transversality condition in (3.83) holds.

See chapter 14 in [8] for further conditions.

In the case of infinite horizon problem of Calculus of Variation, recalling that $\nabla_{\mathbf{u}} f = \nabla_{\dot{\mathbf{x}}} f = -\boldsymbol{\lambda}^*$ (see (2.39)), the sufficient condition in (3.83) and in (3.86) become as follows:

Remark 3.7. *In the case of infinite horizon problem of Calculus of Variation, the transversality condition in vi) becomes*

$$\lim_{t\to\infty} \nabla_{\dot{\mathbf{x}}} f(t, \mathbf{x}^*(t), \dot{\mathbf{x}^*}(t)) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \le 0, \tag{3.87}$$

for all admissible trajectory $\mathbf{x}$. Moreover, if the calculs of variation problem has a final condition on the trajectory of the type (3.85), i.e. $\lim_{t\to\infty} \mathbf{x}(t) = \boldsymbol{\beta}$ with $\boldsymbol{\beta}$ fixed, and there exists a constant c such that

$$\|\nabla_{\dot{\mathbf{x}}} f(t, \mathbf{x}^*(t), \dot{\mathbf{x}^*}(t))\| \le c, \qquad \forall t \ge \tau \tag{3.88}$$

for some τ, then the transversality condition in (3.87) holds.

Finally, it is clear from the proof that

Remark 3.8. *If in (3.82) we replace the max with a min, we have to reverse the inequalities in the previous transversality conditions in (3.83) and (3.87).*

Example 3.6.2. *Let us consider the problem*

$$\begin{cases} \min \displaystyle\int_0^\infty e^{2t}(u^2 + 3x^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 2 \end{cases}$$

It is a calculus of variation problem and in order to guarantee that $\int_0^\infty e^{2t}(u^2 + 3x^2)\,\mathrm{d}t < \infty$, it is clear that we have to require that $\lim_{t\to\infty} x(t) = 0$. Hence we have to solve the problem

$$\begin{cases} \min \displaystyle\int_0^\infty e^{2t}(\dot{x}^2 + 3x^2)\,\mathrm{d}t \\ x(0) = 2 \\ \lim_{t\to\infty} x(t) = 0 \end{cases}$$

The EU gives $\ddot{x} + 2\dot{x} - 3x = 0$ and its general solution is

$$x(t) = ae^{-3t} + be^t,$$

for some constants a and b. The conditions on the trajectory give that the unique extremal is the function $x^*(t) = 2e^{-3t}$.
Now we note that $\frac{\partial f}{\partial \dot{x}} = 2e^{2t}\dot{x}$ and hence

$$\left|\frac{\partial f}{\partial \dot{x}}(t, x^*(t), \dot{x}^*(t))\right| = |2e^{2t}\dot{x}^*(t)| = |-12e^{-t}| \le 12, \ \forall t \ge 0 : \tag{3.89}$$

the concavity of the function $f = e^{2t}(\dot{x}^2 + 3x^2)$, with respect to the variable x and $\dot{x}$, and the transversality condition (3.88), i.e. (3.89), give that x^* is optimal.

3.6.1 The model of Ramsey

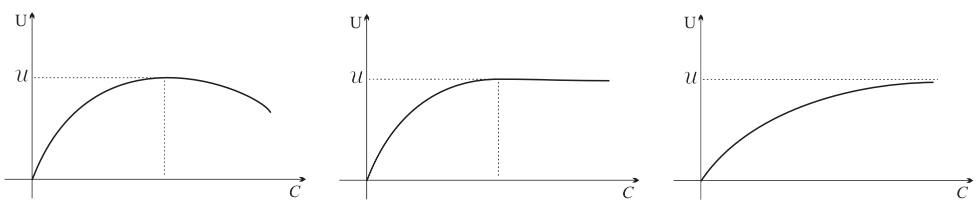
We show the problem of resource allocation in a infinite time (see [23]): we want to determine the combination optimal of consumption and savings from current production. Considering a generic economic system that produces a given level of national product (NP), we must find the fraction of NP that is consumed and what is saved: the first generates utility in the current period, while the second fraction of NP, if invested, will produce a utility in the future.
Suppose then that the control variables are the work $L = L(t)$ and the capital $K = K(t)$. Suppose also that there is only a goods with production function

$$Q = Q(K(t), L(t)). \tag{3.90}$$

So the production is independent, directly, by time and hence we are assuming that there is not progress in the technology. Suppose also that there is no depreciation of the capital and that the population remains stationary. The production is distributed, at every moment, for consumption C and investments: then

$$Q = C + K'. \tag{3.91}$$

The utility function $U = U(C)$ (social utility index) has not increasing marginal utility $\eta = U'$: then $U'' \le 0$. Moreover we suppose that U has an upper bound which we call $\mathcal{U}$.



We note that if $U(C) \to \mathcal{U}$. then $\eta \to 0$.

We introduce also the disutility function $D = D(L)$ arising from work L, with marginal disutility decreasing: then $D'' \ge 0$. The net utility is given by $U(C) - D(L)$. The problem of detecting a dynamic consumption that maximizes the utility of current and future generations can formalize as

$$\max_{(L,C)} \int_0^\infty (U(C) - D(L))\, \mathrm{d}t. \tag{3.92}$$

In general the integral in (3.92) does not exist. This is due in part to the fact that there isn't discount factor, not forgetfulness, but because it is deemed "ethically indefensible", for today's generations who plan, pay the utilities of future generations. Moreover it is reasonable to expect that over the course of time the net utility is positive and grows. Hence, we can assume that there exists a positive B such that

$$\lim_{t\to\infty}(U(C)-D(L))=B;$$

such B is a kind of "ideal situation" (Ramsey called the "Bliss", happiness). Hence we have to minimize the gap between the net utility and the "happiness ":

$$\min_{(L,C)}\int_0^\infty \big[B-U(C)+D(L)\big]\,\mathrm{d}t$$

Taking into account (3.90) and (3.91), we have

$$\begin{cases}\displaystyle\min_{(L,K)}\int_0^\infty\Big[B-U\Big(Q(K(t),L(t))-K'(t)\Big)+D(L(t))\Big]\,\mathrm{d}t\\ K(0)=K_0\\ \displaystyle\lim_{t\to\infty}(U(C)-D(L))=B\end{cases}\tag{3.93}$$

where K_0 is fixed initial capital, while in general it is considered inappropriate fix an initial condition at work. If we denote by

$$F=F(L,K,L',K')=B-U\Big(Q(K(t),L(t))-K'(t)\Big)+D(L(t)),$$

we write the equation of Euler with respect the variables L and K :

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}F_{L'}=F_L\\ \dfrac{\mathrm{d}}{\mathrm{d}t}F_{K'}=F_K\end{cases}\qquad\Rightarrow\qquad\begin{cases}0=-\eta Q_L+D'\\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t}=-\eta Q_K\end{cases}$$

Since

$$D'=\eta Q_L,$$

the marginal disutility of labor must be equal to the product between the marginal utility of consumption and the marginal product of labor. Moreover we have that

$$\frac{\frac{\mathrm{d}\eta}{\mathrm{d}t}}{\eta}=-Q_K$$

provides a "good rule" to consumption: the rate of growth of marginal utility of consumption should be equal, at every moment, to the marginal product of capital changing the sign. Also we note that F does not explicitly depend on t and, from (2.41), we have

$$F-K'F_{K'}=c\qquad\Rightarrow\qquad B-U(C)+D(L)-K'\eta=c,\tag{3.94}$$

for every $t\ge 0$, where c is a constant. Since the net utility tends to B, it is clear that $U(C)$ goes to $\mathcal{U}$ and hence $\eta\to 0$. From (3.94) we have

$$0=\lim_{t\to\infty}[B-U(C)+D(L)-K'\eta-c]=-c.$$

The relations $c = 0$ and (3.94) give us the optimal path of the investment $K^{*\prime}$, i.e.

$$K^{*\prime}(t) = \frac{B - U(C(t)) + D(L(t))}{\eta(t)}.$$

This result is known as "the optimal rule of Ramsey".
Now, if we would like to guarantee that the extremal path (L^*, K^*) is optimal, we study the convexity of the function F :

$$\mathrm{d}^2F(L, K, L', K') = \begin{pmatrix} F_{LL} & F_{LK} & F_{LL'} & F_{LK'} \\ F_{KL} & F_{KK} & F_{KL'} & F_{KK'} \\ F_{L'L} & F_{L'K} & F_{L'L'} & F_{L'K'} \\ F_{K'L} & F_{K'K} & F_{K'L'} & F_{K'K'} \end{pmatrix}$$

$$= \begin{pmatrix} -U''Q_L^2 - U'Q_{LL} + D'' & -U''Q_LQ_K - U'Q_{KL} & 0 & U''Q_L \\ -U''Q_LQ_K - U'Q_{KL} & -U''Q_K^2 - U'Q_{KK} & 0 & U''Q_K \\ 0 & 0 & 0 & 0 \\ U''Q_L & U''Q_K & 0 & -U'' \end{pmatrix}$$

If we consider the quadratic form

$$\mathbf{h} \cdot \left(\mathrm{d}^2F(L, K, L', K') \right) \cdot \mathbf{h}^T,$$

with $\mathbf{h} = (h_L, h_K, \dot{h}_L, \dot{h}_K)$, we have

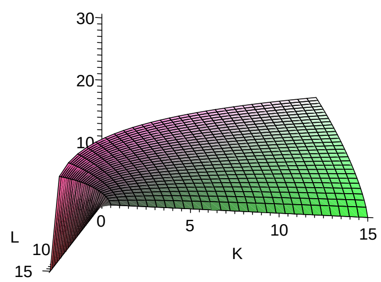
$$\mathbf{h} \cdot \left(\mathrm{d}^2F(L, K, L', K') \right) \cdot \mathbf{h}^T =$$

$$= h_L^2 D''(L) - (h_L, h_K) \cdot \begin{pmatrix} Q_{LL} & Q_{LK} \\ Q_{LK} & Q_{KK} \end{pmatrix} \cdot (h_L, h_K)^T U'(C) +$$

$$-(h_L, h_K, \dot{h}_K) \cdot \begin{pmatrix} Q_L^2 & Q_LQ_K & -Q_L \\ Q_LQ_K & Q_K^2 & -Q_K \\ -Q_L & -Q_K & 1 \end{pmatrix} \cdot (h_L, h_K, \dot{h}_K)^T U''(C).$$

An easy computation shows that the 3×3 matrix of the previous expression is positive semidefinite. Moreover, since $D''(L) \geq 0$ and $U''(C) \leq 0$, if we assume that $U'(C) \geq 0$ and Q is concave in the variable (L, K), then the extremal path (L^*, K^*) is really a minimum for the problem (3.93).

An example of a concave production function Q is given by the Cobb-Douglas $Q(L, K) = aL^{1-b}K^b$, with $a > 0$ and $0 < b < 1$. On the eright, we put the function of Cobb-Douglas $Q(L, K) = 2L^{4/5}K^{1/5}$.



3.7 Current Hamiltonian

In many problems of economic interest, future values of income and expenses are discounted: if $r > 0$ is the discount rate, we have the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} e^{-rt} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \displaystyle\max_{\mathbf{u}\in\mathcal{C}} J(\mathbf{u}) \end{cases} \tag{3.95}$$

where t_1 is fixed and finite; in this situation $H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = e^{-rt} f(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{u})$ and the necessary conditions of Pontryagin are

$$\mathbf{u}^* \in \arg\max_{\mathbf{v}\in U}\Big(f(t,\mathbf{x}^*,\mathbf{v})e^{-rt} + \boldsymbol{\lambda}^*\cdot g(t,\mathbf{x}^*,\mathbf{v})\Big) \tag{3.96}$$

$$\nabla_{\mathbf{x}} H = e^{-rt}\nabla_{\mathbf{x}} f(t,\mathbf{x}^*,\mathbf{u}^*) + \boldsymbol{\lambda}^*\cdot\nabla_{\mathbf{x}} g(t,\mathbf{x}^*,\mathbf{u}^*) = -\dot{\boldsymbol{\lambda}}^* \tag{3.97}$$

$$\boldsymbol{\lambda}^*(t_1) = \mathbf{0}. \tag{3.98}$$

For simplicity and only for few lines, let us consider the case with $n = k = 1$ (i.e. $\mathbf{x} = x_1 = x$, $\mathbf{u} = u_1 = u$) and with the control set $U = \mathbb{R}$; moreover we suppose that $\frac{\partial g}{\partial u} \neq 0$: then (3.96) implies

$$\lambda^* e^{rt} = -\frac{\frac{\partial f}{\partial u}(t,x^*,u^*)}{\frac{\partial g}{\partial u}(t,x^*,u^*)}.$$

Hence, from remark 2.11, $\boldsymbol{\lambda}^*(t)$ gives the marginal value of the state variable at time t discounted ("brought back") at time t_0. It is often convenient to consider the situation in terms of current values, i.e. of values at time t.
Hence, for the generic problem (3.95), let us define the *current Hamiltonian* H^c as

$$H^c(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}_c) = f(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}_c\cdot g(t,\mathbf{x},\mathbf{u}),$$

where $\boldsymbol{\lambda}_c$ is the *current multiplier*. Clearly we obtain

$$H^c = e^{rt} H \tag{3.99}$$

$$\boldsymbol{\lambda}_c^* = e^{rt}\boldsymbol{\lambda}^*. \tag{3.100}$$

If we consider the derivative with the respect the time in (3.100), we have

$$\begin{aligned} \dot{\boldsymbol{\lambda}}_c^* &= re^{rt}\boldsymbol{\lambda}^* + e^{rt}\dot{\boldsymbol{\lambda}}^* \\ \text{(by (3.97) and (3.100))} \quad &= r\boldsymbol{\lambda}_c^* - \nabla_{\mathbf{x}} f(t,\mathbf{x}^*,\mathbf{u}^*) - e^{rt}\boldsymbol{\lambda}^*\cdot\nabla_{\mathbf{x}} g(t,\mathbf{x}^*,\mathbf{u}^*) \\ \text{(by (3.100))} \quad &= r\boldsymbol{\lambda}_c^* - \nabla_{\mathbf{x}} f(t,\mathbf{x}^*,\mathbf{u}^*) - \boldsymbol{\lambda}_c^*\cdot\nabla_{\mathbf{x}} g(t,\mathbf{x}^*,\mathbf{u}^*) \\ &= r\boldsymbol{\lambda}_c^* - \nabla_{\mathbf{x}} H^c(t,\mathbf{x}^*,\mathbf{u}^*,\boldsymbol{\lambda}_c^*). \end{aligned}$$

(3.96) and (3.99) imply

$$\mathbf{u}^*(t) \in \arg\max_{\mathbf{v}\in U} e^{-rt} H^c(t,\mathbf{x}^*(t),\mathbf{v},\boldsymbol{\lambda}_c^*(t)) = \arg\max_{\mathbf{v}\in U} H^c(t,\mathbf{x}^*(t),\mathbf{v},\boldsymbol{\lambda}_c^*(t)).$$

Easily (3.98) becomes $\boldsymbol{\lambda}_c^*(t_1) = \mathbf{0}$. In conclusion

Remark 3.9. *A necessary condition for the problem (3.95) is*

$$\mathbf{u}^* \in \arg\max_{\mathbf{v}\in U} H^c(t, \mathbf{x}^*, \mathbf{v}, \boldsymbol{\lambda}_c^*) \tag{3.101}$$

$$\dot{\boldsymbol{\lambda}}_c^* = r\boldsymbol{\lambda}_c^* - \nabla_{\mathbf{x}} H^c(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}_c^*) \tag{3.102}$$

$$\boldsymbol{\lambda}_c^*(t_1) = \mathbf{0}. \tag{3.103}$$

We will give an interpretation of the current multiplier in remark 5.8. Clearly (3.99) implies the following;

Remark 3.10 ($U = \mathbb{R}^k$)**.** *If in the problem (3.95) we have a control set $U = \mathbb{R}^k$, then in the necessary conditions of remark 3.9 we have to replace (3.101) with*

$$\nabla_{\mathbf{u}} H^c(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}_c^*) = \mathbf{0} \tag{3.104}$$

Recalling that the transversality condition for the infinite horizon problem is delicate, we have the following:

Remark 3.11 (Infinite horizon problem)**.** *If in the problem (3.95) we have $t_1 = \infty$, then in the necessary conditions of remark 3.9 we have to delete (3.103).*

Example 3.7.1. Let us consider[10]

$$\begin{cases} \min \displaystyle\int_0^\infty e^{-rt}(ax^2 + bu^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = x_0 > 0 \\ \lim\limits_{t\to\infty} x(t) = 0 \end{cases} \tag{3.105}$$

where a and b are fixed and positive. The current Hamiltonian is $H^c = ax^2 + bu^2 + \lambda_c u$. Remark 3.10 gives

$$\frac{\partial H^c}{\partial u} = 0 \quad \Rightarrow \quad 2bu^* + \lambda_c^* = 0 \quad \Rightarrow \quad \text{(by (3.108))} \quad \lambda_c^* = -2b\dot{x}^* \tag{3.106}$$

$$\dot{\lambda}_c^* = r\lambda_c^* - \frac{\partial H^c}{\partial x} \quad \Rightarrow \quad \dot{\lambda}_c^* - r\lambda_c^* + 2ax^* = 0 \tag{3.107}$$

$$\frac{\partial H^c}{\partial \lambda_c} = \dot{x}^* \quad \Rightarrow \quad \dot{x}^* = u^* \tag{3.108}$$

(3.106) and (3.107) imply

$$b\ddot{x}^* - br\dot{x}^* - ax^* = 0 \quad \Rightarrow \quad x^*(t) = c_1 e^{\left(br+\sqrt{b^2r^2+4ab}\right)t/(2b)} + c_2 e^{\left(br-\sqrt{b^2r^2+4ab}\right)t/(2b)},$$

with c_1 and c_2 constants. The initial condition implies

$$x^*(t) = c_1 e^{\left(br+\sqrt{b^2r^2+4ab}\right)t/(2b)} + (x_0 - c_1) e^{\left(br-\sqrt{b^2r^2+4ab}\right)t/(2b)}.$$

Now consider the derivative of the previous expression with respect the time to obtain u^* :

$$u^*(t) = c_1 \frac{br + \sqrt{b^2r^2+4ab}}{2b} e^{\left(br+\sqrt{b^2r^2+4ab}\right)t/(2b)} + \\ +(x_0 - c_1)\frac{br - \sqrt{b^2r^2+4ab}}{2b} e^{\left(br-\sqrt{b^2r^2+4ab}\right)t/(2b)}.$$

It is an easy calculation to see that

$$\int_0^\infty e^{-rt}(a(x^*)^2 + b(u^*)^2)\,\mathrm{d}t = \int_0^\infty \left(Ae^{\left(\sqrt{b^2r^2+4ab}\right)t/b} + Be^{\left(-\sqrt{b^2r^2+4ab}\right)t/b} + C \right) \mathrm{d}t,$$

[10] In the example 5.7.1 we solve the same problem with dynamic programming.

for A, B and C constants, converges if and only if $c_1 = 0$. We obtain, using (3.100),

$$\lambda_c^*(t) = x_0\left(\sqrt{b^2r^2+4ab}-br\right)e^{\left(br-\sqrt{b^2r^2+4ab}\right)t/(2b)}, \qquad (3.109)$$

$$\lambda^*(t) = x_0\left(\sqrt{b^2r^2+4ab}-br\right)e^{-\left(br+\sqrt{b^2r^2+4ab}\right)t/(2b)}, \qquad (3.110)$$

$$x^*(t) = x_0 e^{\left(br-\sqrt{b^2r^2+4ab}\right)t/(2b)},$$

$$u^*(t) = x_0\frac{br-\sqrt{b^2r^2+4ab}}{2b}e^{\left(br-\sqrt{b^2r^2+4ab}\right)t/(2b)}.$$

The Hamiltonian is convex and the multiplier λ^* is bounded; from Theorem 3.11 and remark 3.6 u^* is the minimum. In the picture at the end of the next example 3.7.2, there is the optimal tern in a particular case.

Example 3.7.2. *Let us consider a modification of the previous example 3.7.1, in the case $r = 2,\ a = 3,\ b = 1,\ x_0 = 2$:*

$$\begin{cases} \min \int_0^\infty e^{-2t}(3x^2+u^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 2 \\ |u| \le 1 \\ \lim_{t\to\infty} x(t) = 0 \end{cases} \qquad (3.111)$$

This new problem has some similarities with the previous one: we only give an idea of the solution and leave to the reader the details. The current Hamiltonian is $H^c = 3x^2+u^2+\lambda_c u$. Remark 3.9 gives

$$u^* \in \arg\min_{v\in[-1,1]} H^c(t,x^*,u^*,\lambda_c^*) \quad \Rightarrow \quad u^* \in \arg\min_{v\in[-1,1]}(v^2+\lambda_c^* v)$$

$$\Rightarrow \quad u^* = \begin{cases} -1 & \text{if } -\lambda_c^*/2 < -1 \\ -\lambda_c^*/2 & \text{if } -1 \le -\lambda_c^*/2 \le 1 \\ 1 & \text{if } 1 < -\lambda_c^*/2 \end{cases} \qquad (3.112)$$

$$\dot{\lambda}_c^* = r\lambda_c^* - \frac{\partial H^c}{\partial x} \quad \Rightarrow \quad \dot{\lambda}_c^* - 2\lambda_c^* + 6x^* = 0 \qquad (3.113)$$

$$\frac{\partial H^c}{\partial \lambda_c} = \dot{x^*} \quad \Rightarrow \quad \dot{x^*} = u^* \qquad (3.114)$$

Let us suppose that for every $t \in [0,\infty)$ we have $-1 \le -\lambda_c^*/2 \le 1$: we obtain (as in example 3.7.1)

$$u^*(t) = -2e^{-t}, \qquad x^*(t) = 2e^{-t}, \qquad \lambda_c^*(t) = 4e^{-t}, \qquad \forall t \in [0,\infty):$$

this contradicts the assumption $\lambda_c^* \le 2$.

Hence let us suppose that, for some fixed $\tau > 0$, we have $-\lambda_c^*/2 < -1$ for every $t \in [0,\tau)$. Relations (3.112), (3.113) and (3.114) give

$$u^*(t) = -1, \qquad x^* = 2-t, \qquad \lambda_c^*(t) = Ae^{2t} - 3t + \frac{9}{2}, \qquad \forall t \in [0,\tau). \qquad (3.115)$$

Now let us suppose that for every $t \in [\tau,\infty)$ we have $-1 \le -\lambda_c^*/2 \le 1$: we obtain (as in example 3.7.1, taking into account that is an infinite horizon problem)

$$u^*(t) = -c_2e^{-t}, \qquad x^*(t) = c_2e^{-t}, \qquad \lambda_c^*(t) = 2c_2e^{-t}, \qquad \forall t \in [\tau,\infty). \qquad (3.116)$$

The continuity of the multiplier and of the trajectory in $t = \tau$ imply, by (3.115) and (3.116), that

$$x^*(\tau) = 2-\tau = c_2e^{-\tau},$$

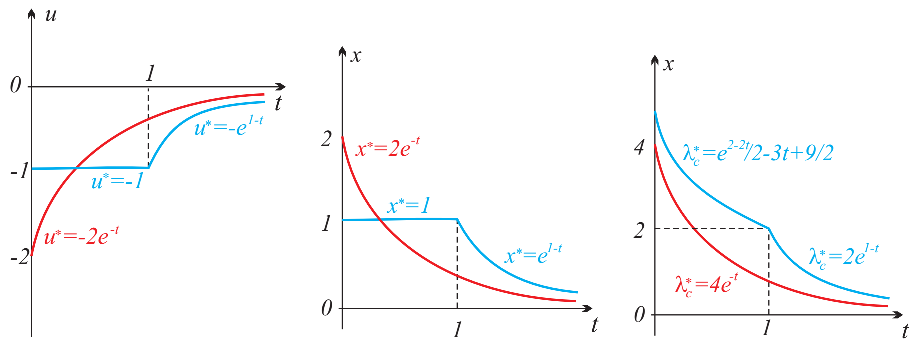
$$\lambda_c^*(\tau) = Ae^{2\tau} - 3\tau + \frac{9}{2} = 2c_2e^{-\tau} = 2.$$

It is an easy calculation to see that

$$u^*(t) = \begin{cases} -1 & \text{if } 0 \le t < 1 \\ -e^{1-t} & \text{if } t \ge 1 \end{cases} \qquad x^*(t) = \begin{cases} 2-t & \text{if } 0 \le t < 1 \\ e^{1-t} & \text{if } t \ge 1 \end{cases}$$

$$\lambda_c^*(t) = \begin{cases} \frac{1}{2}e^{2t-2} - 3t + \frac{9}{2} & \text{if } 0 \le t < 1 \\ 2e^{1-t} & \text{if } t \ge 1 \end{cases}$$

Since the Hamiltonian is convex and the multiplier $\lambda^* = \lambda_c^* e^{-2t}$ is bounded, then u^* is the minimum.



In red: the optimal tern of the problem (3.105), in the case $r = 2,\ a = 3,\ b = 1,\ x_0 = 2$.
In blue: the optimal tern of the problem (3.111).

3.7.1 A model of optimal consumption with log–utility I

We solve[11] the model presented in the example 1.1.5, formulated with (1.6), recalling that here $\delta > r$. Secondly we study a particular case where there is not an excess return, i.e. if where δ is a given discount rate and $r > 0$ is a given rate to return, then we have $\delta = r$. In both these cases, we consider a logarithmic utility function $U(c) = \log c$.

The case $\delta > r$: we have to study (1.6). The current Hamiltonian is $H^c = \ln c + \lambda_c(rx - c)$ and the sufficient condition (3.102) and (3.104) give

$$\dot{\lambda}_c = (\delta - r)\lambda_c \tag{3.117}$$
$$c(t) \in \arg\max_{v\ge 0}(\ln v + \lambda_c(rx - v)). \tag{3.118}$$

Clearly (3.117) gives $\lambda_c = Ae^{(\delta-r)t}$ for some constant A and the max in (3.118) depends on the such A : more precisely

$$c(t) = \begin{cases} \nexists & \text{if } \lambda_c(t) \le 0 \\ \frac{1}{\lambda_c(t)} & \text{if } \lambda_c(t) > 0 \end{cases}$$

Let us suppose[12] that $A > 0$. Hence (3.118) gives $c(t) = \frac{1}{A}e^{(r-\delta)t}$. From the dynamics we obtain $\dot{x} = rx - \frac{1}{A}e^{(r-\delta)t}$ and hence

$$\begin{aligned} x(t) &= e^{\int_0^t r\,\mathrm{d}s}\left(x_0 - \frac{1}{A}\int_0^t e^{(r-\delta)s}e^{-\int_0^s r\,\mathrm{d}v}\,\mathrm{d}s\right) \\ &= \left(x_0 - \frac{1}{A\delta}\right)e^{rt} + \frac{1}{A\delta}e^{(r-\delta)t}. \end{aligned}$$

[11] In subsection 5.7.4 we solve the same problem with the variational approach.
[12] We note that this assumption, taking into account that the multiplier is a shadow price (see section 5.6), is reasonable since if the initial capital x_0 increases, then the value of the max (the total discounted utility) increases.

The condition $\lim_{t\to\infty} x(t) = 0$ implies $A = \frac{1}{x_0\delta}$: note that this result is consistent with the assumption on the sign of A. Hence we obtain

$$c(t) = x_0\delta e^{(r-\delta)t} \qquad \text{and} \qquad x(t) = x_0 e^{(r-\delta)t}.$$

Since the Hamiltonian is convex and the multiplier $\lambda^*(t) = \lambda_c^*(t)e^{-\delta t} = \frac{1}{x_0\delta}e^{-rt}$ is bounded, we have the optimal path of consumption.

The case $\delta = r$: we suppose that the consumption c is bounded with the spending limit $c_{\max} \geq rx_0$ and let us remove the assumption $\lim_{t\to\infty} x(t) = \infty$: then the problem is

$$\begin{cases} \max \displaystyle\int_0^\infty e^{-rt} \ln c \,\mathrm{d}t \\ \dot{x} = rx - c \\ x(0) = x_0 > 0 \\ x \geq 0 \\ 0 \leq c \leq c_{\max} \end{cases}$$

Clearly the current Hamiltonian is $H^c = \ln c + \lambda_c(rx - c)$ and the sufficient condition (3.102) and (3.104) give

$$\begin{aligned} &\dot{\lambda}_c = r\lambda_c - r\lambda_c = 0 \\ &c(t) \in \arg \max_{v\in[0,c_{\max}]} (\ln v + \lambda_c(rx - v)). \end{aligned} \tag{3.119}$$

We obtain that $\lambda_c(t) = \lambda_c(0)$ for every $t \geq 0$. We note that

$$\frac{\partial H^c}{\partial c} = \frac{1}{c} - \lambda_c(0)$$

and hence, taking into account (3.119), we obtain: if $\lambda_c(0) \leq 0$, then H^c increases in $[0, c_{\max}]$ and $c(t) = c_{\max}$; if $0 < \frac{1}{\lambda_c(0)} \leq c_{\max}$, then $c(t) = \frac{1}{\lambda_c(0)}$; if $\frac{1}{\lambda_c(0)} > c_{\max}$, then $c(t) = c_{\max}$. In all the cases we obtain that $c(t) = k$ is a constant: hence the dynamics gives

$$x(t) = e^{\int_0^t r\,\mathrm{d}s} \left(x_0 - \int_0^t k e^{-\int_0^s r\,\mathrm{d}v} \mathrm{d}s \right) = x_0 e^{rt} - k\frac{e^{rt} - 1}{r}. \tag{3.120}$$

We note that $k > rx_0$ implies that $\lim_{t\to\infty} x(t) = -\infty$: this contradicts the requirement on the capital $x > 0$. Hence we consider such constant control $c(t) = k$ with $k \leq rx_0$: we have

$$\max_{\{c(t)=k:\ k\leq rx_0\}} \int_0^\infty e^{-rt} \ln c \,\mathrm{d}t = \left(\max_{k\leq rx_0} \ln k \right) \int_0^\infty e^{-rt}\,\mathrm{d}t;$$

hence $c^*(t) = rx_0 \leq c_{\max}$ is the optimal choice in the constant consumptions. Such control c^* gives a path of the capital constant $x^*(t) = x_0$ and a current multiplier $\lambda_c^*(t) = \frac{1}{rx_0}$.

In order to guarantee that c^* is the max, we note that the Hamiltonian is concave in the variable (x, c) since the dynamics is linear in the variables x and c, the running cost is concave in c. To conclude we have to show that the transversality condition (3.83) in Theorem 3.11 holds: for every admissible trajectory x we have $x(t) > 0$ and hence

$$-x_0 < x(t) - x_0 = x(t) - x^*(t).$$

Taking into account that the multiplier $\lambda^*(t) = \lambda_c^*(t)e^{-rt} = \lambda_c^*(0)e^{-rt}$ is positive, we obtain

$$\lim_{t\to\infty} \lambda^*(t)(x(t) - x^*(t)) \geq \lim_{t\to\infty} -\lambda^*(t)x_0 = -\frac{1}{r} \lim_{t\to\infty} e^{-rt} = 0.$$

Hence c^* is really the optimal path of consumption.

Chapter 4

Constrained problems of OC

The constrained optimal control problems are treated exhaustively in [27] (chapter 8, section C); one can also consult [5] and [19].

4.1 The general case

Let $f, g : [t_0, t_1] \times \mathbb{R}^{n+k} \to \mathbb{R}$ be the running cost and the dynamics respectively; let $h = (h_1, \ldots, h_m) : [t_0, t_1] \times \mathbb{R}^{n+k} \to \mathbb{R}^m$ be the function for the m equality/inequality constraints and $b = (b_1, \ldots, b_r) : [t_0, t_1] \times \mathbb{R}^{n+k} \to \mathbb{R}^r$ be the function for the r integral equality/inequality constraints; let $\psi = (\psi_0, \psi_1, \ldots, \psi_r) : \mathbb{R}^n \to \mathbb{R}^{r+1}$ be the pay off function and $\boldsymbol{\alpha} \in \mathbb{R}^n$ the initial point of the trajectory. Let $0 \le m' \le m,\ 0 \le r' \le r$. We consider the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi_0(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ h_j(t, \mathbf{x}(t), \mathbf{u}(t)) \ge 0 \quad \text{for } 1 \le j \le m' \\ h_j(t, \mathbf{x}(t), \mathbf{u}(t)) = 0 \quad \text{for } m'+1 \le j \le m \\ B_j(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} b_j(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi_j(\mathbf{x}(t_1)) \ge 0 \quad \text{for } 1 \le j \le r' \\ B_j(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} b_j(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi_j(\mathbf{x}(t_1)) = 0 \quad \text{for } r'+1 \le j \le r \\ \displaystyle\max_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}) \\ \mathcal{C} = \{\mathbf{u} : [t_0, t_1] \to \mathbb{R}^k,\ \mathbf{u} \text{ admissible for } \boldsymbol{\alpha} \text{ in } t_0\} \end{cases} \tag{4.1}$$

where t_1 is fixed. We note that we consider a control set U equal to $\mathbb{R}^k$ since all the possible constraints on the value of $\mathbf{u}$ can be write in form of the inequality constraints $h_j(t, \mathbf{x}, \mathbf{u}) \ge 0$: hence all the restrictions on the control is incorporated in these type of constraints. We remark that we require that the every functions h_j depends on the control (see section 4.2 for the case $h_j(t, \mathbf{x}, \mathbf{u}) = h(t, \mathbf{x})$).

As in the static optimization problem with constraints, there are qualifying conditions for the equality/inequality constraints that must be true. Here we are not interested in the arguments of the well–known Arrow-Hurwicz-Uzawa condition (see for example [27]); we only recall a sufficient condition so that the constraints are qualified. The problem to qualify the equality/inequality integral constraints is very different: we will treat this problem in the particular situation of Calculus of Variation in the next section.

Proposition 4.1. *Any one of the following conditions provides the equality/inequality constraint qualification in* $(\mathbf{u}^*, \mathbf{x}^*)$, *where* $\mathbf{u}^*$ *is a control and* $\mathbf{x}^*$ *is the associated trajectory:*

a) the functions $h_j(t, \mathbf{x}, \mathbf{u})$ *are convex in the variable* $\mathbf{u}$, *for all* $\mathbf{x} \in \mathbb{R}^n$, $t \in [t_0, t_1]$ *fixed and* $j = 1, \ldots, m$;

b) the functions $h_j(t, \mathbf{x}, \mathbf{u})$ *are linear in the variable* $\mathbf{u}$, *for all* $\mathbf{x} \in \mathbb{R}^n$, $t \in [t_0, t_1]$ *fixed and* $j = 1, \ldots, m$;

c) the functions $h_j(t, \mathbf{x}, \mathbf{u})$ *are concave in the variable* $\mathbf{u}$, *for all* $\mathbf{x} \in \mathbb{R}^n$, $t \in [t_0, t_1]$ *fixed and* $j = 1, \ldots, m$; *moreover, there exists* $\mathbf{u}' \in U$ *such that* $h(t, \mathbf{x}^*(t), \mathbf{u}') > 0$ *for every* $t \in [t_0, t_1]$;

d) (rank condition) *for every* $t \in [t_0, t_1]$ *fixed, the rank of the* $m \times (k + m)$ *matrix*

$$\begin{pmatrix} \dfrac{\partial h_1(t,\mathbf{x},\mathbf{u})}{\partial u_1} & \ldots & \dfrac{\partial h_1(t,\mathbf{x},\mathbf{u})}{\partial u_k} & h_1(t,\mathbf{x},\mathbf{u}) & \ldots & 0 \\ \dfrac{\partial h_2(t,\mathbf{x},\mathbf{u})}{\partial u_1} & \ldots & \dfrac{\partial h_2(t,\mathbf{x},\mathbf{u})}{\partial u_k} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 & \ldots & 0 \\ \dfrac{\partial h_m(t,\mathbf{x},\mathbf{u})}{\partial u_1} & \ldots & \dfrac{\partial h_m(t,\mathbf{x},\mathbf{u})}{\partial u_k} & 0 & \ldots & h_m(t,\mathbf{x},\mathbf{u}) \end{pmatrix}_{(\mathbf{x}^*,\mathbf{u}^*)}$$

is equal to the number m *of the constraints. This condition is equivalent to require that, for every* $t \in [t_0, t_1]$ *fixed, the rank of the matrix*

$$\left(\frac{\partial h_E}{\partial \mathbf{u}} \right) = \begin{pmatrix} \dfrac{\partial h_{i_1}(t,\mathbf{x},\mathbf{u})}{\partial u_1} & \dfrac{\partial h_{i_1}(t,\mathbf{x},\mathbf{u})}{\partial u_2} & \ldots & \dfrac{\partial h_{i_1}(t,\mathbf{x},\mathbf{u})}{\partial u_k} \\ \dfrac{\partial h_{i_2}(t,\mathbf{x},\mathbf{u})}{\partial u_1} & \dfrac{\partial h_{i_2}(t,\mathbf{x},\mathbf{u})}{\partial u_2} & \ldots & \dfrac{\partial h_{i_2}(t,\mathbf{x},\mathbf{u})}{\partial u_k} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial h_{i_e}(t,\mathbf{x},\mathbf{u})}{\partial u_1} & \dfrac{\partial h_{i_e}(t,\mathbf{x},\mathbf{u})}{\partial u_2} & \ldots & \dfrac{\partial h_{i_e}(t,\mathbf{x},\mathbf{u})}{\partial u_k} \end{pmatrix}_{(\mathbf{x}^*,\mathbf{u}^*)}$$

is equal to the number of effective constraints, where in $\left(\frac{\partial h_E}{\partial \mathbf{u}}\right)$ *we consider the indices* $i_j \in E$ *such that the constraint* h_{i_j} *is effective*[1].

We define the *Hamiltonian function* $H : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ as

$$H(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \lambda_0 f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\nu} \cdot b(t, \mathbf{x}, \mathbf{u})$$

[1] We recall that a constraint $h_j(t, \mathbf{x}, \mathbf{u}) \geq 0$ is effective if $h_j(t, \mathbf{x}, \mathbf{u}) = 0$; hence

$$E = \{j : \ 1 \leq j \leq m, \ h_j(t, \mathbf{x}, \mathbf{u}) = 0\}.$$

and the *Lagrangian function* $L : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}$ as

$$L(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}, \boldsymbol{\nu}, \boldsymbol{\mu}) = H(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}, \boldsymbol{\nu}) + \boldsymbol{\mu} \cdot h(t, \mathbf{x}, \mathbf{u}).$$

We note that the dimensions of the "new multiplier" $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ coincide with the number of the equality/inequality integral constraints r and with the number of the equality/inequality constraints m respectively.

We have the following necessary condition (for the proof see theorem 8.C.4 in [27]):

Theorem 4.1 (di Hestenes)**.** *Let us consider the problem (4.1) with* $f \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $g \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $h \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $b \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$ *and* $\psi \in C^1(\mathbb{R}^n)$.
Let $\mathbf{u}^*$ *be optimal control and* $\mathbf{x}^*$ *be the associated trajectory. Let us suppose that the rank condition for the* m *equality/inequality constraints holds.*
Then, there exists a multiplier $(\lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*)$, *with*

- ◇ λ_0 *constant,*
- ◇ $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*) : [t_0, t_1] \to \mathbb{R}^n$ *continuous,*
- ◇ $\boldsymbol{\nu}^* = (\nu_1^*, \ldots, \nu_r^*)$ *constant,*
- ◇ $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_m^*) : [t_0, t_1] \to \mathbb{R}^m$ *piecewise continuous (but continuous in the discontinuity points of* $\mathbf{u}^*$),

such that

$$(\lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*) \neq (0, \mathbf{0}, \mathbf{0}, \mathbf{0});$$
$$\nu_j^* B_j(\mathbf{u}^*) = 0 \quad \text{for } j = 1, \ldots, r, \quad \text{and} \quad \nu_j^* \geq 0 \quad \text{for } j = 1, \ldots, r';$$
$$\mu_j^* h_j(t, \mathbf{x}^*, \mathbf{u}^*) = 0 \quad \text{for } j = 1, \ldots, m, \quad \text{and} \quad \mu_j^* \geq 0 \quad \text{for } j = 1, \ldots, m'.$$

Such multiplier satisfies the following conditions:

i) (PMP): for all $\tau \in [t_0, t_1]$ *we have*

$$H(\tau, \mathbf{x}^*(\tau), \mathbf{u}^*(\tau), \lambda_0^*, \boldsymbol{\lambda}^*(\tau), \boldsymbol{\nu}^*) = \max_{\mathbf{v} \in U_{\tau, \mathbf{x}^*(\tau)}} H(\tau, \mathbf{x}^*(\tau), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(\tau), \boldsymbol{\nu}^*)$$

where for $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$ *we define*

$$U_{t,\mathbf{x}} = \Big\{ \mathbf{v} \in \mathbb{R}^k : \; h_j(t, \mathbf{x}, \mathbf{v}) \geq 0 \text{ for } 1 \leq j \leq m', \quad h_i(t, \mathbf{x}, \mathbf{v}) = 0 \text{ for } m' + 1 \leq i \leq m \Big\};$$

ii) (adjoint equation): in $[t_0, t_1]$ *we have*

$$\dot{\boldsymbol{\lambda}}^* = -\nabla_{\mathbf{x}} L(t, \mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*);$$

iii) (transversality condition)

$$\boldsymbol{\lambda}^*(t_1) = \nabla_{\mathbf{x}} \Psi(\mathbf{x}^*(t_1)),$$

where $\Psi = \lambda_0^* \psi_0 + \sum_{j=1}^{r} \nu_j^* \psi_j$;

iv) in $[t_0, t_1]$ *we have*

$$\nabla_{\mathbf{u}} L(t, \mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*) = 0.$$

A sufficient condition with a proof very similar to the theorem 2.3 of Mangasarian is the following (for the proof see theorem 8.C.5 in [27])

Theorem 4.2. *Let us consider the maximum problem (4.1) with* $f \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $g \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $h \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $b \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$ *and* $\psi \in C^1(\mathbb{R}^n)$.
Let $\mathbf{u}^*$ *be admissible control in* $\boldsymbol{\alpha}$ *with associated trajectory* $\mathbf{x}^*$ *and associated multiplier* $(\lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ *that satisfies all the thesis of theorem 4.1.*
Moreover, let us suppose that

v) f, g, h, b *and* ψ *are concave functions in the variables* $\mathbf{x}$ *and* $\mathbf{u}$, *for all* $t \in [t_0, t_1]$ *fixed,*

vi) $\lambda_0^* = 1$ *and for all* i, $1 \le i \le n$, $t \in [t_0, t_1]$ *we have* $\lambda_i^*(t) \ge 0$.

Then $\mathbf{u}^*$ *is optimal.*

We say that the problem (4.1) is autonomous if all the functions involved in the statement does not depend on t. In this situation we obtain

Remark 4.1. *Consider the problem (4.1) and let us suppose that it is autonomous. Let* $\mathbf{u}^*$ *be optimal control and let* $\mathbf{x}^*$ *and* $(\lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*, \boldsymbol{\mu}^*)$ *be associated trajectory and multiplier respectively. Then the Hamiltonian is constant along the optimal path* $(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \boldsymbol{\lambda}^*)$, *i.e.*

$$t \mapsto H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)),$$

is constant in $[t_0, t_1]$.

Example 4.1.1. Consider[2]

$$\begin{cases} \max \int_0^1 (v - x)\, dt \\ \dot{x} = u \\ x(0) = \frac{1}{8} \\ u \in [0, 1] \\ v^2 \le x \end{cases}$$

The Hamiltonian H and the Lagrangian L are

$$H = v - x + \lambda u, \qquad L = v - x + \lambda u + \mu_1 u + \mu_2(1 - u) + \mu_3(x - v^2).$$

We have to satisfy the following necessary conditions:

$$(u(t), v(t)) \in \arg \max_{(u,v) \in U_{t,x(t)}} (v - x + \lambda u) \tag{4.2}$$

$$\text{where } U_{t,x} = \{(u, v) \in [0, 1] \times \mathbb{R} : \ v^2 \le x\}$$

$$\dot{\lambda} = -\frac{\partial L}{\partial x} \quad \Rightarrow \quad \dot{\lambda} = 1 - \mu_3 \tag{4.3}$$

$$\lambda(1) = 0 \tag{4.4}$$

$$\frac{\partial L}{\partial u} = 0 \quad \Rightarrow \quad \lambda + \mu_1 - \mu_2 = 0 \tag{4.5}$$

$$\frac{\partial L}{\partial v} = 0 \quad \Rightarrow \quad 1 - 2v\mu_3 = 0 \tag{4.6}$$

$$\mu_1 \ge 0 \quad (= 0 \text{ if } u > 0)$$

$$\mu_2 \ge 0 \quad (= 0 \text{ if } u < 1)$$

$$\mu_3 \ge 0 \quad (= 0 \text{ if } v^2 < x)$$

[2]This example is proposed in [26].

Clearly (4.2) implies

$$(u(t), v(t)) = \begin{cases} (1, \sqrt{x}) & \text{if } \lambda > 0 \\ (?, \sqrt{x}) & \text{if } \lambda = 0 \\ (0, \sqrt{x}) & \text{if } \lambda < 0 \end{cases}$$

If $\lambda > 0$, $(u(t), v(t)) = (1, \sqrt{x})$ implies by the dynamics $x = t + A$ for some constant A. (4.6) gives $\mu_3 = \frac{1}{2\sqrt{t+A}}$ and hence, by (4.3),

$$\dot{\lambda} = 1 - \frac{1}{2\sqrt{t+A}} \quad \Rightarrow \quad \lambda = t - \sqrt{t+A} + B,$$

for some constant B. $\mu_1 = 0$ implies by (4.5) $\mu_2 = t - \sqrt{t+A} + B$.

If $\lambda < 0$, $(u(t), v(t)) = (0, \sqrt{x})$ implies by the dynamics $x = C$ for some constant C. (4.6) gives $\mu_3 = \frac{1}{2\sqrt{C}}$ and hence, by (4.3),

$$\dot{\lambda} = 1 - \frac{1}{2\sqrt{C}} \quad \Rightarrow \quad \lambda = \left(t - \frac{1}{2\sqrt{C}}\right) t + D,$$

for some constant D. $\mu_2 = 0$ implies by (4.5) $\mu_1 = -\left(t - \frac{1}{2\sqrt{C}}\right) t - D$.

Let us suppose that for some $\tau > 0$, we have $\lambda > 0$ in $[0, \tau)$: the initial condition $x(0) = \frac{1}{8}$ implies $A = \frac{1}{8}$. If $\tau > 1$, (4.5) gives $B = \frac{3}{2\sqrt{2}} - 1$ and hence $\lambda(t) = t - \sqrt{t + \frac{1}{8}} + \frac{3}{2\sqrt{2}} - 1$: note that we obtain $\lambda(0) = \frac{1}{\sqrt{2}} - 1 < 0$. Hence $\tau < 1$.

Let us suppose that $\lambda < 0$ in $(\tau, 1]$: (4.5) gives $D = \frac{1}{2\sqrt{2}} - 1$. Now the continuity of μ_1 in τ (note that τ is a discontinuity point for u and hence $\boldsymbol{\mu}$ is continuous) implies

$$\mu_1(\tau^-) = 0 = -\left(1 - \frac{1}{2\sqrt{C}}\right)\tau - \frac{1}{2\sqrt{C}} + 1 = \mu_1(\tau^+) \quad \Rightarrow \quad C = \frac{1}{4}.$$

The continuity of x and λ in τ give $C = \frac{1}{4}$. and $B = -\frac{3}{8}$: in particular we obtain $\lambda = 0$ in $[\frac{1}{8}, 1]$ that contradicts the assumption $\lambda < 0$.

Hence let us suppose that $\lambda = 0$ in $[\frac{1}{8}, 1]$; (4.3) gives $\mu_3 = 1$ and hence, by (4.6) $v = \frac{1}{2}$. The (PMP) gives $v = \sqrt{x}$ and hence $x = \frac{1}{4}$; the dynamics gives $u = 0$ and finally, by (4.5) and the continuity of the multiplier in $\frac{1}{8}$, $\mu_1 = \mu_2 = 0$. Hence we obtain the following situation:

	x	u	v	λ	μ_1	μ_2	μ_3
in $[0, \frac{1}{8})$	$t + \frac{1}{8}$	1	$\sqrt{t + \frac{1}{8}}$	$t - \sqrt{t + \frac{1}{8}} + \frac{3}{8}$	*0*	$t - \sqrt{t + \frac{1}{8}} + \frac{3}{8}$	$\frac{1}{2\sqrt{t+\frac{1}{8}}}$
in $[\frac{1}{8}, 1]$	$\frac{1}{4}$	*0*	$\frac{1}{2}$	0	0	0	1

Let us verify that the rank condition holds:

$$\begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} & h_1 & 0 & 0 \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} & 0 & h_2 & 0 \\ \frac{\partial h_3}{\partial u} & \frac{\partial h_3}{\partial v} & 0 & 0 & h_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & u & 0 & 0 \\ -1 & 0 & 0 & 1-u & 0 \\ 0 & -2v & 0 & 0 & x - v^2 \end{pmatrix};$$

it is to see the for every $t \in [0, 1]$ the rank condition holds in the tern (x, u, v) previous obtained. Finally it is easy to verify that the sufficient conditions of Theorem 4.2 are satisfied. Note that the problem is autonomous and the have $H = v - x + \lambda u = \frac{1}{4}$ in $[0, 1]$. △

4.2 Pure state constraints

An important and particular situation is the case of the equality/inequality constraints where the functions h_j in (4.1) do not depend on the control, i.e. constraints of the type

$$h_j(t, \mathbf{x}(t), \mathbf{u}(t)) = h_j(t, \mathbf{x}(t)) \geq 0 \quad (\text{or } = 0).$$

A simplest example of this situation is $x(t) \geq 0$. We remark that with this type of constraints, the condition of qualification fails since $\frac{\partial h_j}{\partial u_i} = 0$. Hence let us give the fundamental ideas of this constrained problem, called *pure state constraints*: a very exhaustive exposition is in [26].

We consider the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u}) \, \mathrm{d}t + \psi_0(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ h_j(t, \mathbf{x}(t)) \geq 0 \quad \text{for } 1 \leq j \leq m \\ \max_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}) \\ \mathcal{C} = \{\mathbf{u} : [t_0, t_1] \to U, \ \mathbf{u} \text{ admissible for } \boldsymbol{\alpha} \text{ in } t_0\} \end{cases} \tag{4.7}$$

where $t_1 \in \mathbb{R}$ and $U \subset \mathbb{R}^k$ are fixed. We introduce the Hamiltonian and the Lagrangian functions as usual.

In previously discussed constrained problems, the solution is predicated upon the continuity of $\mathbf{x}$ and $\boldsymbol{\lambda}$ variables, so that only the control variable $\mathbf{u}$ is allowed to jump: here, with pure state constraints, the multiplier $\boldsymbol{\lambda}$ can also experience jumps at the junction points where the constraint $h(t, \mathbf{x}(t)) \geq 0$ turns from inactive to active, or vice versa. The condition *v)* in the next theorem checks the jump of $\boldsymbol{\lambda}$ in such discontinuity points τ_l with respect the effective constraints h_j. The following result can be proved (see [26])

Theorem 4.3. *Let us consider the problem (4.7) with* $f \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $g \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$, $h \in C^1([t_0, t_1] \times \mathbb{R}^n)$, *and* $\psi_0 \in C^1(\mathbb{R}^n)$.
Let $\mathbf{u}^*$ *be an admissible control and* $\mathbf{x}^*$ *be the associated trajectory.*
We assume that there exist a multiplier $(\lambda_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, *with*

- ◇ $\lambda_0^* = 1$,
- ◇ $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*) : [t_0, t_1] \to \mathbb{R}^n$ *is piecewise continuous and piecewise continuously differentiable with jump discontinuities at* $\tau_1, \ldots, \tau_N$, *with* $t_0 < \tau_1 < \ldots < \tau_N \leq t_1$,
- ◇ $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_m^*) : [t_0, t_1] \to \mathbb{R}^m$ *piecewise continuous,*

and

- ◇ *numbers* β_s^l, *with* $1 \leq l \leq N$, $1 \leq s \leq m$,

such that the following conditions are satisfied:

i) (PMP): for all $\tau \in [t_0, t_1]$ *we have*

$$\mathbf{u}^*(\tau) \in \arg\max_{\mathbf{v} \in U} H(\tau, \mathbf{x}^*(\tau), \mathbf{v}, \boldsymbol{\lambda}^*(\tau))$$

ii) (adjoint equation): in $[t_0, t_1]$ *we have*

$$\dot{\boldsymbol{\lambda}}^* = -\nabla_{\mathbf{x}} L(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*);$$

iii) (transversality condition)

$$\boldsymbol{\lambda}^*(t_1) = \nabla_{\mathbf{x}}\psi_0(\mathbf{x}^*(t_1));$$

iv) $\mu_j^* h_j(t, \mathbf{x}^*) = 0$ *and* $\mu_j^* \geq 0$, *for* $j = 1, \ldots, m$;

v) the numbers β_j^l *are non negative and such that*

$$\lambda_i^*(\tau_l^-) - \lambda_i^*(\tau_l^+) = \sum_{j=1}^m \beta_j^l \frac{\partial h_j(\tau_l, \mathbf{x}^*(\tau_l))}{\partial x_i} \qquad \text{for } 1 \leq l \leq N, \ 1 \leq i \leq n;$$

moreover

v_1*)* $\beta_j^l = 0$ *if* $h_j(\tau_l, \mathbf{x}^*(\tau_l)) > 0$;

v_2*)* $\beta_j^l = 0$ *if* $h_j(\tau_l, \mathbf{x}^*(\tau_l)) = 0$ *and* $\nabla_{\mathbf{x}} h_j(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t))$ *is discontinuous at* $\tau_l \in (t_0, t_1)$;

vi) for every $\tau \in [t_0, t_1]$, *the function*

$$H^0(\tau, \mathbf{x}, \boldsymbol{\lambda}^*(\tau)) = \max_{\mathbf{v} \in U} H(\tau, \mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}^*(\tau))$$

is concave in $\mathbf{x}$;

vii) h *and* ψ_0 *are concave in* $\mathbf{x}$.

Then $\mathbf{u}^*$ *is optimal.*

We think that the following example and the model in subsection 4.2.1 make clear the assumption of the previous theorem.

Example 4.2.1. *Consider*[3]

$$\begin{cases} \max \displaystyle\int_0^3 (4-t)u \, \mathrm{d}t \\ \dot{x} = u \\ x(0) = 0 \\ x(3) = 3 \\ t + 1 - x \geq 0 \\ u \in [0, 2] \end{cases}$$

The Hamiltonian H *and the Lagrangian* L *are*

$$H = (4-t)u + \lambda u, \qquad L = (4-t)u + \lambda u + \mu(t+1-x).$$

We have to satisfy the following necessary conditions:

$$u(t) \in \arg\max_{v \in [0,2]} (4 - t + \lambda)v \tag{4.8}$$

$$\dot{\lambda} = -\frac{\partial L}{\partial x} \quad \Rightarrow \quad \dot{\lambda} = \mu \tag{4.9}$$

$$\mu \geq 0 \quad (= 0 \text{ if } t + 1 - x > 0)$$

The shape of the running cost function $f(t, x, u) = (4-t)u$ *suggests to put* $u = 2$ *in the first part of the interval* $[0, 3]$. *Since* $x(0) = 0$, *there exists* $\tau_1 > 0$ *such that the constraint* $h(t, x) = t + 1 - x$ *is inactive in* $[0, \tau_1]$: *in such interval we have* $\mu = 0$ *and hence (by (4.9))* $\lambda = A$ *for some constant* A. *We suppose that in* $[0, \tau_1]$

$$t < 4 + A : \tag{4.10}$$

[3]This example is proposed in [26] and it is solved in [9] with a different approach.

This implies that in our interval, by (4.9), $u(t) = 2$ *and, by the dynamics and the initial condition* $x(t) = 2t$*. With this trajectory we have that*

$$h(t, x(t)) = t + 1 - x(t) = 1 - t > 0 \quad \Leftrightarrow \quad t < 1;$$

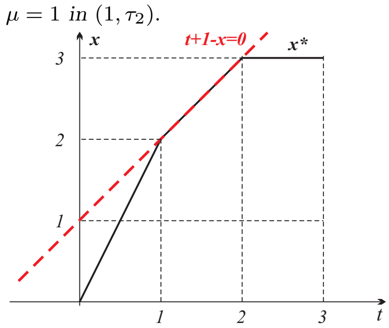
we obtain $\tau_1 = 1$*.*

In order to maximize, it is a good strategy to increase again $f(t, x, u) = (4 - t)u$ *with the trajectory lying in the constraint on the interval* $[1, \tau_2]$*, for some* $\tau_2 > 1$*. In order to do that, let us study condition* $v)$ *of Theorem 4.3 in the point* $\tau_1 = 1$ *:*

$$\lambda(\tau_1^-) - \lambda(\tau_1^+) = \beta_1 \frac{\partial h(\tau_1, x(\tau_1))}{\partial x} = -\beta_1,$$
$$\nabla_x h(t, x(t)) \cdot g(t, x(t), u(t)) = -u(t);$$

since for $t < 1$ *we know that* $u(t) = 2$ *and in order to have* $h(t, x(t)) = 0$ *for* $t \in [1, \tau_2)$ *we have to require* $u(t) = 1$*, the control* u *has a discontinuity point in* $t = 1$ *: condition* v_2 *implies that* $\beta_1 = 0$ *and* λ *continuous in* τ_1*. Hence, in order to satisfy (PMP), for* $t \in [1, \tau_2)$ *we put* $\lambda(t) = t - 4$*; the continuity of such multiplier in* $t = 1$ *implies* $A = -3$*. Note that the previous assumption (4.10) holds. Moreover, by (4.9), we have* $\mu = 1$ *in* $(1, \tau_2)$*.*

Since $u \geq 0$ *and hence the trajectory is a non decreasing function, in order to obtain the final condition* $x(3) = 3$*, we can consider* $\tau_2 = 2$ *and* $u = 0$ *in the final interval* $(2, 3]$*. Clearly, with this choice, we have a discontinuity for the control* u *in the point* $\tau_2 = 2$ *and the same calculations of before give us that* λ *is continuous in* $t = 2$*. For* $t \in (2, 3]$*, the constraint is inactive,* $\mu = 0$ *and again* $\lambda = B$ *is a constant: the continuity of the multiplier in 2 implies* $B = -2$*.*



Hence we obtain the following situation:

	x	u	λ	μ
in $[0, 1)$	$2t$	2	-3	0
in $[1, 2]$	$t + 1$	1	$t - 4$	1
in $(2, 3]$	3	0	-2	0

Since the function H^0 *of condition* $vi)$ *in Theorem 4.3 and the constraint* h *are linear in* x*, then the previous strategy is optimal.* △

4.2.1 Commodity trading

Let us denote by $x_1(t)$ and $x_2(t)$ respectively the money on hand and the amount of wheat owned at time t. Let $x_1(0) = m_0 > 0$ and $x_2(t) = w_0 > 0$. At every time we have the possibility to buy or to sell some wheat: we denote by $\alpha(t)$ our strategy, where $\alpha > 0$ means buying wheat, and $\alpha < 0$ means selling. We suppose that the price of the wheat is a known function $q(t)$ for all the period $[0, T]$, with T fixed (clearly $q > 0$). Let $s > 0$ be the constant cost of storing a unit of amount of wheat for a unit of time. We assume also that the rate of selling and buying is bounded; more precisely $|\alpha| \leq M$, for a given fixed positive constant M.

Our aim is to maximize our holdings at the final time T, namely the sum of the cash on hand and the value of the wheat. Hence we have:

$$\begin{cases} \max\,(x_1(T) + q(T)x_2(T)) \\ \dot{x}_1 = -sx_2 - q\alpha \\ \dot{x}_2 = \alpha \\ x_1(0) = m_0, \qquad x_2(0) = w_0 \\ x_1 \geq 0, \qquad x_2 \geq 0 \\ |\alpha| \leq M \end{cases}$$

Clearly the Hamiltonian H, the Lagrangian L and the pay-off ψ are

$$\begin{aligned} H &= -\lambda_1(sx_2 + q\alpha) + \lambda_2\alpha, \\ L &= -\lambda_1(sx_2 + q\alpha) + \lambda_2\alpha + \mu_1 x_1 + \mu_2 x_2, \\ \psi &= x_1 + qx_2. \end{aligned}$$

We have to satisfy the following necessary conditions:

$$\alpha(t) \in \arg\max_{a\in[-M,M]} [-\lambda_1(t)(sx_2(t) + q(t)a) + \lambda_2(t)a]$$

$$\Rightarrow \quad \alpha(t) \in \arg\max_{a\in[-M,M]} a(\lambda_2(t) - \lambda_1(t)q(t)) \tag{4.11}$$

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial x_1} \quad \Rightarrow \quad \dot{\lambda}_1 = -\mu_1 \tag{4.12}$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial x_2} \quad \Rightarrow \quad \dot{\lambda}_2 = s\lambda_1 - \mu_2 \tag{4.13}$$

$$\lambda_1(T) = \frac{\partial \psi}{\partial x_1} \quad \Rightarrow \quad \lambda_1(T) = 1 \tag{4.14}$$

$$\lambda_2(T) = \frac{\partial \psi}{\partial x_2} \quad \Rightarrow \quad \lambda_2(T) = q(T) \tag{4.15}$$

$$\mu_1 \geq 0 \quad (= 0 \text{ if } x_1 > 0)$$

$$\mu_2 \geq 0 \quad (= 0 \text{ if } x_2 > 0)$$

Now, to solve the model, let us consider a particular situation: we put

$$T = 2, \quad s = 3, \quad q(t) = t^2 + 1, \quad M = 4, \quad m_0 = 2 \quad \text{and} \quad w_0 = 2.$$

The shape of the function q suggests a strategy. In the first part of our time, the cost of storing the wheat is major then its value: hence it seams like a good idea to sell the wheal. In the final part of our time the price of the wheal, and hence the value of the wheat owned, increases: hence it is better to buy wheat. Let us follow this intuition in order to solve the problem.

We start with **the final part of** $[0, 2]$**.** It is reasonable to suppose that $x_1(2)$ and $x_2(2)$ are positive; hence the two constraints $h_1 = x_1 \geq 0$ and $h_2 = x_2 \geq 0$ are sure inactive in $t = 2$. This guarantees that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is continuous in such point $t = 2$. Hence let us suppose that in $(\tau, 2]$, for some $\tau < 2$, the multiplier $\boldsymbol{\lambda}$ is continuous and the constraints are inactive, i.e. $\mu_1 = \mu_2 = 0$. Clearly (4.12)–(4.15) imply

$$\lambda_1(t) = 1 \quad \text{and} \quad \lambda_2(t) = 3t - 1, \qquad \forall t \in (\tau, 2];$$

consequently (4.11) implies, for $t \in (\tau, 2]$

$$\alpha(t) \in \arg \max_{a \in [4,-4]} a(-t^2 + 3t - 2).$$

Since $-t^2 + 3t - 2 > 0$ in $(1, 2)$, we have $\alpha = 4$ in $(\tau, 2]$, for some $\tau \in (1, 2)$ (we recall that we have to check that the constraints are inactive in the interval $(\tau, 2]$).

Let us study **the first part of** $[0, 2]$. We note that $x_1(0)$ and $x_2(0)$ are positive: hence the two constraints are inactive in $t = 2$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is continuous in $t = 0$. Let us suppose that there exists $\tau' > 0$ such that for every $t \in [0, \tau')$ we have

$$\lambda_2(t) - \lambda_1(t)(t^2 + 1) < 0, \quad x_1(t) > 0 \quad \text{and} \quad x_2(t) > 0. \tag{4.16}$$

Then (4.11) implies $\alpha(t) = -4$, for $t \in [0, \tau')$. Using the dynamics and the initial conditions on x_1 and x_2, we obtain

$$x_1(t) = \frac{4}{3}t^3 + 6t^2 - 2t + 2 \quad \text{and} \quad x_2(t) = 2 - 4t.$$

It is easy to see that x_1 is positive in $[0, 2]$ and x_2 is positive only in $[0, 1/2)$. It gives us that

- in $[0, 1/2]$, $\mu_1 = 0$, λ_1 continuous and (by (4.12)) $\lambda_1(t) = A$,
- in $[0, 1/2)$, $\mu_2 = 0$, λ_2 continuous and (by (4.13)) $\lambda_2(t) = 3At + B$,
- the point $\tau' = \tau_1 = 1/2$ can be a jump for the function λ_2,

where A and B are constants. Let us study condition $v)$ of the Theorem 4.3 in the point $\tau_1 = 1/2$:

$$\lambda_1(\tau_1^-) - \lambda_1(\tau_1^+) = \beta_1^1 \frac{\partial h_1(\tau_1, \mathbf{x}(\tau_1))}{\partial x_1} + \beta_2^1 \frac{\partial h_2(\tau_1, \mathbf{x}(\tau_1))}{\partial x_1} = \beta_1^1,$$
$$\lambda_2(\tau_1^-) - \lambda_2(\tau_1^+) = \beta_1^1 \frac{\partial h_1(\tau_1, \mathbf{x}(\tau_1))}{\partial x_2} + \beta_2^1 \frac{\partial h_2(\tau_1, \mathbf{x}(\tau_1))}{\partial x_2} = \beta_2^1;$$

Since h_1 is inactive in τ_1, we have $\beta_2^1 = 0$ that confirms the continuity of λ_1 in τ_1. Since

$$\nabla_{\mathbf{x}} h_2(t, \mathbf{x}(t)) \cdot g(t, \mathbf{x}(t), \alpha(t)) = (0, 1) \cdot (-3x_2(t) - q(t)\alpha(t), \alpha(t)) = \alpha(t)$$

has a discontinuity point in τ_1 (for $t < \tau_1$ we know that $\alpha(t) = -4$ and in order to have $x_2(t) \geq 0$ for $t > \tau_1$ we have to require $\alpha(t) \geq 0$), condition v_2 implies that $\beta_2^1 = 0$: hence λ_2 is continuous in τ_1. The assumption (4.16) becomes

$$\lambda_2(t) - \lambda_1(t)(t^2 + 1) = -At^2 + 3At + B - A < 0 \qquad \text{for } t \in [0, 1/2);$$

moreover, in order to construct the discontinuity for α in $t = 1/2$ and to guarantee the PMP (4.11) in $t = 1/2$, it is necessary to have

$$\lambda_2(t) - \lambda_1(t)(t^2 + 1) = -At^2 + 3At + B - A = 0 \qquad \text{for } t = 1/2.$$

These last two conditions give

$$A = -4B \quad \text{and} \quad A > 0.$$

Now we pass to study **the middle part of** $[0,2]$, i.e. the set $[1/2,\tau]$. The idea is to connect the trajectory x_2 along the constraint $h_2 = 0$: in order to do this we put

$$u(t) = 0, \qquad \text{for } t \in [1/2,\tau]. \tag{4.17}$$

This clearly gives, in $[1/2,\tau]$,

$$x_2(t) = 0 \quad \Rightarrow \quad \dot{x}_1(t) = 0 \quad \Rightarrow \quad x_1(t) = 11/3,$$

since $x_1(1/2^-) = 11/3$. In $[1/2,\tau]$, since $x_1(t) > 0$ we have $\mu_1 = 0$. By (4.12) and the continuity of λ_1 in $t = 1/2$, we have $\lambda_1(t) = A$ in $[1/2,\tau]$. From (4.17) and (4.11) we have

$$0 \in \arg\max_{a\in[-4,4]} a(\lambda_2(t) - Aq(t)), \qquad \text{for } t \in [1/2,\tau].$$

This implies $\lambda_2(t) = Aq(t)$ in $[1/2,\tau]$. Since λ_2 is continuous in $t = 1/2$ and $\lambda_2(1/2^-) = 5A/4$, we have

$$\lambda_2(t) = At^2 + A.$$

The previous study of the point $\tau_1 = 1/2$ suggests that the multipliers is continuous where the control is discontinuous. Now to connect this second part $[0,\tau]$ with the final part $[\tau,1]$, we have to put

$$A = \lambda_1(\tau^-) = \lambda_1(\tau^+) = 1 \qquad At^2 + A = \lambda_2(\tau^-) = \lambda_2(\tau^+) = 3t - 1 :$$

we obtain $A = 1$ and $\tau = 1$. The dynamic and the continuity of x_1 and x_2 in the point $\tau = 1$ imply

$$x_2(t) = 4t - 4, \qquad \text{and } x_1(t) = \frac{4}{3}t^3 - 6t^2 + 16t - \frac{23}{3}, \qquad \text{for } t \in [1,2].$$

Note that $x_1(t) > 0$ in this final interval that guarantees $\mu_1 = 0$. Finally (4.12) implies $\mu_2 = 2t + 1$ in $[1/2,\tau]$. Hence we obtain the following situation:

	x_1	x_2	α	λ_1	λ_2	μ_1	μ_2
in $[0,1/2)$	$\frac{4}{3}t^3 + 6t^2 - 2t + 2$	$2 - 4t$	-4	1	$3t - \frac{1}{4}$	0	0
in $[1/2,1]$	$\frac{11}{3}$	0	0	1	$t^2 + 1$	0	$2t + 1$
in $(1,2]$	$\frac{4}{3}t^3 - 6t^2 + 16t - \frac{23}{3}$	$4t - 4$	4	1	$3t - 1$	0	0

Let us calculate the function H^0 of condition *vi)* in Theorem 4.3:

$$\begin{aligned}
H^0(t,x_1,x_2,\lambda_1^*,\lambda_2^*) &= \max_{a\in[-M,M]} [-\lambda_1(sx_2 + qa) + \lambda_2 a] \\
&= -3x_2 + \max_{a\in[-4,4]} \begin{cases} (-t^2 + 3t - 5/4)a & \text{if } t \in [0,1/2) \\ 0 & \text{if } t \in [1/2,1] \\ (-t^2 + 3t - 2)a & \text{if } t \in (1,2] \end{cases} \\
&= \begin{cases} -3x_2 - 4(-t^2 + 3t - 5/4) & \text{if } t \in [0,1/2) \\ -3x_2 & \text{if } t \in [1/2,1] \\ -3x_2 + 4(-t^2 + 3t - 2) & \text{if } t \in (1,2] \end{cases} .
\end{aligned}$$

Clearly, for every fixed t, the function H^0 is concave in (x_1, x_2). Since the constraints h_1 and h_2 and the pay–off function ψ are linear in x, then the previous strategy is optimal.

4.3 Isoperimetric problems in CoV

In this section we are interested to specialized the results of the previous section to the calculus of variation problems with only equality integral constraints for trajectory in $\mathbb{R}$ (i.e. with $n = 1$), with fixed initial and final points. This type of problems is in the big family of the *isoperimetric problems of calculus of variation*. More precisely, let $f : [t_0, t_1] \times \mathbb{R}^2 \to \mathbb{R}$, $r \geq 1$, and $b = (b_1, \ldots, b_r) : [t_0, t_1] \times \mathbb{R}^2 \to \mathbb{R}$. Let us consider the problem

$$\begin{cases} J(x) = \displaystyle\int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\,\mathrm{d}t \\ x(t_0) = \alpha \\ x(t_1) = \beta \\ B_j(x) = \displaystyle\int_{t_0}^{t_1} b_j(t, x(t), \dot{x}(t))\,\mathrm{d}t - \tilde{b}_j = 0 \quad \text{with } j = 1, \ldots, r \\ \mathrm{Ott}_{x \in \mathcal{A}_{iso}} J(x) \end{cases} \tag{4.18}$$

where α, β and $\tilde{b}_j$ are fixed constants and $\mathcal{A}_{iso}$ clearly is defined as

$$\mathcal{A}_{iso} = \{x \in C^1([t_0, t_1]);\ x(t_0) = \alpha,\ x(t_1) = \beta, B_j(x) = 0 \text{ for } 1 \leq j \leq r\}.$$

Since in problem 4.18 $u = \dot{x}$, we have $H = \lambda_0 f(t, x, u) + \lambda u + \boldsymbol{\nu} \cdot b$: as usual, we obtain

$$\begin{aligned} &\text{(PMP)} \ \Rightarrow\ \lambda_0^* \frac{\partial f}{\partial u} + \lambda^* + \boldsymbol{\nu}^* \cdot \frac{\partial b}{\partial u} = 0 \\ &\text{(adjoint equation)} \ \Rightarrow\ \lambda_0^* \frac{\partial f}{\partial x} + \boldsymbol{\nu}^* \cdot \frac{\partial b}{\partial x} = -\dot{\lambda}^*. \end{aligned}$$

Considering a derivative with respect to the time in the first relation, and replacing λ^* we obtain the EU for a new functions: more precisely we have that an immediate consequence of theorem 4.1 is the following

Theorem 4.4. *Let us consider (4.18) with f and b in C^2. Let $x^* \in C^1$ be a minimum (or a maximum).*

Then, there exists a constant multiplier $(\lambda_0^*, \boldsymbol{\nu}^*) \neq \mathbf{0}$ such that, in $[t_0, t_1]$, we have[4]

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{0\dot{x}}(t, x^*, \dot{x^*}, \lambda_0^*, \boldsymbol{\nu}^*) = L_{0x}(t, x^*, \dot{x^*}, \lambda_0^*, \boldsymbol{\nu}^*),$$

where L_0 is the generalized Lagrangian function $L_0 : [t_0, t_1] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ defined as

$$L_0(t, x, \dot{x}, \lambda_0, \boldsymbol{\nu}) = \lambda_0 f(t, x, \dot{x}) + \boldsymbol{\nu} \cdot b(t, x, \dot{x}).$$

We remark that, in the "language of optimal control", the function L_0 is an Hamiltonian; however, in the classical "language of calculus of variation" it is called Lagrangian: we prefer this second approach.

[4] We denote by L_{0x} and by b_{jx} the derivative with respect to x of the functions L_0 and b_k respectively.

4.3.1 Necessary conditions with regular constraints

As we will see, the study of isoperimetric problems in CoV has some similarities with the static optimization problem with equality constraints. Let us start with the following definition

Definition 4.1. *We consider the problem (4.18) with f and b in C^2 and let $x^* \in C^1$. We say that the constraints are regular in x^* if the r functions*

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{j\dot{x}}(t, x^*, \dot{x^*}) - b_{jx}(t, x^*, \dot{x^*}), \quad j = 1, \ldots, r$$

are linearly independent.

Moreover, we note that the definition of regularity of the constraints is related to the choice of the function x^*. The following example clarify the situation

Example 4.3.1. Consider

$$\begin{cases} \int_0^1 b_1(t, x, \dot{x})\,\mathrm{d}t = \int_0^1 x\dot{x}^2\,\mathrm{d}t = \tilde{b}_1 \\ \int_0^1 b_2(t, x, \dot{x})\,\mathrm{d}t = \int_0^1 -tx\,\mathrm{d}t = \tilde{b}_2 \end{cases}$$

Clearly

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{1\dot{x}}(t, x, \dot{x}) - b_{1x}(t, x, \dot{x}) = \dot{x}^2 + 2x\ddot{x},$$
$$\frac{\mathrm{d}}{\mathrm{d}t} b_{2\dot{x}}(t, x, \dot{x}) - b_{2x}(t, x, \dot{x}) = t.$$

It is easy to see that if we consider the function $x_1^* = 0$, the constraints are not regular since, for a_1 and a_2 constants, we have, for every $t \in [0, 1]$,

$$a_1 \left(\frac{\mathrm{d}}{\mathrm{d}t} b_{1\dot{x}}(t, x_1^*, \dot{x_1^*}) - b_{1x}(t, x_1^*, \dot{x_1^*}) \right) + a_2 \left(\frac{\mathrm{d}}{\mathrm{d}t} b_{2\dot{x}}(t, x_1^*, \dot{x_1^*}) - b_{2x}(t, x_1^*, \dot{x_1^*}) \right) = 0$$
$$\Leftrightarrow \ a_2 t = 0.$$

Choosing $a_1 \in \mathbb{R}$ and $a_2 = 0$, the last relation is satisfied. Hence the functions $\frac{\mathrm{d}}{\mathrm{d}t} b_{1\dot{x}}(t, x_1^*, \dot{x^*}_1) - b_{1x}(t, x^*, \dot{x^*}_1)$ and $\frac{\mathrm{d}}{\mathrm{d}t} b_{2\dot{x}}(t, x_1^*, \dot{x^*}_1) - b_{2x}(t, x^*, \dot{x^*}_1)$ are not linearly independent.
A similar computation shows that for the function x_2^* defined by $x_2^*(t) = t$, the constraints are regular. △

We remark that in the case of only one constraint (i.e. $r = 1$), such constraint is regular in x^* if

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{\dot{x}}(t, x^*, \dot{x^*}) \neq b_x(t, x^*, \dot{x^*}).$$

In other words

Remark 4.2. *In a isoperimetric problem of calculus of variation with a unique constraint, such constraint is regular in x^* if x^* does not satisfy the Euler equation for the function b.*

We define the *Lagrangian function* $L : [t_0, t_1] \times \mathbb{R}^2 \times \mathbb{R}^r \to \mathbb{R}$ by

$$L(t, x, \dot{x}, \boldsymbol{\nu}) = f(t, x, \dot{x}) + \boldsymbol{\nu} \cdot b(t, x, \dot{x}). \tag{4.19}$$

We have the following necessary condition

Theorem 4.5. *Let us consider (4.18) with f and b in C^2. Let $x^* \in C^1$ be a minimum (or a maximum). Moreover, let us suppose that the constraints are regular in x^*.*

Then, there exists a constant multiplier $\boldsymbol{\nu}^*$ such that, in $t \in [t_0, t_1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}}(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) = L_x(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*). \tag{4.20}$$

A function x^* that satisfies (4.20) (i.e. the EU for the Lagrangian) is called *extremal for the Lagrangian*.
It is possible to prove theorem 4.5 as an application of the Dini's Theorem, as in the static optimization problem with equality constraints: this approach does not follow the idea of "variation" of the CoV; a different proof, using a variational approach, is in [7].

Example 4.3.2. *We consider*

$$\begin{cases} \text{Ott} \displaystyle\int_0^1 \dot{x}^2 \, \mathrm{d}t \\ x(0) = 0 \\ x(1) = 0 \\ \displaystyle\int_0^1 x \, \mathrm{d}t = \frac{1}{12} \\ \displaystyle\int_0^1 tx \, \mathrm{d}t = \frac{1}{20} \end{cases}$$

First of all, let us study the regularity of the constraints: since $b_1(t, x, \dot{x}) = x$ and $b_2(t, x, \dot{x}) = tx$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{1\dot{x}}(t, x^*, \dot{x^*}) - b_{1x}(t, x^*, \dot{x^*}) = -1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} b_{2\dot{x}}(t, x^*, \dot{x^*}) - b_{2x}(t, x^*, \dot{x^*}) = -t.$$

For every x^*, the functions $\frac{\mathrm{d}}{\mathrm{d}t} b_{1\dot{x}}(t, x^*, \dot{x^*}) - b_{1x}(t, x^*, \dot{x^*})$ and $\frac{\mathrm{d}}{\mathrm{d}t} b_{2\dot{x}}(t, x^*, \dot{x^*}) - b_{2x}(t, x^*, \dot{x^*})$ are linearly independent since

$$\alpha_1(-1) + \alpha_2(-t) = 0, \quad \forall t \in [0, 1] \qquad \Leftrightarrow \qquad \alpha_1 = \alpha_2 = 0.$$

Hence the constraints are regular for every x^*.

The Lagrangian is $L = \dot{x}^2 + \nu_1 x + \nu_2 tx$; the EU for L is $2\ddot{x} = \nu_1 + \nu_2 t$ and the general solution is

$$x^*(t) = at + b + \frac{\nu_1}{4} t^2 + \frac{\nu_2}{12} t^3,$$

with $a, b \in \mathbb{R}$. The initial condition and the final condition on the trajectory, and the two constraints give

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad b = 0 \\ x(1) = 0 \quad &\Rightarrow \quad a + b + \frac{\nu_1}{4} + \frac{\nu_2}{12} = 0 \\ \int_0^1 x \, \mathrm{d}t = \frac{1}{12} \quad &\Rightarrow \quad \int_0^1 (at + b + \nu_1 t^2/4 + \nu_2 t^3/12) \, \mathrm{d}t = \\ &\qquad = a/2 + b + \nu_1/12 + \nu_2/48 = 1/12 \\ \int_0^1 tx \, \mathrm{d}t = \frac{1}{20} \quad &\Rightarrow \quad \int_0^1 (at^2 + bt + \nu_1 t^3/4 + \nu_2 t^4/12) \, \mathrm{d}t = \\ &\qquad = a/3 + b/2 + \nu_1/16 + \nu_2/60 = 1/20. \end{aligned}$$

Hence $a = b = 0$, $\nu_1 = 4$ and $\nu_2 = -12$. The unique extremal for the Lagrangian is the function $x^* = t^2 - t^3$.

Let us show that x^* is really the minimum of the problem. We want to prove that for every function $x = x^* + h$, that satisfies the initial and the final conditions and the two integral *constraints, we have* $J(x^*) \le J(x)$*, where* $J(x) = \int_0^1 \dot{x}^2\,\mathrm{d}t$*. Let* $x = x^* + h$ *: hence*

$$x(0) = x^*(0) + h(0) \quad \Rightarrow \quad h(0) = 0 \tag{4.21}$$

$$x(1) = x^*(1) + h(1) \quad \Rightarrow \quad h(1) = 0 \tag{4.22}$$

$$\int_0^1 x\,\mathrm{d}t = \int_0^1 (h + x^*)\,\mathrm{d}t = \frac{1}{12} \quad \Rightarrow \quad \int_0^1 h\,\mathrm{d}t = 0 \tag{4.23}$$

$$\int_0^1 tx\,\mathrm{d}t = \int_0^1 t(x^* + h)\,\mathrm{d}t = \frac{1}{20} \quad \Rightarrow \quad \int_0^1 th\,\mathrm{d}t = 0. \tag{4.24}$$

The four previous conditions give

$$\begin{aligned}
J(x) &= \int_0^1 (\dot{x^*} + \dot{h})^2\,\mathrm{d}t \\
&= \int_0^1 (\dot{x^*}^2 + 2\dot{x^*}\dot{h} + \dot{h}^2)\,\mathrm{d}t \\
&= \int_0^1 (\dot{x^*}^2 + 2(2t - 3t^2)\dot{h} + \dot{h}^2)\,\mathrm{d}t \\
\text{(by part)} \quad &= \int_0^1 (\dot{x^*}^2 + \dot{h}^2)\,\mathrm{d}t + 2\Big((2t - 3t^2)h(t)\Big]_0^1 - 2\int_0^1 (2 - 6t)h\,\mathrm{d}t \\
\text{(by (4.21) e la (4.22))} \quad &= \int_0^1 (\dot{x^*}^2 + \dot{h}^2)\,\mathrm{d}t - 4\int_0^1 h\,\mathrm{d}t + 12\int_0^1 th\,\mathrm{d}t \\
\text{(by (4.23) e la (4.24))} \quad &= \int_0^1 (\dot{x^*}^2 + \dot{h}^2)\,\mathrm{d}t \\
&\ge \int_0^1 \dot{x^*}^2\,\mathrm{d}t \\
&= J(x^*).
\end{aligned}$$

Hence x^* *is a minimum.* △

In this note we are not interested to study the sufficient conditions for the problem (4.18) (see for example [7]).

4.3.2 The multiplier $\boldsymbol{\nu}$ as shadow price

We consider the problem (4.18); let x^* be a minimum in C^2 and let us suppose that the r integral constraints are regular in x^*. Theorem 4.5 guarantees that there exists a constant multiplier $\boldsymbol{\nu}^* = (\nu_1^*, \ldots, \nu_r^*)$ such that EU holds for the Lagrangian $L = f + \boldsymbol{\nu}^* \cdot b$ in x^*. Let us show an interpretation of the role of this multiplier $\boldsymbol{\nu}^*$. If we define

$$J(x) = \int_{t_0}^{t_1} f(t, x, \dot{x})\,\mathrm{d}t,$$

clearly $J(x^*)$ is the minimum value for the problem (4.18). Since x^* satisfy the constraints we have

$$\sum_{j=1}^{r} \nu_j^* \left(\int_{t_0}^{t_1} b_j(t, x^*, \dot{x^*})\,\mathrm{d}t - \tilde{b}_j \right) = 0.$$

Hence

$$\begin{aligned}
J(x^*) &= \int_{t_0}^{t_1} f(t, x^*, \dot{x^*})\,\mathrm{d}t + \sum_{j=1}^{r} \nu_j^* \int_{t_0}^{t_1} b_j(t, x^*, \dot{x^*})\,\mathrm{d}t - \sum_{j=1}^{r} \nu_j^* \tilde{b}_j \\
&= \int_{t_0}^{t_1} L(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*)\,\mathrm{d}t - \sum_{j=1}^{r} \nu_j^* \tilde{b}_j
\end{aligned} \tag{4.25}$$

Our aim is to study the "variation" of the minimum value $J(x^*)$ of the problem when we consider a "variation"of the value $\tilde{b}_k$ of the k–th constraints; taking into account (4.25), such "variation" is

$$\begin{aligned}
\frac{\partial J(x^*)}{\partial \tilde{b}_k} &= \int_{t_0}^{t_1} \left(L_x(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) \frac{\partial x^*}{\partial \tilde{b}_k} + L_{\dot{x}}(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) \frac{\partial \dot{x^*}}{\partial \tilde{b}_k} \right) \mathrm{d}t - \nu_k^* \\
\text{(by part)} \quad &= \int_{t_0}^{t_1} \left(L_x(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) - \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}}(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) \right) \frac{\partial x^*}{\partial \tilde{b}_k} \mathrm{d}t + \\
&\quad + \left(L_{\dot{x}}(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) \frac{\partial x^*}{\partial \tilde{b}_k} \right]_{t_0}^{t_1} - \nu_k^*. \qquad (4.26)
\end{aligned}$$

Since we have $x(t_0) = x^*(t_0) = \alpha$, clearly $\frac{\partial x^*}{\partial \tilde{b}_k}(t_0) = 0$; a similar argument implies $\frac{\partial x^*}{\partial \tilde{b}_k}(t_1) = 0$. By (4.26) and since x^* is extremal for the Lagrangian, we have

$$\begin{aligned}
\frac{\partial J(x^*)}{\partial \tilde{b}_k} &= \int_{t_0}^{t_1} \left(L_x(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) - \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}}(t, x^*, \dot{x^*}, \boldsymbol{\nu}^*) \right) \frac{\partial x^*}{\partial \tilde{b}_k} \mathrm{d}t - \nu_k^* \\
&= -\nu_k^*.
\end{aligned}$$

Finally

$$\frac{\partial J(x^*)}{\partial \tilde{b}_k} = -\nu_k^*,$$

and ν_k^* measures the sensitivity of the optimal value of the problem with respect to a variation of the k-th integral constraints; this is the notion of *shadow price.*

4.3.3 The foundation of Cartagena

Let us consider the problem (1.2)

$$\begin{cases} \max \displaystyle\int_0^1 x \, \mathrm{d}t \\ x(0) = 0 \\ x(1) = 0 \\ \displaystyle\int_0^1 \sqrt{1 + \dot{x}^2} \, \mathrm{d}t = A > 1 \end{cases}$$

and, for symmetry, let us check solution with $x(t) \geq 0$. For the function $b(t, x, \dot{x}) = \sqrt{1 + \dot{x}^2}$ of the integral constraint we have

$$b_x - \frac{\mathrm{d}}{\mathrm{d}t} b_{\dot{x}} = 0 \quad \Rightarrow \quad \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} = d \quad \Rightarrow \quad x(t) = \pm t \sqrt{\frac{d^2}{1 - d^2}} + e,$$

with $d,\ e \in \mathbb{R}$ and $d \neq \pm 1$. Since the unique function that satisfies the previous relation and the conditions $x(0) = x(1) = 0$ is the null function, the constraint is regular since $A > 1$.
The Lagrangian is $L = x + \nu\sqrt{1 + \dot{x}^2}$: since $L_x = 1$, its Euler equation $\frac{\mathrm{d}L_{\dot{x}}}{\mathrm{d}t} = L_x$ is $L_{\dot{x}} + c = t$, i.e.

$$\frac{\nu \dot{x}}{\sqrt{1 + \dot{x}^2}} = t - c.$$

Solving for $\dot{x}$ we obtain

$$\dot{x}(t) = \frac{t-c}{\sqrt{\nu^2 - (t-c)^2}}$$

and hence

$$x(t) = -\sqrt{\nu^2 - (t-c)^2} + k \qquad \Rightarrow \qquad (x(t)-k)^2 + (t-c)^2 = \nu^2.$$

This solution is a circle and the constants k, c and ν are found using the two endpoint conditions and the integral constraint. We are not interested to discuss the sufficient conditions.

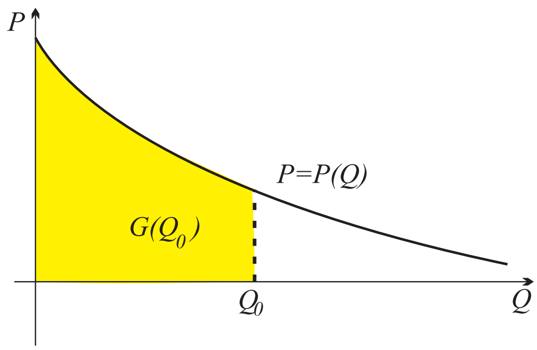
4.3.4 The Hotelling model of socially optimal extraction

One of the assumptions implicit in the classical theory of production is that all inputs are inexhaustible: in reality this is often not true. The model of Hotelling (see [16], [17]) arises to the problem of considering a dynamic of consumption of a good whose production is linked to a finite resource. The notion of "the social value" of an exhaustible resource is used for judging the desirability of any extraction pattern of the resource. If we denote by $Q(t)$, with $Q(t) \geq 0$, the quantity of extraction of the resource, since it is exhaustible we have

$$\int_0^\infty Q \,\mathrm{d}t = S_0,$$

with $S_0 > 0$ fixed. The cost to extract a quantity Q of such resource is $C = C(Q)$.

The gross social value G of a marginal unit of output of extraction of the resource is measured by the price P which society is willing to pay for such particular unit of output. If the price of the resource P is negatively related to the quantity demanded, then the gross social value $G(Q_0)$ of an output Q_0 is measured by the yellow area under the curve, i.e.



$$G(Q_0) = \int_0^{Q_0} P(Q)\,\mathrm{d}Q.$$

Hence to find the net social value, we subtract from the gross social value the total cost of extraction $C(Q_0)$. Hence, the net social value is given by

$$N(Q) = \int_0^Q P(x)\,\mathrm{d}x - C(Q).$$

We suppose that P is continuous. The problem is to find an optimal path of extraction $Q(t)$, the solution of

$$\begin{cases} \max_Q \int_0^\infty N(Q)e^{-rt}\,\mathrm{d}t \\ Q(0) = Q_0 < S_0 \qquad\qquad Q_0 \geq 0 \\ \int_0^\infty Q\,\mathrm{d}t = S_0 \end{cases} \tag{4.27}$$

Clearly $r > 0$ is a rate of discount.

First of all, let us note that the constraint is regular for every function: indeed, $b = b(t, Q, Q') = Q$ implies

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{Q'} \neq b_Q \qquad \Leftrightarrow \qquad 0 \neq 1.$$

We set the Lagrangian $L(t, Q, Q', \nu) = N(Q)e^{-rt} + \nu Q$. The continuity of P and the fundamental theorem of integral calculus give

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{Q'} = L_Q \qquad \Rightarrow \qquad P(Q) - C'(Q) = -\nu e^{rt}. \tag{4.28}$$

Hence

$$\big(P(Q) - C'(Q)\big)e^{-rt} = c,$$

with c constant. Along the optimal extraction path, the present difference $P(Q) - C'(Q)$ has a uniform value for at every time: this relation is called the "social optimal condition".

Consider **the particular case of n firms of small size** compared to the market and therefore are not able to influence the price. Let $P(t) = P_0$ be the price of the resource, let Q_i be the rate of extraction of the i-th firm and let S_i be the quantity of resource available for the i-th firm. The problem of the i-th firm is to maximize

$$\begin{cases} \max \int_0^\infty N_i(Q_i)e^{-rt}\,\mathrm{d}t \\ Q_i(0) = Q_i^0 < S_i \qquad\qquad Q_i^0 \geq 0 \\ \int_0^\infty Q_i\,\mathrm{d}t = S_i \end{cases}$$

where

$$N_i(Q_i) = \int_0^{Q_i} P_0\,\mathrm{d}x - C_i(Q_i) = Q_i P_0 - C_i(Q_i).$$

EU for the Lagrangian gives

$$P_0 - C_i'(Q_i) = -\nu e^{rt}.$$

This optimality condition is obtained under conditions of pure competition, is perfectly consistent with the social optimal condition (4.28).

However, if we consider **the case of a monopoly system** in which the company has the power to influence the market price, we define $R = R(Q)$ the input of the monopolist for the quantity Q of resource. The problem of the monopolist is to maximize (4.27), where

$$N(Q) = R(Q) - C(Q).$$

EU for the Lagrangian gives

$$R'(Q) - C'(Q) = -\nu e^{rt};$$

this is very different from the extraction rule of the social optimal condition: here, the difference between the marginal inputs and the marginal costs grows at a rate r.

Chapter 5

OC with dynamic programming

5.1 The value function: necessary conditions

Let $f : [t_0, t_1] \times \mathbb{R}^{n+k} \to \mathbb{R}$, $g : [t_0, t_1] \times \mathbb{R}^{n+k} \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ continuous functions and $\boldsymbol{\alpha} \in \mathbb{R}^n$ be fixed. We consider the optimal control

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \displaystyle\max_{\mathbf{u} \in \mathcal{C}_{t_0,\boldsymbol{\alpha}}} J(\mathbf{u}) \end{cases} \tag{5.1}$$

where t_0 and t_1 are fixed. We recall (see subsection 1.2.1) that $\mathcal{C}_{t_0,\boldsymbol{\alpha}}$ denotes the class of admissible control for $\boldsymbol{\alpha}$ at time t_0, i.e. the set of all such controls $\mathbf{u} : [t_0, t_1] \to U$ that have a unique associated trajectory defined on $[t_0, t_1]$ with $\mathbf{x}(t_0) = \boldsymbol{\alpha}$.
We define the *value function* $V : [t_0, t_1] \times \mathbb{R}^n \to [-\infty, \infty]$ for the problem (5.1) as[1]

$$V(\tau, \boldsymbol{\xi}) = \begin{cases} \displaystyle\sup_{\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}} \left(\int_{\tau}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1)) \right) & \text{if } \mathcal{C}_{\tau,\boldsymbol{\xi}} \neq \emptyset; \\ -\infty & \text{if } \mathcal{C}_{\tau,\boldsymbol{\xi}} = \emptyset. \end{cases} \tag{5.2}$$

Clearly, in (5.2), $\mathbf{x}$ is the trajectory associated to the control $\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}$. The idea of Bellman [6] and of dynamic programming is to study the properties of such value function. Let us consider $(\tau, \boldsymbol{\xi}) \in [t_0, t_1] \times \mathbb{R}^n$ and the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{\tau}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \\ \displaystyle\max_{\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}} J(\mathbf{u}) \end{cases} \tag{5.3}$$

[1] If in the problem (5.1) we replace the max with a min, in definition (5.2) clearly we have to replace sup and $-\infty$ with inf and $+\infty$ respectively. In subsection 5.5.2 and in example 5.5.4 we will see a min-problem with a value function that admits the value ∞.

Remark 5.1. *If there exists the optimal control* $\mathbf{u}^*_{\tau,\boldsymbol{\xi}}$ *for (5.3), then*

$$V(\tau,\boldsymbol{\xi}) = \int_\tau^{t_1} f(t,\mathbf{x}^*_{\tau,\boldsymbol{\xi}},\mathbf{u}^*_{\tau,\boldsymbol{\xi}})\,\mathrm{d}t + \psi(\mathbf{x}^*_{\tau,\boldsymbol{\xi}}(t_1)),$$

where $\mathbf{x}^*_{\tau,\boldsymbol{\xi}}$ *denotes the trajectory associated to* $\mathbf{u}^*_{\tau,\boldsymbol{\xi}}$.

Example 5.1.1. *Let us consider the problem*

$$\begin{cases} \min \int_0^2 (u^2+x^2)\,\mathrm{d}t \\ \dot{x} = x+u \\ x(0)=1 \\ u \geq 0 \end{cases} \tag{5.4}$$

In the example 2.7.1, we have found that, for every $(\tau,\xi) \in [0,2]\times(0,\infty)$ *fixed, the problem*

$$\begin{cases} \min \int_\tau^2 (u^2+x^2)\,\mathrm{d}t \\ \dot{x} = x+u \\ x(\tau)=\xi \\ u \geq 0 \end{cases}$$

has the optimal tern, see (2.97),

$$(u^*_{\tau,\xi}, x^*_{\tau,\xi}, \lambda^*_{\tau,\xi}) = (0, \xi e^{t-\tau}, \xi(e^{4-t-\tau} - e^{t-\tau})).$$

Moreover, the value function $V : [0,2]\times[0,\infty) \to \mathbb{R}$ *for the problem (5.4) is, as in (2.98),*

$$V(\tau,\xi) = \int_0^2 ((u^*_{\tau,\xi})^2 + (x^*_{\tau,\xi})^2)\,\mathrm{d}t = \frac{\xi^2}{2}(e^{4-2\tau}-1).$$

△

5.1.1 The final condition on V: an first necessary condition

Consider the problem (5.1) and its value function V. In the particular case of $\tau = t_1$, from the definition (5.2) we have

$$V(t_1,\boldsymbol{\xi}) = \sup_{\mathbf{u}\in\mathcal{C}_{t_1,\boldsymbol{\xi}}} \left(\int_{t_1}^{t_1} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1)) \right) = \psi(\boldsymbol{\xi}).$$

Hence we have

Remark 5.2.

$$V(t_1,\mathbf{x}) = \psi(\mathbf{x}), \qquad \text{for every } \ \mathbf{x}\in\mathbb{R}^n. \tag{5.5}$$

The condition (5.5) is called the *final condition on the value function:* clearly, it is a necessary condition for a function $V : [t_0,t_1]\times\mathbb{R}^n \to [-\infty,\infty]$ to be the value function for the problem (5.1).

If in the problem (5.1) we add a final condition on the trajectory, i.e. $\mathbf{x}(t_1) = \boldsymbol{\beta}$ with $\boldsymbol{\beta}\in\mathbb{R}^n$ fixed, then the final condition on the value function is

$$V(t_1,\boldsymbol{\beta}) = \psi(\boldsymbol{\beta}).$$

We will see in section 5.4 the condition for the value function when the trajectory has a more general condition at the final time.

5.1.2 Bellman's Principle of optimality

Theorem 5.1 (Bellman's Principle of optimality)**.** *The second part of an optimal trajectory is optimal.*

More precisely: let us consider the problem (5.1) and let $\mathbf{u}^*_{t_0,\boldsymbol{\alpha}}$ *and* $\mathbf{x}^*_{t_0,\boldsymbol{\alpha}}$ *be the optimal control and the optimal trajectory respectively. Let us consider the problem (5.3) with* $(\tau, \boldsymbol{\xi})$ *such that* $\mathbf{x}^*_{t_0,\boldsymbol{\alpha}}(\tau) = \boldsymbol{\xi}$. *Let* $\mathbf{u}^*_{\tau,\boldsymbol{\xi}}$ *be the optimal control for (5.3). Then*

$$\mathbf{u}^*_{t_0,\boldsymbol{\alpha}} = \mathbf{u}^*_{\tau,\boldsymbol{\xi}}, \qquad \text{in } [\tau, t_1]$$

and, consequently, $\mathbf{x}^*_{t_0,\boldsymbol{\alpha}} = \mathbf{x}^*_{\tau,\boldsymbol{\xi}}$ *in* $[\tau, t_1]$.

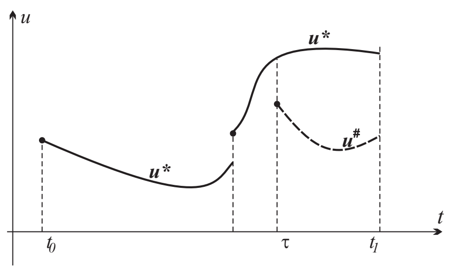
Proof. Let $\mathbf{u}^*$ be the optimal control for the problem (5.1) and $\tau \in [t_0, t_1]$: we prove that the optimal control $\tilde{\mathbf{u}} \in \mathcal{C}_{\tau,\mathbf{x}^*(\tau)}$, defined as the restriction of $\mathbf{u}^*$ on the interval $[\tau, t_1]$, is optimal for the problem (5.3) with $\boldsymbol{\xi} = \mathbf{x}^*(\tau)$. By contradiction, let us suppose that there exists $\mathbf{u}^\sharp \in \mathcal{C}_{\tau,\mathbf{x}^*(\tau)}$, $\mathbf{u}^\sharp \neq \tilde{\mathbf{u}}$, optimal for the problem (5.3) with initial data $\boldsymbol{\xi} = \mathbf{x}^*(\tau)$ and such that

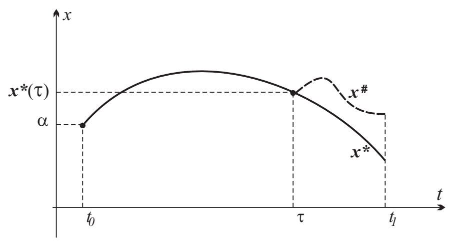
$$\int_\tau^{t_1} f(t, \tilde{\mathbf{x}}, \tilde{\mathbf{u}})\,\mathrm{d}t + \psi(\tilde{\mathbf{x}}(t_1)) < \int_\tau^{t_1} f(t, \mathbf{x}^\sharp, \mathbf{u}^\sharp)\,\mathrm{d}t + \psi(\mathbf{x}^\sharp(t_1)), \tag{5.6}$$

where $\tilde{\mathbf{x}}$ and $\mathbf{x}^\sharp$ are the trajectories associated to $\tilde{\mathbf{u}}$ and $\mathbf{u}^\sharp$ respectively. We consider the control $\mathbf{u}$ defined by

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}^*(t) & \text{for } t_0 \le t < \tau, \\ \mathbf{u}^\sharp(t) & \text{for } \tau \le t \le t_1 \end{cases} \tag{5.7}$$

and $\mathbf{x}$ be the corresponding trajectory.





Clearly $\mathbf{u} \in \mathcal{C}_{t_0,\boldsymbol{\alpha}}$; hence

$$\begin{aligned}
V(t_0, \boldsymbol{\alpha}) &= \int_{t_0}^{t_1} f(t, \mathbf{x}^*, \mathbf{u}^*)\,\mathrm{d}t + \psi(\mathbf{x}^*(t_1)) \\
&= \int_{t_0}^{\tau} f(t, \mathbf{x}^*, \mathbf{u}^*)\,\mathrm{d}t + \int_\tau^{t_1} f(t, \mathbf{x}^*, \mathbf{u}^*)\,\mathrm{d}t + \psi(\mathbf{x}^*(t_1)) \\
&= \int_{t_0}^{\tau} f(t, \mathbf{x}^*, \mathbf{u}^*)\,\mathrm{d}t + \int_\tau^{t_1} f(t, \tilde{\mathbf{x}}, \tilde{\mathbf{u}})\,\mathrm{d}t + \psi(\tilde{\mathbf{x}}(t_1)) \\
\text{(by 5.6)} \quad &< \int_{t_0}^{\tau} f(t, \mathbf{x}^*, \mathbf{u}^*)\,\mathrm{d}t + \int_\tau^{t_1} f(t, \mathbf{x}^\sharp, \mathbf{u}^\sharp)\,\mathrm{d}t + \psi(\mathbf{x}^\sharp(t_1)) \\
\text{(by 5.7)} \quad &= \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1))
\end{aligned}$$

that is absurd for the definition of value function. Hence such $\mathbf{u}^\sharp$ does not exist.
□

5.2 The Bellman-Hamilton-Jacobi equation

5.2.1 Necessary condition of optimality

The Bellman's Principle of optimality plays a fundamental role in the proof of this crucial property of the value function.

Theorem 5.2. *Let us consider the problem (5.1) with f and g continuous and let us suppose that for every $(\tau, \boldsymbol{\xi}) \in [t_0, t_1] \times \mathbb{R}^n$ there exists the optimal control $\mathbf{u}^*{}_{\tau,\boldsymbol{\xi}}$ for the problem (5.3). Let V be the value function for the problem (5.1) and let V be differentiable. Then, for every $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$, we have*

$$\frac{\partial V}{\partial t}(t, \mathbf{x}) + \max_{\mathbf{v} \in U} \Big(f(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} V(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) = 0. \tag{5.8}$$

The equation (5.8) is called *Bellman-Hamilton-Jacobi equation* (shortly BHJ equation). Clearly (5.8) is a necessary condition for a generic function V to be the value function for the problem (5.1). The main difficulty of dynamic programming is that such equation in general is a Partial Differential Equation (shortly PDE). One of the fundamental property of dynamic programming is that it is possible to generalize such approach to a stochastic context.

Proof. Let $(\tau, \boldsymbol{\xi}) \in [t_0, t_1] \times \mathbb{R}^n$ be fixed. For the assumptions, there exists the optimal control $\mathbf{u}^*{}_{\tau,\boldsymbol{\xi}}$ for the problem (5.3); we drop to the notation "$\tau, \boldsymbol{\xi}$" and we set $\mathbf{u}^* = \mathbf{u}^*{}_{\tau,\boldsymbol{\xi}}$. We divide the proof in two steps.
First step: we will prove (5.14). For the Remark 5.1 and for every $h > 0$ we have

$$\begin{aligned} V(\tau, \boldsymbol{\xi}) &= \int_\tau^{t_1} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t + \psi(\mathbf{x}^*(t_1)) \\ &= \int_\tau^{\tau+h} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t + \int_{\tau+h}^{t_1} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t + \psi(\mathbf{x}^*(t_1)) \end{aligned} \tag{5.9}$$

where $\mathbf{x}^*$ is the trajectory associated to $\mathbf{u}^*$ with initial point $(\tau, \boldsymbol{\xi})$.
For the Bellman's Principle of optimality (theorem 5.1), the problem (5.3) with initial point $(\tau + h, \mathbf{x}^*(\tau + h))$ has as optimal control the function $\mathbf{u}^*$ restricted to the interval $[\tau + h, t_1]$; hence

$$V(\tau + h, \mathbf{x}^*(\tau + h)) = \int_{\tau+h}^{t_1} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t + \psi(\mathbf{x}^*(t_1)). \tag{5.10}$$

Equation (5.9), for (5.10), now is

$$V(\tau, \boldsymbol{\xi}) = \int_\tau^{\tau+h} f(t, \mathbf{x}^*, \mathbf{u}^*)\, \mathrm{d}t + V(\tau + h, \mathbf{x}^*(\tau + h)). \tag{5.11}$$

Now, let us prove that

$$\max_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}}\left(\int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + V(\tau+h,\mathbf{x}(\tau+h))\right) = \int_\tau^{\tau+h} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + V(\tau+h,\mathbf{x}^*(\tau+h)) : \tag{5.12}$$

by contradiction let us suppose that there exists a control $\tilde{\mathbf{u}}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}$ (with associated trajectory $\tilde{\mathbf{x}}$) such that

$$\int_\tau^{\tau+h} f(t,\tilde{\mathbf{x}},\tilde{\mathbf{u}})\,\mathrm{d}t + V(\tau+h,\tilde{\mathbf{x}}(\tau+h)) > \int_\tau^{\tau+h} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + V(\tau+h,\mathbf{x}^*(\tau+h)), \tag{5.13}$$

taking into account that there exists an optimal control ${\mathbf{u}^*}_{\tau+h,\tilde{\mathbf{x}}(\tau+h)}$ for the problem (5.3) with initial point $(\tau+h,\tilde{\mathbf{x}}(\tau+h))$, then the function $\mathbf{u}^\sharp$, defined by

$$\mathbf{u}^\sharp(t) = \begin{cases} \tilde{\mathbf{u}}(t) & \text{for } \tau\le t<\tau+h,\\ {\mathbf{u}^*}_{\tau+h,\tilde{\mathbf{x}}(\tau+h)}(t) & \text{for } \tau+h\le t\le t_1\end{cases},$$

is in $\mathcal{C}_{\tau,\boldsymbol{\xi}}$ with associated trajectory $\mathbf{x}^\sharp$. Hence (5.11) and (5.13) give

$$\begin{aligned}
&\int_\tau^{t_1} f(t,\mathbf{x}^\sharp,\mathbf{u}^\sharp)\,\mathrm{d}t + \psi(\mathbf{x}^\sharp(t_1)) =\\
&= \int_\tau^{\tau+h} f(t,\tilde{\mathbf{x}},\tilde{\mathbf{u}})\,\mathrm{d}t + \int_{\tau+h}^{t_1} f(t,{\mathbf{x}^*}_{\tau+h,\tilde{\mathbf{x}}(\tau+h)},{\mathbf{u}^*}_{\tau+h,\tilde{\mathbf{x}}(\tau+h)})\,\mathrm{d}t + \psi({\mathbf{x}^*}_{\tau+h,\tilde{\mathbf{x}}(\tau+h)}(t_1))\\
&> \int_\tau^{\tau+h} f(t,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}t + V(\tau+h,\mathbf{x}^*(\tau+h))\\
&= V(\tau,\boldsymbol{\xi})
\end{aligned}$$

that contradicts the definition of value function. Hence (5.12) holds and, by (5.11), we obtain

$$V(\tau,\boldsymbol{\xi}) = \max_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}}\left(\int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + V(\tau+h,\mathbf{x}(\tau+h))\right). \tag{5.14}$$

Second step: we will conclude the proof. Since V is differentiable, for h sufficiently small,

$$\begin{aligned}
V(\tau+h,\mathbf{x}(\tau+h)) &= V(\tau,\mathbf{x}(\tau)) + \frac{\partial V}{\partial t}(\tau,\mathbf{x}(\tau))\big(\tau+h-\tau\big) +\\
&\quad +\nabla_{\mathbf{x}} V(\tau,\mathbf{x}(\tau))\cdot\big(\mathbf{x}(\tau+h)-\mathbf{x}(\tau)\big) + o(h)\\
&= V(\tau,\boldsymbol{\xi}) + \frac{\partial V}{\partial t}(\tau,\boldsymbol{\xi})h + \nabla_{\mathbf{x}} V(\tau,\boldsymbol{\xi})\cdot\big(\mathbf{x}(\tau+h)-\boldsymbol{\xi}\big) + o(h),
\end{aligned}$$

since $\mathbf{x}(\tau)=\boldsymbol{\xi}$. Then (5.14) and the previous relation give

$$\max_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}}\left(\int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \frac{\partial V}{\partial t}(\tau,\boldsymbol{\xi})h + \nabla_{\mathbf{x}} V(\tau,\boldsymbol{\xi})\cdot\big(\mathbf{x}(\tau+h)-\boldsymbol{\xi}\big) + o(h)\right) = 0.$$

If we divide the two members of the previous relation for $h > 0$ and we consider the limit for $h \to 0^+$, we obtain

$$\lim_{h\to 0^+} \left\{ \max_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}} \Big(\frac{1}{h} \int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \frac{\partial V}{\partial t}(\tau,\boldsymbol{\xi}) + \right.$$
$$\left. + \nabla_{\mathbf{x}} V(\tau,\boldsymbol{\xi}) \cdot \frac{\mathbf{x}(\tau+h)-\boldsymbol{\xi}}{h} + o(1) \Big) \right\} = 0. \tag{5.15}$$

Now, we note that for a given fixed $\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}$ we have that in (5.14)

$$\int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + V(\tau+h,\mathbf{x}(\tau+h)) \tag{5.16}$$

depends only on the value of $\mathbf{u}$ in the set $[\tau,\tau+h]$: in fact, given $(\tau,\boldsymbol{\xi})$ and $\mathbf{u}$, we construct $\mathbf{x}$ in $[\tau,\tau+h]$ using the dynamics and hence the value of function V in $(\tau+h,\mathbf{x}(\tau+h))$. Hence, for $h \to 0^+$, we have that (5.16) depends only on the set of the values that $\mathbf{u}$ can assume in the point τ, i.e. in the control set U.

Now let us suppose that $\mathbf{u}$ is continuous in a neighborhood I of the point τ. Hence the function $t \mapsto f(t,\mathbf{x}(t),\mathbf{u}(t))$ is continuous in $[\tau,\tau+h]$, for h small: hence, for the mean value theorem[2], we have

$$\inf_{t\in[\tau,\tau+h]} f(t,\mathbf{x}(t),\mathbf{u}(t)) \le \frac{1}{h}\int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \le \sup_{t\in[\tau,\tau+h]} f(t,\mathbf{x}(t),\mathbf{u}(t))$$

Hence, for the continuity in $[\tau,\tau+h]$,

$$\lim_{h\to 0^+} \frac{1}{h}\int_\tau^{\tau+h} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t = \lim_{t\to\tau^+} f(t,\mathbf{x}(t),\mathbf{u}(t)) = f(t,\mathbf{x}(\tau),\mathbf{u}(\tau)). \tag{5.17}$$

Now, since $t \mapsto g(t,\mathbf{x}(t),\mathbf{u}(t))$ is continuous in I, we have $\mathbf{x} \in \mathbf{C}^1(I)$ and hence

$$\lim_{h\to 0^+} \frac{\mathbf{x}(\tau+h)-\boldsymbol{\xi}}{h} = \dot{\mathbf{x}}(\tau). \tag{5.18}$$

If $\mathbf{u}$ is not continuous in a neighborhood of τ, i.e. $\mathbf{u}$ is only measurable and admissible, we have only tedious calculations, using the previous ideas, but no really problems: hence we omit this part of the proof.

Equation (5.15), using (5.17) and (5.18), gives

$$\max_{\mathbf{v}\in U}\Big(f(\tau,\mathbf{x}(\tau),\mathbf{v}) + \frac{\partial V}{\partial t}(\tau,\boldsymbol{\xi}) + \nabla_{\mathbf{x}} V(\tau,\boldsymbol{\xi}) \cdot \dot{\mathbf{x}}(\tau)\Big) = 0. \tag{5.19}$$

Using the dynamics, the initial condition $\mathbf{x}(\tau) = \boldsymbol{\xi}$ and remarking that $\frac{\partial V}{\partial x}(\tau,\boldsymbol{\xi})$ does not depend on $\mathbf{v}$, we obtain easily (5.8). □

[2] We recall that if $\phi : [a,b] \to \mathbb{R}$ is integrable in $[a,b]$, then

$$\inf_{t\in[a,b]} \phi(t) \le \frac{1}{b-a}\int_a^b \phi(s)\,\mathrm{d}s \le \sup_{t\in[a,b]} \phi(t);$$

moreover, if ϕ is continuous in $[a,b]$, then there exists $c \in [a,b]$ such that

$$\int_a^b \phi(s)\,\mathrm{d}s = \phi(c)(b-a).$$

Let us consider the problem (5.1); we define the *Hamiltonian of Dynamic Programming* $H_{DP} : [t_0, t_1] \times \mathbb{R}^{2n} \to (-\infty, +\infty]$ defined by

$$H_{DP}(t, \mathbf{x}, \mathbf{p}) = \max_{\mathbf{v} \in U} \Big(f(t, \mathbf{x}, \mathbf{v}) + \mathbf{p} \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) \tag{5.20}$$

It is clear that, in the assumption of Theorem 5.2, the value function solves the system

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, \mathbf{x}) + H_{DP}(t, \mathbf{x}, \nabla_{\mathbf{x}} V(t, \mathbf{x})) = 0 & \text{for } (t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n \\ V(t_1, \mathbf{x}) = \psi(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^n \end{cases} \tag{5.21}$$

We mention that the conclusion of the first step in the proof of Theorem 5.2 holds with very different assumptions: since we will use these different assumptions and this conclusion in the following, let us give the precise statement:

Remark 5.3. *Let us consider the problem (5.1) in the autonomous case (see (5.35)) with the assumptions* 1., 2. *and* 3. *in the beginning of section 5.3. Then*

$$V(\tau, \boldsymbol{\xi}) = \max_{\mathbf{u} \in \mathcal{C}_{\tau, \boldsymbol{\xi}}} \left(\int_\tau^{\tau+h} f(\mathbf{x}, \mathbf{u}) \,\mathrm{d}t + V(\tau + h, \mathbf{x}(\tau + h)) \right).$$

For a proof of this result see Theorem 1 in subsection 10.3.2 in [11].

It is clear that the previous result in Theorem 5.2 leads naturally to the question "when the value function $V(\tau, \boldsymbol{\xi})$ is differentiable". We will give an idea with Theorem 5.4.

5.2.2 Sufficient condition of optimality

At this point the question is to suggest sufficient conditions such that a function $W : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$, that satisfies the Bellman-Hamilton-Jacobi equation (5.8) and the final condition (5.5), is really the value function for the problem (5.1). Moreover, we hope that the value function gives us some information about the optimal control. This is the content of the next result.

Theorem 5.3. *Let us consider the problem (5.1). Let* $W : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ *be a differentiable function that satisfies the BHJ system (5.21), i.e.*

$$\begin{cases} \dfrac{\partial W}{\partial t}(t, \mathbf{x}) + H_{DP}(t, \mathbf{x}, \nabla_{\mathbf{x}} W(t, \mathbf{x})) = 0 & \forall (t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n \\ W(t_1, \mathbf{x}) = \psi(\mathbf{x}) & \forall \mathbf{x} \in \mathbb{R}^n \end{cases} \tag{5.22}$$

Let $w : [t_0, t_1] \times \mathbb{R}^n \to U$ *be a piecewise continuous function with respect to* $t \in [t_0, t_1]$ *and a* C^1 *function with respect to* $\mathbf{x} \in \mathbb{R}^n$*. Moreover, let* w *be such that in* $[t_0, t_1] \times \mathbb{R}^n$ *we have*

$$w(t, \mathbf{x}) \in \arg\max_{\mathbf{v} \in U} \Big(f(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} W(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) \tag{5.23}$$

Finally, let $\mathbf{x}^*$ *be the solution of the ODE*

$$\begin{cases} \dot{\mathbf{x}}(t) = g\big(t, \mathbf{x}, w(t, \mathbf{x})\big) & \text{in } [t_0, t_1] \\ \mathbf{x}(t_0) = \boldsymbol{\alpha}. \end{cases} \tag{5.24}$$

Then $\mathbf{x}^*$ *is the optimal trajectory and* $\mathbf{u}^*$*, defined by*

$$\mathbf{u}^*(t) = w(t, \mathbf{x}^*(t)), \tag{5.25}$$

is the optimal control for the problem (5.1). Moreover, W *is the value function for the problem (5.1).*

Proof. Let $\mathbf{x}^*$ be the solution of (5.24) and let $\mathbf{u}^*$ be defined as in (5.25); we prove that $\mathbf{u}^*$ is optimal. By (5.22), (5.23), (5.24) and (5.25) we have, for every fixed t,

$$\begin{aligned}
\frac{\partial W}{\partial t}(t, \mathbf{x}^*(t)) &= -\max_{\mathbf{v}\in U}\Big(f(t, \mathbf{x}^*(t), \mathbf{v}) + \nabla_{\mathbf{x}} W(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{v})\Big) \\
&= -f(t, \mathbf{x}^*(t), w(t, \mathbf{x}^*(t))) - \nabla_{\mathbf{x}} W(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), w(t, \mathbf{x}^*(t))) \\
&= -f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) - \nabla_{\mathbf{x}} W(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \\
&= -f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) - \nabla_{\mathbf{x}} W(t, \mathbf{x}^*(t)) \cdot \dot{\mathbf{x}}^*(t) \qquad (5.26)
\end{aligned}$$

Since W is differentiable, the fundamental theorem of integral calculus implies

$$\begin{aligned}
W(t_1, \mathbf{x}^*(t_1)) - W(t_0, \mathbf{x}^*(t_0)) &= \int_{t_0}^{t_1} \frac{\mathrm{d}W(t, \mathbf{x}^*(t))}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_{t_0}^{t_1} W_t(t, \mathbf{x}^*(t)) + \nabla_{\mathbf{x}} W(t, \mathbf{x}^*(t)) \cdot \dot{\mathbf{x}}^*(t)\,\mathrm{d}t \\
\text{(by (5.26))} \quad &= -\int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))\,\mathrm{d}t. \qquad (5.27)
\end{aligned}$$

Now, let $\mathbf{u}$ be an admissible control, with $\mathbf{u} \neq \mathbf{u}^*$, and let $\mathbf{x}$ be the associated trajectory. The definition of the function w, (5.22) and the dynamics give, for every fixed t,

$$\begin{aligned}
\frac{\partial W}{\partial t}(t, \mathbf{x}(t)) &= -\max_{\mathbf{v}\in U}\Big(f(t, \mathbf{x}(t), \mathbf{v}) + \nabla_{\mathbf{x}} W(t, \mathbf{x}(t)) \cdot g(t, \mathbf{x}(t), \mathbf{v})\Big) \\
&\le -f(t, \mathbf{x}(t), \mathbf{u}(t)) - \nabla_{\mathbf{x}} W(t, \mathbf{x}(t)) \cdot g(t, \mathbf{x}(t), \mathbf{u}(t)) \\
&= -f(t, \mathbf{x}(t), \mathbf{u}(t)) - \nabla_{\mathbf{x}} W(t, \mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) \qquad (5.28)
\end{aligned}$$

Again we have

$$\begin{aligned}
W(t_1, \mathbf{x}(t_1)) - W(t_0, \mathbf{x}(t_0)) &= \int_{t_0}^{t_1} \frac{\mathrm{d}W(t, \mathbf{x}(t))}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_{t_0}^{t_1} W_t(t, \mathbf{x}(t)) + \nabla_{\mathbf{x}} W(t, \mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)\,\mathrm{d}t \\
\text{(by (5.28))} \quad &\le -\int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\,\mathrm{d}t. \qquad (5.29)
\end{aligned}$$

We remark that $\mathbf{x}^*(t_0) = \mathbf{x}(t_0) = \boldsymbol{\alpha}$; if we subtract the two expressions in (5.27) and in (5.29), then we obtain

$$W(t_1, \mathbf{x}^*(t_1)) - W(t_1, \mathbf{x}(t_1)) \ge -\int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))\,\mathrm{d}t + \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\,\mathrm{d}t.$$

Using the final condition in (5.22), the previous inequality becomes

$$\int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))\,\mathrm{d}t + \psi(\mathbf{x}^*(t_1)) \geq \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\,\mathrm{d}t + \psi(\mathbf{x}(t_1)),$$

for every $\mathbf{u} \in \mathcal{C}_{t_0,\boldsymbol{\alpha}}$ and $\mathbf{x}$ associated trajectory: hence $\mathbf{u}^*$ is optimal for the problem (5.1).

Now we consider the value function V defined by (5.2). By (5.27), since $\mathbf{u}^*$ is optimal for the problem (5.1) and using (5.22) and the final condition on the trajectory

$$\begin{aligned} V(t_0, \boldsymbol{\alpha}) &= \int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))\,\mathrm{d}t + \psi(\mathbf{x}^*(t_1)) \\ &= \int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))\,\mathrm{d}t + W(t_1, \mathbf{x}^*(t_1)) = W(t_0, \boldsymbol{\alpha}). \end{aligned}$$

Hence the function W in the point $(t_0, \boldsymbol{\alpha})$ coincides with the value of the value function V in $(t_0, \boldsymbol{\alpha})$. Now, if we replace the initial data $\mathbf{x}(t_0) = \boldsymbol{\alpha}$ in (5.24) with the new initial data $\mathbf{x}(\tau) = \boldsymbol{\xi}$, then the same proof gives $V(\tau, \boldsymbol{\xi}) = W(\tau, \boldsymbol{\xi})$. Hence W is really the value function. □

At this point let V be a regular value function that satisfies the BHJ equation for every $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$,

$$\frac{\partial V}{\partial t}(t, \mathbf{x}) + \max_{\mathbf{v} \in U}\Big(f(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} V(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) = 0;$$

moreover, if we consider the function $w = w(t, \mathbf{x})$ that realizes the max in the previous BHJ equation, then we know that the optimal control is $\mathbf{u}^*(t) = w(t, \mathbf{x}^*(t))$, i.e.

$$\begin{aligned} &\max_{\mathbf{v} \in U}\Big(f(t, \mathbf{x}^*(t), \mathbf{v}) + \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{v}) \Big) = \\ &\qquad = f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) + \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t)), \end{aligned}$$

for every $t \in [t_0, t_1]$. Essentially we obtain

Remark 5.4. *If V is a regular value function and $\mathbf{u}^*$ is the optimal control (with trajectory $\mathbf{x}^*$) for the problem (5.1), we have*

$$\frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) + \nabla_{\mathbf{x}}(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) = 0$$

for every $t \in [t_0, t_1]$.

5.2.3 Affine Quadratic problems

We now consider a very important class of optimal control problems; let

$$\begin{cases} \displaystyle\max_{\mathbf{u} \in \mathcal{C}} \frac{1}{2} \int_{t_0}^{t_1} \left(\mathbf{x}' Q(t) \mathbf{x} + 2\mathbf{x}' S(t) + \mathbf{u}' R(t) \mathbf{u}\right)\,\mathrm{d}t + \frac{1}{2} \mathbf{x}(t_1)' P \mathbf{x}(t_1) \\ \dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u} + C(t) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \mathcal{C} = \{\mathbf{u} : [t_0, t_1] \to \mathbb{R}^k,\ \text{admissible}\} \end{cases} \tag{5.30}$$

where $\mathbf{v}'$ is the transpose of the matrix $\mathbf{v}$; in (5.30) we denotes the trajectory $\mathbf{x}$ and the control $\mathbf{u}$ such that $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$ and $\mathbf{u} = (u_1, u_2, \ldots, u_k)'$ respectively. We assume that $Q = Q(t)$, $R = R(t)$, $S = S(t)$, $A = A(t)$, $B = B(t)$ and $C = C(t)$ are matrices of appropiate dimensions with continuous entries on $[t_0, t_1]$; moreover P is a constant matrix.
These type of problems are called Affine Quadratic problems; if in particular $C = 0$, the problem is called Linear Quadratic problem (LQ problem). In this type of maximization problem it is reasonable to add the following assumption:

A. the matrices $Q(t)$, for every t, and P are nonpositive defined (i.e. $Q(t) \le 0$, $P \le 0$);

B. the matrix $R(t)$, for every t, is negative defined (i.e. $R(t) < 0$);

it is clear that if we consider a min problem, we have to change the previous inequalities.

For sake of simplicity, let us consider the case $n = k = 1$; we looking for a function

$$V(t, x) = \frac{1}{2} Z x^2 + W x + Y$$

with $Z = Z(t)$, $W = W(t)$ and $Y = Y(t)$ in C^1, such that V is the value function for the problem (5.30): the necessary conditions (5.5) and (5.8) give for such choice of V that, for all $(t, x) \in [t_0, t_1] \times \mathbb{R}$

$$\begin{cases} \frac{1}{2}\dot{Z}x^2 + \dot{W}x + \dot{Y} + \max_{v \in \mathbb{R}} \Big[\frac{1}{2} Q x^2 + Sx + \\ \qquad\qquad + \frac{1}{2} R v^2 + (Zx + W)(Ax + Bv + C) \Big] = 0, \\ \frac{1}{2} Z(t_1) x^2 + W(t_1) x + Y(t_1) = \frac{1}{2} P x^2, \end{cases}$$

In order to realize a maximum, it clear that the assumption B. $R(t) < 0$ plays a fundamental role. Hence we realize the max for

$$v(t, x) = -\frac{(Z(t)x + W(t))B(t)}{R(t)},$$

an easy computation gives

$$\dot{Z} - \frac{B^2}{R} Z^2 + 2AZ + Q = 0, \qquad Z(t_1) = P \tag{5.31}$$

$$\dot{W} + \left(A - \frac{B^2}{R} Z \right) W + S + CZ = 0, \qquad W(t_1) = 0 \tag{5.32}$$

$$\dot{Y} - \frac{B^2}{2R} W^2 + CW = 0, \qquad Y(t_1) = 0 \tag{5.33}$$

If there exists a unique solution Z of the Riccati ODE (5.31) in $[t_0, t_1]$ with its initial condition, then we put such solution in (5.32) and we obtain a linear ODE in the variable W with its initial condition; again, putting such solution W in (5.33) we obtain an easy ODE in the variable Y with its initial condition. These arguments give the following:

Remark 5.5. *Let us suppose that there exists a unique solution for the system*

$$\begin{cases} \dot{Z} - ZBR^{-1}B'Z + ZA + A'Z + Q = 0, & Z(t_1) = P \\ \dot{W} + \left(A - BR^{-1}B'Z\right)' W + S + ZC = 0, & W(t_1) = 0 \\ \dot{Y} - \frac{1}{2}W'BR^{-1}B'W + W'C = 0, & Y(t_1) = 0 \end{cases} \tag{5.34}$$

in the variable (Z, W, Y) such that

$$v(t, \mathbf{x}) = -R^{-1}B'(Z\mathbf{x} + W)$$

realizes the max *in*

$$\max_{\mathbf{v} \in \mathbb{R}^k} \left(\frac{1}{2}\mathbf{v}'R\mathbf{v} + (Z\mathbf{x} + W)'B\mathbf{v} \right)$$

for all $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$, then

$$V(t, \mathbf{x}) = \frac{1}{2}\mathbf{x}'Z\mathbf{x} + W\mathbf{x} + Y$$

is the value function for the problem (5.30).

A more precise result, that we are not interested to prove (see Proposition 5.3 in [4]), is

Proposition 5.1. *Let us assume A. and B. Then there exists a unique solution of the system (5.34).*

Let us consider now the particular situation of a linear homogeneous quadratic problem, i.e. $C = 0$ and $S = 0$; it is clear that (5.32) implies the solution $W = W(t) = 0$; consequently, in (5.33) we obtain the solution $Y = Y(t) = 0$. Clearly we obtain the following (see [15]):

Remark 5.6. *Let us consider the problem (5.30) in the linear and homogeneous case, with the assumption A. and B. Then*

$$V(t, \mathbf{x}) = \frac{1}{2}\mathbf{x}'Z\mathbf{x}$$

is the value function for the problem.

5.3 Regularity of V and viscosity solution

The results of the previous section require some regularity of the value function. More precisely in order to give a sense to a BHJ equation, we need the differentiability of the value function V. In this section we focus our attention of the problem (5.1) in the autonomous case, i.e.

$$\begin{cases} \max\limits_{\mathbf{u}} \displaystyle\int_{t_0}^{t_1} f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(\mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \end{cases} \tag{5.35}$$

where t_0 and t_1 are fixed. In all that follows in this section, let us assume that for the problem (5.35) we have that

1. the control set U is compact;

2. $f : \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R}^n \times U \to \mathbb{R}^n$ are bounded and uniformly continuous with

$$\begin{array}{ll} |f(\mathbf{x},\mathbf{u})| \le C & |f(\mathbf{x},\mathbf{u}) - f(\mathbf{x}',\mathbf{u})| \le C\|\mathbf{x}-\mathbf{x}'\| \\ \|g(\mathbf{x},\mathbf{u})\| \le C & \|g(\mathbf{x},\mathbf{u}) - g(\mathbf{x}',\mathbf{u})\| \le C\|\mathbf{x}-\mathbf{x}'\| \end{array}$$

for some constant C and for every $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$, $\mathbf{u} \in U$;

3. $\psi : \mathbb{R}^n \to \mathbb{R}$ is bounded and uniformly continuous with

$$|\psi(\mathbf{x})| \le C \qquad |\psi(\mathbf{x}) - \psi(\mathbf{x}')| \le C\|\mathbf{x}-\mathbf{x}'\|$$

for some constant C and for every $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$.

We only mention that many results of this section hold in weaker assumptions w.r.t. *1.*, *2.* and *3.*.

The next result fixes an important idea about the regularity of the value function V.

Theorem 5.4. *Consider the problem autonomous (5.35) with the assumptions* 1., 2. *and* 3.. *Then the value function V is bounded and Lipschitz continuous, i.e. there exists a constant $\widetilde{C}$ such that*

$$|V(\tau,\boldsymbol{\xi})| \le \widetilde{C}, \qquad |V(\tau,\boldsymbol{\xi}) - V(\tau',\boldsymbol{\xi}')| \le \widetilde{C}\left(|\tau-\tau'| + \|\boldsymbol{\xi}-\boldsymbol{\xi}'\|\right)$$

for every $\tau,\tau' \in [t_0,t_1]$ and $\boldsymbol{\xi},\boldsymbol{\xi}' \in \mathbb{R}^n$.

An immediate but fundamental consequence of the previous theorem and of the Rademacher's theorem[3] is the following

Corollary 5.1. *In the assumption of Theorem 5.4, the value function V is differentiable except on a set of Lebesgue measure zero.*

Proof of Theorem 5.4[4] In all the proof we denote by C a generic constant, that in general can be different in every situation.

First, it is easy to see that the Lipschitz assumption on g guarantees (see Theorem 1.2) that for every initial data $(\tau,\boldsymbol{\xi})$ and for every control $\mathbf{u} : [\tau,t_1] \to U$ we have that

$$\begin{cases} \dot{\mathbf{x}} = g(\mathbf{x},\mathbf{u}) & \text{in } [\tau,t_1] \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \end{cases}$$

admits a unique solution in $[\tau,t_1]$: this implies that $\mathcal{C}_{\tau,\boldsymbol{\xi}} \ne \emptyset$ and hence $V(\tau,\boldsymbol{\xi}) \ne -\infty$.

[3] Let us recall this fundamental result

Theorem 5.5 (Rademacher)**.** *Let $U \subset \mathbb{R}^n$ be open and let $\phi : U \to \mathbb{R}^m$ be Lipschitz continuous. Then ϕ is differentiable almost everywhere in U.*

[4] This proof as in [11]; for a proof in the non autonomous case, see, for example, [13].

Second, the boundedness assumption on f and ψ guarantee that

$$\begin{aligned} |V(\tau,\boldsymbol{\xi})| &= \left| \sup_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}} \int_\tau^{t_1} f(\mathbf{x},\mathbf{u})\,\mathrm{d}t + \psi(\mathbf{x}(t_1)) \right| \\ &\le \sup_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}} \left(\int_\tau^{t_1} |f(\mathbf{x},\mathbf{u})|\,\mathrm{d}t + |\psi(\mathbf{x}(t_1))| \right) \\ &\le |t_1-\tau|C + C < C. \end{aligned}$$

Now let us fix $\widetilde{\boldsymbol{\xi}}$ and $\widehat{\boldsymbol{\xi}}$ in $\mathbb{R}^n$, and $\tau \in [t_0, t_1]$. For every $\epsilon > 0$ there exists a control $\widehat{\mathbf{u}} \in \mathcal{C}_{\tau,\widehat{\boldsymbol{\xi}}}$ (with trajectory $\widehat{\mathbf{x}}$) such that

$$V(\tau,\widehat{\boldsymbol{\xi}}) - \epsilon \le \int_\tau^{t_1} f(\widehat{\mathbf{x}},\widehat{\mathbf{u}})\,\mathrm{d}t + \psi(\widehat{\mathbf{x}}(t_1)).$$

It is clear that for this control we have $\widehat{\mathbf{u}} \in \mathcal{C}_{\tau,\widetilde{\boldsymbol{\xi}}}$ (with trajectory $\widetilde{\mathbf{x}}$): by the definition of value function,

$$\begin{aligned} V(\tau,\widehat{\boldsymbol{\xi}}) - V(\tau,\widetilde{\boldsymbol{\xi}}) &\le \int_\tau^{t_1} \left(f(\widehat{\mathbf{x}},\widehat{\mathbf{u}}) - f(\widetilde{\mathbf{x}},\widehat{\mathbf{u}})\right)\mathrm{d}t + \psi(\widehat{\mathbf{x}}(t_1)) - \psi(\widetilde{\mathbf{x}}(t_1)) + \epsilon \\ &\le \int_\tau^{t_1} |f(\widehat{\mathbf{x}},\widehat{\mathbf{u}}) - f(\widetilde{\mathbf{x}},\widehat{\mathbf{u}})|\,\mathrm{d}t + |\psi(\widehat{\mathbf{x}}(t_1)) - \psi(\widetilde{\mathbf{x}}(t_1))| + \epsilon. \end{aligned} \tag{5.36}$$

The Lipschitz assumption on g implies that, for a.e. $t \in [\tau, t_1]$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t)\|\right) &= \frac{\left(\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t), \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{x}}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{x}}(t)\right)}{\|\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t)\|} \\ &\le \left\| \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{x}}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{x}}(t) \right\| \\ &= \|g(\widehat{\mathbf{x}}(t),\widehat{\mathbf{u}}(t)) - g(\widetilde{\mathbf{x}}(t),\widehat{\mathbf{u}}(t))\| \\ &\le C\,\|\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t)\| \end{aligned}$$

The Gronwall's inequality (see section 3.5 or the appendix in [11]) implies, for every $t \in [\tau, t_1]$,

$$\|\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t)\| \le \|\widehat{\mathbf{x}}(\tau) - \widetilde{\mathbf{x}}(\tau)\| \exp\left(\int_\tau^t C\,\mathrm{d}s\right) \le C\left\|\widehat{\boldsymbol{\xi}} - \widetilde{\boldsymbol{\xi}}\right\|. \tag{5.37}$$

Hence, using (5.37) and the Lipschitz assumptions, we obtain by (5.36)

$$\begin{aligned} V(\tau,\widehat{\boldsymbol{\xi}}) - V(\tau,\widetilde{\boldsymbol{\xi}}) &\le C\int_\tau^{t_1} \|\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t)\|\,\mathrm{d}t + C\,\|\widehat{\mathbf{x}}(t_1) - \widetilde{\mathbf{x}}(t_1)\| + \epsilon \\ &\le C(t_1-t_0)\left\|\widehat{\boldsymbol{\xi}} - \widetilde{\boldsymbol{\xi}}\right\| + C\left\|\widehat{\boldsymbol{\xi}} - \widetilde{\boldsymbol{\xi}}\right\| + \epsilon \\ &= C\left\|\widehat{\boldsymbol{\xi}} - \widetilde{\boldsymbol{\xi}}\right\| + \epsilon. \end{aligned}$$

The same argument with the role of $\widehat{\boldsymbol{\xi}}$ and $\widetilde{\boldsymbol{\xi}}$ reversed implies $V(\tau,\widetilde{\boldsymbol{\xi}}) - V(\tau,\widehat{\boldsymbol{\xi}}) \le C\left\|\widehat{\boldsymbol{\xi}} - \widetilde{\boldsymbol{\xi}}\right\| + \epsilon$ and hence

$$|V(\tau,\widehat{\boldsymbol{\xi}}) - V(\tau,\widetilde{\boldsymbol{\xi}})| \le C\left\|\widehat{\boldsymbol{\xi}} - \widetilde{\boldsymbol{\xi}}\right\|.$$

Now let us fix $\boldsymbol{\xi}$ in $\mathbb{R}^n$, and $t_0 \le \tau < \widehat{\tau} \le t_1$. For every $\epsilon > 0$ there exists a control $\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}$ (with trajectory $\mathbf{x}$) such that

$$V(\tau, \boldsymbol{\xi}) - \epsilon \le \int_{\tau}^{t_1} f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(\mathbf{x}(t_1)).$$

Consider the function $\widehat{\mathbf{u}} : [\widehat{\tau}, t_1] \to U$ defined by

$$\widehat{\mathbf{u}}(s) = \mathbf{u}(s + \tau - \widehat{\tau}), \qquad \forall s \in [\widehat{\tau}, t_1].$$

It is clear that $\widehat{\mathbf{u}} \in \mathcal{C}_{\widehat{\tau},\boldsymbol{\xi}}$ with trajectory $\widehat{\mathbf{x}}$ such that, since g does not depend on t, $\widehat{\mathbf{x}}(s) = \mathbf{x}(s + \tau - \widehat{\tau})$ for $s \in [\widehat{\tau}, t_1]$: hence, by the definition of value function,

$$\begin{aligned} V(\tau, \boldsymbol{\xi}) - V(\widehat{\tau}, \boldsymbol{\xi}) &\le \int_{\tau}^{t_1} f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(\mathbf{x}(t_1)) + \\ &\quad - \int_{\widehat{\tau}}^{t_1} f(\widehat{\mathbf{x}}, \widehat{\mathbf{u}})\, \mathrm{d}t - \psi(\widehat{\mathbf{x}}(t_1)) + \epsilon \\ &= \int_{t_1 + \tau - \widehat{\tau}}^{t_1} f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(\mathbf{x}(t_1)) - \psi(\widehat{\mathbf{x}}(t_1)) + \epsilon \end{aligned}$$

Since f is bounded and ψ is Lipschitz we obtain

$$V(\tau, \boldsymbol{\xi}) - V(\widehat{\tau}, \boldsymbol{\xi}) \le |\tau - \widehat{\tau}| C + C \|\mathbf{x}(t_1) - \widehat{\mathbf{x}}(t_1)\| + \epsilon; \tag{5.38}$$

since g is bounded we have

$$\begin{aligned} \|\mathbf{x}(t_1) - \mathbf{x}(t_1 + \tau - \widehat{\tau})\| &= \left\| \boldsymbol{\xi} + \int_{\tau}^{t_1} g(\mathbf{x}, \mathbf{u})\, \mathrm{d}s - \boldsymbol{\xi} - \int_{\tau}^{t_1 + \tau - \widehat{\tau}} g(\mathbf{x}, \mathbf{u})\, \mathrm{d}s \right\| \\ &= \left\| \int_{t_1 + \tau - \widehat{\tau}}^{t_1} g(\mathbf{x}, \mathbf{u})\, \mathrm{d}s \right\| \\ &\le C |\tau - \widehat{\tau}|. \end{aligned} \tag{5.39}$$

Clearly (5.38) and (5.39) give

$$V(\tau, \boldsymbol{\xi}) - V(\widehat{\tau}, \boldsymbol{\xi}) \le C |\tau - \widehat{\tau}| + \epsilon. \tag{5.40}$$

Now, with the same $\boldsymbol{\xi}$, $t_0 \le \tau < \widehat{\tau} \le t_1$ and ϵ, let us consider a new control $\widehat{\mathbf{u}} \in \mathcal{C}_{\widehat{\tau},\boldsymbol{\xi}}$ (with trajectory $\widehat{\mathbf{x}}$) such that

$$V(\widehat{\tau}, \boldsymbol{\xi}) - \epsilon \le \int_{\widehat{\tau}}^{t_1} f(\widehat{\mathbf{x}}, \widehat{\mathbf{u}})\, \mathrm{d}t + \psi(\widehat{\mathbf{x}}(t_1)).$$

Consider the function $\mathbf{u} : [\tau, t_1] \to U$ defined by

$$\mathbf{u}(s) = \begin{cases} \widehat{\mathbf{u}}(s + \widehat{\tau} - \tau) & \text{for } s \in [\tau, t_1 - \widehat{\tau} + \tau] \\ \widehat{\mathbf{u}}(t_1) & \text{for } s \in (t_1 - \widehat{\tau} + \tau, t_1] \end{cases}$$

It is clear that $\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}$ (with trajectory $\mathbf{x}$) and that $\mathbf{x}(s) = \widehat{\mathbf{x}}(s + \widehat{\tau} - \tau)$ for $s \in [\tau, t_1 - \widehat{\tau} + \tau]$: hence, by the definition of value function,

$$\begin{aligned} V(\widehat{\tau}, \boldsymbol{\xi}) - V(\tau, \boldsymbol{\xi}) &\le \int_{\widehat{\tau}}^{t_1} f(\widehat{\mathbf{x}}, \widehat{\mathbf{u}})\, \mathrm{d}t + \psi(\widehat{\mathbf{x}}(t_1)) + \\ &\quad - \int_{\tau}^{t_1} f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t - \psi(\mathbf{x}(t_1)) + \epsilon \\ &\le -\int_{t_1 - \widehat{\tau} + \tau}^{t_1} f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(\mathbf{x}(t_1 - \widehat{\tau} + \tau)) - \psi(\mathbf{x}(t_1)) + \epsilon \end{aligned}$$

The same arguments of before, give

$$V(\widehat{\tau}, \boldsymbol{\xi}) - V(\tau, \boldsymbol{\xi}) \leq C|\tau - \widehat{\tau}| + \epsilon. \tag{5.41}$$

The inequalities (5.40) and (5.41) guarantee that

$$|V(\widehat{\tau}, \boldsymbol{\xi}) - V(\tau, \boldsymbol{\xi})| \leq C|\tau - \widehat{\tau}|$$

and the proof is finished. □

5.3.1 Viscosity solution

Now if we consider the problem (5.1) with the assumption of Theorem 5.4, such theorem and Corollary 5.1 guarantee that the value function $V : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ is bounded and for a.e. $(\bar{t}, \overline{\mathbf{x}}) \in [t_0, t_1] \times \mathbb{R}^n$ exists

$$\left(\frac{\partial V}{\partial t}(\bar{t}, \overline{\mathbf{x}}),\ \nabla_{\mathbf{x}} V(\bar{t}, \overline{\mathbf{x}}) \right). \tag{5.42}$$

At this point we have a problem: if there exists a point $(\bar{t}, \overline{\mathbf{x}}) \in [t_0, t_1] \times \mathbb{R}^n$ such that V is not differentiable, i.e. the vector in (5.42) does not exist, how should we interpret the BHJ equation in $(\bar{t}, \overline{\mathbf{x}})$, i.e. the relation $\frac{\partial V}{\partial t}(\bar{t}, \overline{\mathbf{x}}) + H_{DP}(\bar{t}, \overline{\mathbf{x}}, \nabla_{\mathbf{x}} V(\bar{t}, \overline{\mathbf{x}})) = 0$?

A more general and important result of regularity for the value function passes through the definition of viscosity solution (see Chapter 10 in [11] and [3] for more details):

Definition 5.1. *Let us consider a bounded and uniformly continuous function* $V : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ *with* $V(t_1, \mathbf{x}) = \psi(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{R}^n$.
We say that V *is a* ***viscosity subsolution*** *of BHJ system (5.21) if whenever* v *is a test function in* $C^\infty((t_0, t_1) \times \mathbb{R}^n)$ *such that* $V - v$ *has a local minimum in the point* $(\bar{t}, \overline{\mathbf{x}}) \in (t_0, t_1) \times \mathbb{R}^n$, *then we have that*

$$\frac{\partial v}{\partial t}(\bar{t}, \overline{\mathbf{x}}) + H_{DP}(\bar{t}, \overline{\mathbf{x}}, \nabla_{\mathbf{x}} v(\bar{t}, \overline{\mathbf{x}})) \leq 0. \tag{5.43}$$

We say that V *is a* ***viscosity supersolution*** *of BHJ system (5.21) if whenever* v *is a test function in* $C^\infty((t_0, t_1) \times \mathbb{R}^n)$ *such that* $V - v$ *has a local maximum in the point* $(\bar{t}, \overline{\mathbf{x}}) \in (t_0, t_1) \times \mathbb{R}^n$, *then we have that*

$$\frac{\partial v}{\partial t}(\bar{t}, \overline{\mathbf{x}}) + H_{DP}(\bar{t}, \overline{\mathbf{x}}, \nabla_{\mathbf{x}} v(\bar{t}, \overline{\mathbf{x}})) \geq 0. \tag{5.44}$$

A function that is both a viscosity subsolution and a viscosity supersolution is called ***viscosity solution****.*

The theory of viscosity solution is very important and wide, but it is not the focus of this note. We are interested to give only an idea. We only remark that the previous definition allow us to give an interpretation of the BHJ equation in the point $(\bar{t}, \overline{\mathbf{x}})$ where V is not regular enough; on the other hand, in the points $(\bar{t}, \overline{\mathbf{x}})$ where V is regular then nothing change with respect to the classical notion of solution of BHJ equation:

Remark 5.7. *Let $V : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function such that is C^1 in a neighborhood of a point $(\bar{t}, \overline{\mathbf{x}}) \in (t_0, t_1) \times \mathbb{R}^n$. Then the two requirements (5.43)–(5.44) in Definition 5.1 are equivalent to*

$$\frac{\partial V}{\partial t}(\bar{t}, \overline{\mathbf{x}}) + H_{DP}(\bar{t}, \overline{\mathbf{x}}, \nabla_{\mathbf{x}} V(\bar{t}, \overline{\mathbf{x}})) = 0. \tag{5.45}$$

Proof. Let us start by proving (5.43)–(5.44) imply (5.45): let us consider $v \in C^\infty((t_0, t_1) \times \mathbb{R}^n)$ such that $V - v$ has a local minimum in the point $(\bar{t}, \overline{\mathbf{x}}) \in (t_0, t_1) \times \mathbb{R}^n$; then inequality (5.43) holds. However, since the function $V - v$ is regular and $(\bar{t}, \overline{\mathbf{x}})$ is a minimum point for such function, then

$$\left(\frac{\partial V}{\partial t}(\bar{t}, \overline{\mathbf{x}}),\ \nabla_{\mathbf{x}} V(\bar{t}, \overline{\mathbf{x}}) \right) = \left(\frac{\partial v}{\partial t}(\bar{t}, \overline{\mathbf{x}}),\ \nabla_{\mathbf{x}} v(\bar{t}, \overline{\mathbf{x}}) \right). \tag{5.46}$$

Inequality (5.43) becomes

$$\frac{\partial V}{\partial t}(\bar{t}, \overline{\mathbf{x}}) + H_{DP}(\bar{t}, \overline{\mathbf{x}}, \nabla_{\mathbf{x}} V(\bar{t}, \overline{\mathbf{x}})) \le 0.$$

Similar argument proves the reverse inequality and hence (5.45).

Now suppose that (5.45) holds and let us consider $v \in C^\infty((t_0, t_1) \times \mathbb{R}^n)$ such that $V - v$ has a local minimum in the point $(\bar{t}, \overline{\mathbf{x}}) \in (t_0, t_1) \times \mathbb{R}^n$. Clearly (5.46) holds and hence (5.43). Similar argument proves that if we have a local maximum for $V - v$ then (5.44) is true. □

Now we have the following fundamental result:

Theorem 5.6. *Consider the autonomous problem (5.35) with the assumptions* 1., 2. *and* 3.. *Then the value function V is the unique viscosity solution of the BHJ system (5.21).*

Proof. We prove only that V is a viscosity solution (see Theorem 3 in section 10.3 of [11]); the proof of the uniqueness is very hard and we omit it (see for example Theorem 1 in section 10.2 of [11]).

Theorem 5.4 and Remark 5.2 give that V is bounded, uniformly continuous and satisfies the final condition. Now let $v \in C^\infty((t_0, t_1) \times \mathbb{R}^n)$ such that the function $V - v$ has a local minimum in the point $(\tau_0, \boldsymbol{\xi}_0) \in (t_0, t_1) \times \mathbb{R}^n$: we have to prove that

$$\frac{\partial v}{\partial t}(\tau_0, \boldsymbol{\xi}_0) + \max_{\mathbf{u} \in U} \left(f(\boldsymbol{\xi}_0, \mathbf{u}) + \nabla_{\mathbf{x}} v(\tau_0, \boldsymbol{\xi}_0) \cdot g(\boldsymbol{\xi}_0, \mathbf{u}) \right) \le 0. \tag{5.47}$$

Suppose that (5.47) is not true; then there exists $\theta > 0$ and a point $\widetilde{\mathbf{u}}$ in U such that

$$\frac{\partial v}{\partial t}(\tau_0, \boldsymbol{\xi}_0) + f(\boldsymbol{\xi}_0, \widetilde{\mathbf{u}}) + \nabla_{\mathbf{x}} v(\tau_0, \boldsymbol{\xi}_0) \cdot g(\boldsymbol{\xi}_0, \widetilde{\mathbf{u}}) \ge \theta.$$

The continuity of the functions involved in the previous line implies that there exists $\delta > 0$ such that

$$\frac{\partial v}{\partial t}(\tau, \boldsymbol{\xi}) + f(\boldsymbol{\xi}, \widetilde{\mathbf{u}}) + \nabla_{\mathbf{x}} v(\tau, \boldsymbol{\xi}) \cdot g(\boldsymbol{\xi}, \widetilde{\mathbf{u}}) \ge \theta \tag{5.48}$$

for every $(\tau, \boldsymbol{\xi}) \in I_\delta := \{(\tau, \boldsymbol{\xi}) \in (t_0, t_1) \times \mathbb{R}^n : \ |\tau - \tau_0| + \|\boldsymbol{\xi} - \boldsymbol{\xi}_0\| \le \delta\}$. Since $V - v$ has a local minimum in $(\tau_0, \boldsymbol{\xi}_0)$, we can consider δ such that

$$V(\tau_0, \boldsymbol{\xi}_0) - v(\tau_0, \boldsymbol{\xi}_0) \le V(\tau, \boldsymbol{\xi}) - v(\tau, \boldsymbol{\xi}), \qquad \forall (\tau, \boldsymbol{\xi}) \in I_\delta. \tag{5.49}$$

Let as consider the constant control equal to $\widetilde{\mathbf{u}}$, denoting it by $\widetilde{\mathbf{u}}$, and let us consider $h > 0$, with $\delta > h$, such that the solution $\mathbf{x}_{\widetilde{\mathbf{u}}}$ of the problem

$$\begin{cases} \dot{\mathbf{x}} = g(\mathbf{x}, \widetilde{\mathbf{u}}) & \text{in } [\tau_0, \tau_0 + h] \\ \mathbf{x}(\tau_0) = \boldsymbol{\xi}_0 \end{cases}$$

is such that $\|\mathbf{x}_{\widetilde{\mathbf{u}}}(t) - \boldsymbol{\xi}_0\| \le \delta$ for every $t \in [\tau_0, \tau_0 + h]$: note that such h there exists since, using the boundedness assumption on g,

$$\|\mathbf{x}_{\widetilde{\mathbf{u}}}(t) - \boldsymbol{\xi}_0\| = \left\| \int_{\tau_0}^t g(\mathbf{x}_{\widetilde{\mathbf{u}}}(s), \widetilde{\mathbf{u}}(s)) \, \mathrm{d}s \right\| \le hC, \qquad \forall t \in [\tau_0, \tau_0 + h].$$

Hence by (5.49), we have

$$\begin{aligned} V(\tau_0 + h, \mathbf{x}_{\widetilde{\mathbf{u}}}(\tau_0 + h)) - V(\tau_0, \mathbf{x}_{\widetilde{\mathbf{u}}}(\tau_0)) &\ge \\ &\ge v(\tau_0 + h, \mathbf{x}_{\widetilde{\mathbf{u}}}(\tau_0 + h)) - v(\tau_0, \mathbf{x}_{\widetilde{\mathbf{u}}}(\tau_0)) \\ &= \int_{\tau_0}^{\tau_0 + h} \frac{\mathrm{d}}{\mathrm{d}s} v(s, \mathbf{x}_{\widetilde{\mathbf{u}}}(s)) \, \mathrm{d}s \\ &= \int_{\tau_0}^{\tau_0 + h} \frac{\partial v}{\partial t}(s, \mathbf{x}_{\widetilde{\mathbf{u}}}(s)) + \nabla_{\mathbf{x}} v(s, \mathbf{x}_{\widetilde{\mathbf{u}}}(s)) \cdot g(\mathbf{x}_{\widetilde{\mathbf{u}}}(s), \widetilde{\mathbf{u}}(s)) \, \mathrm{d}s \end{aligned} \tag{5.50}$$

Now Remark 5.3 implies that

$$V(\tau_0, \boldsymbol{\xi}_0) \ge \int_{\tau_0}^{\tau_0 + h} f(\mathbf{x}_{\widetilde{\mathbf{u}}}(s), \widetilde{\mathbf{u}}(s)) \, \mathrm{d}s + V(\tau_0 + h, \mathbf{x}_{\widetilde{\mathbf{u}}}(\tau_0 + h)).$$

The previous inequality and (5.50), taking into account (5.48), give

$$\begin{aligned} 0 \ &\ge \ \int_{\tau_0}^{\tau_0 + h} \left(\frac{\partial v}{\partial t}(s, \mathbf{x}_{\widetilde{\mathbf{u}}}) + f(\mathbf{x}_{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{u}}) + \nabla_{\mathbf{x}} v(s, \mathbf{x}_{\widetilde{\mathbf{u}}}) \cdot g(\mathbf{x}_{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{u}}) \right) \mathrm{d}s \\ &\ge \ \int_{\tau_0}^{\tau_0 + h} \theta \, \mathrm{d}s = \theta h : \end{aligned}$$

this is impossible and V is really a viscosity subsolution.

Now let $v \in C^\infty((t_0, t_1) \times \mathbb{R}^n)$ such that the function $V - v$ has a local maximum in the point $(\tau_0, \boldsymbol{\xi}_0) \in (t_0, t_1) \times \mathbb{R}^n$: we have to prove that

$$\frac{\partial v}{\partial t}(\tau_0, \boldsymbol{\xi}_0) + \max_{\mathbf{u} \in U} \left(f(\boldsymbol{\xi}_0, \mathbf{u}) + \nabla_{\mathbf{x}} v(\tau_0, \boldsymbol{\xi}_0) \cdot g(\boldsymbol{\xi}_0, \mathbf{u}) \right) \ge 0. \tag{5.51}$$

Suppose that (5.51) is not true; then there exists $\theta > 0$ such that

$$\frac{\partial v}{\partial t}(\tau_0, \boldsymbol{\xi}_0) + f(\boldsymbol{\xi}_0, \mathbf{u}) + \nabla_{\mathbf{x}} v(\tau_0, \boldsymbol{\xi}_0) \cdot g(\boldsymbol{\xi}_0, \mathbf{u}) \le -\theta$$

for every $\mathbf{u} \in U$. The continuity of the functions involved in the previous line implies that there exists $\delta > 0$ such that

$$\frac{\partial v}{\partial t}(\tau, \boldsymbol{\xi}) + f(\boldsymbol{\xi}, \mathbf{u}) + \nabla_{\mathbf{x}} v(\tau, \boldsymbol{\xi}) \cdot g(\boldsymbol{\xi}, \mathbf{u}) \le -\theta \tag{5.52}$$

for every $\mathbf{u} \in U$ and $(\tau, \boldsymbol{\xi}) \in I_\delta$, with I_δ as before. Since $V - v$ has a local maximum in $(\tau_0, \boldsymbol{\xi}_0)$, we can consider δ such that

$$V(\tau_0, \boldsymbol{\xi}_0) - v(\tau_0, \boldsymbol{\xi}_0) \geq V(\tau, \boldsymbol{\xi}) - v(\tau, \boldsymbol{\xi}), \qquad \forall (\tau, \boldsymbol{\xi}) \in I_\delta. \tag{5.53}$$

Let us consider $h > 0$, with $\delta > h$, such that for every control $\mathbf{u}$ the solution $\mathbf{x_u}$ of the problem

$$\begin{cases} \dot{\mathbf{x}} = g(\mathbf{x}, \mathbf{u}) & \text{in } [\tau_0, \tau_0 + h] \\ \mathbf{x}(\tau_0) = \boldsymbol{\xi}_0 \end{cases}$$

is such that $\|\mathbf{x_u}(t) - \boldsymbol{\xi}_0\| \leq \delta$ for every $t \in [\tau_0, \tau_0 + h]$: such h exists again for the boundedness assumption on g. Hence, for every control $\mathbf{u}$, by (5.53), we have

$$\begin{aligned} V(\tau_0 + h, \mathbf{x_u}(\tau_0 + h)) - V(\tau_0, \mathbf{x_u}(\tau_0)) &\leq \\ &\leq v(\tau_0 + h, \mathbf{x_u}(\tau_0 + h)) - v(\tau_0, \mathbf{x_u}(\tau_0)) \\ &= \int_{\tau_0}^{\tau_0 + h} \frac{\mathrm{d}}{\mathrm{d}s} v(s, \mathbf{x_u}(s))\, \mathrm{d}s \\ &= \int_{\tau_0}^{\tau_0 + h} \left(\frac{\partial v}{\partial t}(s, \mathbf{x_u}(s)) + \nabla_{\mathbf{x}} v(s, \mathbf{x_u}(s)) \cdot g(\mathbf{x_u}(s), \mathbf{u}(s)) \right) \mathrm{d}s \end{aligned} \tag{5.54}$$

Now Remark 5.3 implies that there exists a control $\widehat{\mathbf{u}}$ such that

$$V(\tau_0, \boldsymbol{\xi}_0) \leq \int_{\tau_0}^{\tau_0 + h} f(\mathbf{x}_{\widehat{\mathbf{u}}}(s), \widehat{\mathbf{u}}(s))\, \mathrm{d}s + V(\tau_0 + h, \mathbf{x}_{\widehat{\mathbf{u}}}(\tau_0 + h)) + \frac{\theta h}{2}.$$

The previous inequality and (5.54), taking into account (5.52), give

$$\begin{aligned} 0 &\leq \int_{\tau_0}^{\tau_0 + h} \left(\frac{\partial v}{\partial t}(s, \mathbf{x}_{\widehat{\mathbf{u}}}) + f(\mathbf{x}_{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}) + \nabla_{\mathbf{x}} v(s, \mathbf{x}_{\widehat{\mathbf{u}}}) \cdot g(\mathbf{x}_{\widehat{\mathbf{u}}}, \widehat{\mathbf{u}}) \right) \mathrm{d}s + \frac{\theta h}{2} \\ &\leq \int_{\tau_0}^{\tau_0 + h} -\theta\, \mathrm{d}s + \frac{\theta h}{2} = -\frac{\theta h}{2} : \end{aligned}$$

this is impossible and V is really a viscosity supersolution. □

Let us give an example in order to show what can happen:

Example 5.3.1. *Let us consider*

$$\begin{cases} \max \displaystyle\int_{-1}^{0} -\frac{(|u| + 2)^2}{4}\, \mathrm{d}t + |x(0)| \\ \dot{x} = u \\ |u| \leq 2 \\ x(-1) = 1 \end{cases}$$

Note that the assumptions of Theorem 5.4 hold. We are looking for a function $V : [-1, 0] \times \mathbb{R} \to \mathbb{R}$ *that satisfies the BHJ equation (5.8) and the final condition (5.5):*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, x) + H_{DP}\left(\dfrac{\partial V}{\partial x}(t, x) \right) = 0 & \forall (t, x) \in [-1, 0] \times \mathbb{R} \\ V(0, x) = |x| & \forall x \in \mathbb{R} \end{cases} \tag{5.55}$$

where

$$H_{DP}(t, x, p) = H_{DP}(p) = \max_{u \in [-2, 2]} h_p(u)$$

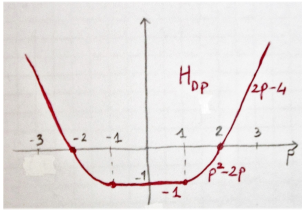
and

$$h_p(u) = -\frac{(|u| + 2)^2}{4} + up.$$

For every fixed p, the function h_p has a maximum point: let us call such point $\widetilde{w}(p)$. We note that $\widetilde{w}(-p) = -\widetilde{w}(p)$ and hence that $H_{DP}(p) = H_{DP}(-p)$. Now it is clear that we can restrict our attention, for a fixed $p \geq 0$, to the $\max_{\{u\in[0,2]\}} h_p(u)$. It is easy to see that, for $u \geq 0$, $h'_p(u) \geq 0$ if and only if $u \leq 2p - 2$. Hence we obtain

$$\widetilde{w}(p) = \begin{cases} 0 & \text{if } |p| \leq 1 \\ 2|p| - 2 & \text{if } 1 < |p| \leq 2 \\ 2 & \text{if } 2 < |p| \end{cases}$$

$$H_{DP}(p) = \begin{cases} -1 & \text{if } |p| \leq 1 \\ p^2 - 2|p| & \text{if } 1 < |p| \leq 2 \\ 2|p| - 4 & \text{if } 2 < |p| \end{cases}$$



Let us prove that

$$V(t, x) = t + |x| \tag{5.56}$$

is a viscosity solution of (5.55). In order to do that, we have to use Definition 5.1.
First, our V is continuous and satisfies the final condition, i.e. $V(0, x) = |x|$: it is bounded only on compact sets.
If we consider a point $(\bar{t}, \bar{x}) \in (-1, 0) \times \mathbb{R}$ with $\bar{x} \neq 0$, our V is C^1 in such point and (we recall Remark 5.7 for the "regular" of V) we obtain

$$\frac{\partial V}{\partial t}(\bar{t}, \bar{x}) + H_{DP}\left(\frac{\partial V}{\partial x}(\bar{t}, \bar{x})\right) = 1 + H_{DP}(\pm 1) = 0.$$

Now, if we consider $(\bar{t}, \bar{x}) \in (-1, 0) \times \mathbb{R}$ with $\bar{x} = 0$, formally we obtain

$$\frac{\partial V}{\partial t}(\bar{t}, \bar{x}) + H_{DP}\left(\frac{\partial V}{\partial x}(\bar{t}, \bar{x})\right) = 1 + H_{DP}\left(\frac{\partial(|x|)}{\partial x}(0)\right).$$

Let us use the definition of viscosity solution in details. If a function v is in C^∞ such that $(t, x) \mapsto V(t, x) - v(t, x) = t + |x| - v(t, x)$ has a local minimum in $(\bar{t}, 0)$, this implies that[5]

$$\bar{t} - v(\bar{t}, 0) \leq t + |x| - v(t, x), \qquad \forall (t, x) \in B((\bar{t}, 0), \varepsilon), \tag{5.57}$$

for some $\varepsilon > 0$. In particular setting $x = 0$ in (5.57) we necessarily have for v that

$$\bar{t} - v(\bar{t}, 0) \leq t - v(t, 0), \qquad \forall t \in B(\bar{t}, \varepsilon):$$

taking into account that $\frac{\partial v}{\partial t}(\bar{t}, 0)$ exists, then the previous relation implies

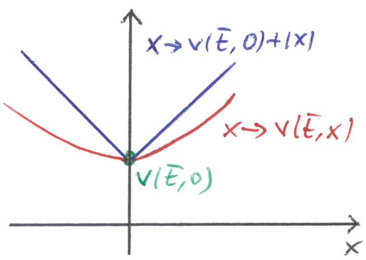
$$\frac{\partial v}{\partial t}(\bar{t}, 0) = \lim_{t \to \bar{t}} \frac{v(t, 0) - v(\bar{t}, 0)}{t - \bar{t}} \leq \lim_{t \to \bar{t}^+} \frac{t - \bar{t}}{t - \bar{t}} = 1. \tag{5.58}$$

Now setting $t = \bar{t}$ in (5.57) we require

$$v(\bar{t}, x) \leq v(\bar{t}, 0) + |x|, \qquad \forall x \in B(0, \varepsilon);$$

taking into account that $\frac{\partial v}{\partial x}(\bar{t}, 0)$ exists, then the previous relation implies (see the picture)

$$\left|\frac{\partial v}{\partial x}(\bar{t}, 0)\right| \leq 1. \tag{5.59}$$



Hence (5.58) and (5.59) give that

$$\frac{\partial v}{\partial t}(\bar{t}, 0) + H_{DP}\left(\frac{\partial v}{\partial x} v(\bar{t}, 0)\right) \leq 1 - 1 = 0$$

and condition (5.43) of Definition 5.1 is satisfied.
Now if a function v is in C^∞ such that $(t, x) \mapsto V(t, x) - v(t, x) = t + |x| - v(t, x)$ has a local maximum in $(\bar{t}, 0)$, this implies that

$$\bar{t} - v(\bar{t}, 0) \geq t + |x| - v(t, x), \qquad \forall (t, x) \in B((\bar{t}, 0), \varepsilon), \tag{5.60}$$

for some $\varepsilon > 0$. Setting $t = \bar{t}$ in (5.60) we require for v that

$$v(\bar{t}, x) \geq v(\bar{t}, 0) + |x|, \qquad \forall x \in B(0, \varepsilon);$$

[5] We denote by $B(y, r)$ a ball of center $y \in \mathbb{R}^k$ and radius $r > 0$.

taking into account that $\frac{\partial v}{\partial x}(\bar{t},0)$ *exists, then the previous relation implies (see again the picture) that such function* v *does not exist. Hence condition (5.44) of Definition 5.1 is satisfied.*
This concludes the proof that V *in (5.56) is a viscosity solution for (5.55).*

At this point Theorem 5.6 guarantees that V *in (5.56) is the value function. Now let us solve our initial problem. Clearly, for* $x \neq 0$, *we have*

$$w(t,x) = \widetilde{w}\left(\frac{\partial V}{\partial x}(t,x)\right) = \widetilde{w}(\pm 1) = 0.$$

Hence

$$\begin{cases} \dot{x} = w(t,x) = 0 & t \in [-1,0] \\ x(-1) = 1 \end{cases}$$

gives $x^*(t) = 1$*: note that* $x^*(t) > 0$. *Hence the optimal control is* $u^*(t) = w(t, x^*(t)) = 0$.

△

5.4 More general problems of OC

Let us consider the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{T} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + \psi(T,\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ (T,\mathbf{x}(T)) \in \mathcal{T} \\ \displaystyle\max_{\mathbf{u}\in\mathcal{C}_{t_0,\boldsymbol{\alpha}}} J(\mathbf{u}), \end{cases} \tag{5.61}$$

with a control set $U \subset \mathbb{R}^k$, with the target set $\mathcal{T} \subset (t_0,\infty)\times\mathbb{R}^n$ and the class of admissible control defined by, for $(\tau,\boldsymbol{\xi})$,

$$\mathcal{C}_{\tau,\boldsymbol{\xi}} = \Big\{\mathbf{u} : [\tau,T] \to U \subset \mathbb{R}^k,\ \mathbf{u} \text{ measurable},\ \exists!\ \mathbf{x} \in C([\tau,T]) \\ \text{with } \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \text{ a.e.},\ \mathbf{x}(\tau) = \boldsymbol{\xi},\ \mathbf{x}(T) \in \mathcal{T}\Big\}.$$

Let us consider the *reachable set* for the target set $\mathcal{T}$ defined by

$$R(S) = \{(\tau,\boldsymbol{\xi}) :\ \mathcal{C}_{\tau,\boldsymbol{\xi}} \neq \emptyset\},$$

i.e. as the set of the points $(\tau,\boldsymbol{\xi})$ from which it is possible to reach the terminal target set $\mathcal{T}$ with some trajectory. We have the following generalization of the necessary condition of Theorem 5.2 (see [13]):

Theorem 5.7. *Let us consider the problem (5.61) with value function* V. *Let* f, g *and* ψ *be continuous. Let the target set* S *be closed. Let* $(\tau,\boldsymbol{\xi})$ *be a point in the interior of the reachable set* $R(S)$; *let us suppose that* V *is differentiable in* $(\tau,\boldsymbol{\xi})$ *and that exists the optimal control for the problem (5.61) with initial data* $\mathbf{x}(\tau) = \boldsymbol{\xi}$. *Then we have*

$$\frac{\partial V}{\partial t}(\tau,\boldsymbol{\xi}) + \max_{\mathbf{v}\in U}\Big(f(\tau,\boldsymbol{\xi},\mathbf{v}) + \nabla_{\mathbf{x}} V(\tau,\boldsymbol{\xi})\cdot g(\tau,\boldsymbol{\xi},\mathbf{v})\Big) = 0.$$

A sufficient conditions for the problem (5.61) holds and it is a modification of Theorem 5.3: note that for this problem the final time is not fixed.

Theorem 5.8. *Let us consider the problem (5.61) with S closed. Let $W : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 solution of the BHJ equation*

$$\frac{\partial W}{\partial t}(t,\mathbf{x}) + \max_{\mathbf{v}\in U}\Big(f(t,\mathbf{x},\mathbf{v}) + \nabla_{\mathbf{x}} W(t,\mathbf{x}) \cdot g(t,\mathbf{x},\mathbf{v})\Big) = 0,$$

for every $(t, \mathbf{x})$ in the interior of the reachable set $R(S)$. Suppose that the final condition

$$W(t,\mathbf{x}) = \psi(t,\mathbf{x}), \qquad \forall (t,\mathbf{x}) \in S \tag{5.62}$$

holds. Let $(t_0, \boldsymbol{\alpha})$ be in the interior of $R(S)$ and let $\mathbf{u}^* : [t_0, T^*] \to U$ be a control in $\mathcal{C}_{t_0,\boldsymbol{\alpha}}$ with corresponding trajectory $\mathbf{x}^*$ such that

$$\frac{\partial W}{\partial t}(t,\mathbf{x}^*(t)) + f(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) + \nabla_{\mathbf{x}} W(t,\mathbf{x}^*(t)) \cdot g(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) = 0,$$

for every $\in [t_0, T^*]$. Then $\mathbf{u}^*$ is the optimal control with exit time T^*.

A result with weaker assumptions with respect to the previous theorem can be found for example in Theorem 7.1 in [13] (page 97).

In subsection 5.5.2 we will see an example of a min-problem where the target S is a single point, i.e. $\mathbf{x}(t_1) = \boldsymbol{\beta}$ where t_1 is the final and fixed time, and $\boldsymbol{\beta} \in \mathbb{R}^n$ is fixed: for such example, we will prove that V achieves the value ∞ and hence the regularity of V is very delicate !!!

5.4.1 On minimum problems

Let us consider a problem (5.61) where we replace the maximum with a minimum problem. It is clear that in the previous arguments, we have only to replace

$$\max \qquad \to \quad \min .$$

5.5 Examples and applications

Example 5.5.1. *Let us consider*[6] *the problem*[7]

$$\begin{cases} \max \displaystyle\int_0^1 (x - u^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = 2 \end{cases}$$

We are looking for a function $V : [0,1] \times \mathbb{R} \to \mathbb{R}$ that satisfies the BHJ equation (5.8) and the final condition (5.5):

$$\frac{\partial V}{\partial t} + \max_{v\in\mathbb{R}}\Big(x - v^2 + v\frac{\partial V}{\partial x}\Big) = 0$$

$$\Rightarrow \quad \frac{\partial V}{\partial t} + x + \max_{v\in\mathbb{R}}\Big(-v^2 + v\frac{\partial V}{\partial x}\Big) = 0 \tag{5.63}$$

$$V(1,x) = 0, \ \forall x \in \mathbb{R} \tag{5.64}$$

[6] In the example 2.5.1 we solve the same example with a variational approach.

[7] Suggestion: In order to solve $x + A(\frac{\partial F}{\partial x})^2 + \frac{\partial F}{\partial t} = 0$ with A constant, we suggest to find the solution in the family of functions

$$\mathcal{F} = \{F(t,x) = at^3 + bt^2 + ct + dx + fxt + g, \ \text{with } a,b,c,d,f,g \text{ constants}\}.$$

We obtain the max *in (5.63) when* $v = \frac{\partial V}{\partial x}/2$: *hence the function* $w(t,x)$ *defined in (5.23) is, in this situation,*

$$w(t,x) = \frac{1}{2}\frac{\partial V}{\partial x}(t,x). \tag{5.65}$$

In (5.63) we obtain

$$\frac{\partial V}{\partial t}(t,x) + x + \frac{1}{4}\left(\frac{\partial V}{\partial x}(t,x)\right)^2 = 0.$$

Using the suggestion and with easy calculations we obtain that the solution is

$$V(t,x) = -\frac{1}{12}t^3 + \frac{d}{4}t^2 - \frac{d^2}{4}t + dx - xt + g.$$

The condition (5.64) implies that

$$V(t,x) = -\frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{4}t + x - xt + \frac{1}{12}. \tag{5.66}$$

The optimal control is defined by (5.25): using (5.65) and (5.66) we obtain

$$u^*(t) = w(t,x^*(t)) = \frac{1}{2}\frac{\partial V}{\partial x}(t,x^*(t)) = \frac{1-t}{2}.$$

The dynamics and the condition $x(0) = 2$ *give* $x^*(t) = (2t - t^2)/4 + 2$. △

Example 5.5.2. *Let us consider*[8] *the problem*[9] *(5.4)*

$$\begin{cases} \min \int_0^2 (u^2 + x^2)\,\mathrm{dt} \\ \dot{x} = x + u \\ x(0) = x_0 \\ u \geq 0 \end{cases}$$

We are looking for a function $V : [0,2] \times \mathbb{R} \to \mathbb{R}$ *that satisfies the BHJ equation (5.8) and the final condition (5.5):*

$$\frac{\partial V}{\partial t} + \min_{v\in[0,\infty)}\left(v^2 + x^2 + \frac{\partial V}{\partial x}(x+v)\right) = 0 \Rightarrow$$

$$\Rightarrow \quad \frac{\partial V}{\partial t} + x^2 + x\frac{\partial V}{\partial x} + \min_{v\in[0,\infty)}\left(v^2 + \frac{\partial V}{\partial x}v\right) = 0 \tag{5.67}$$

$$V(2,x) = 0, \ \forall x \in \mathbb{R} \tag{5.68}$$

The point v *that realizes the* min *in (5.67) is given by the function* $w(t,x)$ *defined in (5.23):*

$$w(t,x) = \begin{cases} -\frac{1}{2}\frac{\partial V}{\partial x}(t,x) & \text{if } \frac{\partial V}{\partial x}(t,x) < 0 \\ 0 & \text{if } \frac{\partial V}{\partial x}(t,x) \geq 0 \end{cases} \tag{5.69}$$

We note that (5.68) implies

$$\frac{\partial V}{\partial x}(2,x) = 0, \quad \forall x \in \mathbb{R}$$

Let us assume that there exists a point $\tau \in [0,2)$ *and an interval* $I \subset \mathbb{R}$ *such that*

$$\frac{\partial V}{\partial x}(t,x) \geq 0, \qquad \forall (t,x) \in (\tau,2] \times I \tag{5.70}$$

Condition (5.69) implies that in $(\tau,2] \times I$ *we have* $w(t,x) = 0$ *and the BHJ equation (5.67) is*

$$\frac{\partial V}{\partial t}(t,x) + x^2 + x\frac{\partial V}{\partial x}(t,x) = 0.$$

[8] In the example 2.7.1 and in the example 5.1.1 we solve the problem and construct the value function using a variational approach.

[9] *First suggestion:* In order to solve $x\frac{\partial F}{\partial x} + Ax^2 + \frac{\partial F}{\partial t} = 0$ and $x\frac{\partial F}{\partial x} + Ax^2 + B(\frac{\partial F}{\partial x})^2 + \frac{\partial F}{\partial t} = 0$ with $A,\ B$ constants, we suggest to find the solution in the family of functions

$$\mathcal{F} = \{F(t,x) = x^2G(t), \text{ with } G \text{ function}\}.$$

Second suggestion: the case $x_0 < 0$ is difficult ...

Using the suggestion to looking for a function $V(x,t) = x^2 G(t)$ we obtain

$$G'(t) + 1 + 2G(t) = 0 \qquad \Rightarrow\ G(t) = ae^{-2t} - 1/2,\ \forall a \in \mathbb{R}$$
$$\Rightarrow\ V(t,x) = x^2(ae^{-2t} - 1/2),\ \forall a \in \mathbb{R}$$

The condition (5.68) implies that

$$V(t,x) = \frac{x^2}{2}(e^{4-2t} - 1). \tag{5.71}$$

Since by (5.71) we have

$$\frac{\partial V}{\partial x}(t,x) = x(1 - e^{4-2t}) \geq 0 \quad \text{if } (t,x) \in [0,2] \times [0,\infty),$$

all these last arguments hold if we put $I = [0,\infty)$ in the assumption (5.70). Hence the function V defined in (5.71) satisfies the BHJ equation in $[0,2] \times [0,\infty)$ and the final condition. Hence the optimal control for every initial data of the trajectory in $[0,2] \times [0,\infty)$ is defined by (5.25): using (5.69) we obtain

$$u^*(t) = w(t,x) = 0.$$

The dynamics and the condition $x(0) = x_0 \geq 0$ give the candidate to be the optimal trajectory, i.e. $x^*(t) = x_0 e^t$.
At this point, we have to construct the function V in $[0,2) \times (-\infty,0)$. Hence, let us assume that there exists a point $\tau' \in [0,2)$ such that

$$\frac{\partial V}{\partial x}(t,x) < 0, \qquad \forall (t,x) \in (\tau',2) \times (-\infty,0) \tag{5.72}$$

Condition (5.69) implies that in $(\tau',2) \times (-\infty,0)$ we have $w(t,x) = -\frac{\partial V}{\partial x}(t,x)/2$ and the BHJ equation (5.67) is

$$\frac{\partial V}{\partial t}(t,x) + x^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x}(t,x)\right)^2 + x\frac{\partial V}{\partial x}(t,x) = 0.$$

Using the suggestion to looking for a function $V(t,x) = x^2 G(t)$ we obtain

$$G'(t) = -1 - 2G(t) + G^2(t).$$

To solve this Riccati differential equation[10]*, we consider the new variable $G = -\frac{z'}{z}$ obtaining*

$$z'' + 2z' - z = 0$$

and hence

$$z(t) = c_1 e^{(\sqrt{2}-1)t} + c_2 e^{-(\sqrt{2}+1)t}, \qquad \textit{with } c_1,\ c_2 \textit{ constants} \tag{5.73}$$

The condition (5.68), i.e. $V(2,x) = -x^2 \frac{z'(2)}{z(2)} = 0$ for all $x < 0$, implies $z'(2) = 0$, hence

$$c_1 = \frac{\sqrt{2}+1}{\sqrt{2}-1} e^{-4\sqrt{2}} c_2.$$

Noticing that the choice of the constant c_2 is irrelevant to construct G, putting $c_2 = \sqrt{2} - 1$, by (5.73) we obtain the function

$$\widetilde{z}(t) = (\sqrt{2}+1)e^{(\sqrt{2}-1)t - 4\sqrt{2}} + (\sqrt{2}-1)e^{-(\sqrt{2}+1)t}. \tag{5.74}$$

---

[10]Let us consider the *Riccati differential equation* in $y = y(t)$

$$y' = P + Qy + Ry^2$$

where $P = P(t),\ Q = Q(t)$ and $R = R(t)$ are functions, we introduce a new variable $z = z(t)$ putting

$$y = -\frac{z'}{Rz}$$

and solve the new ODE. In the particular case where $P,\ Q$ and R are constants, we obtain the new ODE

$$z'' - Qz' + PRz = 0.$$

and hence

$$V(t,x) = -x^2 \frac{\widetilde{z}'(t)}{\widetilde{z}(t)} = -x^2 \frac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}}. \tag{5.75}$$

It is easy to verify that

$$\frac{\partial V}{\partial x}(t,x) = -2x \frac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}} < 0 \quad \text{if } (t,x) \in [0,2) \times (-\infty, 0)$$

that is coherent with assumption (5.72); hence these last arguments hold. Using (5.69) and (5.75) we obtain

$$w(t,x) = -\frac{1}{2}\frac{\partial V}{\partial x}(t,x) = x\frac{\widetilde{z}'(t)}{\widetilde{z}(t)} = x \frac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}} \tag{5.76}$$

In order to find x^*, we have to solve the ODE (5.24)

$$\begin{cases} \dot{x}(t) = x(t) + w(t, x(t)) = x\left(1 + \dfrac{\widetilde{z}'(t)}{\widetilde{z}(t)}\right) & \text{in } [0,2] \\ x(0) = x_0. \end{cases}$$

From the previous system we have

$$\int \frac{1}{x} dx = \int \left(1 + \frac{\widetilde{z}'(t)}{\widetilde{z}(t)}\right) dt + k \qquad \Rightarrow \qquad x(t) = \widetilde{k}\widetilde{z}(t)e^t$$

with k and $\widetilde{k}$ constants. Using the condition $x(0) = x_0$ we obtain that the unique solution of the ODE is

$$x^*(t) = x_0 \frac{\widetilde{z}(t)}{\widetilde{z}(0)} e^t = x_0 \frac{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1) + (\sqrt{2}-1)e^{4\sqrt{2}}}$$

and is the optimal trajectory (for $x_0 < 0$). The optimal control is defined by (5.25), i.e. using the previous expression of x^* and (5.76)

$$u^*(t) = w(t, x^*(t)) = x_0 \frac{\widetilde{z}'(t)}{\widetilde{z}(0)} e^t = x_0 \frac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1) + (\sqrt{2}-1)e^{4\sqrt{2}}}$$

is the optimal control for the problem (5.1).

As conclusion, we have that the value function of the problem is

$$V(t,x) = \begin{cases} \dfrac{x^2}{2}(e^{4-2t} - 1) & \text{for } t \in [0,2] \times (0, \infty) \\ 0 & \text{for } x = 0 \\ -x^2 \dfrac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}} & \text{for } t \in [0,2] \times (-\infty, 0) \end{cases}$$

The optimal control and the optimal trajectory of initial problem are

$$u^*(t) = \begin{cases} 0 & \text{for } x_0 \geq 0 \\ x_0 \dfrac{e^{\sqrt{2}t} - e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1) + (\sqrt{2}-1)e^{4\sqrt{2}}} & \text{for } x_0 < 0 \end{cases}$$

$$x^*(t) = \begin{cases} x_0 e^t & \text{for } x_0 \geq 0 \\ x_0 \dfrac{(\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{\sqrt{2}(4-t)}}{(\sqrt{2}+1) + (\sqrt{2}-1)e^{4\sqrt{2}}} & \text{for } x_0 < 0 \end{cases}$$

△

The next example gives an idea of what happens in a situation where the optimal control is discontinuous.

Example 5.5.3. *Let us consider*[11] *the problem* [12]

$$\begin{cases} \max \int_0^2 (2x - 4u)\,\mathrm{d}t \\ \dot{x} = x + u \\ x(0) = 5 \\ 0 \le u \le 2 \end{cases}$$

We are looking for a function $V : [0,2] \times \mathbb{R} \to \mathbb{R}$ *such that*

$$\frac{\partial V}{\partial t} + \max_{v\in[0,2]} \left(2x - 4v + \frac{\partial V}{\partial x}(x+v) \right) = 0$$

$$\Rightarrow \quad \frac{\partial V}{\partial t} + 2x + x\frac{\partial V}{\partial x} + \max_{v\in[0,2]} v \left(\frac{\partial V}{\partial x} - 4 \right) = 0 \tag{5.77}$$

$$V(2,x) = 0, \ \forall x \in \mathbb{R} \tag{5.78}$$

Clearly the max *in (5.77) depends on the sign of* $\left(\frac{\partial V}{\partial x} - 4\right)$. *Let us suppose that* V *is differentiable: condition (5.78) guarantees that*

$$\frac{\partial V}{\partial x}(2,x) = 0, \qquad \forall x \in \mathbb{R};$$

hence let us suppose that there exists $\tau \in [0,2)$ *such that*

$$\frac{\partial V}{\partial x}(t,x) < 4, \qquad \forall (t,x) \in (\tau, 2] \times \mathbb{R} \tag{5.79}$$

The function $w(t,x)$ *defined in (5.23) here is*

$$w(t,x) = 0 \tag{5.80}$$

and (5.77) becomes

$$\frac{\partial V}{\partial t}(t,x) + 2x + x\frac{\partial V}{\partial x}(t,x) = 0.$$

Using the suggestion, an easy computation gives

$$V(t,x) = -2x + bxe^{-t} + c, \qquad \forall (t,x) \in (\tau,2] \times \mathbb{R}. \tag{5.81}$$

In particular, for $t = 2$, *the function* V *must satisfy the condition (5.78):*

$$V(2,x) = -2x + bxe^{-2} + c = 0, \qquad \forall x,$$

this implies $c = 0$ *and* $b = 2e^2$. *Hence, by (5.81) we obtain*

$$V(t,x) = -2x + 2xe^{2-t}, \qquad \forall (t,x) \in (\tau,2] \times \mathbb{R}.$$

Now we are in the position to calculate $\frac{\partial V}{\partial x}$ *and to determinate the point* τ *such that the condition (5.79) holds:*

$$\frac{\partial V}{\partial x}(t,x) = -2 + 2e^{2-t} < 4 \qquad \forall (t,x) \in (\tau,2] \times \mathbb{R}$$

implies $\tau = 2 - \log 3$. *The function* V *is*

$$V(t,x) = 2x(e^{2-t} - 1), \qquad \text{for } t \in (2 - \log 3, 2]. \tag{5.82}$$

The control candidates to be optimal is defined by (5.25): using (5.80) we have

$$u^*(t) = w(t, x^*(t)) = 0, \qquad \text{for } t \in (2 - \log 3, 2].$$

---

[11] In the example 2.5.2 we solve the same problem with the variational approach.

[12] Suggestion: In order to solve $Ax + x\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} = 0$ with A constant, we suggest to find the solution in the family of functions

$$\mathcal{F} = \{F(t,x) = ax + bxe^{-t} + c, \text{ with } a,b,c \text{ constants}\}.$$

To solve $Ax + x\frac{\partial F}{\partial x} + B\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + C = 0$ with A, B, C constants, we suggest to find the solution in the family of functions

$$\mathcal{F} = \{F(t,x) = ax + bt + ce^{-t} + dxe^{-t} + f, \text{ with } a,b,c,d,f \text{ constants}\}.$$

Now let us suppose that there exists $\tau' \in [0, 2-\log 3)$ such that

$$\frac{\partial V}{\partial x}(t,x) > 4, \qquad \text{for } t \in [\tau', 2-\log 3). \tag{5.83}$$

If we consider the function $w(t,x)$ defined in (5.23), we have

$$w(t,x) = 2, \qquad \text{for } t \in [\tau', 2-\log 3) \tag{5.84}$$

and the BHJ equation in 5.77) is

$$\frac{\partial V}{\partial t}(t,x) + 2x + x\frac{\partial V}{\partial x}(t,x) + 2\frac{\partial V}{\partial x}(t,x) - 8 = 0.$$

Using the suggestion we obtain

$$V(t,x) = -2x + 12t + 2de^{-t} + dxe^{-t} + f. \tag{5.85}$$

The function V is continuous: hence, by (5.82) and (5.85), we have

$$\lim_{t\to(2-\log 3)^+} V(t,x) = \lim_{t\to(2-\log 3)^-} V(t,x) \quad \Longrightarrow$$

$$\Longrightarrow \quad 4x = -2x + 12(2-\log 3) + 6de^{-2} + 3dxe^{-2} + f$$

for every $x \in \mathbb{R}$. Hence $d = 2e^2$ and $f = 12(\log 3 - 3)$. We obtain by (5.85) that

$$V(t,x) = -2x + 12t + 4e^{2-t} + 2xe^{2-t} + 12(\log 3 - 3), \qquad \text{for } t \in (\tau', 2-\log 3].$$

Let us check that such result is coherent with the assumption (5.83):

$$\frac{\partial V}{\partial x}(t,x) = -2 + 2e^{2-t} > 4 \qquad \Longleftrightarrow \qquad t < 2-\log 3.$$

This implies that $\tau' = 0$. Hence we obtain

$$V(t,x) = \begin{cases} 2x(e^{2-t}-1) + 12t + 4e^{2-t} + 12(\log 3 - 3) & \text{for } t \in [0, 2-\log 3] \\ 2x(e^{2-t}-1) & \text{for } t \in (2-\log 3, 2] \end{cases}$$

and it is easy to verify that $V \in C^1$. The control candidate to be optimal is given by (5.25) that, using (5.84), is

$$u^*(t) = w(t, x^*(t)) = 2, \qquad \text{for } t \in [0, 2-\log 3).$$

We obtain (2.52), i.e.

$$u^*(t) = \begin{cases} 2 & \text{if } 0 \le t < 2-\log 3, \\ 0 & \text{if } 2-\log 3 \le t \le 2. \end{cases}$$

Finally, we have to show that we are able to obtain the trajectory associated to this control: this computation is similar to the situation of example 2.5.2. Since all the sufficient conditions of theorem 5.3 are satisfied, then u^* is optimal. △

5.5.1 A problem of business strategy II

Let us recall[13] the problem 1.1.2, formulated with (1.3):

$$\begin{cases} \displaystyle\max_{u\in\mathcal{C}} \int_0^T (1-u)x \,\mathrm{d}t \\ \dot{x} = ux \\ x(0) = \alpha, \\ \mathcal{C} = \{u : [0,T] \to [0,1] \subset \mathbb{R},\ u \in KC\} \end{cases}$$

with α and T positive and fixed constants. Clearly, we are looking for a function $V : [0,T] \times \mathbb{R} \to \mathbb{R}$ that satisfies the necessary condition of Bellman-Hamilton-Jacobi (5.8) and the final condition (5.5).

[13]In subsection 2.5.2 we solve the same problem with the variational approach.

Since $x(t)$ denotes the quantity of good product (at time t,) it is reasonable in $V(\tau,\xi)$ to assume that ξ (i.e. the production at time τ) is not negative. Hence, we are looking for $V : [0,T]\times(0,\infty)\to\mathbb{R}$ with

$$\frac{\partial V}{\partial t} + \max_{v\in[0,1]}\left((1-v)x + \frac{\partial V}{\partial x}xv\right) = 0 \qquad \forall(t,x)\in[0,T]\times\mathbb{R} \quad (5.86)$$

$$V(T,x) = 0 \qquad \forall x > 0 \quad (5.87)$$

As in subsection 2.5.2, we are able to check that $x(0)=\alpha>0$ and $\dot{x}=ux\ge 0$ imply that $x(t)\ge\alpha$, for every $t\in[0,T]$. Hence we have $x>0$ and the BHJ equation in (5.86) becomes

$$\frac{\partial V}{\partial t} + x + x\max_{v\in[0,1]}\left[v\left(\frac{\partial V}{\partial x}-1\right)\right] = 0, \qquad \forall(t,x)\in[0,T]\times\mathbb{R}^+. \quad (5.88)$$

Hence, if $\frac{\partial V}{\partial x}-1>0$, then we obtain the max in (5.88) for $v=1$; on the other hand, if $\frac{\partial V}{\partial x}-1<0$, then the max in (5.88) is realized for $v=0$.
Now we note that equation (5.87) gives $\frac{\partial V}{\partial x}(T,x)=0$, for all $x>0$. Hence it is reasonable to suppose that there exists a point $\tau\in[0,T)$ such that

$$\frac{\partial V}{\partial x}(t,x) < 1, \qquad \text{for all } t\in[\tau,T]. \quad (5.89)$$

With this assumption, equation (5.88) in the set $[\tau,T]$ gives

$$\frac{\partial V}{\partial t} + x = 0 \quad\Rightarrow\quad V(t,x) = -xt + g(x),$$

where $g:\mathbb{R}\to\mathbb{R}$ is a differentiable function. Using (5.87), we are able to show that

$$V(x,T) = -xT + g(x) = 0 \qquad\Rightarrow\qquad g(x) = xT.$$

Hence

$$V(t,x) = x(T-t), \qquad \forall(t,x)\in[\tau,T]\times(0,\infty)$$

and $\frac{\partial V}{\partial x}=T-t$, for all $t\in[\tau,T]$. Since the previous arguments hold in the assumption (5.89),i.e. $T-t<1$, for the time τ we have

$$\tau = \max\{T-1,0\}.$$

Now we have to consider two different situations: $T\le 1$ and $T>1$.
Case A: $T\le 1$.
In this situation we have $V:[0,T]\times(0,\infty)\to\mathbb{R}$ definited by $V(t,x)=x(T-t)$, that satisfies BHJ and the final condition. The function w that realizes the max in (5.86) is identically zero and theorem 5.3 guarantees that the optimal control is $u^*=0$. Moreover we obtain the optimal trajectory by

$$\dot{x^*} = u^*x^* \quad\Rightarrow\quad \dot{x^*}=0 \quad\Rightarrow\quad x^*=\alpha.$$

In a situation where the corporate strategy is to consider on short period of time, the best choice is to sell the entire production.
Case B: $T>1$.

Since in this situation we have $\tau = T-1$, the function $V : [0,T] \times (0,\infty) \to \mathbb{R}$ in $[T-1,T] \times \mathbb{R}$ is defined by

$$V(t,x) = x(T-t)$$

and satisfies BHJ and the final condition. We have to construct V in $[0,T-1] \times (0,\infty)$.
For the continuity of V (we recall that we suppose V differentiable) we have

$$V(T-1,x) = x, \qquad \text{for all } x > 0. \tag{5.90}$$

Let us suppose[14] that there exists $\tau' < T-1$ such that $\frac{\partial V}{\partial x}(t,x) > 1$ in $[\tau', T-1] \times (0,\infty)$. Hence

$$(5.86) \quad \Rightarrow \quad \frac{\partial V}{\partial t} + x\frac{\partial V}{\partial x} = 0.$$

A solution of this PDE is given by[15] $V(t,x) = axe^{-t}$ with $a \in \mathbb{R}$. By condition (5.90) we have $V(t,x) = xe^{-t+T-1}$ in $[\tau', T-1] \times \mathbb{R}$.
We remark that $\frac{\partial V}{\partial x} = e^{-t+T-1} > 1$ if and only if $t < T-1$: hence we are able to chose $\tau' = 0$. Hence the function $V : [0,T] \times (0,\infty) \to \mathbb{R}$ defined as

$$V(t,x) = \begin{cases} xe^{-t+T-1} & \text{for } (t,x) \in [0,T-1) \times (0,\infty), \\ x(T-t) & \text{for } (t,x) \in [T-1,T] \times (0,\infty). \end{cases}$$

satisfies BHJ equation and the final condition. We note that V is differentiable.

Using theorem 5.3, the optimal control is defined via the function w in (5.25) that realizes the max in (5.23) (i.e. in (5.86)): in our situation we obtain, taking into account $w(t, x^*(t)) = u^*(t)$,

$$w(t,x) = \begin{cases} 1 & \text{if } 0 \le t < T-1, \\ ? & \text{if } t = T-1, \\ 0 & \text{if } T-1 < t \le T. \end{cases} \qquad \Rightarrow \qquad u^*(t) = \begin{cases} 1 & \text{if } 0 \le t < T-1, \\ ? & \text{if } t = T-1, \\ 0 & \text{if } T-1 < t \le T. \end{cases}$$

---

[14] For the reader who wants to see what happens with the other assumptions:
• Let us suppose that there exists $\tau' < T-1$ such that $\frac{\partial V}{\partial x}(t,x) < 1$ in $[\tau', T-1] \times (0,\infty)$. Hence

$$(5.86) \quad \Rightarrow \quad \frac{\partial V}{\partial t} + x = 0 \quad \Rightarrow \quad V(t,x) = -xt + h(x),$$

where $h : \mathbb{R} \to \mathbb{R}$ is a generic differentiable function. Relation (5.90) guarantees that for all $x \in \mathbb{R}$

$$V(x,T) = -xT + h(x) = x \qquad \Rightarrow \qquad h(x) = xT.$$

Hence we obtain that $V(t,x) = x(T-t)$ for $(t,x) \in [\tau', T-1] \times (0,\infty)$: clearly we have $\frac{\partial V}{\partial x} = T - t$ and condition $\frac{\partial V}{\partial x}(t,x) < 1$ is false. Hence τ' does not exist.
• Now, let us suppose that there exists $\tau' < T-1$ such that $\frac{\partial V}{\partial x}(t,x) = 1$ in $[\tau', T-1] \times (0,\infty)$. Hence

$$\frac{\partial V}{\partial x} = 1 \text{ and } (5.86) \quad \Rightarrow \quad \frac{\partial V}{\partial t} + x = 0 \quad \Rightarrow \quad V(t,x) = -xt + k_1(x) \tag{5.91}$$

$$\frac{\partial V}{\partial x} = 1 \quad \Rightarrow \quad V(t,x) = x + k_1(t) \tag{5.92}$$

where $k_1,\ k_2 : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Clearly (5.91) and (5.92) are in contradiction: hence τ' does not exist.

[15] A not expert reader in PDE will be convinced by checking that the function $V = axe^{-t}$ satisfies the equation $\frac{\partial V}{\partial t} + x\frac{\partial V}{\partial x} = 0$.

We construct the optimal trajectory by the dynamics and the initial condition as in subsection 2.5.2:

$$x^*(t) = \begin{cases} \alpha e^t & \text{for } 0 \le t \le T-1, \\ \alpha e^{T-1} & \text{for } T-1 < t \le T \end{cases}$$

In a situation where the business strategy is of medium or long time, the best choice is to invest the entire production to increase it up to time $T-1$ and then sell everything to make profit.

5.5.2 A problem of inventory and production II.

Let us consider the problem presented in subsection 2.5.4[16] with a different initial condition on the inventory accumulated at the initial time, i.e.

$$\begin{cases} \min\limits_u \displaystyle\int_0^T (\alpha u^2 + \beta x)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = A \\ x(T) = B \\ u \ge 0 \end{cases}$$

where $T > 0,\ 0 \le A < B$ are all fixed constants.[17]

We are looking for a value function $V : [0,T] \times \mathbb{R} \to [-\infty, \infty]$. First let us consider $0 \le \tau < T$: we note that V admits the value ∞ in some points (τ, ξ) of its domain since an initial data the trajectory $x(\tau) = \xi > B$ and the dynamic $\dot{x} = u \ge 0$ give that $x(T) = B$ is impossible: hence $\mathcal{C}_{\tau,\xi} = \emptyset$. Second we note that $(T, \xi) \ne (T, B)$ implies $\mathcal{C}_{T,\xi} = \emptyset$. Hence the effective domain[18] of V is the set $\big([0,T) \times (-\infty, B]\big) \cup \{(T, B)\}$ and such set coincides with the reachable set. Finally note the final condition for the value function (5.62) is

$$V(T, B) = 0.$$

Now we study the points in the effective domain of V. First for the point (τ, B) with $\tau \in [0, T)$: we have that the unique admissible control is the zero function and its trajectory in constant: hence for such points the point (τ, B) we have

$$V(\tau, B) = \min_{u \in \mathcal{C}_{\tau,B}} \int_\tau^T (\alpha u^2 + \beta x)\,\mathrm{d}t = \int_\tau^T \beta B\,\mathrm{d}t = \beta B(T - \tau.)$$

We study the points of the effective domain with $\xi < B$. The Bellman-Hamilton-Jacobi equation is

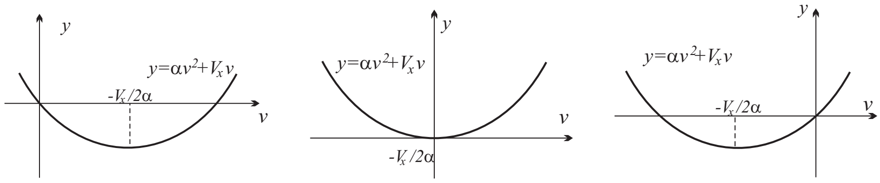
$$\frac{\partial V}{\partial t} + \beta x + \min_{v \ge 0} \left(\alpha v^2 + \frac{\partial V}{\partial x} v \right) = 0 \tag{5.93}$$

The min in (5.93) depends on the value $-\frac{\partial V}{\partial x}(t, x)/(2\alpha)$: more precisely

[16] In the mentioned subsection 5.5.2 we solve the same model with the variational approach.

[17] Suggestion: to solve the PDE $Cx + D(\frac{\partial V}{\partial x})^2 + E\frac{\partial V}{\partial t} = 0$ with $C,\ D$ and E constants, we suggest to consider the family of functions $F(t,x) = a(t-T)^3 + b(x+B)(t-T) + c\frac{(x-B)^2}{t-T}$, with a, b, c non zero constants.

[18] Let $f : \Omega \to [-\infty, \infty]$ be a function with $\Omega \subset \mathbb{R}^n$; the effective domain is the set $\Omega' = \{x \in \Omega : f(x) \text{ is finite }\}$.



The cases $\frac{\partial V}{\partial x}(t,x) < 0$, $\frac{\partial V}{\partial x}(t,x) = 0$ and $\frac{\partial V}{\partial x}(t,x) > 0$.

Hence the function $w(t,x)$ defined by (5.23) is

$$w(t,x) = \begin{cases} 0 & \text{if } \frac{\partial V}{\partial x}(t,x) \geq 0 \\ -\frac{1}{2\alpha}\frac{\partial V}{\partial x}(t,x) & \text{if } \frac{\partial V}{\partial x}(t,x) < 0 \end{cases} \tag{5.94}$$

First let us suppose that there exists a open set in the set $[0,T) \times (-\infty, B]$ such that

$$\frac{\partial V}{\partial x}(t,x) \geq 0. \tag{5.95}$$

Note that $x(\tau) < B$ and $x(T) = B$ imply that $u = 0$ in $[\tau, T]$ is impossible: hence the previous assumption (5.95) cannot be true for every t. In the set where (5.95) is satisfied, (5.93) gives

$$\frac{\partial V}{\partial t} + \beta x = 0;$$

hence

$$V(t,x) = -\beta x t + F(x), \tag{5.96}$$

for some function F. However since for the optimal control we have $u^*(t) = w(t, x^*(t)) = 0$, in the set where (5.95) is guarantee, we have that the optimal trajectory $x^*(t)$ is constant.

Now let us assume that exists a open set in the set $[0,T) \times (-\infty, B]$ such that

$$\frac{\partial V}{\partial x}(t,x) < 0; \tag{5.97}$$

In this set (5.93) gives

$$\frac{\partial V}{\partial t} + \beta x - \frac{1}{4\alpha}\left(\frac{\partial V}{\partial x}\right)^2 = 0;$$

the suggestion and some easy calculations gives that

$$V(t,x) = \frac{\beta^2}{48\alpha}(t-T)^3 - \frac{\beta}{2}(x+B)(t-T) - \alpha\frac{(x-B)^2}{t-T} \tag{5.98}$$

satisfies BHJ. Clearly we have to guarantee that (5.97) holds, i.e.

$$\frac{\partial V}{\partial x}(t,x) = -\frac{\beta}{2}(t-T) - 2\alpha\frac{(x-B)}{t-T} < 0$$

This implies

$$x < -\frac{\beta}{4\alpha}(t-T)^2 + B.$$

Since we are looking for a continuous value function, in the point where it is finite, equations (5.96) and (5.98) along the line $x = -\frac{\beta}{4\alpha}(t-T)^2 + B$, i.e. $t = T - 2\sqrt{\frac{\alpha(B-x)}{\beta}}$, gives

$$\begin{aligned}
-\beta x \left(T - 2\sqrt{\frac{\alpha(B-x)}{\beta}} \right) + F(x) = \\
= -\frac{\beta^2}{6\alpha}\sqrt{\frac{\alpha^3(B-x)^3}{\beta^3}} + \beta(x+B)\sqrt{\frac{\alpha(B-x)}{\beta}} + \alpha\frac{(x-B)^2}{2\sqrt{\frac{\alpha(B-x)}{\beta}}}
\end{aligned}$$

and hence, with a simple calculation, $F(x) = \beta T x + \frac{4}{3}\sqrt{\alpha\beta(B-x)^3}$. By (5.96) we have

$$V(t,x) = \beta x(T-t) + \frac{4}{3}\sqrt{\alpha\beta(B-x)^3}. \tag{5.99}$$

Since for this function we require that assumption (5.95) holds, we note that

$$\frac{\partial V}{\partial x} = \beta(T-t) - 2\sqrt{\alpha\beta(B-x)} \geq 0 \quad \Leftrightarrow \quad x \geq -\frac{\beta}{4\alpha}(t-T)^2 + B.$$

We obtain that

$$V(t,x) = \begin{cases} \infty & \text{if } 0 \leq t < T \text{ and } x > B \\ \infty & \text{if } t = T \text{ and } x \neq B \\ 0 & \text{if } t = T \text{ and } x = B \\ \beta x(T-t) + \frac{4}{3}\sqrt{\alpha\beta(B-x)^3} & \text{if } 0 \leq t < T,\ x < B \\ & \quad \text{and } x \geq -\frac{\beta}{4\alpha}(t-T)^2 + B \\ \frac{\beta^2}{48\alpha}(t-T)^3 - \frac{\beta}{2}(x+B)(t-T) - \alpha\frac{(x-B)^2}{t-T} & \text{if } 0 \leq t < T,\ x < B \\ & \quad \text{and } x < -\frac{\beta}{4\alpha}(t-T)^2 + B \end{cases}$$

Now let us construct the trajectory solving the ODE

$$\begin{cases} \dot{x} = w(t,x) & \text{for } t \in [0,2] \\ x(0) = A \end{cases}$$

In order to do that we have that (5.94) becomes

$$w(t,x) = \begin{cases} 0 & \text{if } 0 \leq t < T,\ x < B \text{ and } x \geq -\frac{\beta}{4\alpha}(t-T)^2 + B \\ \frac{\beta}{4\alpha}(t-T) + \frac{x-B}{t-T} & \text{if } 0 \leq t < T,\ x < B \text{ and } x < -\frac{\beta}{4\alpha}(t-T)^2 + B \end{cases}$$

Let us define $\widetilde{T} = \max\left(T - 2\sqrt{\frac{\alpha(B-A)}{\beta}}, 0 \right)$. We have

$$\begin{cases} \dot{x} = w(t,x) = 0 & \text{for } t \in [0,\widetilde{T}) \\ x(0) = A \end{cases}$$

Hence $x(t) = A$ for every $t \in [0,\widetilde{T}]$. Now we have to solve the linear ODE

$$\begin{cases} \dot{x} = w(t,x) = \frac{1}{t-T}x + \frac{\beta}{4\alpha}(t-T) - B\frac{1}{t-T} & \text{for } t \in [\widetilde{T},T] \\ x(\widetilde{T}) = A \end{cases}$$

The general solution is

$$\begin{aligned} x(t) &= e^{\int \frac{1}{t-T}\,\mathrm{d}t} \left[\int \left(\frac{\beta}{4\alpha}(t-T) - \frac{B}{t-T} \right) e^{\int \frac{1}{T-t}\,\mathrm{d}t}\,\mathrm{d}t + k \right] \\ &= (T-t)\left(\int -\frac{\beta}{4\alpha} + \frac{B}{(T-t)^2}\,\mathrm{d}t + k \right) \\ &= \frac{\beta}{4\alpha}t^2 - t\left(\frac{\beta T}{4\alpha} + k \right) + B + Tk, \end{aligned} \tag{5.100}$$

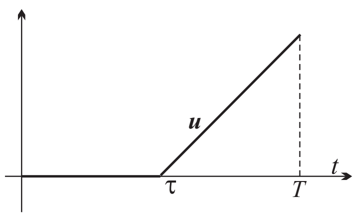
where k is a constant (note that $x(T) = B$). It is clear that this solution exists in all $[\widetilde{T}, T]$ and hence x is really the optimal path.

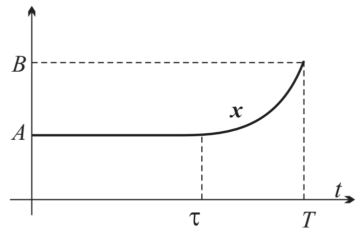
• Now let us consider the case $\widetilde{T} > 0$, i.e. $T > 2\sqrt{\frac{\alpha(B-A)}{\beta}}$: the condition $x(\widetilde{T}) = A$ in (5.100) gives, with a tedious calculation,

$$k = \frac{\beta T}{4\alpha} - \sqrt{\frac{\beta(B-A)}{\alpha}}.$$

Hence for $\tau = T - 2\sqrt{\frac{\alpha(B-A)}{\beta}}$ we obtain

$$u^*(t) = \begin{cases} 0 & \text{if } 0 \le t < \tau \\ \frac{\beta}{2\alpha}(t-\tau) & \text{if } \tau \le t \le T \end{cases} \quad \text{and} \quad x^*(t) = \begin{cases} 0 & \text{if } 0 \le t < \tau \\ \frac{\beta}{4\alpha}(t-\tau)^2 + A & \text{if } \tau \le t \le T \end{cases}$$



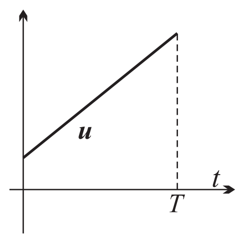


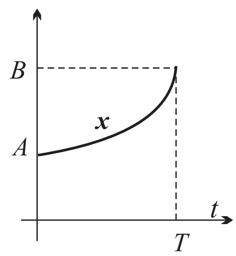
• Now let us consider the case $\widetilde{T} = 0$, i.e. $T \le 2\sqrt{\frac{\alpha(B-A)}{\beta}}$: the condition $x(0) = A$ in (5.100) gives

$$k = -\frac{B-A}{T}.$$

Then

$$u^*(t) = \frac{\beta}{2\alpha}t + \frac{4\alpha(B-A) - \beta T^2}{4\alpha T} \quad \text{and} \quad x^*(t) = \frac{\beta}{4\alpha}t^2 + \frac{4\alpha(B-A) - \beta T^2}{4\alpha T}t + A$$





Example 5.5.4. *Consider the previous model in the particular case $T = B = 2$, $\alpha = 1$, and $\beta = 4$. More precisely we consider*[19]

$$\begin{cases} \min_u \int_0^2 (u^2 + 4x)\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = A \\ x(2) = 2 \\ u \geq 0 \end{cases}$$

where $A < 2$.

In this case we obtain that

$$V(t,x) = \begin{cases} \infty & \text{if } 0 \leq t < 2 \text{ and } x > 2 \\ \infty & \text{if } t = 2 \text{ and } x \neq 2 \\ 0 & \text{if } t = 2 \text{ and } x = 2 \\ 4x(2-t) + \frac{8}{3}\sqrt{(2-x)^3} & \text{if } 0 \leq t < 2,\ x < 2 \\ & \quad \text{and } x \geq 2 - (t-2)^2 \\ \frac{1}{3}(t-2)^3 - 2(x+2)(t-2) - \frac{(x-2)^2}{t-2} & \text{if } 0 \leq t < 2,\ x < 2 \\ & \quad \text{and } x < 2 - (t-2)^2 \end{cases}$$

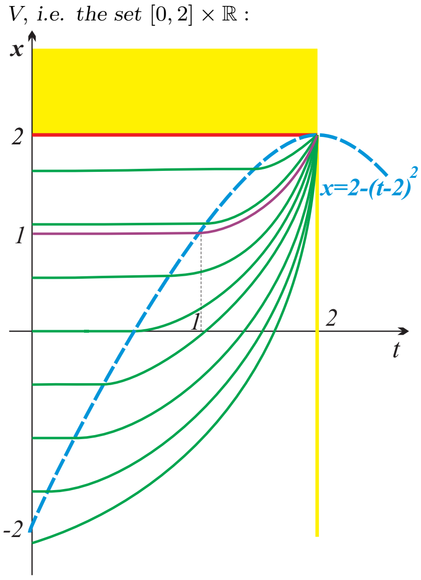
Here $\tau = 2 - \sqrt{2-A}$ and the optimal trajectory is

$$x^*(t) = \begin{cases} A & \text{for } t \in [0, \tau] \\ (t-\tau)^2 + A & \text{for } t \in (\tau, 2] \end{cases}$$

The optimal control is given by

$$u^*(t) = \begin{cases} 0 & \text{for } t \in [0, \tau] \\ 2(t-\tau) & \text{for } t \in (\tau, 2] \end{cases}$$

In the figure, the domain of the value function V, i.e. the set $[0,2] \times \mathbb{R}$:



- *in the yellow subset of* $\mathrm{dom}(V)$, *i.e. the set*

  $$[0,2] \times (2,\infty) \cup (\{2\} \times (-\infty, 2))$$

  the value function is equal to ∞ since there are no controls whose trajectory starts from to a point of this set and arrive in the point $(2,2)$;
- *the red line is an optimal trajectory; on this red points, i.e. the set $[0,2] \times \{2\}$, the value function is equal to zero;*
- *in the points (t,x) of the blue line, i.e. such that $x = 2-(t-2)^2$, we have $\frac{\partial V}{\partial x}(t,x) = 0$. This blue line divides the set $[0,2] \times (-\infty, 2)$, in two regions:*
  - *in the upper, we have $\frac{\partial V}{\partial x}(t,x) > 0$,*
  - *in the lower, we have $\frac{\partial V}{\partial x}(t,x) < 0$;*
- *every green line is an optimal trajectory: recalling that the second part of an optimal trajectory is again an optimal trajectory, starting from generic point a (τ, ξ) on a green line, we arrive in the point $(2,2)$ with the optimal path lying on the same green line;*
- *the blue line divides every trajectory in two parts:*
  - *the first part is a segment, i.e. the control is equal to zero,*

[19] Suggestion: to solve the PDE $Cx + D(\frac{\partial V}{\partial x})^2 + E\frac{\partial V}{\partial t} = 0$ with C, D and E constants, we suggest to consider the family of functions $F(t,x) = a(t-2)^3 + b(x+2)(t-2) + c\frac{(x-2)^2}{t-2}$, with a, b, c non zero constants.

  - *the second part is a parabola with vertex on the blue line;*

- *the violet curve is the optimal trajectory with $x(0) = 1$.*

△

5.6 The multiplier as shadow price II: the proof

Let us consider the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_\tau^{t_1} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \\ \displaystyle\max_{\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}} J(\mathbf{u}), \end{cases} \tag{5.101}$$

with t_1 fixed. At this point we know that (with some assumptions)

- by the PMP (2.2) in the Pontryagin theorem 2.1

$$H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*, \boldsymbol{\lambda}^*(t)) = \max_{\mathbf{v} \in U} H(t, \mathbf{x}^*(t), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(t)), \tag{2.2}$$

  for all $t \in [t_0, t_1]$, obtained via a variational approach;

- the BHJ equation (5.8) of theorem 5.1

$$\frac{\partial V}{\partial t}(t, \mathbf{x}) + \max_{\mathbf{v} \in U} \Big(f(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} V(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) = 0, \tag{5.8}$$

  for all $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$, obtained via the approach of the dynamic programming.

Since in the problem (5.101) the final point of trajectory is free and the final time is fixed, the optimal control in normal and the Hamiltonian is $H = f + \boldsymbol{\lambda} \cdot g$; moreover $\mathbf{u}^*$ is defined as the function that, for every t, associates the value of $\mathbf{v}$ such that realizes the max in (2.2). Taking into account (5.23), the function that, for every $(t, \mathbf{x})$, associates the value $\mathbf{v}$ that realizes the max in (5.8) is given by the function $w(t, \mathbf{x})$; such function allow us to define the optimal control, as in (5.25),

$$\mathbf{u}^*(t) = w(t, \mathbf{x}^*(t)). \tag{5.102}$$

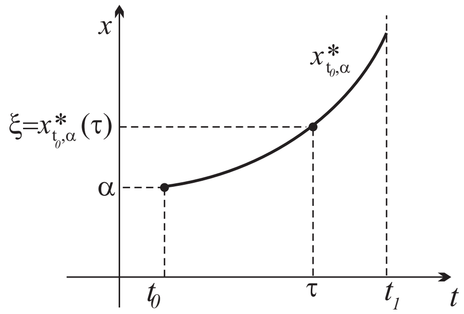
A comparison between (2.2) and (5.8) suggests the following result, that we have announced in remark 2.11: the multiplier $\boldsymbol{\lambda}^*$, at every time and along its optimal trajectory, provides the sensitivity of the problem (5.101) at the variation of the initial data $\boldsymbol{\xi}$:

Theorem 5.9. *Let $\mathbf{x}^*_{t_0,\boldsymbol{\alpha}}$ be the optimal trajectory, $\boldsymbol{\lambda}^*_{t_0,\boldsymbol{\alpha}}$ be the optimal multiplier and let V be the value function for the problem 5.101 with initial data $\mathbf{x}(t_0) = \boldsymbol{\alpha}$. If V is differentiable, then*

$$\nabla_{\mathbf{x}} V(t, \mathbf{x}^*_{t_0,\boldsymbol{\alpha}}(t)) = \boldsymbol{\lambda}^*_{t_0,\boldsymbol{\alpha}}(t), \tag{5.103}$$

for every $t \in [t_0, t_1]$.

The equation (5.103) implies that for a given $(\tau, \boldsymbol{\xi}) \in [t_0, t_1] \times \mathbb{R}^n$ on the optimal trajectory $x^*_{t_0,\boldsymbol{\alpha}}$, i.e. $x^*_{t_0,\boldsymbol{\alpha}}(\tau) = \boldsymbol{\xi}$



we obtain

$$\nabla_{\mathbf{x}} V(\tau, \boldsymbol{\xi}) = \boldsymbol{\lambda}^*_{t_0,\boldsymbol{\alpha}}(\tau).$$

Hence, as in remark 2.11, the multiplier λ^*, at time τ, expresses the sensitivity, the "shadow price", of the optimal value of the problem when we modify the initial data $\boldsymbol{\xi}$, along the optimal trajectory.

Proof. *(with the additional assumption that $V \in C^2$).* Let V be the value function for (5.101) and let $\mathbf{x}^* = \mathbf{x}^*_{t_0,\boldsymbol{\alpha}}$ be the optimal trajectory. We will prove that if we define the function $\boldsymbol{\lambda} : [t_0, t_1] \to \mathbb{R}^n$ as

$$\boldsymbol{\lambda}(t) = \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)), \tag{5.104}$$

then such function coincides with the multiplier $\boldsymbol{\lambda}^*$, i.e. as the unique function that solves the ODE in (2.23) in the proof of theorem of Pontryagin 2.1. The final condition (5.5) gives that the value of $V(t_1, \mathbf{x})$ does not vary if one modifies $\mathbf{x}$: hence, by the definition (5.104) of $\boldsymbol{\lambda}$, we have

$$\boldsymbol{\lambda}(t_1) = \nabla_{\mathbf{x}} V(t_1, \mathbf{x}^*(t_1)) = 0 :$$

the tranversality condition is proved.

Remark 5.4 gives, for every fixed t,

$$\begin{aligned}\frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) &= -f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) - \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \\ &= -f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) - \sum_{k=1}^n \Big(\frac{\partial V}{\partial x_k}(t, \mathbf{x}^*(t))\, g_k(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \Big).\end{aligned}$$

Considering a derivative with respect to x_j we have

$$\begin{aligned}\frac{\partial^2 V}{\partial t \partial x_j}(t, \mathbf{x}^*(t)) &= -\frac{\partial f}{\partial x_j}(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) + \\ &\quad -\sum_{k=1}^n \Big(\frac{\partial^2 V}{\partial x_k \partial x_j}(t, \mathbf{x}^*(t))\, g_k(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) + \\ &\qquad + \frac{\partial V}{\partial x_k}(t, \mathbf{x}^*(t))\, \frac{\partial g_k}{\partial x_j}(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \Big).\end{aligned} \tag{5.105}$$

Since V is in C^2, the theorem of Schwartz and a derivative with respect to the time of (5.104) give

$$\begin{aligned}
\dot\lambda_j(t) &= \frac{\partial^2 V}{\partial t \partial x_j}(t,\mathbf{x}^*(t)) + \sum_{i=1}^n \frac{\partial^2 V}{\partial x_j \partial x_i}(t,\mathbf{x}^*(t))\,\dot x_i^*(t) \\
\text{(by (5.105))} \quad &= -\frac{\partial f}{\partial x_j}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) + \\
&\quad -\sum_{k=1}^n \bigg(\frac{\partial^2 V}{\partial x_k \partial x_j}(t,\mathbf{x}^*(t))\, g_k(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) + \\
&\qquad + \frac{\partial V}{\partial x_k}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) \frac{\partial g_k}{\partial x_j}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) \bigg) + \\
&\quad + \sum_{i=1}^n \frac{\partial^2 V}{\partial x_j \partial x_i}(t,\mathbf{x}^*(t))\,\dot x_i^*(t) \\
\text{(by dynamics)} \quad &= -\frac{\partial f}{\partial x_j}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) + \\
&\quad -\sum_{k=1}^n \frac{\partial V}{\partial x_k}(t,\mathbf{x}^*(t)) \frac{\partial g_k}{\partial x_j}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) \\
\text{(by (5.103))} \quad &= -\frac{\partial f}{\partial x_j}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) + \\
&\quad -\sum_{k=1}^n \lambda_k(t) \frac{\partial g_k}{\partial x_j}(t,\mathbf{x}^*(t),\mathbf{u}^*(t)).
\end{aligned} \tag{5.106}$$

Hence the function $\boldsymbol{\lambda}$ solves the ODE

$$\begin{cases} \dot{\boldsymbol{\lambda}}(t) = -\boldsymbol{\lambda}(t) \cdot \nabla_{\mathbf{x}} g(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) - \nabla_{\mathbf{x}} f(t,\mathbf{x}^*(t),\mathbf{u}^*(t)) \\ \boldsymbol{\lambda}(t_1) = 0 \end{cases}$$

that is exactly the ODE (2.23) in the theorem of Pontryagin. The uniqueness of the solution of such ODE implies $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$. The relation (5.106) is the adjoint equation.

□

5.7 Infinite horizon problems

Let us consider the problem, for $r > 0$,

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_0^\infty e^{-rt} f(\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ \dot{\mathbf{x}} = g(\mathbf{x},\mathbf{u}) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \\ \displaystyle\max_{\mathbf{u}\in\mathcal{C}_{0,\boldsymbol{\alpha}}} J(\mathbf{u}) \end{cases} \tag{5.107}$$

If we consider the value function $V : [0,\infty) \times \mathbb{R}^n \to [-\infty,\infty]$ of this problem, it satisfies the BHJ equation

$$\frac{\partial V}{\partial t}(t,\mathbf{x}) + \max_{\mathbf{v}\in U} \left(e^{-rt} f(\mathbf{x},\mathbf{v}) + \nabla_{\mathbf{x}} V(t,\mathbf{x}) \cdot g(\mathbf{x},\mathbf{v}) \right) = 0, \quad \forall (t,\mathbf{x}) \in [0,\infty) \times \mathbb{R}^n \tag{5.108}$$

where $U \subset \mathbb{R}^k$ is, as usual, the control set. We remark that

$$\begin{aligned} V(\tau, \boldsymbol{\xi}) &= \max_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}} \int_\tau^\infty e^{-rt} f(\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ &= e^{-r\tau} \max_{\mathbf{u}\in\mathcal{C}_{\tau,\boldsymbol{\xi}}} \int_\tau^\infty e^{-r(t-\tau)} f(\mathbf{x},\mathbf{u})\,\mathrm{d}t \\ (\text{with } s = t-\tau) \quad &= e^{-r\tau} \max_{\widetilde{\mathbf{u}}\in\mathcal{C}_{0,\boldsymbol{\xi}}} \int_0^\infty e^{-rs} f(\widetilde{\mathbf{x}},\widetilde{\mathbf{u}})\,\mathrm{d}s. \end{aligned}$$

Two comments on the last equality. Since g does not depend explicitly by t, the function $\widetilde{\mathbf{u}} : [0,\infty) \to U$ is such that $\widetilde{\mathbf{u}} \in \mathcal{C}_{0,\boldsymbol{\xi}}$ if and only if the function $\mathbf{u} : [\tau,\infty) \to U$, with $\widetilde{\mathbf{u}}(t) = \mathbf{u}(\tau+t)$, is such that $\mathbf{u} \in \mathcal{C}_{\tau,\boldsymbol{\xi}}$. Essentially, the classes of functions $\mathcal{C}_{\tau,\boldsymbol{\xi}}$ and $\mathcal{C}_{0,\boldsymbol{\xi}}$ contain the same object. Moreover, for the respectively trajectory $\widetilde{\mathbf{x}}$ and $\mathbf{x}$ we have $\widetilde{\mathbf{x}}(t) = \mathbf{x}(\tau+t)$.

The last integral depends on the initial value $\mathbf{x}$, but it does not depend on the initial time τ. Hence we define the *current value function*[20] $V^c : \mathbb{R}^n \to [-\infty,\infty]$ as

$$V^c(\boldsymbol{\xi}) = \max_{\mathbf{u}\in\mathcal{C}_{0,\boldsymbol{\xi}}} \int_0^\infty e^{-rt} f(\mathbf{x},\mathbf{u})\,\mathrm{d}t;$$

hence

$$V^c(\mathbf{x}) = e^{rt} V(t,\mathbf{x}). \tag{5.109}$$

From (5.108) we have

$$-rV^c(\mathbf{x}) + \max_{\mathbf{v}\in U} \left(f(\mathbf{x},\mathbf{v}) + \nabla V^c(\mathbf{x}) \cdot g(\mathbf{x},\mathbf{v}) \right) = 0, \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{5.110}$$

Such new BHJ is called *Bellman–Hamilton–Jacobi equation for the current value function*. The BHJ equation for the current value function is very useful since it is not a PDE, but an ODE.

The final condition on the value function for the problem is

$$\lim_{t\to\infty} V(t,\mathbf{x}) = 0;$$

clearly, using (5.109), we obtain that the final condition is automatically guaranteed for the function V^c.

If we define $w^c : \mathbb{R}^n \to \mathbb{R}^k$ as the value $\mathbf{v}$ such that realizes the max in the previous equation, i.e.

$$w^c(\mathbf{x}) \in \arg\max_{\mathbf{v}\in U} \left(f(\mathbf{x},\mathbf{v}) + \nabla V^c(\mathbf{x}) \cdot g(\mathbf{x},\mathbf{v}) \right), \tag{5.111}$$

it is easy to see that

$$w(t,\mathbf{x}) = w^c(\mathbf{x}), \qquad \forall t \in [t_0,t_1],\ \mathbf{x} \in \mathbb{R}^n$$

where w is defined in (5.23). Hence, in order to guarantee some sufficient condition of optimality and to find the optimal control, we have to guarantee the existence of $\mathbf{x}^*$ solution of the ODE (5.24), i.e.

$$\begin{cases} \dot{\mathbf{x}}(t) = g\big(t,\mathbf{x},w^c(\mathbf{x})\big) & \text{in } [t_0,t_1] \\ \mathbf{x}(t_0) = \boldsymbol{\alpha}. \end{cases} \tag{5.112}$$

[20]We remark that V^c depends only on $\mathbf{x}$ and hence $\nabla_{\mathbf{x}} V^c(\mathbf{x}) = \nabla V^c(\mathbf{x})$.

Then $\mathbf{x}^*$ is the optimal trajectory and $\mathbf{u}^*$, defined by (5.25), i.e.

$$\mathbf{u}^*(t) = w(t, \mathbf{x}^*(t)), \tag{5.113}$$

is the optimal control.

Example 5.7.1. *Let us consider*[21]

$$\begin{cases} \min \int_0^\infty e^{-rt}(ax^2 + bu^2)\,\mathrm{d}t \\ \dot{x} = u \\ x(0) = x_0 > 0 \\ a, b \text{ fixed and positive} \end{cases}$$

The current value function $V^c = V^c(x)$ must satisfy (5.110), i.e.

$$\begin{aligned} & -rV^c + \min_{v\in\mathbb{R}} \left(ax^2 + bv^2 + (V^c)'v\right) = 0 \\ \Longrightarrow \quad & -rV^c + ax^2 + \min_{v\in\mathbb{R}} \left(bv^2 + (V^c)'v\right) = 0. \end{aligned} \tag{5.114}$$

The function $v \mapsto bv^2 + (V^c)'v$ is, for every fixed x, a parabola; since b is positive

$$w^c(x) = -(V^c)'/(2b). \tag{5.115}$$

Hence (5.114) becomes

$$4brV^c - 4abx^2 + [(V^c)']^2 = 0. \tag{5.116}$$

We looking for the solution in the homogenous polynomials on x of degree two, i.e. as the functions $V^c(x) = Ax^2$, with $A \in \mathbb{R}$: replacing this expression in the (5.116) we have

$$4brAx^2 - 4abx^2 + 4A^2x^2 = 0 \quad \Longrightarrow \quad A^2 + brA - ab = 0 \quad \Longrightarrow \quad A_\pm = \frac{-br \pm \sqrt{b^2r^2 + 4ab}}{2}.$$

From the problem it is clear that the current value function V^c and the value function $V = V^c e^{-rt}$ are non negative: hence we consider only A_+, i.e.

$$V^c(x) = \frac{-br + \sqrt{b^2r^2 + 4ab}}{2} x^2. \tag{5.117}$$

and, from (5.115) we have

$$w^c(x) = \frac{br - \sqrt{b^2r^2 + 4ab}}{2b} x.$$

In order to find the optimal trajectory, the ODE (5.112) is

$$\begin{cases} \dot{x}(t) = \frac{br - \sqrt{b^2r^2+4ab}}{2b} x(t) \\ x(0) = x_0. \end{cases}$$

and its unique solution is

$$x^*(t) = x_0 e^{\left(br - \sqrt{b^2r^2+4ab}\right)t/(2b)}. \tag{5.118}$$

The (5.113) gives us the optimal control

$$u^*(t) = w^c(x^*(t)) = \frac{br - \sqrt{b^2r^2 + 4ab}}{2b} x_0 e^{\left(br - \sqrt{b^2r^2+4ab}\right)t/(2b)}.$$

△

Consider the problem (5.107). Using theorem 5.9 it is very easy to see that the interpretation of the current multiplier is similar: recalling (3.100) and (5.109), i.e.

$$\boldsymbol{\lambda}_c^* = e^{rt}\boldsymbol{\lambda}^* \qquad \text{and} \qquad V(t, \mathbf{x}) = e^{-rt}V^c(\mathbf{x}),$$

then (5.103) gives, dropping the apex $t_0, \boldsymbol{\alpha}$,

[21] In the example 3.7.1 we solve the same example with the variational approach.

Remark 5.8.

$$\nabla V^c(\mathbf{x}^*(t)) = \boldsymbol{\lambda}_c^*(t),$$

for every $t \in [t_0, t_1]$. The current multiplier is the sensitivity of the current value function if we chance the initial state, along the optimal trajectory.

Example 5.7.2. *Let us consider* [22]

$$\begin{cases} \min \int_0^\infty e^{-rt}(ax^2 + bu^2)\, \mathrm{d}t \\ \dot{x} = u \\ x(0) = x_0 > 0 \\ a, b \text{ fixed and positive} \end{cases}$$

We want to verify the relation in Remark 5.8.

The solution of example 3.7.1 and example 5.7.1 give (see (3.109), (3.110), (5.117) and (5.118))

$$\begin{aligned} x^*(t) &= x_0 e^{\left(br - \sqrt{b^2r^2+4ab}\right)t/(2b)}, \\ \lambda_c^*(t) &= x_0 \left(\sqrt{b^2r^2 + 4ab} - br\right) e^{\left(br - \sqrt{b^2r^2+4ab}\right)t/(2b)}, \\ V^c(x) &= \frac{-br + \sqrt{b^2r^2 + 4ab}}{2} x^2. \end{aligned}$$

Clearly these equations give

$$\nabla V^c(x^*(t)) = \frac{\mathrm{d}V^c}{\mathrm{d}x}(x^*(t)) = 2x^*(t)\frac{-br + \sqrt{b^2r^2 + 4ab}}{2} = \lambda_c^*(t).$$

△

5.7.1 Particular infinite horizion problems

We note that many of the arguments in the first part of this section hold when we consider the case $r = 0$. In particular

Remark 5.9. *Let us consider the problem*

$$\begin{cases} \max_{\mathbf{u}\in\mathcal{C}_{0,\boldsymbol{\alpha}}} \int_0^\infty f(\mathbf{x}, \mathbf{u})\, \mathrm{d}t \\ \dot{\mathbf{x}} = g(\mathbf{x}, \mathbf{u}) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \end{cases}$$

Then we value function does not depend directly by t, i.e. $V(t, \mathbf{x}) = V(\mathbf{x})$. Moreover the BHJ equation is

$$\max_{\mathbf{v}\in U} \left(f(\mathbf{x}, \mathbf{v}) + \nabla V(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{v})\right) = 0.$$

In the spirit of Proposition 5.5, taking into account the previous remark, we have that

Remark 5.10. *Let us consider the autonomous Affine Quadratic infinite horizon problem*

$$\begin{cases} \max_{\mathbf{u}\in\mathcal{C}_{0,\boldsymbol{\alpha}}} \frac{1}{2}\int_0^\infty \left(\mathbf{x}'Q\mathbf{x} + 2\mathbf{x}'S + \mathbf{u}'R\mathbf{u}\right)\, \mathrm{d}t \\ \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + C \\ \mathbf{x}(0) = \boldsymbol{\alpha} \end{cases}$$

[22]In the example 3.7.1 and in example 5.7.1 we solve the example with the variational and the Dynamic Programming approach respectively.

where the matrices Q, S, R, A, B and C are constants. Moreover we assume $Q \le 0$ and $R < 0$. Then the value function is of the type

$$V(t, \mathbf{x}) = V(\mathbf{x}) = \frac{1}{2}\mathbf{x}' Z \mathbf{x} + W\mathbf{x} + Y$$

where Z, W and Y are constant matrices.

5.7.2 A model of consumption with HARA–utility

We solve the model presented in the example 1.1.5, formulated with (1.6) with a utility function

$$U(c) = \frac{1}{\gamma} c^\gamma,$$

where γ is fixed in $(0, 1)$: this is a commonly used utility function, of so called HARA type[23]. Our problem[24] is

$$\begin{cases} \max \displaystyle\int_0^\infty \frac{1}{\gamma} c^\gamma e^{-\delta t}\, \mathrm{d}t \\ \dot{x} = rx - c \\ x(0) = x_0 > 0 \\ x \ge 0 \\ c \ge 0 \end{cases} \tag{5.119}$$

We will show that two situations occur depending on the fixed constants r, γ and δ: the case $0 < r\gamma < \delta$ and the case $0 < \delta \le r\gamma$.

A generalization of this model is the fundamental Merton model that we will introduce in subsection 5.7.3.

The case $\delta > r\gamma$: the current value function $V^c = V^c(x)$ must satisfy (5.110), i.e.

$$-\delta V^c + \max_{v \ge 0} \left(\frac{1}{\gamma} v^\gamma + (V^c)'(rx - v) \right) = 0$$

$$\Longrightarrow \quad -\delta V^c + (V^c)' rx + \max_{v \ge 0} \left(\frac{1}{\gamma} v^\gamma - (V^c)' v \right) = 0. \tag{5.120}$$

Now, since for definition

$$V^c(\xi) = \max_c \int_0^\infty \frac{1}{\gamma} c^\gamma e^{-\delta t}\, \mathrm{d}t \qquad \text{with } x(0) = \xi,$$

if the initial wealth ξ increases, then it is reasonable that the utility increases, i.e. we can suppose that $(V^c)' > 0$. Hence, recalling the definition (5.111),

$$w^c(x) = [(V^c)']^{\frac{1}{\gamma - 1}} = \arg\max_{v \ge 0} \left(\frac{1}{\gamma} v^\gamma - (V^c)' v \right) \tag{5.121}$$

and the BHJ equation for V^c (5.120) becomes

$$-\delta V^c + (V^c)' rx + \frac{1 - \gamma}{\gamma} [(V^c)']^{\frac{\gamma}{\gamma - 1}} = 0, \qquad \forall x \ge 0.$$

[23]In economics Hyperbolic Absolute Risk Aversion (HARA) refers to a type of risk aversion that is particularly convenient to model mathematically and to obtain empirical prediction.

[24]Note that here we remove from the model the assumption $\lim_{t \to \infty} x(t) = 0$.

In order to solve the previous ODE, let us consider a function of the type

$$V^c(x) = Ax^\gamma, \qquad x \geq 0,$$

where A is a positive constant (coherent with the assumption $(V^c)' > 0$) : we obtain

$$-\delta A x^\gamma + A\gamma r x^\gamma + \frac{1-\gamma}{\gamma}[\gamma A x^{\gamma-1}]^{\frac{\gamma}{\gamma-1}} = 0, \qquad \forall x \geq 0.$$

An easy calculation, together with the assumption $\delta > r\gamma$, gives

$$A = \frac{1}{\gamma}\left(\frac{1-\gamma}{\delta-\gamma r}\right)^{1-\gamma} > 0.$$

Now (5.121) implies that the function w in Theorem 5.3 is given by (recalling the $w^c(x) = w(t,x)$)

$$w^c(x) = [(V^c)']^{\frac{1}{\gamma-1}} = \frac{\delta-\gamma r}{1-\gamma}x.$$

In our contest, the ODE (5.112) becomes

$$\begin{cases} \dot{x} = \dfrac{r-\delta}{1-\gamma}x \\ x(0) = x_0 \end{cases}$$

Its solution is $x^*(t) = x_0 e^{\frac{r-\delta}{1-\gamma}t}$: let us note that the condition $x^*(t) \geq 0$ is satisfied. Hence Theorem 5.3 guarantees that x^*,

$$c^*(t) = w^c(x^*(t)) = x_0\frac{\delta-\gamma r}{1-\gamma}e^{\frac{r-\delta}{1-\gamma}t} \quad \text{and} \quad V(t,x) = \frac{1}{\gamma}\left(\frac{1-\gamma}{\delta-\gamma r}\right)^{1-\gamma} e^{-\beta t}x^\gamma,$$

for every $x \geq 0,\ t \geq 0$, are the optimal trajectory, the optimal consumption plain and the value function for the investor's problem respectively. Finally we note that for such optimal solution we have

$$\frac{c^*(t)}{x^*(t)} = \frac{\delta-\gamma r}{1-\gamma}, \qquad \forall t \geq 0,$$

i.e. in the optimal plain the ratio between consumption and wealth is constant.

The case $\delta \leq r\gamma$: Let us show that in this second case we have that the value function is equal to ∞ and hence an optimal path of consumption does not exist.

Let us consider a fixed constant $A > 0$ and the path of consumption $c_A(t) = Ae^{rt}$, for $t \geq 0$: let us show that this control is not admissible. The dynamics and the initial condition on the wealth give

$$\begin{cases} \dot{x} = rx - Ae^{rt} \\ x(0) = x_0 \end{cases}$$

and its solution is $x_A(t) = e^{rt}(x_0 - At)$: note that the condition $x(t) \geq 0$ is satisfied for $t \leq x_0/A$: hence c_A is not and admissible control. Now we consider a modification of the previous path of consumption (that we denote again with c_A):

$$c_A(t) = \begin{cases} Ae^{rt} & \text{for } 0 \le t \le \frac{x_0}{A}, \\ 0 & \text{for } t > \frac{x_0}{A} \end{cases} \tag{5.122}$$

The dynamics and the initial condition give now

$$x_A(t) = \begin{cases} e^{rt}(x_0 - At) & \text{for } 0 \le t \le \frac{x_0}{A}, \\ 0 & \text{for } t > \frac{x_0}{A} \end{cases}$$

Hence, for every initial wealth $x_0 > 0$ and for every $A > 0$, the control c_A given by (5.122) is admissible and

$$\begin{aligned} V^c(x_0) = V(0, x_0) &= \sup_c \int_0^\infty \frac{1}{\gamma} c^\gamma e^{-\delta t}\,\mathrm{d}t \\ &\ge \lim_{A \to 0^+} \int_0^\infty \frac{1}{\gamma} c_A^\gamma e^{-\delta t}\,\mathrm{d}t \\ &= \lim_{A \to 0^+} \int_0^{x_0/A} \frac{A^\gamma}{\gamma} e^{(\gamma r - \delta)t}\,\mathrm{d}t \\ &\ge \lim_{A \to 0^+} \int_0^{x_0/A} \frac{A^\gamma}{\gamma}\,\mathrm{d}t = \infty, \end{aligned}$$

where, using the assumption, we use the fact that $e^{(\gamma r - \delta)t} \ge 1$ for every $t \ge 0$. Hence $V^c(x_0) = \infty$, for every $x_0 > 0$, implies $V(t, x) = \infty$, for every $(t, x) \in [0, \infty) \times (0, \infty)$.

5.7.3 Stochastic consumption: the idea of Merton's model

A generalization of the problem (5.119) is the fundamental Merton's model (see [21]), where an investor divides the wealth between consumption, a riskless asset with rate r and a risk asset with uncertain rate return: it is a stochastic model in the context of stochastic optimal control. The aim of this subsection is only to give and idea of the problem (see [18], [13], [12] for details).

In this model, the stock portfolio consists of two assets:

- the price $p_1 = p_1(t)$ for the "risk free" asset changes according to $\dot{p}_1 = rp_1$, i.e.

$$\mathrm{d}p_1 = rp_1\,\mathrm{d}t, \tag{5.123}$$

  where r is a positive constant;

- the price $p_2 = p_2(t)$ for the "risk " asset changes according to

$$\mathrm{d}p_2 = sp_2\,\mathrm{d}t + \sigma p_2\,\mathrm{d}B_t, \tag{5.124}$$

  where B_t is a Brownian motion, s and σ are positive constants: s formalizes the expected profit for the risk investment and σ is its variance.

It is reasonable, for the investor's point of view, to require

$$0 < r < s.$$

According to (5.123) and (5.124), the total wealth $x = x(t)$, evolves as

$$\mathrm{d}x = [r(1-w)x + swx - c]\,\mathrm{d}t + wx\sigma\,\mathrm{d}B_t, \tag{5.125}$$

where $c = c(t)$ is, as in (5.119), the consumption and $w = w(t)$ is the fraction (i.e. $0 \le w \le 1$) of the remaining wealth invested in the risk asset. We note that if we put $w(t) = 0$ in (5.125), then we obtain the dynamic in problem (5.119).

Again, we have a HARA utility function for the consumption to maximize, in the sense of the expected value since w, x and c are all random variables: hence we obtain

$$\begin{cases} \max\limits_{(c,w)} \mathbb{E}\left(\displaystyle\int_0^\infty \frac{c^\gamma}{\gamma} e^{-\delta t}\,\mathrm{d}t\right) \\ \mathrm{d}x = [r(1-w)x + swx - c]\,\mathrm{d}t + wx\sigma\,\mathrm{d}B_t \\ x(0) = x_0 > 0 \\ c \ge 0 \\ 0 \le w \le 1 \end{cases} \tag{5.126}$$

with γ constant in $(0,1)$.

5.7.4 A model of consumption with log–utility II

We solve[25] the model presented in the example 1.1.5, formulated with (1.6), in the case $\delta > r$ and with a logarithmic utility function $U(c) = \log c$. The current value function $V^c = V^c(x)$ must satisfy (5.110), i.e.

$$\begin{aligned} &-\delta V^c + \max_{v \ge 0} \left(\ln v + (V^c)'(rx - v)\right) = 0 \\ \Longrightarrow \quad &-\delta V^c + (V^c)'rx + \max_{v \ge 0} \left(\ln v - (V^c)'v\right) = 0. \end{aligned} \tag{5.127}$$

Now, since for definition

$$V^c(\xi) = \int_0^\infty e^{-\delta t} \ln c\,\mathrm{d}t \qquad \text{with } x(0) = \xi,$$

if the initial capital ξ increases, then it is reasonable that the utility increases, i.e. we can suppose that $(V^c)' > 0$. Hence

$$w^c(x) = \frac{1}{(V^c)'} = \arg\max_{v \ge 0} \left(\ln v - (V^c)'v\right) \tag{5.128}$$

and the BHJ equation for V^c (5.127) becomes

$$-\delta V^c + (V^c)'rx - \ln[(V^c)'] - 1 = 0, \qquad \forall x \ge 0. \tag{5.129}$$

In order to solve the previous ODE, let us consider a derivative with respect to x of it; we obtain

$$-\delta(V^c)' + (V^c)''rx + (V^c)'r - \frac{(V^c)''}{(V^c)'} = 0, \qquad \forall x \ge 0.$$

[25] In subsection 3.7.1 we solve the same problem with the variational approach.

Now suppose that $(V^c)'$ is a homogeneous polynomial of degree k in the variable x, i.e. $(V^c)' = Ax^k$ with A constant. Then we obtain

$$-\delta A x^k + krAx^k + rAx^k - k\frac{1}{x} = 0, \qquad \forall x \geq 0;$$

for $k = -1$, the previous equation is homogeneous and we obtain $A = \frac{1}{\delta}$ that implies $V^c(x) = \frac{\ln(\delta x)}{\delta} + B$, for some constant B. If we replace this expression for V^c in (5.129) we obtain

$$-\delta\left(\frac{\ln(\delta x)}{\delta} + B\right) + \frac{r}{\delta} + \ln(\delta x) - 1 = 0, \qquad \forall x \geq 0,$$

that implies $B = \frac{r-\delta}{\delta^2}$, i.e.

$$V^c(x) = \frac{1}{\delta}\left(\ln(\delta x) + \frac{r-\delta}{\delta}\right), \qquad \forall x \geq 0.$$

We don't know if it's the general solution for the BHJ equation (5.127), but sure it is a solution. Now (5.128) implies that

$$w^c(x) = \delta x.$$

In our contest, the ODE (5.112) becomes

$$\begin{cases} \dot{x} = (r-\delta)x \\ x(0) = x_0 \end{cases}$$

Its solution is $x(t) = x_0 e^{(r-\delta)t}$: let us note that the condition $x(t) \geq 0$ and $\lim_{t\to\infty} x(t) = 0$ are satisfied. Hence Theorem 5.3 guarantees that

$$c(t) = \delta x_0 e^{(r-\delta)t} \quad \text{and} \quad V(t,x) = \frac{e^{-\delta t}}{\delta}\left(\ln(\delta x) + \frac{r-\delta}{\delta}\right), \quad \forall x \geq 0,\ t \geq 0,$$

are the optimal consumption plain and the value function for the investor's problem.

5.8 Problems with discounting and salvage value

Let us consider the problem (see [28]), for a fixed final time $T > 0$,

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_0^T e^{-rt} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + e^{-rT}\psi(x(T)) \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \\ \displaystyle\max_{\mathbf{u}\in U} J(\mathbf{u}) \end{cases} \tag{5.130}$$

where $r > 0$ is a given discount rate and ψ is the pay-off function (or salvage value). We define the function $\widehat{f}$ by

$$\widehat{f}(t,\mathbf{x},\mathbf{u}) = e^{-rt} f(t,\mathbf{x},\mathbf{u}) + e^{-rT}\nabla\psi(x)\cdot g(t,\mathbf{x},\mathbf{u}). \tag{5.131}$$

It is easy to see that for the new functional $\widehat{J}$ we have

$$\begin{aligned}
\widehat{J}(\mathbf{u}) &= \int_0^T \widehat{f}(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \\
&= \int_0^T e^{-rt} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + e^{-rT}\int_0^T \nabla\psi(\mathbf{x}(t))\cdot g(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t \\
&= \int_0^T e^{-rt} f(t,\mathbf{x},\mathbf{u})\,\mathrm{d}t + e^{-rT}\int_0^T \frac{\mathrm{d}\psi(\mathbf{x}(t))}{\mathrm{d}t}\,\mathrm{d}t \\
&= J(\mathbf{u}) - e^{-rT}\psi(\boldsymbol{\alpha});
\end{aligned}$$

hence the new objective function $\widehat{J}$ differs from the original objective functional J only by a constant. So the optimization problem remains unchanged when substituting $\widehat{f}$ with f (i.e. the optimal controls of the two problems are the same). The BHJ-equation of the problem "$\widehat{\ }$"(with value function $\widehat{V}$) is

$$\begin{aligned}
-\frac{\partial \widehat{V}}{\partial t}(t,\mathbf{x}) &= \max_{\mathbf{v}\in U}\left(\widehat{f}(t,\mathbf{x},\mathbf{v}) + \nabla_{\mathbf{x}}\widehat{V}(t,\mathbf{x})\cdot g(t,\mathbf{x},\mathbf{v})\right) \\
&= e^{-rt}\max_{\mathbf{v}\in U}\left(f(t,\mathbf{x},\mathbf{v}) + e^{rt}(\nabla_{\mathbf{x}}\widehat{V}(t,\mathbf{x}) + e^{-rT}\nabla\psi(\mathbf{x}))\cdot g(t,\mathbf{x},\mathbf{v})\right),
\end{aligned}$$

and the final condition is $\widehat{V}(T,\mathbf{x}) = 0$. Let us define

$$V^c(t,\mathbf{x}) = e^{rt}\left(\widehat{V}(t,\mathbf{x}) + e^{-rT}\psi(\mathbf{x})\right); \tag{5.132}$$

we obtain

$$-rV^c(t,\mathbf{x}) + \frac{\partial V^c}{\partial t}(t,\mathbf{x}) + \max_{\mathbf{v}\in U}\left(f(t,\mathbf{x},\mathbf{v}) + \nabla_{\mathbf{x}}V^c(t,\mathbf{x})\cdot g(t,\mathbf{x},\mathbf{v})\right) = 0, \tag{5.133}$$

$$V^c(T,\mathbf{x}) = \psi(\mathbf{x}). \tag{5.134}$$

It is clear that

$$\begin{aligned}
&\arg\max_{\mathbf{v}\in U}\left(\widehat{f}(t,\mathbf{x},\mathbf{v}) + \nabla_{\mathbf{x}}\widehat{V}(t,\mathbf{x})\cdot g(t,\mathbf{x},\mathbf{v})\right) = \\
&\qquad = \arg\max_{\mathbf{v}\in U}\Big(f(t,\mathbf{x},\mathbf{v}) + \nabla_{\mathbf{x}}V^c(t,\mathbf{x})\cdot g(t,\mathbf{x},\mathbf{v})\Big).
\end{aligned}$$

It is easy to see that, given an optimal control $\mathbf{u}^*$ for the initial problem (5.130), we have by (5.131) and (5.132)

$$\begin{aligned}
V^c(t,\mathbf{x}^*(t)) &= e^{rt}\left(\widehat{V}(t,\mathbf{x}^*(t)) + e^{-rT}\psi(\mathbf{x}^*(t))\right) \\
&= e^{rt}\left(\int_t^T \widehat{f}(s,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}s + e^{-rT}\psi(\mathbf{x}^*(t))\right) \\
&= \int_t^T e^{-r(s-t)} f(s,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}s + \\
&\qquad + e^{-r(T-t)}\left(\int_t^T \nabla\psi(\mathbf{x}^*)\cdot g(s,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}s + \psi(\mathbf{x}^*(t))\right) \\
&= \int_t^T e^{-r(s-t)} f(s,\mathbf{x}^*,\mathbf{u}^*)\,\mathrm{d}s + e^{-r(T-t)}\psi(\mathbf{x}^*(T)).
\end{aligned}$$

Hence $V^c(t, \mathbf{x}^*(t))$ is the optimal discounted (at time t) value or, in analogy with (5.109), the *current value function* for (5.130). The equation (5.133) is called *Bellman–Hamilton–Jacobi equation with discounting and salvage value.*

5.8.1 A problem of selecting investment

Consider the firm's problem of selecting investment in the fixed period $[0, T]$. The profit rate, exclusive of capital costs, that can be earned with a stock of productive capital k is proportional to k^2. The capital stock decays at a constant proportionate rate[26] α, so $\dot{k} = \imath - \alpha k$, where $\imath$ is gross investment, that is, gross additions of capital. The cost of gross additions of capital at rate $\imath$ is proportional to $\imath^2$. At the end of the investment period, the firm receives an additional profit on an asset (for example, a coupon) of a value proportional to the square of the final capital. We seek to maximize the present value of the net profit stream over the period $[0, T]$:

$$\begin{cases} \max \displaystyle\int_0^T e^{-rt}(\rho k^2 - \sigma \imath^2)\,\mathrm{d}t + \pi e^{-rT} k(T)^2 \\ \dot{k} = \imath - \alpha k \\ k(0) = k_0 > 0 \\ \imath \in I \end{cases}$$

where α, σ, π and r are fixed positive constants such that $\sigma\alpha^2 > \rho$ and $I \subset \mathbb{R}$ is convex, compact and large enough (so as to allow for an unconstrained minimization).

Conditions (5.133) and (5.134) give, for the optimal discounted value function $V^c = V^c(t, k)$,

$$-rV^c + \frac{\partial V^c}{\partial t} + \max_{v \in I}\left(\rho k^2 - \sigma v^2 + (v - \alpha k)\frac{\partial V^c}{\partial k}\right) = 0,$$
$$V^c(T, k) = \pi k^2.$$

Let us assume that $V^c(t, k) = q(t)k^2$; we obtain

$$-(2\alpha + r)qk^2 + q'k^2 + \rho k^2 + \max_{v \in I}\left(-\sigma v^2 + 2kqv\right) = 0,$$
$$q(T) = \pi.$$

The assumption on I implies that

$$\imath = \arg\max_{v \in I}\left(-\sigma v^2 + 2kqv\right) = \frac{kq}{\sigma} \tag{5.135}$$

and the BHJ now is, after a division by k^2,

$$q' = -\rho + (2\alpha + r)q - \frac{q^2}{\sigma} \tag{5.136}$$

In order to solve this Riccati equation in q (see the footnote in example 5.5.2) with the condition $q(T) = \pi$, let us introduce the new variable $z = z(t)$ with

$$q = \sigma\frac{z'}{z}, \qquad z(T) = \sigma, \qquad z'(T) = \pi.$$

[26]If a stock k decays at a constant proportionate rate $\beta > 0$ and if it is not replenished, then $\dot{k}(t)/k(t) = -\beta$. Since the solution of this ODE is $k(t) = k(0)e^{-\beta t}$, we sometimes say that the stock k decays exponentially at rate β.

This implies $q' = \sigma \frac{z''z-(z')^2}{z^2}$ and, by (5.136),

$$z'' - 2\left(\alpha + \frac{r}{2}\right) z' + \frac{\rho}{\sigma} z = 0.$$

Let us set, by assumption, $\theta = \sqrt{(\alpha + r/2)^2 - \rho/\sigma} > 0$: hence

$$z(t) = e^{\left(\alpha+\frac{r}{2}+\theta\right)(t-T)} \left(c_1 + c_2 e^{-2\theta(t-T)}\right).$$

This implies

$$z'(t) = e^{\left(\alpha+\frac{r}{2}+\theta\right)(t-T)} \left(c_1 \left(\alpha + \frac{r}{2} + \theta\right) + c_2 \left(\alpha + \frac{r}{2} - \theta\right) e^{-2\theta(t-T)}\right)$$

and conditions $z(T) = \sigma,\ z'(T) = \pi$ allow us to determinate the two constants c_1 and c_2 :

$$c_1 = \frac{\pi - \left(\alpha + \frac{r}{2} - \theta\right)\sigma}{2\theta}, \qquad c_2 = \frac{\left(\alpha + \frac{r}{2} + \theta\right)\sigma - \pi}{2\theta}.$$

Condition (5.135) gives, for every $t \in [0, T]$, that the candidate to be the optimal investment (i.e. control) is $\imath = kq/\sigma$ and using $q = \sigma z'/z$ we obtain $\imath = k\frac{z'}{z}$. If we substitute this expression of $\imath$ in the dynamics we obtain

$$\begin{aligned} k' = k\frac{z'}{z} - \alpha k \qquad &\Rightarrow \quad \int \frac{1}{k}\,\mathrm{d}k = \int \left(\frac{z'}{z} - \alpha\right) \mathrm{d}t + c \\ &\Rightarrow \quad k = zfe^{-\alpha t} \end{aligned}$$

with c and f constants. The initial condition on the capital gives

$$k(t) = \frac{k_0}{z(0)} z(t) e^{-\alpha t}$$

and hence the optimal path of investment is given by

$$\imath^*(t) = \frac{k_0}{z(0)} e^{-\alpha t} z'(t) = \frac{k_0}{z(0)} e^{\left(\frac{r}{2}+\theta\right)t} \left(c_1 \left(\alpha + \frac{r}{2} + \theta\right) + c_2 \left(\alpha + \frac{r}{2} - \theta\right) e^{-2\theta t}\right)$$

Index

Bibliography

[1] A. Agracev and Y. Sachkov. *Control theory from the geometric viewpoint.* Springer, 2003.

[2] K. J. Arrow. Applications of control theory of economic growth. *Mathematics of Decision Sciences, part 2, AMS*, page 92, 1968.

[3] M. Bardi and I. Capuzzo Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations.* Birkhäuser, Boston, 1997.

[4] T. Basar and G.J. Olsder. *Dynamic Noncooperative Game Theory.* SIAM Classic in Applied Mathematics, 1998.

[5] B. Beavis and I. Dobbs. *Optimization and stability for economic analysis.* Cambridge university press.

[6] R. Bellman. *Dynamic programming.* Princeton University Press, 1957.

[7] A. Calogero. *Appunti di calcolo delle variazioni e controllo ottimo.* avaible on the web site www.matapp.unimib.it/∼calogero/.

[8] M.C. Caputo. *Foundation of dynamic economic analysis: optimal control theory and applications.* Cambridge University Press, 2005.

[9] A.C. Chiang. *Dynamic optimization.* McGraw-Hill international editions.

[10] L.C. Evans. *An Introduction to Mathematical Optimal Control Theory.* available in the web site of Prof. L.C. Evans (Berkley University).

[11] L.C. Evans. *Partial Differental Equation.* American Mathematical Society, 2010.

[12] L.C. Evans. *An introduction to stochastic differential equations.* avaible on the web site http://math.berkeley.edu/evans, version 1.2.

[13] W.H. Fleming and P.W. Rishel. *Deterministic and stochastic optimal control.* Springer-Verlag, 1975.

[14] R. Gamkrelidze. *Principles of Optimal Control Theory.* Plenum Press, 1978.

[15] A. Haurie, J.B. Krawczyk, and G. Zaccour. *Games and dynamic games.* World Scientific, 2012.

[16] H. Hotteling. The economics of exhaustible resources. *Journal of Political Economic*, pages 137–175, April 1931.

[17] M. I. Kamien and N.L. Schwartz. *Dynamic optimization.* Elsevier, 2006.

[18] B. Øksendal. *Stochastic differential equations.* Springer, 2007.

[19] D. Lèonard and N. Van Long. *Static optimization in economics.* Cambridge university press.

[20] O. L. Mangasarian. Sufficient conditions for the optimal control nonlinear systems. *SIAM Journal on control*, 4:139–152, 1966.

[21] R. C. Merton. Lifetime portofolio selection under uncertainty: the continuous–time case. *Rev. Econ. Statist.*, pages 247–257, August 1969.

[22] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko. *The mathematical theory of optimal process.* Interscience publishers, John Wiley & Sons.

[23] F. P. Ramsey. A mathematical theory of saving. *Economic Journal*, pages 543–559, December 1928.

[24] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, 1969.

[25] S. Salsa and A. Squellati. *Modelli dinamici e controllo ottimo.* Egea.

[26] A. Seierstad and K. Sydsæter. *Optimal control theory with economics applications.* Elsevier Science, 1987.

[27] A. Takayama. *Mathematical economics.* second edition, Cambridge university press.

[28] T. A. Weber. *Optimal control theory with applications in economics.* The MIT Press, 2011.