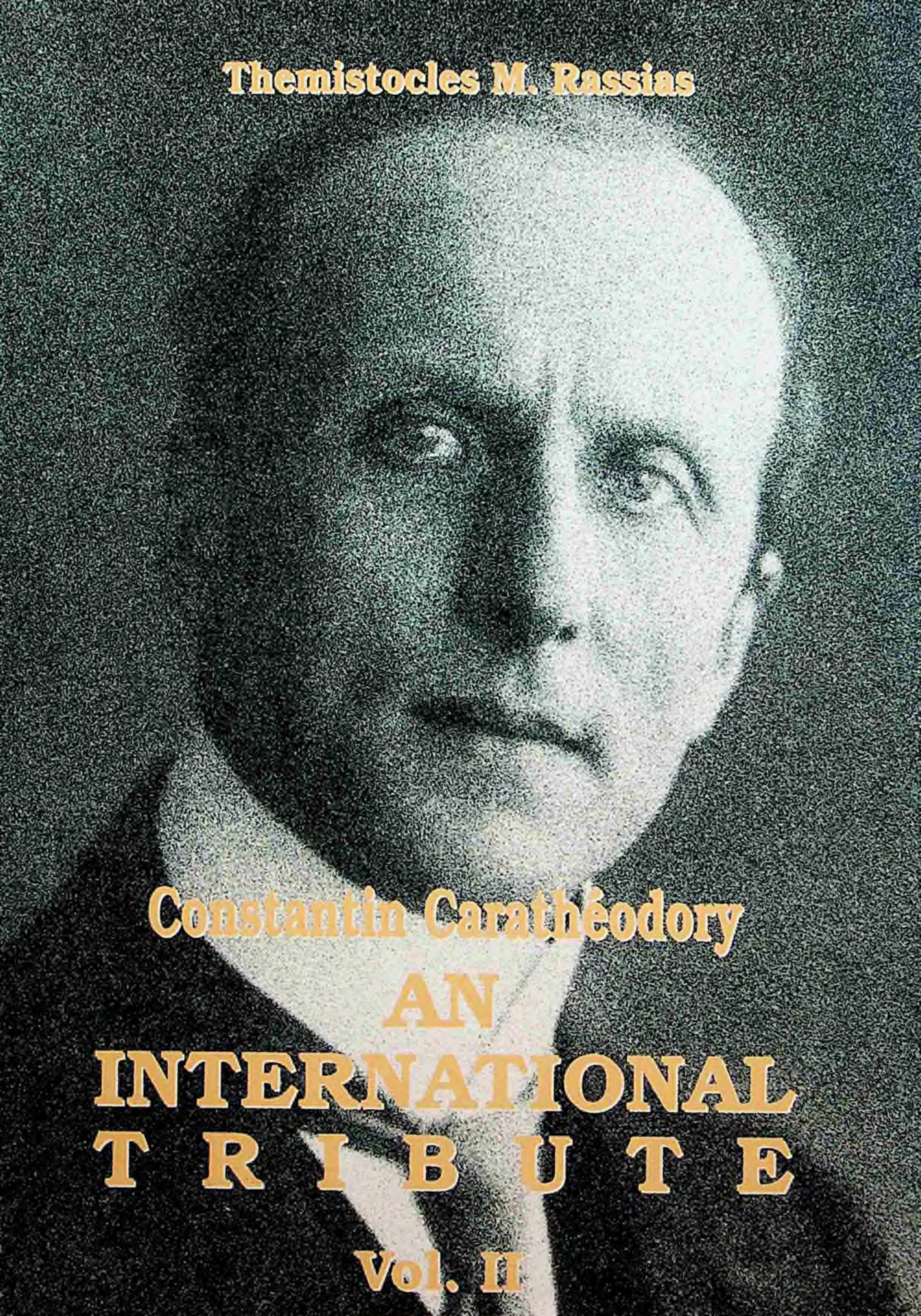
Themistocles M. Rassias



Constantin Carathéodory

AN INTERNATIONAL TRIBUTE

Vol. II

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Editor
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FIXPOINT APPROACH IN MATHEMATICS

Đuro R. Kurepa

Some fixpoint methods and results will be presented with a particular aim to show how the matter developed. Some author's results are included. The matter consists of sections 0,1,2,3,4 with positional subdivisions. Universality character of the fixpoint approach is shown in no. 4,5.

Notations and Terminology

Antichain: no 2 distict comparable points
Branch or clique: the most extensive subchain
Chain or complete subgraph: each 2 points are comparable
$I = R[0,1]$
Inaccessible number: each infinite non-countable regular limit number.
KARD [KARD_∞]: the class of all [infinite] cardinal numbers.
$(n)_1((n)_2)$is the first (second) part of 2-relation (n).
N_0: = set of 0 and all finite cardinal (ordinal) numbers.
ω or ω_0 is the first infinite ordinal number.
ON (ON_∞): the class of all (infinite) ordinal numbers.
Ordered: = partially ordered
pX := power of X
$R[R_0]$:= the set of all real numbers [≥ 0].
$R(i)$ is the set of all complex numbers.
S-un: any procedure f such that fx ($x \in S$) is a point, set, structure,...;
if p$S = k$, one has k-un; 2-un: = ordered pair; 3-un: = ordered triplet.
v : vacuous, empty, void.
Wn or W_n or $W(n)$: set of all ordinal (cardinal) numbers $< n$;
n is a given ordinal (cardinal) number.

$X \subset Y$ embraces $X = Y$ as well.

0. A Heuristic Approach

0.0. If a pupil in primary school were asked to find x from

(0) $x^2 + x - 2 = 0$, it is quite possible that he would write the same equation (0) as

(1) $x(1+x) = 2$ thus $x = 2/(1+x)$ and write

(2) $x = 2/(1+x) = 2/(1+(2/(1+x))) = \ldots$ and get

(3) $x = 2/(1+(2/(1+(2/(1+\ldots$; in this edifice of x the sign x does not appear but the edifice itself does appear after each 1, in particular after the first 1, in such a way that from (3) one gets (2). Similarly, if

(4) $x = fx$ (f is given and fixed, x is varying in R or else), then the pupil would write $x = ffx = fffx = \ldots = ffff\ldots$ thus

(5) $x = fff\ldots$

In the "spear" for x the symbol x does not appear at all but the edifice for x does appear again as the whole section coming after each f; in particular from (5) one has (4). Such puerile heuristic approaches are elaborated and founded as a whole mathematical discipline – Iteration procedures, a very interesting and very large field of researches.

0.1. *Definition of fixpoint*

For a given function f which is defined in a set S, each $x \in S$ such that

(0) $x = fx$ (if f is single-valued) or $x \in fx$ (if f is set-valued) is said to be a fixed (invariant, immuable, reproductive) point of f. The relation (0) is read also as x is f-fixed or x is an f-fixpoint or f-fixvalue, or x is f-invariant. The set of all invariant points of $f|S$ is denoted

(1) Inv $(S, f) := \{x : x \in S$ and $x = fx\}$ and Inv $(S, f) :=$ $\{x : x \in S$ and $x \in fx\}$ respectively.

For abbreviation, one simply writes $I(f)$ or If instead of Inv (S, f).

0.2. *Task*

A major task is the following one: given $f|S$, determine Inv (S, f). In general case, it is not needed to know completely the set Inv. Anyway, one has to determine whether the fixpoint set (1) is v (vacuous, empty, void) or $\neq v$ (nonvacuous) i.e., whether the power pI of I is 0 or $\neq 0$. There are many degrees and nuances in the knowledge of I and of the members of I. In practice it is frequently sufficient to know approximately some member(s) of Inv.

Example. $I(R(i); p(x)) =?$ Here, $p(x)$ is any algebraic polynomial over the field $R(i)$ of complex numbers. Inv $(p, R(i)) = v$, if and only if $p(x) = x + c$, where $c \in R(i) \setminus \{0\}$. Except this case, 0 is a fixpoint for every polynomial over $R(i)$ and $I(p(x))$ has gp members, where gp denotes the grade or degree of $p(x)$. The preceding fact is the content of the Fundamental Theorem of Algebra, each fixpoint z of p being counted with its multiplicity; one has $I(p(x))$ = spectre of $p(x) - x$; thus, each p-fixpoint z is counted with its multiplicity in such a way that if $p(x) - x \equiv p_0 + p_1 x + \ldots + p_n x^n, p_n \neq 0, \{p_1, \ldots, p_n\} \subset R(i)$, then $f(x) - x = p_n \Pi(x - z)^{m(z)}$ (z running over the spectre of $p(x) - x$).

Exercise. How does $I(R(i), \sin x)$ look like?

For each particular function $f : R(i) \twoheadrightarrow R(i)$ it is worth to consider $I(R(i), f)$.

0.3. *On the oldest open mathematical problem*

It is interesting that the oldest not yet resolved mathematical problem is connected with fixpoints. For any natural number $n > 1$ let $s(n)$ denote the sum of all divisors d of n such that $d < n$. In classical Greek mathematics one partitioned N into $N_<, N_=$ and $N_>$ consisting of all natural numbers $n > 1$ for which $s(n) < n, s(n) = n, s(n) > n$ respectively. One knows that $N_<, N_>$ are infinite; but still at present time, in 1990, one does not know whether the set $N_=$ of all "perfect" numbers is finite or infinite. (Any $1 < n \in N$ such that $n = s(n) := \Sigma d$ (d divides n and $n > d \in N$) is called perfect; cf. Euclid, Stoicheia IX: Def 23., and Theorem IX:36: Let $n \in N$; if $2^n - 1$ is prime, then the product $2^{n-1}(2^n - 1) := E_n$ is perfect. Euclid mentions no particular perfect number). Ancient mathematicians Nikomedes (cca 180), Boetius (480?–524) knew the following 4 perfect numbers: 6, 28, 496, 8128; all these numbers are E_n for $n = 2, 3, 5, 7$ respectively. The question on whether Inv (N, s) is infinite is the oldest open mathematical problem; the same is true for the problem whether there exists any odd perfect number.

0.3.1. *A very instructive remark.* The set N has 2 interesting order structures: $(N, \leq)$ and $(N, |)$, where $a|b$ means "a divides b"; for any $n \in N$ one has the corresponding strict left-cone L_n consisting of all $x \in N$ which are $< n$ and "strictly less than" n; in either of the cases one forms the sum $s(n)$ of all members of L_n; the question is to find the set Inv of all invariant points of $s(n)$.

In the case of $(N, \leq)$ one has $s(n) = 1 + 2 + \ldots + (n-1) = n(n-1)/2$; the requested Inv $= \{3\}$; thus 3 is the unique $n \in N$ which is the sum of its predecessors 1,2, in $(N, \leq)$.

Really, a trivial solution! The corresponding situation in $(N, |)$ is completely different because, in this case, the set Inv is precisely the set of all perfect numbers. Thus, transfering the problem of determining Inv $((N, \leq), s(n)) = \{3\}$ to the problem to determine

(1) Inv $((N, |), s(n))$

one encounters the oldest not yet resolved mathematical problems: Does the set (1) contain infinitely many even numbers? Does the set (1) contain some odd number?

1. Iteration Procedures

In this section (M, d) will denote any metric space; thus $d(x, y) (\in R^0_+ := [0, \infty) = R(0, .))$ denotes the distance between x, y.

1.0. If one has a function $F : R \twoheadrightarrow R$ it matters to find one zero z of F, i.e., a $z \in R$ such that $F(z) = 0$. In general, a relation like

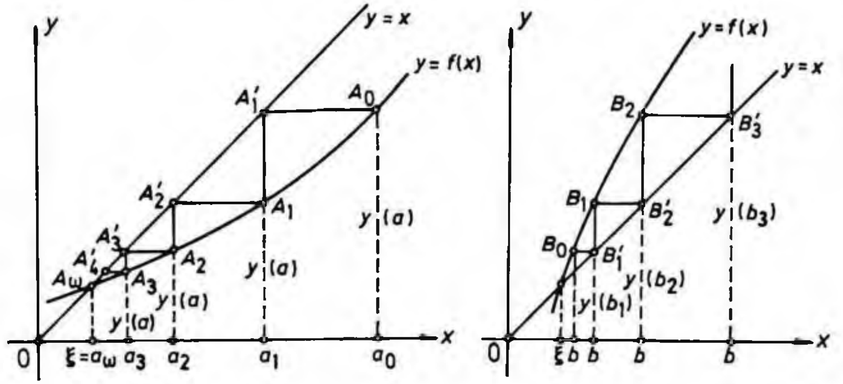
(0) $F(x) = 0$ could be equivalently written as

(1) $x = fx$.

If then one starts with a special value a_0 for x as a possible approximative value for a requested solution of (1), then one gets $a_1 := fa_0$ as a possible solution of (1); by iteration one gets $a_2 := fa_1, \ldots, a_{n+1} := fa_n (n = 0, 1, \ldots)$. If a_n is convergent and if $\xi := \lim a_n$ and if f is continuous at ξ, then one gets a valid relation $\xi = f\xi$; i.e., ξ is a root of (1) and of (0). Thus ξ is a zero of the given function F.

1.1. Graphically we have the following picture and procedure. In the first picture, the procedure is converging; in the second picture the procedure is diverging.

1.2. If instead of f in the second case one considers the inverse function g, the picture of which is the symmetrical map of f with respect to the symmetry axis $y = x$, one gets a converging procedure yielding a solution of $gx = x$ thus also of $fx = x$ because both equations have a common root.



1.3. *A case of convergence*

Theorem. If I is a closed segment $R[a, b]$ of R and $f : I \rightarrowtail\!\!\!\twoheadrightarrow I$ is such that the derivative $f'|I$ exists and satisfies $\sup|f'| := B < 1$, then the equation $x = fx$ has a unique solution ξ; the solution is the limit point of the above sequence $a_{n+1} = fa_n$, taking for a_0 any point of I.

A proof of the theorem is easy because if $a, b \in I$, then $fa - fb = (a - b)f'(c)$ for some $c \in R(a, b)$ (mean value theorem) and one gets

$$|fa - fb| \leq |a - b|B\,, \qquad \text{for some } 1 > B \geq |f'c|\,.$$

Now, we have the following general

1.4. Theorem. q-Contraction Principle (Banach, 1922; cf. Picard, 1890) on q-contraction in any complete metric space. Let (M, d) be any complete metric space and T be any selfmapping of M such that some number $q \in R[0, 1)$ satisfies $d(Tx, Ty) \leq qd(x, y)$, whenever $x, y \in M$; then there is just one point $x \in M$ such that $Tx = x$; if $x_0 \in M$, then $T^n x_0 \rightarrowtail\!\!\!\twoheadrightarrow x$ as $n \rightarrowtail\!\!\!\twoheadrightarrow \infty$. In other words, T has a unique fixpoint in M; in addition it is obtained as $\lim T^n x_0$ for every $x_0 \in M$.

At first we have the following

1.4.1. Lemma. If $T|(M,d)$ is any selfretraction, i.e., if $d(Tx,Ty) < d(x,y)$ whenever $x, y \in M$ and $x \neq y$, then the mapping T is continuous (proof is easy).

Proof of Theorem 1.4. Let $x_0 \in M$ and

(0) $x_n = f^n x_0$ $(n \in N)$, where $f^1 := f$ and f^{n+1} is the compound ff^n. The sequence (0) is Cauchy. As a matter of fact, if n, s are natural numbers and ϵ any given real number > 0, then $d(x_n, x_s) < \epsilon$ for any sufficiently great natural numbers n, s. Namely, we have $d(x_n, x_s) := d(Tx_{n-1}, Tx_{s-1}) \leq qd(x_{n-1}, x_{s-1})$, thus $d(x_n, x_s) \leq qd(x_{n-1}, x_{s-1})$ and for the same reason,

$$d(x_{n-1}, x_{s-1}) \leq qd(x_{n-2}, x_{s-2}),$$

i.e., $d(x_n, x_s) \leq q^2(x_{n-2}, x_{s-2})$; analogously, if $n < s$,

(1) $d(x_n, x_s) \leq q^n d(x_0, x_{s-n})$. Now, the last factor satisfies

$$\begin{aligned} d(x_0, x_{s-n}) &\leq d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{s-n-1}, x_{s-n}) \\ &\leq d(x_0, x_1) + qd(x_0, x_1) + q^2(x_0, x_1) + \ldots + q^{s-n-1} d(x_0, x_1) \\ &\leq (1 + q + q^2 + \ldots) d(x_0, x_1) = (1-q)^{-1} d(x_0, x_1). \end{aligned}$$

This relation jointly with (1) yields

(2) $d(x_n, x_s) < q^n (1-q)^{-1} d(x_0, x_1)$.

Since, $|q| < 1$ and $q^n \gg\!\!\to 0$, the evaluation (2) says that $x_n (n \in N)$ is a Cauchy sequence in the space (M, d); since this one is complete, the limit of x_n is a determined point x in M. Therefore the relation $x_{n+1} = Tx_n$ yields $\lim x_{n+1} = \lim Tx_n =$ (because T is continuous) $T \lim x_n$ i.e., $x = Tx$. We say that x does not depend on $x_0 \in M$, because if $x_0 \neq y_0 \in M$ then the limit of $T^n y_0$ yields a definite value $y \in M$ such that $Ty = y$. Again the supposition $0 \neq d(x,y) = d(fx, fy)$ contradicts the condition $d(Tx, Ty) \leq qd(x,y) < d(x,y)$.

1.5. The above considerations are typical in approximation procedures by tangents (I. Newton 1669, J. Raphson 1697), Regula falsi or Method of secants and Mixed methods. Emile Picard used and developed the "method of successive approximations" in the theory of differential equations (including partial derivatives) and integral equations.

1.6. The idea of contraction mappings with various nuances and variants was very much examined, used and generalized. Hundreds papers were written on the subject. It matters to stress that also the contraction coefficient(s) allow great generalizations, in particular that they may be members of ordered sets instead of to be in R (cf. the idea of pseudometric spaces; uniform spaces, general metric spaces,... cf. Collatz 1964; further one may deal with system of mappings etc.) We are going to indicate some generalizations.

As illustration of such a trend let us quote the following facts 1.7, 1.8, 1.9.

Iseki (1965) has transfered Theorem 1.4 to general metric spaces ≡ uniform spaces and proved the following

1.7. Theorem (S.Iseki 1965). Let (M, d, E) be a sequentially complete metric space over a topological semifield E and $T|M$ be a selfmapping such that $d(Tx, Ty) \ll cd(x, y)$, where c is a positive number < 1 and $\ll$ denotes the order in R; then $T|M$ has a unique fixpoint u; one has $u = \lim T^n x$ for every $x \in M$. He pointed out that the condition $c < 1$ is not replaceable by $c \ll 1$.

Ćirić (1987) has extended Iseki's result and proved the following

1.8. Theorem (Th. 1 in Ćirić 1987). Like Theorem 1.7, for $c \in K$ with $c < 1$.

Ćirić has furnished a space for which 1.7 does not hold and Theorem 1.8 does hold.

The terminology is like in Antonowski-Boltjanski-Sarymsakov 1960.

1.9. A very general Fixpoint Theorem in pseudometric spaces with determined approximations was proved in [§11, pp. 160–171, Collatz 1964].

1.10. As to the terminology, one has: uniform spaces = pseudometric spaces = spaces with ordered ecart = spaces over topological semifields = Kurepa spaces = Weil spaces = generalized metric spaces, g-spaces (cf. also p. 184, Nagata, Jun-Iti 1985).

2. T-Orbits for Any Selfmapping $T|S$

2.0. Given a Set$\neq \emptyset$ and a selfmapping $T|S$; it is natural to consider the ω-sequence of iterates: $T^0 := 1_s, T^1 := T, T^2 := TT, \ldots, T^{n+1} := TT^n$ $(n \in N)$ and to examine how they behave.

2.1. *T-orbit*

For any $x \in S$ and any $T : S \twoheadrightarrow S$ the set $\{T^n x : n < \omega\}$ is called the T-orbit of x and is denoted by $O(T, x)$; thus $O(T, x) := \{T^n x; n \in N_0\}$. If the power of the orbit is $\mathrm{p}O = 2$, then $\{x, Tx\}$ is a fixed edge; if $\mathrm{p}O = 3$, then $\{x, Tx, T^2x\}$ is a fixed triangle, etc. If $\mathrm{p}O(T, x) = n \in N$, then $T|O(T, x)$ is a cyclic permutation of $O(T, x)$.

Obviously, if $a = Ta$ then $Ta = TTa$ thus $a = T^2a$, similarly $a = T^n a$.

2.2. Lemma. If $1 < n \in N$ and if u is the unique fixpoint of $T^n|S$, the same u is the unique fixpoint of $T|S$ as well:

(0) $\quad \mathrm{Inv}\,(S, T^n) = \{u\} \Rightarrow \mathrm{Inv}\,(S, T) = \{u\}$.

Proof is trivial because $u = T^n u \Rightarrow Tu = T(T^n u) =$ (because $TT^n = T^nT$) $T^n(Tu)$, i.e., $Tu = T^n(Tu)$, thus Tu is T^n-fixpoint and by (0) equals u, i.e., $Tu = u$.

2.3. It matters to consider $O(T, x)$ also as infinite sequence $T^n x\,(n \in N)$ and to say:

(i) a metric space (M, d) is orbitally complete $\Longleftrightarrow$ Each Cauchy subsequence of $O(T, x)$ $(x \in M)$ converges in the space;

(ii) a space (S, cl) is orbitally continuous $\Longleftrightarrow y \in \mathrm{cl}\ O(T, x) \Rightarrow Ty \in \mathrm{cl}\ TO(S, x)$ $(x, y \in S)$.

2.4. A natural complete graph tied with $T|S$ and $(x, y) \in S^2$ is a graph $G(T, x, y)$ such that the members of the orbits $O(T, x), O(T, y)$ constitute the vertex set of $G(T, x, y)$ and that all $\{x, y\}_{\neq} \subset G$ are edges of the graph. For a given 2-un (r, s) of natural ordinal numbers one has the subgraph $G(T, x, r, y, s)$ consisting of all vertices $x_i\,(i < r)$ and $y_j\,(j < s)$.

2.5. Theorem (q-Contraction Principle for orbitally complete metric spaces). Let (M, d) be any T-orbitally complete metric space for some selfmapping $T|M$; let $r \in N, q \in R[0, 1)$ exist such that

(0) $\quad d(T^r x, T^r y) < qd(x, y) \quad (x, y \in M)$.

Then T has one and only one fixpoint u in M; one has $u = \lim O(T,x)$ whenever $x \in M$.

Proof. If $r = 1$, then the proof of Theorem 1.4 is transferable to the present situation, thus $V := T^r$ has a unique fixpoint u. In virtue of Lemma 2.2 the same u is the unique fixpoint of $T|(M,d)$.

2.6. Theorem (= Th. 2 in Rassias 1985). Let (M,d) be a complete metric space, T_n be, for each $n \in N$, a c_n-selfcontraction; if $\sup c_n := c < 1$ and if $Tx := \lim T_n(x)$ $(x \in M)$ exists then $d(Tx,Ty) \leq cd(x,y)$ $(x,y \in M)$ and the selfmapping $T|M$ has a unique fixpoint u; one has $u = \lim u_n$, where u_n is the fixpoint of $T_n|M$.

2.7. Theorem. Let (M,d) be a metric space, H be a system of selfmappings $T|M$; if (M,d) is T-orbitally complete for each $T \in H$ and if there exist $r \in N$ and a positive constant $0 < c < 1$ such that

(0) $d(T^r x, V^r y) \leq cd(x,y)$ $(T,V \in H)$,

then each $T \in H$ has a unique fixpoint $u_T \in M$; moreover $u_T = u_v$ for $T,V \in H$ (cf. Kurepa 1972(3) Th. 2).

Proof. In virtue of Theorem 2.5, if $T \in H$, one has $I(T,M) = \{u_T\}$ and $u_T = \lim T^n x$, where x is any given point in M. Now, one has $u_T = u_V$ whenever $T,V \in H$. In the opposite case, there would be some $T,V \in H$ such that $u_T \neq u_V$, thus $d(u_T,u_V) > 0$. But, in virtue of (0) one has $d(Tu_T,Vu_V) \leq cd(u_T,u_V)$, i.e., $d(u_T,u_V) \leq cd(u_T,u_V)$ and therefore (divide by $d(u_T,u_V) \neq 0$)$1 < c$, contrary to the assumption $0 < c < 1$.

Remark. Theorem 2.7 may fail if both $pH > 1$ and $c = 0$.

2.8. Theorem. Let (M,d) be a nonempty metric space, T be a selfmapping of M and (r,s) be a 2-un of natural numbers such that $x, x_1 := Tx, x_2 := T^2x, \ldots, x_r := T^r x, y, y_1, \ldots, y_s$ are pairwise distinct and

(0) $d(x_r,y_s) = 0$, whenever $x,y \in M$; then the set $I(M,T)$ of all T-fixpoints in M is nonempty; more precisely:

(i) If $r = s = 1$, then the function $T|M$ is a constant $u \in M$ and u is the unique T-fixpoint in M.

(ii) If $1 = r < s$, or if $1 < r = s$, then $I(M,T) = T^s M :=$ the range of the function $T^s|M$.

(iii) If $1 < r < s$, then $I(M,T) = T^{s-1}M$.

Proof of (i) is trivial.

Proof of (ii) for the subcase $1 = r < s$. In this case the relation (0) says that $x = y_s$ is a T-fixpoint because $d(Ty_s, y_s) = 0$ $(y \in M)$; in other words $T^s M \subset I(M,T)$. The dual relation holds as well, i.e., if some $v \in M$ satisfies $v = Tv$, then also $v = T^s y$ for some $y \in M$, and even $v = v_s$. Namely, $v = Tv$ implies $Tv = v_2$, thus $v = Tv = v_2$ and inductively $v = v_n$ $(n \in N)$, thus in particular $v = v_s$.

Proof of (ii) for the subcase $r = s > 1$. The assumed relation (i) $d(x_s, y_s) = 0$ $(x, y \in M)$ for $y = Tx$ becomes $d(x_s, T^s(Tx)) = 0$ i.e., $d(u; Tu) = 0, u = Tu$ for $u = T^s x$ $(x \in M)$ because $T_s(Tx) = T(Tx)$. Thus $T^s M \subset I(M,T)$. The dual inclusion holds also, as one sees, like in the previous first subcase.

Proof of (iii). In this case $1 \leq s - r \in N$; if $x = y_{s-r} (y \in M)$, then (0) is fulfilled because $T^r(T^{s-r}y) = T^s y$. If $x = T^{s-r-1}y$, then $x_r = y_{s-1}$ and (0) yields $d(T^{s-1}y, T^s y) = 0$, and consequently $d(u, Tu) = 0, u = Tu$ for $u := T^{s-1}y$, whenever $y \in M$. Thus the sign $\supset$ for $=$ in (iii) is correct. As in other cases, one proves that the sign $=$ in (iii) is replaceable by $\subset$ as well.

2.8.1. Corollary to 2.8 Theorem. If (0) is satisfied, then for every $x \in M$ the sequence $T^n x$ $(n \in N)$ is not only a Cauchy sequence but is almost constant, i.e., there is some $m(x) \in N$ such that $T^{m(x)}x = T^n x$ $(m \leq n \in N)$; therefore $T|M$ is orbitally continuous and (M,d) is T-orbitally complete.

2.9. Theorem. Let (M,d) be a complete metric space; for a nonempty set I of indices let $m|I$ be an I-un of natural numbers m_i, $T|I$ be an I-un of selfmappings $T_i : M \twoheadrightarrow M$, $c|I^2$ be an I^2-un of real numbers c_{ij}, such that

(0) $c_{ij} \in R(0,1), d(T_i^{m_i}x, T_j^{m_j}y) \leq c_{ij}\, d(x,y)$ $(i,j \in I; x, y \in M)$; then T_i has a unique fixpoint $u_i \in M$; one has

(1) $u_i = u_j$ for $i, j \in I$.

Proof. At first, the mapping $V_i := T_i^{m_i}$ being a c_{ii}-contraction has a unique fixpoint u_i and for every $x \in M$ one has

(2) $V_i^n x \twoheadrightarrow u_i$. We claim that

(3) $T_i u_i = u_i$ $(i \in I)$.

In fact, the relation (2) for $x = Tu_i$ yields $V_i^n Tu_i \twoheadrightarrow u_i$ thus (V_i^n and T commute)

(4) $TV_i^n u_i \twoheadrightarrow u_i (n \in N)$.

Since u_i is a V_i-fixpoint, we have $V_i^n u_i = u_i$; therefore, $(4)_1$ is a constant sequence Tu_i which, by (4), converges to u_i, thus we have (3).

Finally, one has (1). In the opposite case there would be $i, j \in I$ such that $u_i \neq u_j$. For $x = u_i, y = u_j$, the relation (0) would yield

(5) $d(T_i^{mi} u_i, T_j^{mj} u_j) \leq c_{ij}\, d(u_i, u_j)$. But (5) is not possible because $(5)_1 = d(u_i, u_j)$ and $(5)_2 < d(u_i, , u_j)$, the number c_{ij} being in $[0, 1)$. This completes the proof of theorem 2.9.

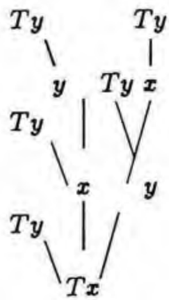
2.10. ***Paths contracting selfmappings*** $T|M$

We shall illustrate one case, i.e., how graph theoretical considerations are useful in fixpoint considerations.

2.10.1. *Graph* g. For a given 2-point-set $\{x, y\} \in M$ we consider the corresponding complete graph $\{x, y, Tx, Ty\} := g$.

2.10.1.1. There are paths joining the points Tx, Ty; there are just 5 such paths, viz.

(0) $L_0 = TxTy, L_1 = TxxTy, T_2 = TxxyTy, T_3 = TxyTy, T_4 = TxyxTy$; all these paths could be visualized in the following way:



2.10.1.2. *Length of a path.* The length lL of a path L is the sum of lengths of all its edges; thus

(1) $lL_0 = d(Tx, Ty)$, $lL_1 = d(Tx, x) + d(x, Ty), \ldots,$
$lL_4 = d(Tx, y) + d(y, x) + d(x, Ty)$.

To each of these oriented paths corresponds the sequence of edges of the path.

2.10.1.3. *Contracted length.*

Definition. Contracted length cL of an oriented path L is the scalar product of the sequence of the lengths of the corresponding edges and of a sequence of numbers $\in R[0,1)$. In other words, contracted length of the paths (0) are of the following form respectively:

$$c_0L_0 := c_{00}\, d(Tx,Ty) \text{ for } L_0$$
$$c_1L_1 := c_{11}\, d(Tx,x) + c_{12}\, d(x,Ty) \text{ for } L_1$$
$$(2) \quad c_2L_2 := c_{21}\, d(Tx,x) + c_{22}\, d(x,y) + c_{23}\, d(y,Ty) \text{ for } L_2$$
$$c_3L_3 := c_{31}\, d(Tx,y) + c_{32}\, d(y,Ty) \text{ for } L_3$$
$$c_4L_4 := c_{41}\, d(Tx,y) + c_{42}\, d(y,x) + c_{43}\, d(xTy) \text{ for } L_4.$$

Of course c_{ij} depends on x, y and T. In this way for given $T : M \twoheadrightarrow M$ we have 11 functions

$$(3) \quad c_{ij}(x,y) \in R[0,1) \quad ((x,y) \in M^2).$$

2.10.2. Theorem. Let (M,d) be a metric space and $T : M \twoheadrightarrow M$ a c-contracting path mapping in the sense that there are functions like (3) satisfying for every $x, y \in M$:

$$(4) \quad d(Tx,Ty) \leq \sum_{i=0}^{4} c_iL_i$$
$$= c_{00}\, d(Tx,Ty) + c_{11}\, d(Tx,x)$$
$$+c_{12}\, d(x,Ty) + c_{21}\, d(Tx,x) + c_{22}\, d(x,y)$$
$$+c_{23}\, d(y,Ty) + c_{31}\, d(Tx,y) + c_{32}\, d(y,Ty)$$
$$+c_{41}\, d(Tx,y) + c_{42}\, d(y,x) + c_{43}\, d(x,Ty)$$

and such that

$$(5) \quad \sup_{x,y \in M} [c_{12} + c_{43} + \Sigma_{ij} c_{ij}(xy)] := c < 1.$$

If the space (M,d) is T-orbitally complete, then there exists one and only one fixed point u of $T|M$, and for every $x \in M$ one has

(6) $\lim_n T^n x = u$,

(7) $d(T^n, u) \leq c^n(1-c)^{-1}\, d(x,Tx)$, and even

(8) $(T^n x, u) \leq m^n(1-m)^{-1}\, d(x,Tx)$, where

(9) $m = a(1-b)^{-1}, 0 \leq m \leq c < 1$,

(10) $a = c_{11} + c_{12} + c_{22} + c_{42} + c_{43}$

(11) $b = c_{00} + c_{12} + c_{23} + c_{32} + c_{43}$ (cf. Kurepa 1973(8) Th. 2.2).

2.10.3. Proof. The case $c = 0$ being obvious, because $T|M$ is constant, let us assume $0 < c < 1$. Let us majorate

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Putting x_{n-1} instead of x and x_n instead of y in (4) we get

$$(12)\quad \begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq c_{00}\, d(x_{n+1}, x_n) \\ &\quad + c_{11}\, d(x_n, x_{n-1}) + c_{12}\, d(x_{n-1}, x_{n+1}) \\ &\quad + c_{21}\, d(x_n, x_{n-1}) + c_{22}\, d(x_{n-1}, x_n) \\ &\quad + c_{23}\, d(x_n, x_{n+1}) + c_{31}\, d(x_n, x_n) \\ &\quad + c_{32}\, d(x_n, x_{n+1}) + c_{41}\, d(x_n, x_n) \\ &\quad + c_{42}\, d(x_n, x_{n-1}) + c_{43}\, d(x_{n-1}, x_{n+1}). \end{aligned}$$

On the other hand, by the triangular relation,

$$(13)\quad d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

Writing the expression $(13)_2$ instead of $(13)_1$ where $(13)_1$ occurs in (12) we get the following relation after transfering on left side the terms containing $d(x_n, x_{n+1})$ as factor:

$(14)\quad d(x_n, x_{n+1})(1-b) \leq a\,.d(x_{n-1}, x_n)$ where a, b are defined by (10) and (11) respectively.

One has either $a = 0$ or $0 < a \leq c < 1$.

A. Case $a = 0$.

The relation (14) implies $d(x_n, x_{n+1}) = 0$ $(x \in M, n \in N)$ thus in particular $x_1 = Tx_1$, i.e., Tx is a fixpoint, whenever $x \in M$. We claim that $Tx = Ty$ for every $x, y \in M$. In the opposite case, there would be 2 distinct points $x, y \in M$ and $Tx \neq Ty$, thus $0 < d(Tx, Ty)$; since $Tx = x, Ty = y$, the relation (4) would yield

$$\begin{aligned} 0 < d(x,y) &\leq c_{00}\, d(x,y) + c_{12} d(x,y) \\ &\quad + c_{22} d(x,y) + c_{31} d(x,y) + c_{41} d(x,y) + c_{42} d(x,y) + c_{43} d(x,y) \end{aligned}$$

and therefore, dividing by $d(x,y) \neq 0$, one would have $0 < 1 \leq c_{00} + c_{12} + c_{22} + c_{31} + c_{41} + c_{42} + c_{43}$; the last sum being $\leq c$, one would have $0 < 1 \leq c$, in contradiction to the condition (5). This contradiction proves that the assumption $Tx \neq Ty$ does not hold; thus one has $Tx = Ty$, whenever $x, y \in M$, i.e., the mapping $T|M$ is constant, $u \in M$, thus in particular $Tu = u$, and therefore $T^n u = u$ $(n \in N)$ etc.

B. Case $0 < a$.

According to (5) we have $a + b + c_{31} + c_{41} = c$, that jointly with $0 \leq c < 1$ yields $a + b \leq c, a + cb \leq c$

(15) $a \leq c(1-b)$.

Since $0 \leq b \leq c < 1$, one has $0 < 1-b \leq 1$, and the relation (15) yields

(15′) $0 \leq e \leq c$ with $e = a/(1+b)$.

Consequently, the relation (7) reads

(16) $d(x_n, x_{n+1}) \leq ed(x_{n-1}, x_n)$ for some e satisfying

(17) $0 \leq e \leq c < 1$.

2.10.3.2. The classical argument is applicable to the sequence $(x_n)_n$ yielding the limit point $u = \lim_n x^n$.

2.10.3.3. Moreover $Tu = u$. As a matter of fact

(18) $d(Tu, u) \leq d(x_n, u) + d(Tu, Tx_{n-1})$ for every $n \in N$.

In virtue of (4) we have

$$d(Tu, Tx_{n-1}) \leq \sum_{i=0}^{4} c_i L_i \text{ putting there } x = u, y = x_{n-1}.$$

Therefore (18) yields

$$(19) \quad d(Tu, u) \leq d(x_n, u) + \sum_{i=0}^{4} c_i L_i \text{ or explicitly:}$$

$$\begin{aligned} d(Tu, u) \leq\ & d(x_n, u) + c_{00}\, d(Tu, Tx_{n-1}) + c_{11}\, d(Tu, u) \\ & + c_{12}\, d(u, Tx_{n-1}) + c_{21}\, d(Tu, u) + c_{22}\, d(u, x_{n-1}) \\ & + c_{23}\, d(x_{n-1}, Tx_n) + c_{31}\, d(Tu, x_{n-1}) \\ & + c_{32}\, d(x_{n-1}, Tx_{n-1}) + c_{41}\, d(Tu, x_{n-1}) \\ & + c_{42}\, d(x_{n-1}, u) + c_{43}\, d(u, Tx_{n-1}). \end{aligned}$$

Applying the triangular relations $d(Tu, x_k) \leq d(Tu, u) + d(u, x_k)$ for $k = n, n-1$ the relation (19) yields

$$\begin{aligned} & (1 - c_{00} - c_{21} - c_{31} - c_{41}) d(Tu, u) \\ & \leq d(x_n, u) + c_{00}\, d(u, x_n) + c_{12}\, d(u, x_n) + c_{22}\, d(u, x_{n-1}) \\ & \quad + c_{23}\, d(x_{n-1}, x_{n+1}) + c_{31}\, d(u, x_{n-1}) + c_{32}\, d(x_{n-1}, x_n) \\ & \quad + c_{41}\, d(u, x_{n-1}) + c_{42}\, d(x_{n-1}, u) + c_{43}\, d(u, x_n). \end{aligned}$$

Applying in this identity the operator lim, each term at the right side yields 0 and therefore

$$(1 - c_{00} - c_{21} - c_{31} - c_{41}) d(Tu, u) \leq 0\,.$$

Since

$$0 \leq c_{00} + c_{21} + c_{31} + c_{41} < c < 1\,,$$

one concludes that $d(Tu, u) \leq 0$, i.e., $d(Tu, u) = 0$ and $Tu = u$.

2.10.3.4. By a similar argument,, taking any point $x' \in M$ and the corresponding $u' = \lim T^n x'$, and applying the majoration (4) to the ordered pair $(x,y) := (u,u')$, one proves that necessarily $u = u'$. In other words $IT = \{u\}$.

2.10.3.5. Let us prove still the evaluations (7), (8). Now, for every $n \in N$ we had the relation (16) from where obviously

$$(20) \quad d(x_n, x_{n+1}) < e^n d(x, Tx), \quad (n \in N).$$

Using this evaluation and the formula

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}),\ n \in N$$

we get

$$d(x_n, x_{n+p}) \leq (e^n + e^{n+1} + \ldots + e^{n+p-1})\, d(x, Tx)\,, \text{i.e.,}$$
$$d(x_n, x_{n+p}) \leq e^n (1-e)^{-1} d(x, Tx) \quad (p \in N)\,.$$

Applying here operator lim one gets precisely the requested formula (8). The Theorem 2.10.2 is completely proved.

2.10.4. Some particular cases of the Theorem 2.10.2.

2.10.4.1. All quantities c_{ij} are vanishing: $c_{ij} = 0$. Then the mapping $T : x \in M \gg\!\!\rightarrow M$ is constant (this occurs in particular if $c_{ij} = 0$ for $(i,j) \neq (0,0)$).

2.10.4.2. $c_{22} \neq 0 \vee c_{42} \neq 0$ and $c_{ij} = 0$ for all other (i,j). One gets the classical Banach case.

2.10.4.3. $0 < c_{21} = c_{23} < 1/2$, and all other $c_{ij} = 0$. One gets a theorem of R. Kannan (1968, Theorem 3).

2.10.4.4. The functions $c_{11}, c_{12}, c_{32}, c_{41}, c_{42}, c_{43}$ are identically 0. One gets a theorem of Lj. Ćirić (1971, Theorem 2.5).

2.11. *Diametral f-contractions in* (M,d)

Let us prove the following

2.11.1. Lemma (L.1 in Tasković 1980). Let $f : R_+ \gg\!\!\rightarrow R_+$ be such that $(\forall t \in R_0) ft < t$ and

(0) $\limsup_{x \gg\!\rightarrow t+0} fx < t$ whenever $t \in R_+$;

let a sequence

(1) x_n $(n \in N)$ of reals > 0 satisfy

(2) $x_{n+1} \leq fx_n$ $(n \in N)$; then x_n is a 0-sequence i.e., $\lim_n x = 0$.

Proof. Since the sequence (1) is decreasing and its terms are > 0 the limit t of (0) exists and is ≥ 0. We claim that $t = 0$. In the opposite case there would be $t > 0$. Thus there would be $0 < t := \lim\ \sup x_{n+1} \leq$ (by (2)) $\lim\ \sup_n fx_n \leq$ (obviously, because $(x_n)(n \in N)$ is a particular sequence $\twoheadrightarrow t+0) \lim \sup_{x \to t+0} fx < t$ (by (0)); thus one has the contradiction $t < t$, which proves that the Lemma is true.

2.11.2. Definition. Let f be a mapping like in 2.11.1 Lemma, then a selfmapping $T|M$ is said to be a diametral f-contraction if and only if $d(Tx, Ty) \leq f\delta(O(x, y, T))$ and $\delta O(T, x) \in R_+; \delta X := \sup d(x, y)$ $(x, y \in M)$.

On basis of the Lemma 2.11.1 one can prove the following.

2.11.3. Theorem on diametral f-contractions. Let (M, d) be T-orbitally complete for some diametral f-contraction selfmapping T of M. Then T has a unique fixpoint u. In addition, this u is the limit of the orbit $T^n x$, whenever $x \in M$ (cf. Theorem 1, p. 250, Tasković 1980).

2.12. Theorem. Let (M, d) be a complete g-metric (= uniform) space and $T : M \twoheadrightarrow M$ a continuous (U, q, k)-contraction with some $(\emptyset \neq U \subset M^2, 0 \leq q < 1, k \in N)$. If $M \times TM \subset \text{Hull } U := U \cup U^2 \cup U^3 \cup, \ldots,$ where $U^n = U \circ U^{n-1}$ $(n = 2, 3, \ldots)$, then for every $x \in M$ the iterates $T^n x$ converge to a T-fixpoint $u \in M$. IF (M, d) is U-chainable (i.e., $M^2 = \text{Hull } U$), then u is the unique fixpoint of $T|M$.
Terminology: $T : M \twoheadrightarrow M$ is a (U, q, k)-contraction $\Longleftrightarrow$ if $(x, y) \in U$ then $d(T^k x, T^k y) \leq q \max d(T^i x, T^i y)$ $(i < k)$. (v. Marjanović 1968, Naimpally 1965 Th. 3.7).

2.13. Case of $T : M^k \twoheadrightarrow M$. One can prove the following interesting

2.13.1. Theorem. Let there be given: a natural number k, a point $x := (x_1, \ldots, x_k) \in M^k$ and a mapping $T : M^k \twoheadrightarrow M$; if there exists a k-un $c := (c_1, c_2, \ldots, c_k)$ of reals ≥ 0 such that $c_1 + c_2 + \ldots + c_k < 1$ and

$$d(T(u_1, \ldots, u_k), T(u_2, u_3, \ldots, u_{k+1}) \leq c_1\, d(u_1, u_2) + a_2\, d(u_2, u_3) + \ldots + a_k\, d(u_k, u_{k+1})$$

for each $(u_1, \ldots, u_{k+1}) \in M^{k+1}$, then sequence $x_{k+n} := T(x_n, x_{n+1}, \cdots, x_{n+k-1})$ $(n \in N)$ is a Cauchy sequence. If the space (M, d) is T-orbitally complete, then there exists one and only one $y \in M$ such that $y = f(y, y, \ldots, y) \in M^k$; one has $y = \lim O(T, x)$ for every $x \in M^k$ (cf. Theorem 1 Prešić 1965).

2.13.2. The wording of the theorem is transferable to any sequencially complete g-metric space.

2.14. Quasicontractions of (M,d).

2.14.0. Definition. A selfmapping $T|M$ is a quasicontraction $\Longleftrightarrow$ there exists a number $c \in R[0,1)$ such that

$$d(Tx,Ty) \leq c\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

$(x,y \in M)$ (cf. Ćirić 1974).

2.14.1. Theorem. Every quasicontraction T of any T-orbitally complete metric space (M,d) has just one fixpoint u in M; one has

(i) $u = \lim T^n x$ $(x \in M)$ and

(ii) $d(T^n x, u) \leq c^n(1-c)^{-1}\, d(x,Tx)$ $(x \in M)$ (Th. 1 Ćirić 1974).

Using the graph theoretical terminology, Theorem 2.14.1. becomes

2.14.2. Theorem. If a selfmapping $T|(M,d)$ satisfies $d(Tx,Ty) \leq c\delta Vg$, where δVg denotes the diameter of the vertex set $\{x,y,Tx,Ty\}$ of the complete graph $g(x,y,Tx,Ty)$, and if (M,d) is T-orbitally complete, then T has a unique fixpoint $u \in M$ with the properties (i), (ii).

It matters to notify the following

2.14.3. Lemma. Any $0 \leq c < 1$, any c-quasicontraction T of (M,d) and any $x \in M, n \in N$ satisfy $d(T^i x, T^j x) \leq c\delta O(T,x,n]$, where $O(T,x,n] := \{T^i x, i = 0,1,\dots,n\}$; thus $\delta O(T,x,n] = d(x,T^m x)$ for some $m \in [1,n]$ (Lemma 1 p. 269, Ćirić 1974).

2.15. Theorem: $\equiv$ "Theorem 1 (Monotone principle of F.P.). Let T be a mapping of metric space (X,ρ) into itself and let X be T-orbitally complete with the condition of AT-type. Suppose that there exists a mapping $\gamma : R^0_+ \twoheadrightarrow R^0_+$ such that (γ) and (T) $A(Tx,Ty) \leq \gamma(A(x,y))$ for any $x,y \in X$, where $A : X \times X \twoheadrightarrow R^0_+, x \twoheadrightarrow A(x,Tx)$ is T-orbitally lower semicontinuous (or A is continuous and $A(x,x) = 0$) and $\rho(x,,y) \leq A(x,y)$ for all $x,y \in X$. Then T has unique fixed point $\xi \in X$ and $T^n x \twoheadrightarrow \xi$ for each $x \in X$" (p. 128, Tasković 1985).

Terminology: Condition of AT-type in a metric space X means: if $x \in X$ and $A(T^n x, T^{n+1}x) \twoheadrightarrow 0$ $(n \twoheadrightarrow \infty)$, then $\{A(T^n x, T^m x)$ is a bounded double sequence, where $A : X \times X \twoheadrightarrow R_+^0, x \twoheadrightarrow A(x, Tx)$ is T-orbitally lower semi-continuous.

For a $\gamma : R_+ \twoheadrightarrow R_+$ the (γ) condition means $(\forall t \in R_+)$ $(\gamma(t) < t)$ and $\lim\sup \gamma(z) < t$ for $z \twoheadrightarrow t + 0$. A $g : X \twoheadrightarrow R$ is said to be T-orbitally semicontinuous at p if (x_n) is a subsequence in the orbit $(x, Tx, T^2x \dots)$ and if $x_n \twoheadrightarrow p$ then $g(p) \leq \lim\ \inf\ g(x_n)$.

The quoted Theorem 1 embraces many known theorems on fixpoints.

For other Taskovič's results, the readers are referred to his bibliography and his book 1986.

2.16. c-contractions; c-expansions in (M, d).

2.16.0. Definition. Let c be a given member of $R_+ := R(0, .)$. If a selfmapping $T|M$ satisfies

(0) $(Tx, Ty) < cd(x, y) \quad (x, y \in M, x \neq y)$,

then T is called a c-contraction of (M, d). In particular, 1-retraction means $d(Tx, Ty) < d(x, y)$ in M^2 for $x \neq y$.

If $d(Tx, Ty) = 0 (x, y \in M), T$ is said a 0-contraction in (M, d). If

(0′) $d(Tx, Ty) > cd(x, y) \quad (x, y \in M, x \neq y), T$ is said a c-expansion in (M, d); in particular, 1-expansion means that $d(Tx, Ty) > d(x, y)$ for $x \neq y$ in M^2. If in (0), (0′) one has respectively $\leq$ instead of $<$ and $\geq$ instead of $>$, T is said to be a weak c-retraction and a weak c-expansion respectively.

The preceding definitions are of a global character; of course, they could be localized as well in the sense that the corresponding relations hold in a certain neighborhood $V(x)$ for every $x \in M$ with a given c.

2.16.1. Theorem. If (M, d) is T-orbitally complete for some c-contraction $T|M$ for some $0 \leq c \leq 1$ and if M contains a point x such that the T-orbit of x contains a convergent subsequence s, then $\lim s := u$ is the unique fixpoint of $T|M$ (v. Th. 1 in Edelstein 1962).

2.16.1.1. **Corollary.** Every 1-contractive selfmapping of every compact metric space has a unique fixpoint.

If in Theorem 1 one assumes that $T|M$ is a local c-contraction, then in the conclusion one has the following: u is a periodic point of $T|M$, i.e., for some $k \in N$ the point u satisfies $T^k u = u$ (Theorem 2 in Edelstein 1962).

2.16.3. Theorem (see Rosenholtz 1976). Let (M, d) be compact and connected; if a selfmapping $T|M$ is a continuous weak c-expansion for some

$c > 1$ and if the T-image TG of every open set G in M in an open set, then T has a fixpoint in M.

The proof uses covering space techniques.

3. Further Fixpoint Theorems in Various Structures

In this section we list some important results on fixpoints concerning various structures (spaces, ordered sets,...).

3.0. *The last Poincaré's geometric theorem*

One could write a drama on what is called Poincaré's last G. Theorem., especially if one bears in mind that Poincaré was one of the greatest mathematicians and that he had presentiment of his proper death soon. In his study of the problem of three bodies Poincaré arrived in 1912 at the following statement.

3.1. *Poincarés last geometric theorem* (Poincaré 1912)

Let R be a ring formed by 2 concentric circles c_a of radius a, c_b of radius b $(a > b > 0)$; let $T|R$ be a one-to-one continuous selftransformation such that it advances the points on c_a, c_b in opposite directions; if T preserves area, then $T|R$ has at least 2 fixpoints.

Poincaré knew that there were ≥ 2 fixpoints, provided that there is at least one; he proved the theorem for various special cases, but had no time to settle the general case. The proof for general case was given by George Birkhoff 1913, very soon after Poincaré's death!

3.2. *Brouwer's fixpoint theorem* (1912 Satz 4)

"A one valued continuous transformation of n-dimensional elements into itself has surely a fixpoint" and on p. 97 one reads: "Under an n-dimensional element S we understand a one-valued continuous image of n-dimensional number space". Of course, all these happen in R^n.

The proof given by Brouwer is not simple. Afterwards, much simpler proofs were found, especially founded on Sperner's lemma. A very interesting proof was discovered by John Milnor 1978 and is backed on the fact that the function $(1+x^2)^{n/2}|R$ is not a polynomial over R, whenever n is odd.

3.2.1. It is extremely interesting that the great french mathematician Poincaré Henri (1956.04.29–1912.07.17) in his fundamental researches on qualitative solutions of differential equations, in the period 1883–1886 and later on used tools which are equivalent to the Brouwer's theorem (cf. Miranda, Carlo 1940; for more details and a relevant bibliography cf.F.E. Browder 1983).

3.2.2. It is worth-while to notice that the Poincaré's last geometric theorem 3.1 is not (is) covered by the Brower's (Schauder's and *a fortiori* by Tychonoff's) F.P.Th. 3.2.

In connection with the Poincaré's last Geometrical Theorem it is interesting to quote the following

3.2.3. Theorem (Rassias 1982). Let D_2 be the open unit ball in R^2. If T is Lebesgue measure preserving and orientation preserving homeomorphism of D_2 onto D_2; then $T|D_2$ has at least one fixpoint. If one deletes the condition of the orientation preserving, then $T^2|D_2$ has some fixpoint (the statement is not true for $n > 2$).

3.2.4. Brouwer's Theorem deals with particular subsets of Euclidean spaces $R^n (n \in N)$; the theorem was studied and generalized replacing simplexes in R^n by more general subsets of more general spaces. Typical results were obtained by Schauder, Tychonoff, Kakutani (s.3.3.1, 3.3.2, 3.4.1).

3.3. In 1930 J. Schauder proved the following two theorems I, II.

3.3.1. Theorem I. Let (M, d) be linear, complete and such that:

1° $d(x, y) = d(x - y, \emptyset)$ ($x, y \in M$: $\emptyset$ is the zero of the space).

2° $\lim d(x_n, x) = 0, \lim d(y_n, y) = 0$ imply $\lim d(x_n + y_n, x + y) = 0$.

3° If λ_n is a sequence of real numbers and x_n a sequence of elements of M then $\lambda_n \twoheadrightarrow \lambda$ and $d(x_n, x) \twoheadrightarrow 0$ imply $d(\lambda_n x_n, \lambda x) \twoheadrightarrow 0$.

Let H be a convex closed and compact subset. Then every continuous selfmap of H has a fixpoint in H.

Here is a translation of a section from the same paper. "For linear normed and complete spaces, considered by Mr Banach in his dissertation — we call them shortly "B"-spaces — the preceding theorem could still be generalized. Namely one need not assume the compacity of the convex closed set H. It suffices to know that the image $F(H)$ is compact. Thus one has

3.3.2. "**Theorem II.** Let H be a convex and closed set in a "B"-space. Let the continuous functional operation $F(x)$ map H into itself. Further let $F(H) \subset H$ be compact. Then there exists a fixpoint".

In Collatz [1968] p. 281 this theorem is labelled as "a far-reaching generalization [of the Brouwer's theorem] which is very suitable to applications".

3.3.3. Theorem (Th. 1, Rassias 1977). Let (M, d) be linear topological space with convex balls; let $A \subset M$ be complete and convex; let $T : A >\gg M$ be continuous. Then there exists at least one $u \in A$ such that

$$d(Tu, u) = d(Tu, A),$$

where $d(x, A) := \inf d(x, y) \quad (y \in A)$, whenever $x \in M$. In particular, if in addition $TA \subset A$, then $Tu = u$.

The Theorem 3.3.3 is a generalization of Schauder's theorem because Rassias gives an example of a space (M, d) which satisfies the conditions of Theorem 3.3.3 but the space is not normed.

3.4. Tychonoff's paper 1935 was reviewed by J. Schauder in Zbl. 12 (1936) 308 where one reads "On generalizing a reviewer's fixpoint theorem the proof of which yields the existence of a fixpoint only when the space is linear, metric and locally convex (J. Schauder, Studia Math. 2, 171–180, Theorem I) the author proved the following"

3.4.1. Tychonoff's Fixpoint Theorem. "For each continuous selfmapping of a convex bicompact set in a linear topological locally convex space there exists at least one fixpoint"...

To be noticed that the quoted text 3.4.1 is an English translation by the exact reproduction of the original Tychonoff's wording [1935].

The proofs of fixpoint theorems of Schauder and Tychonoff were founded on the Brouwer's Fixpoint Theorem.

3.5. *Case of multivalued mappings*

3.5.0. If T is a set function, i.e., if values of T are sets, then every $x \in \text{Dom}\, T$ such that $x \in Tx$ is called a fixpoint of T. The set of all fixpoints of T is denoted also by Fix T.

A very great job on fixpoint problematics of multivalued mappings was done. The following Theorem is well known.

3.5.1. Theorem (Kakutani 1941). Let S be a closed r-dimensional simplex; let to every $x \in S$ be associated a closed convex subset Tx of S; if the mapping $T|S$ is such that $x_n \twoheadrightarrow x_0, y_n \twoheadrightarrow y_0, y_n \in T(x_n)$ imply $y_0 \in Tx_0$, then $x \in Tx$ for at least one $x \in S$.

What a beautiful generalization of Brouwer's theorem (1912) found 29 years after Brouwer's result. If Tx is a singleton whenever $x \in S$, oen gets Brouwer's result.

From Olga Hadžić's numerous results on fixpoints of multivalued mappings let us quote the following

3.5.2. Theorem. "Let (E, τ) be a Hausdorff locally convex space, K be a nonempty closed convex subset of E, T a continuous mapping from K into E, S be a compact mapping from K into the class $B(K)$ of all nonempty closed and convex subsets of K such that for every $y \in S(K)$ there exists one and only one solution $x(y) \in K$ of the equation $x = Tx + y$ and the set $\overline{\{x(y)\}y} \in \overline{S(K)}$ is compact. If the mapping T is affine, then Fix $(T + S) \neq \emptyset$" (cf. Theorem 1. Hadžić 1980).

For some other Hadžić's results cf. her books 1972*, 1984*.

3.6. Theorem (Th. 1, Ćirić 1978). Let X be a topological space and $T : X \twoheadrightarrow X$ a strongly nonperiodic orbitally continuous selfmapping. If for some $x_0 \in X$ the set

(1) cl $O(T, x_0)$ is compact, then (1) contains a T-fixpoint u; in particular, if L is any maximal $\supset$-chain in the system F of all closed subsets Z of (1) such that

(2) $TZ \subset Z$, then $\cap L := L_0$ satisfies

(3) $v \neq L_0 \subset \text{Inv}(X, T)$.

Proof. First, (1) is the initial member in the chain $(L, \supset)$; further, $L_0 \neq v$, because in the opposite case L_0 would be the closed empty set v; thus the system CL:= $\{(1)\backslash Y; Y \in L\}$ would be an open cover of (1); since (1) is compact, CL would contain a finite subcover M of (1), thus $\cup M = (1)$. Now, CL is a chain; therefore, M is a finite $\supset$-chain and its union $\cup M$ would be the initial member I in $(M, \supset)$; hence, $I = (1)$, i.e., $(1)\backslash I = v$, contradicting the fact that for each $D \in L$ thus also for $D = L$ the complement $(1)\backslash D$ is a nonvoid closed subset of (1). So $L_0 \neq v$. It still remains to prove the second relation in (2), i.e., that every $u \in L_0$ is a T-fixpoint. Assume, on the contrary $u \neq Tu$ for some $u \in L_0$. Since T is strongly non-periodic one would have

(4) $u \notin$ Cl $O(T^2u) := E$; now, E is a Z and satisfies (2) when $\subset$

means = and not $\subset\neq$ because otherwise $L \cup \{E\}$ would be a subchain of F, more extensive than the maximal subchain L. Thus $L_0 = E$ and $u \in E$, contrary to (4).

Theorem 3.6 implies various known results like the following ones for metric spaces (M, d).

3.6.1. Corollary (Edelstein 1962). Let T be a contractive selfmapping of (M, d), i.e., $d(Tx, Ty) < d(x, y)$ for each $(x, y) \in M^2$ such that $x \neq y$. If M contains a point x such that the sequence $T^n x$ $(n \in N)$ contains a convergent infinite subsequence $\twoheadrightarrow u \in M$, then $\{u\} = \text{Inv}\,(M, T)$.

3.6.2. Corollary (Ćirić 1971). Let $T : M \twoheadrightarrow M$ be orbitally continuous and M be T-orbitally complete. If T is a contraction type mapping (:= there are functions $q : M^2 \twoheadrightarrow R[0, 1)$ such that sup $q(x, y) = 1$ and $d(T^n x, T^n y) \leq q(x, y)^n d(x, y)$ $(n \in N)$), then T has a unique fixpoint u in M and one has $u = \lim T^n x$, for every $x \in M$.

3.7. *Some continuation theorems for A-proper maps*

3.7.0. The famous Leray-Schauder continuation theorem for compact perturbations of the identity map proved to be very useful in proving the existence of solutions of nonlinear operator and differential equations. Presently, there are many extensions of it to more general classes of maps (condensing, L-compact, A-proper etc.). For L-compact maps we refer to Mawhin 1979 and to a survey paper [Mawhin – Rybakowski 1987]. In what follows, we shall briefly discuss some extensions to A-proper maps.

3.7.1. Let X and Y be Banach spaces, $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of X and Y respectively with $\cup X_n$ dense in X, dim X_n = dim Y_n, and $Q_n : Y \twoheadrightarrow Y_n$ be linear projections with $\|Q_n\| \leq M < \infty$ for all n. Let $D \subset X$.

Definition. A map $H : [0, 1] \times D \twoheadrightarrow Y$ is said to be an A-proper homotopy w.r.t. $\Gamma = \{X_n, Y_n, Q_n\}$ if whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $t_k \in [0, 1]$ with $t_k \twoheadrightarrow t$ are such that $Q_{n_k} H(t_k, x_{n_k}) \twoheadrightarrow f$, then a subsequence $x_{n_{k_i}} \twoheadrightarrow x$ and $H(t, x) = f$. For such homotopies, we have the following general continuation theorem, whose proof is based solely on the Brouwer degree theory.

3.7.1.1. **Theorem** (Milojević 1982, 1983). Let $D \subset X$ be an open and bounded subset, $V \subset X$ be a dense subspace, $f \in Y$ and $H : [0, 1] \times (\overline{D \cap V})$

$\twoheadrightarrow Y$ be an A-proper homotopy w.r.t. Γ such that

(i) $H(t,x) \neq f$ for $x \in \partial D \cap V, \quad t \in [0,1]$

(ii) $H(0,x) \neq tf$ for $x \in \partial D \cap V, \quad t \in [0,1]$

(iii) the Brouwer degree $\deg(Q_n H(0,.), D \cap X_n, 0) \neq 0$ for all large n.

Then the equation $H(1,x) = f$ is feably approximation-solvable (i.e., its solutions are limits of some subsequences of solutions of $Q_n H(1,x) = Q_n f, x \in X_n$).

In applications (cf. 1982, 1983), $H(t,x)$ takes various forms depending on the type of equations considered. For example, when studying semilinear equations with Fredholm maps, we can take $H(t,x) = Ax + F(t,x)$, where $A : D(A) \subset X \twoheadrightarrow Y$ is a Fredholm map of index $i(A) = 0$. Let $X_0 = \ker A$ and $\tilde{X} \subset X$ and $Y_0 \subset Y$ be such that $X = X_0 \oplus \tilde{X}$ and $Y = Y_0 \oplus R(A)$. Let $\overline{D} = \{x_0 + x_1 \in X_0 \oplus \tilde{X} | \|x_0\| \leq r, \|x_1\| \leq R\}$ for some $r, R > 0$ and $Q : Y \twoheadrightarrow Y_0$ a linear projection.

3.7.1.2. **Theorem** (Milojević 1982, 1983). Let $H(t,x) = Ax + F(t,x)$ be an A-proper homotopy on $[0,1] \times (\overline{D} \cap D(A))$ w.r.t. Γ and

(i) $Ax + F(t,x) \neq f$ for $x \in \partial D \cap D(A), t \in [0,1]$

(ii) $F(0,.)(\overline{D}) \subset Y_0$

(iii) $F(0,x) \neq tQf$ for $x \in \partial D \cap X_0, t \in [0,1]$

(iv) $\deg(F(0,.), D \cap X_0, 0) \neq 0$.

Then the equation $Ax + F(I,x) = f$ is f.a. solvable.

The following corollary is useful in applications.

3.7.1.3. **Corollary** (Milojević 1982, 1983). Let $A + tN : \overline{D} \cap D(A) \subset X \twoheadrightarrow Y, t \in [0,1]$, be A-proper w.r.t. Γ with $Q_n Ax = Ax$ on X_n, N be nonlinear and bounded and

(i) $Ax + tNx \neq 0$ for $x \in \partial D \cap D(A), t \in (0,1)$

(ii) $QNx \neq 0$ for $x \in \partial D \cap X_0$

(iii) $\deg(QN, D \cap X_0, 0) \neq 0$.

Then the equation $Ax + Nx = 0$ is f.a. solvable.

3.7.2. *Positive solutions of operator equations*

When $Y = X = V$, K is a cone in X and $H : [0,1] \times \overline{D} \cap K \twoheadrightarrow K$, then a version of Theorem 3.7.1.1, based on the index theory, gives the existence of positive solutions, i.e., $x \in K$, of $H(I,x) = f$. In particular, if $H(t,x) = x - tNx$, we get positive fixed points of N. We refer to [Milojević 1977] for details. However, in Milojević 1986 a method was introduced, based on topological transversality and approximation-essentiality, which gives more general results. For example, one has

3.7.2.1. **Theorem** (Nonlinear alternative) (Milojević 1986). Let $C \subset X$ be convex, D an open subset of G with $0 \in D$ and $N : \overline{D} \rightarrowtail\!\!\!\!\twoheadrightarrow C$ such that $I - tN, 0 \leq t \leq 1$, is A-proper at 0 w.r.t. $\Gamma = \{X_n, P_n\}$ with $P_n C \subset C$. Then

(i) $0 \in (I - N)\,(\overline{D})$ and, if $0 \notin (I - N)(\partial D)$,

the equation $Nx = x$ is f.a. solvable; and/or

(ii) there exists an $x \in \partial D$ such that $x = tNx$ for some $t \in (0,1)$.

Imposing conditions on N which prevent (ii) to hold, one gets various types of fixed point results. For example:

3.7.2.2. **Theorem** (Milojević 1986). Let $C \subset X$ be convex, $0 \in C, N : C \rightarrowtail\!\!\!\!\twoheadrightarrow C$ be such that $N(C \cap D)$ is bounded for each open neighborhood D of 0 and $I - tN, 0 \leq t \leq 1$, be A-proper at 0 w.r.t. $\Gamma = \{X_n, P_n\}$ with $P_n C \subset C$. If $S = \{x \in C | x = tNx \text{ for some } t \in (0,1)\}$, then either S is unbounded, or the equation $Nx = x$ is f.a. solvable.

When N is compact, Theorems 3.7.2.1 and 3.7.2.2 are due to Granas 1976.

3.8. For locally convex normed spaces very much has been done about fixpoint problematics (Leray – Schauder 1934, Krasnosel'ski,...).

3.8.1. Theorem (Hadžić O.- Stanković S., 1970). Let S be a sequentially complete subset of a locally convex vector space E and $|\quad|_\alpha (\alpha \in J)$ a saturated system of seminorms. If for every $(\alpha, k) \in J \times N$ there is a $q_\alpha(k) > 0$ such that $|T^k x - T^k y|_\alpha < q_\alpha(k)|x - y|_{\varphi(\alpha,k)} (x, y \in S), \Sigma q_\alpha(k) < \infty$ $(k \in N)$ and if for every $(\alpha, x, y) \in J \times S \times S$ there is $p_\alpha(x, y) \in [0, \infty)$ such that $|x - y|_{\varphi(\alpha,k)} \leq p_\alpha(x, y), k \geq 1$, then $T|S$ has unique fixpoint u; in addition $u = \lim T^n x$ whenever $x \in S$.

In the same paper, the Theorem 3.8.1 was applied for solving some differential equations in the field of Mikusiński's operators.

3.9. *Fixpoints is probablistic spaces*

3.9.0. In 1942 K. Menger replaced the Fréchet's distance $d(p, q)$ between $p, q \in M$ by a real-valued function $F_{pq} : x \in R \rightarrowtail\!\!\!\!\twoheadrightarrow F_{pq}(x) \in I := R[0,1]$; he interpreted $F_{pq}(x)$ as the probability that the distance between p, q be $\leq x$. If $F_{pq}(\cdot)$ is left continuous and $F_{pq}(-\infty) = 0, F_{pq}(\infty) = 1$, then $F_{pq}(\cdot)$ is called a probability distribution function (pdf). He considered any set

P of pdf's each of which is: 0 at 0, 1 on the diagonal $p = q$ and such that if $p \neq q$, then $F_{pq}(x) < 1$ for some $x > 0$. Menger introduced statistical metrical space as any (S, F) where $F : S \times S \twoheadrightarrow P$ is such that

(0) $F_{pq}(x+y) \geq T(F_{pq}(x), F_{qr}(y))$ whenever $(p, q, r) \in S^3$ and $(x, y) \in R^2$ and where $T : I^2 \twoheadrightarrow I$ satisfies

(1) $T(a, b) = T(b, a)\,(a, b \in I)$

(2) $T(a, b) \leq T(c, d)$ whenever $a, b, c, d \in I$ and $a \leq c, b \leq d$

(3) $T(a, 1) > 0$ for $a > 0$ and $T(1, 1) = 1$.

3.9.2. This was a generalization of metric spaces because if there is a mapping $d : S \times S \twoheadrightarrow R^0_{\neq}$ such that

$$(4) \quad F_{pr}(x) = \begin{cases} 0\,, & x \leq d(p, q) \\ \text{for}\,, & x \in R \\ 1\,, & x > d(p, q) \end{cases}$$

then (S, d) is a metric space. And conversely, if (S, d) is a given metric space and if one defines F by (4), then (S, F) is a statistical metric space for every $T : I^2 \twoheadrightarrow I$ satisfying (1), (2), (3).

3.9.3. Any $T : I^2 \twoheadrightarrow I$ such that (1), (2), (3) and the associative law $T(T(a, b), c) = T(a, T(b, c)) \quad ((a, b, c) \in I^3)$ hold is called a triangular and more specifically the T-norm.

3.9.4. Menger space is any 3-un (M, F, T), where M is a set, F is a mapping of M^2 into P as above and T is a t-norm. One can prove the following (p. 45, Istracescu 1974*):

3.9.5. Theorem. If T is a continuous t-norm, then one has $I^2 = (\cup J_k^2) \cup C(\cup J_k)^2$ ($k \in K$; the index set K is at most countable); the sets J_k ($k \in K$) are disjoint open intervals of I and the restrictions $T_k := T|J_k$ ($k \in K$) are Archimedean semigroups, i.e., $T_k(x, x) < x\,(x \in J_k)$.

3.9.6. Theorem (= Theorem p. 108, Hadžić 1979). Let (M, F, T) be a complete Menger space with a continuous t-norm T such that the system $T_1(x) = T(x), T_{n+1}(x) := T(T_n(x))\,(x \in I, n \in N)$ is equicontinuous at $x = 1$ and for every $k \in K$ and whenever $y < z$ one has $T_k(x, y) < T_k(x, z)$.

Let H be a selfmapping of M such that for every 5-un $r := (r_1, r_2, r_3, r_4, r_5) \in R^5_+$ and every 2-un $(u, v) \in M^2$ one has some 5-un $(a, b, c, d, e) \in R^5_+$ such that $a + b + c + d + e < 1$ and

$$F_{Hu,Hv}(\Sigma_1^5 r_i) > T(T(T(T(F_{u,Hv} t_5/a), F_{v,Hu}(r_4/b)), F_{v,Hv}(r_3/c)), F_{u,Hu}(r_2/d)), F_{u,v}(r_1/e)).$$

Then H has a unique fixpoint.

3.10. *Invariant points of continuous self-similarities of well-ordered sets*

It is worthy to know that historically the first paper concerning invariant points was Veblen's paper 1908 dealing with finite or infinite ordinal numbers or equivalently with well-ordered sets W. Selfmappings $T|W$ which he studied were continuous self-similarities, i.e., such ones that $x < y$ in $(W, <)$ implies $Tx < Ty$ and that for every nonvacuous $S \subset W$ one has $\sup\ TS = T \sup\ S$.

3.10.1. Theorem. Let ω_σ be any regular non-countable ordinal initial number and $W := W\omega_\sigma := \{\mathrm{ON}(n), n < \omega_\sigma\}$. For every continuous self-similarity $s|W$ the set $I := \mathrm{Inv}\,(W, s)$ of invariant points is order-similar to the whole set W.

Proof. First of all, one has $sx \geq x$ for every $x \in W$ because if there were an $x \in W$ such that $sx < x$ one would have $ssx < sx < x$, i.e., $s^2x < sx$ etc. One would get an infinite regression $\ldots < s^{n+1} < s^n x < \ldots < sx < x$ in the well-ordered set W — which is contrary to the definition of well-order.

1. **Lemma.** For every $x \in W$ and every limit ordinal $\lambda < \omega_\sigma$ the point $x' := \sup\ s^\alpha x\,(\alpha < \lambda)$ where $s^{\alpha+1}s := s(s^\alpha x)$ and $s^\alpha x := \sup\ s^\beta x\ (\beta < \alpha$ if α is limit) is a point in $\mathrm{Inv}(W, s)$.

As a matter of fact, $s^\alpha x\,(\alpha < \lambda)$ is a strictly increasing λ-sequence in W; since $\lambda < \omega_\sigma := \text{type } W$, the point $x' = \sup\ s^\alpha x\,(\alpha < \beta)$ is a point in W. Now, $sx' = s(\sup\ s^\alpha x) =$ (by the continuity of s) $\sup\ ss^\alpha x = \sup\ s^{\alpha+1}x(\alpha < \lambda) = \sup\ s^{\alpha+1}\,x\,(\alpha + 1 < \lambda) = x'$. Thus $sx' = x'$.

2. **Lemma.** If $y \in I$, then $(y+1)'$ is the immediate successor of y in I, i.e., $y \in I$ and $(y+1)'$ are consecutive fixed points of s (proof is obvious).

Now, there is a similarity-mapping g of W onto I.

Put $go := \sup\ s^n o\,(n < \omega)$; let $0 < \beta < \omega_\sigma$; assume that $g\alpha(\alpha < \beta)$ is defined as strictly increasing; let us define $g\beta$ as well. If $\beta - 1$ exists, let $g\beta := g(g(\alpha - 1) + 1)$; if β is limit, we define $g\beta := \sup\ g\alpha(\alpha < \beta)$.

By transfinite induction the function g is defined in W. Obviously, g is strictly increasing continuous and maps W onto I. This completes the proof of Theorem 3.10.1.

3.10.2. Theorem. There are exactly $2^{p\omega_\sigma}$ continuous self-similarities of the set $W\omega_\sigma$; in other words, the set B of all continuous self-similarities s of the set $W\omega_\sigma$ is equinumerous to the set of all selfmappings of the set $W\omega_\sigma$.

Proof. In the representation $s = s_0 < s_1 < \ldots < s_\alpha < \ldots (\alpha < \omega_\partial)$ the unique restrictions are $s_0 \in W$ and $s_\alpha \in W \backslash \{s_0, \ldots, s_\beta, \ldots\} (\beta < \alpha)$; thus each member of s is running independently through a set of power χ_σ; therefore pB equals $\chi_\sigma, \ldots, \chi_\sigma$ (the number of factors is χ_σ); thus $\mathrm{p}B = \chi_\sigma^{\chi_\sigma} = 2^{\chi_\sigma} = \mathrm{p}\,P(W)$, what was to be shown.

Consequently, the number pB is the maximal number of all self-mappings of W. It is interesting and surprising that everyone of this immense set of $2^{\mathrm{p}W}$ selfmappings of W has an invariant set which is isomorphic to the whole basic set W.

Is there any another structure S of a similar bizzare property? Yes, because by similar arguments used in the proof of Theorem 3.10.1. one proves the following

3.10.3. Theorem. Let W be an ordered set such that for every $x \in W$ the cone $W(x)$ consisting of all members of W each comparable to x is a well-ordered set of some regular non-countable initial type. Then for every continuous self-similarity $s : W \twoheadrightarrow W$ such that whenever $x \in W$ the points a, sx are comparable, the set $I(W, s)$ of all invariant points of $s|W$ is order isomorphic to the set W itself. If pW is regular and equal to pL for some subchain L of W, then the system B' of all continuous self-similarities of W is equinumerous to the system of all selfmappings of L.

Remark. The continuity of $s : W \twoheadrightarrow W$ is defined by the implication $X \subset W$ and $\sup X \in W \Rightarrow s(\sup X) = \sup(sX)$ where $\sup X \in W$ means that $\sup X$ is an element x of W such that $X \leq x$ and that $X \leq y \in W \Rightarrow x \leq y$. Therefore, for a given $X \subset W$ either $\sup X$ exists as a unique member of W or $\sup X$ does not exist at all. In particular, for the empty set v one convenes that $\sup v$ exists and denotes the first element of W provided W has such an element.

Proof of Theorem 3.10.3. We restrict ourselves to prove the last sentence in the Theorem. Now, by assumption, W is equinumerous to a subchain L; thus L is well-ordered and of regular power $\mathrm{p}W > \chi_\sigma$; since W is degenerate, we can assume that L is a branch in W. Now, in virtue of Theorem 1 the system B of all continuous self-similarities $s|L$ is equinumerous to the system L^L of all selfmappings of L; thus $pB = pL^L = 2^{pL} = 2^{pW} = pPW$, i.e., $pB = pPW$.

Now, every continuous self-similarity $s|L$ is extendable to some continuous self-similarity $s|W$; it is sufficient to consider for every branch $L' \neq L$ of W a continuous self-similarity $s(L')|L'$; then the union of $s|L$ and of $\cup s(L')|L'$ (L' running through the set of all branches of W) is a continuous self-similarity of W (remark that $W[a,.)$ $(a \in R_0W)$ coincides with the system of all branches of W, because supposedly W is degenerate). Therefore, B' of Theorem 3 is of a power $\geq pB = pPW = pW^W$, thus $pB' \geq pW^W$ and *a fortiori* $pB' = pW^W$. This completes the proof of Theorem 3.10.3.

3.11. *Fixpoints of permutations*

Let S be a given set and $S!$ be the set of all permutations of S. This means that every $T \in S!$ is a bijection of S onto S, thus, in particular, $TS = S$.

A special kind of permutations are *transpositions* T in S defined by the property of T-invariance of every point of S except just two-ones. In other words if x, y are 2 distinct members of S then the corresponding transposition is defined by $Tx = y, Ty = x$ and $Tz = z$ $(z \in S\backslash\{x, y\})$.

3.11.1. An interesting kind of permutations are *cyclic* ones. A permutation c of S is quoted to be cyclic if for each $x \in S$ the corresponding c-orbit of x coincides with S.

3.11.2. Lemma. A given nonempty set S admits some cyclic permutation, $c \in S!$ if and only if the power $n := pS \in N$.

The proof is obvious. If $n \in N$, it is sufficient to consider any $x \in S$ and to consider as cx any member in $S\backslash\{x\}$ and inductively if $c^i x$ $(i < k)$ is defined for every ordinal $i < k$ (putting $c^0 x := x$), then $c^k x$ would denote any member of S such that $c^k x \neq c^i x$ $(i < k)$.

If n is finite, the procedure of forming $c^k x$ stops for $k = n$. But n is necessarily finite, because in the opposite case one would have an infinite bijective sequence $c^i x$ $(i = 0, 1, 2, \ldots)$ and thus in particular the c-orbit of cx; this orbit is obviously $\{cx, c^2 x, \ldots\}$ and does not contain x, contrarily to the cyclicity of c that the c-orbit of each member of S coincides with S.

3.11.2.1. **Corollary**. Let S be any nonempty set and $T|S$ be any selfmapping; if r is an ordinal number such that some $x \in S$ satisfies $T^r x = x$ and that the r-sequence $T^i x\,(i < r)$ is bijective, then r is finite and the subfunction $T|O(T,x)$ on the T-orbit of x is a cyclic permutation on the orbit.

3.11.3. We assume that $n := pS$ is not infinite. One knows that $0! := 1$ and $n! = 1 \cdot 2 \cdots n\,(n \in N)$.

3.11.3.1. Let $n!_0$ denote the number of all $T \in S!$ having no fixpoint; and $0! := 1$; one can prove that

(0) $n!_0 = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \ldots + (-1)^n \binom{n}{n} = n!\Sigma_0^n(-1)^i/i!$
(cf. Vituškin, p. 73).

(1) $n!_0/n! = \Sigma(-1)^i/i!\ (i = 0,1,2,\ldots,n)\ (n \in N_0)$.

3.11.3.2. The number $n!_>$ of all $T \in S!$ having at least one fixpoint equals

(2) $n!_> = n! - n!_0$.

One has

(3) $\lim n!_0/n! = \lim \Sigma(-1)^i/i!\ (i \leq n) = e^{-1} = 0{,}367879441$ and

(4) $\lim n!_>/n!_0 = e - 1 = 1{,}718281828\ldots$ for $(n \in N)$.

The equality (4) is a very nice occurrence of e.

3.11.3.3. The equalities (1) show that for even (odd) integers the corresponding subsequence of (1) is decreasing (increasing). The number $n!_>$ of members of $S!$ equipped with some fixpoint is almost 2 times the number $n!_0$ of members of $S!$ having no fixpoint at all.

3.11.3.4. The number $n!_0$ (resp. $n!_>$) is called the subfactorial (cofactorial) of n. Corresponding to the formula

$$n! = (n-1)[(n-1)! + (n-2)!]$$

one has

$$n!_0 = (n-1)((n-1)!_0 + (n-2)!_0).$$

3.11.3.5. It is interesting that $(n-1)!_0/(n-1)! = -\mathrm{res}\,K(-n)\ (n = 2,3,\ldots)$ where the left factorial $!z = K(z) = \int_0^\infty e^{-t}(t^z - 1)/(t-1)dt$ for $\mathrm{Re}z > 0$ and stepwise one extends $K(z)$ in the field $R(i)$ of complex numbers using the difference equation $K(z) = K(z+1) - \Gamma(z+1)$ (cf. Kurepa 1973 (2)). The relation (0) yields

3.11.6. If $n!_k$ denotes the number of members of $S!$ having just k fixpoints one proves without difficulty that

$$n!_k = \binom{n}{k} n!_{n-k}\,,\; n!_k = (n!/k!)\Sigma(-1)^i/i!\; (0 \leq i \leq n-k, k \leq n)\,.$$

For $k = 0$ one gets (0). Of course, $n!_> = \Sigma_{k=1}^{n} n!_k$.

3.11.7. If $\mathrm{p}S = n$ is transfinite, we are not able to establish without choice axiom that $n!_0 > 0$, although one has $\mathrm{p}(S!) > \mathrm{p}S$ for each set S.

3.11.8. For any alef n one has $n! = 2^n = n!_0$ (cf. Kurepa 1953(4), 1954(16)).

Therefore it is natural to put forward the following question.

3.11.9. Problem. What is the position of the following statement F_n?

F_n If n is infinite cardinal, then $n!_0 = n!$ where $n!_0$ denotes the power of the set $S!_0$ of all permutations f of S such that $fx \neq x$ for each $x \in S$; S is any set of power n. Of course AC$\Rightarrow$ F_n. We convene that $S!_> := S!\backslash S!_0$.

We have also the following consequence of AC.

3.11.10. If I is any nonempty set let $f|I$ be any I-un of sets of power > 1 each; then there is an I-un of permutations $p_i \in (fi)_0\,(i \in I)$.

3.12. *Fixpoints of self-mappings of ordered sets*

3.12.0. We denote by (0) $(O, \leq)$ any ordered set and by L any linearly ordered set or chain; extremal cases of (0) are well-ordered sets and antichains, i.e., ordered sets in which there are no distinct comparable points. Sections 3.10, 3.11 dealt exactly with such kinds of ordered sets because every set S could be considered as an antichain or free set.

In this section we shall discuss on Inv $((O, \leq), T)$ for 2 particular important cases that $T|(O, \leq)$ is isotone (= increasing: if $x \leq y$ in (0) then $Tx \leq Ty$) or decreasing: (= if $x \leq y$ in $(O, \leq)$, then $Tx \geq Ty$).

3.12.1. Definitions. $(O, \leq)$ is (condionally) left complete: $\Longleftrightarrow$ For every nonempty subset X (such that $a \leq X$ for some $a \in O$) the infimum of X relative to $(O, \leq)$ exists and is a member of O; it is denoted as inf X rel$(O, \leq)$ or simply inf X and defined by inf $X = y :\Longleftrightarrow$ 1) $y \in O$, 2) $y \leq X$ and 3) if $x \in O$ and $x \leq X$ then $x \leq y$. Dually one defines the (conditional) right completeness of $(O, \leq)$ as the left completeness of the dual $(O, \geq)$ of $(O, \leq)$: sup x in $(O, \leq)$:= inf x in $(O, \geq)$. Completeness of $(O, \leq)$:= left complete and right complete. Conditional [c.] completeness:= c. left and c. right completeness. The empty set is considered to be complete. A lattice is any $(O, \leq)$ such that $x, y \in O$ implies $\inf\{x,y\}, \sup\{x,y\} \in O$.

3.12.2. Lemma. For any selfmapping $f|(O, \leq)$, one has the partition

$O = (O, \leq)_f \cup (O, \leq)^f \cup (O, \leq)(f)$, where

(i) $(O, \leq)_f := \{x : x \in O \text{ and } fx \leq x\}, (O, \leq)^f := \{x : x \in O$ and $x \leq fx\}$;

$(O, \leq)(f) := \{x : x \in O \text{ and } x \| fx\}$.

(ii) One has $(O, \leq)_f \cap (O, \leq)^f = \operatorname{Inv}(O, f) := I$.

(iii) If $f|O$ is increasing, then $fO^f \subset O^f$; if $S := \sup O^f$ exists, then either $fS \leq S$ or $fS \| S$.

(iv) If $f|O$ is decreasing, then (1) $fO_f \subset O^f, fO^f \subset O_f; O^f(O_f)$ is a left (right) piece in $(O, \leq)$; (2) moreover, if $\sup O_f := s$ exists, then $\neg(s < fs)$. If $s \leq fs$, then $s = fs$.

Proof of (iv). Let us prove (2) and that O^f is a left piece. Now, if $x < y \leq fy$, then $fx \geq fy \geq f^2y$, hence $x < fx$ and (2) is true. Analogously, if $fy \leq y < x$, then $f^2y \geq fy \geq fx$, thus (1) is true and O_f is a right piece of $(O, \leq)$.

Further let us assume that $\sup O_f := s$ exists as a member of O. One does not have $s < fs$ because this relation implies $fs \geq ffs$, i.e., fs would be an element of O_f greater than the supremum s of the same set-absurdity. If $s \in O^f$, i.e., if $s \leq fs$ one has $s = fs$ because the relation $s < fs$ was proved to be impossible.

3.12.3. Theorem (= 1 Th., Kurepa 1964(4)). Let $(O, \leq)$ be any ordered set (totally or non-totally ordered) and f any mapping of O into itself.

(i) If 1.1. The set $(O, \leq)$ is left complete,

1.2. f is increasing in $(O, \leq)$,

1.3. the set $O_f := \{x; x \in O \wedge fx \leq x\}$ is not empty.

then the ordered set $I = (\operatorname{Inv}(O, f), \leq)$ is non-empty and is left complete; in particular the point

(1) $\inf O_f := I_m$

is the minimal point of $(I, \leq)$ in the sense that

(2) $I_m \in I[I_m, .) = I$ where $X[I_m, .) = \{x; x \in X \wedge I_m \leq x\}$.

(ii) Dually, if:

1.1.[d] $(O, \leq)$ is right complete,

1.2.[d] f is an increasing function on $(O, \leq)$ to $(O, \leq)$,

1.3.[d] $O^f := \{x; x \in O \wedge fx \geq x\} \neq \emptyset$,

then the set $(I(O, f) \leq)$ is a non-empty right complete ordered set; in particular the point

(1)[d] $\sup O^f := I_M$

is the maximal point of I in the sense that

(2)[d] $I_M \in I(., I_M] := \{x; x \in I \wedge x \leq I_M\} = I$.

If $f : (O, \leq) \twoheadrightarrow (O, \leq)$ is decreasing (antitone, order-reversing), the set $I(O, f)$ is an (empty or non-empty) antichain; if moreover, $(O, \leq)$ is left or right complete, or both, the set I might be of any cardinality; in the particular case when $I \neq \emptyset$, then both

(3) $m = \inf I$, $M = \sup I$

do exist and are elements of O, satisfying

(4) $fO(., M] \subset O[m, .)$,

(5) $fO[m, .) \subset (., M]$;

in particular

(6) $fm \geq M$,

(7) $fM \leq m$.

3.12.3.1. **Corollary**. Let $(O, \leq)$ be any complete non-void lattice.

(i) If f is any increasing mapping of $(O, \leq)$ into itself, then $I((O, \leq), f) = \{x; x \in O \wedge fx = x\}$ is a non-empty complete lattice, in which in particular $\inf I = \inf O_f = \inf\{x; x \in O \wedge fx \leq x\}$, $\sup I = \sup O^f = \sup\{x; x \in O \wedge fx \geq x\}$. (A. Tarski (1955), p. 286, Theorem 1; cf. G. Birkhoff (1948), p. 54, Theorem 8 and Exercise 5; cf. also V. Devide (1964)).

(ii) If f is any decreasing mapping of $(O, \leq)$ into itself, then the set $I(O, f)$ is an antichain (empty or non-empty) with properties as in Theorem 1 (ii). In particular, a decreasing mapping of a chain into itself may have at most one fixpoint.

(iii) If $(O, \leq)$ is any nonvoid complete lattice and $f|(O, \leq)$ is a decreasing self-map, then $I = \mathrm{Inv}((O, \leq), g)$ for $g = f^2$ is a nonempty complete lattice, in which in particular $\inf I (= \inf O_g)$ and $\sup I (= \sup O^g)$ are permuted by f.

3.12.3.2. **Remarks on Theorem 1(i).**

1. In Theorem 1(i) the three conditions 1.1, 1.2, 1.3 are valid. If anyone is dropped the main conclusion that Inv $\neq v$ may fail. This is verified by the following examples:

$(Q, \leq, fx = x + 1)$; here the condition 1.1 is violated;

$(R, \leq, fx = x + 1)$; here the condition 1.3 is violated

$(R[0,3; \leq; f|[0,1] = 2, f|R(1,3] = 2)$; here the condition 1.2 is violated. In all three cases, f has no fixpoint.

2. If incidentally, $(O, \leq)$ has a maximal point 1, then every increasing self-function $f|O$ satisfies $f1 = 1$; since completeness of $(O, \leq) \Longleftrightarrow$ left completeness and Max $(O, \leq) := 1$ exists, the case of 1.3 when $\sup(O, \leq) := 1$ exists and of course belongs to $(O, \leq)_f$ yields precisely the Tarski's Complete Lattice Fixpoint theorem. The generalization of this theorem is just the case when $\sup(O, \leq)$ does not exist, whenever some $x \in O$ verifies $fx \leq x$, thus it is not true that each $x \in O$ verifies $x < fx$ or $x||fx$.

3. It matters to take note that the set $I(O, \leq, f)$ in Theorem 1(i) is non-empty and left [right] complete but that in general case if $\emptyset \neq X \subset$ Inv the point inf X as an element of Inv is not the infimum of X relative to the whole set $(O, \leq)$; in general, one has $\inf_{(I,\leq)} X < \inf_{(O,\leq)} X$. One has the following

4. **Lemma.** If $A \subset O$ and if $(A, \leq), (O, \leq)$ are left (right) complete, then for any nonvoid $X \subset A$ one has inf X (relative to $(O, \leq) \geq$ inf X (relative to A), where in general $\geq$ stands for $>$; and dually for sup X.

4.1. Example. Let $A := \{\omega_1\alpha : \alpha < \omega_\omega, cf\alpha = \omega_1\}$ and $X = \{\omega_1\omega_1 n : n < \omega_0)$; then sup $X(\text{mod}\, A) = \omega_1\omega_1\omega_1 >$ sup $X(\text{mod ON}\,(< \omega_0) = w_1 w_1 w_0$; where ON$(< \beta)$:= the initial segment of the ordered class ON of all ordinal numbers $< \beta$ ordered by ordinal magnitude $\leq$, i.e., ON(β) <ON(γ) means that a well ordered set of type β is order similar to a proper initial segment of a well ordered set of order type γ.

3.12.4. Theorem (= 2 Th., Kurepa 1975(2)). Let $(O, \leq)$ be a non-empty right conditionally complete ordered set and f a decreasing selfmapping of $(O, \leq)$ such that for at least one member $x \in O$ we have

(2.1) $x \leq fx \vee x \geq fx$, i.e., $\neg\,(\forall x \in O\,,\, x||fx)$.

Let us assume that;

(2.2) $f \sup = \inf\ f, f \inf = \sup\ f$

(2.3) Each point of O_f is comparable with each point of O^f

(2.4) If $S := \sup\ O^f \in O$ exists then $S \geq fS$;

Then

(2.5) $O^f \leq O_f$ (i.e., if $fx \geq x \in O$ and $fy \leq y \in O$, then $x \leq y$);

(2.6) the points $S := \sup\ O^f, i := \inf\ O_f$ exist and satisfy

(2.7) $fS = S = \inf\ O_f$.

(2.8) $S := \sup O^f = \inf O_f =: i \in O.$

3.12.4.1. **Remark.** A typing error in Kurepa 1975(2): in Theorem 2 the second condition $f \inf = \sup f$ in (2.2) is missed; the same condition under the code 2.2 was explicitly used in the paper in 2.11 lines 2,3; 2.15.2 line 2; p. 116, line 9.

3.12.4.2. In the same paper the following conditions $(\leq), (\geq)$ were considered:

$(\leq)$ If a set $A \subset (O, \leq)$ satisfies $fa \leq a (a \in A)$ and if $\sup A$ (resp. $\inf A$) exists, then also $f(\sup A) \leq \sup A$ (resp. $f \inf A \leq \inf A$).

$(\geq)$ = the dual of $(\leq)$.

In no. 3 "A way to get some solution of $fa^2 = a$" (of course fa^2 should be f^2a) the following was proved.

Let $(O, \leq)$ be σ-complete and $f|(O, \leq)$ be a decreasing selfmapping such that $f \sup = \inf f$ and $f \inf = \sup f$. If $(O, \leq)^f$ and $(O, \leq)_f$ satisfy $(\leq), (\geq)$, then for every $a \in O^f$ the element $s \in fO$ defined by $s = s(a) := \sup\{f^{(0)}a = a, f^2a, f^4a, \ldots\}$ exists, the sequence $f^{(2k)}s (k = 0, 1, \ldots)$ is a decreasing sequence of members in O^f such that $I := \inf f^{(2k)}s \in O^f$ $(k = 0, 1, \ldots)$; the sequence $f^{(2k+1)}s (k = 0, 1, \ldots)$ is increasing in O_f such that $S := \sup f^{(2k+1)}s \in O_f (k \in N)$; one has $fI = S, fS = I$ and $\{I, S\} \subset \mathrm{Inv}((O, \leq), f^2)$. Thus, $\{I, S\}$ is a fixed edge for f.

3.12.5. Theorem (= 2 Th., Kurepa 1988(2)) For any non-empty ordered set $(O, \leq)$ and any decreasing selfmapping d in $(O, \leq)$, such that

(PS) $$dO^d = O_d, \quad dO_d = O^d \quad \text{and}$$

(2.1) $$O^d \neq v \ (= \text{vacuous set})$$

the following four statements are pairwise equivalent:

(F) d has a unique fixed point in O, i.e., the equality $dx = x$ has a unique solution in O: the set $I((O, \leq), d)$ is a singleton;

(S = i) $S := \sup O^d$ and $i := \inf O_d$ exist in $(O, \leq)$ and are equal;

(S) $S := \sup O^d$ exists in $(O, \leq)$ and satisfies $S \geq dS$; thus $S \in O_d$;

(i) $i := \inf O_d$ exists in $(O, \leq)$ and satisfies $i \leq di$, thus $i \in O^d$.

3.12.6. Theorem (= Fixed Edge Theorem. = Th. 1, Klimeš 1981). For any non-empty complete lattice L and decreasing selfmapping $f|L$ there exists a fixed edge $\{x, y\} \subsetneq L$, i.e., $fx = y, fy = x$. In particular, the edge $\{u, v\}$

where $u := \inf L_g, v := \sup L^g$ for $g = f^2$ is fixed; u is the least element in L such that (u, fu) is invariant.

3.12.7. Theorem (= Th. 5, Klimeš 1981). For every complete lattice $(L, \leq)$ and every non-empty commuting family F of decreasing self-mappings of L, the set $(E(F), \leq')$, is a complete atomic lattice of power > 1; $E(F) := \{\emptyset\}+$ Inv (L, F); for members $(a, b), (c, d)$ of the non-empty set Inv:= Inv(L, F) of all common fixed (invariant) edges for all members of F one introduces the ordering $(a, b) \leq' (c, d) \iff c \leq a$ and $b \leq d$; $\emptyset$ is a thing not belonging to Inv; one defines $\emptyset \leq'$ Inv.

3.12.8. Theorem (= Th. 8, Klimeš 1981). Let L be any non-empty complete lattice and F be any non-empty family of commuting set-valued decreasing mappings from L to $P'(L) := \{X : \emptyset \neq X \subset L\}$ such that $\sup fx \in fx (x \in L)$ for every $f \in F$; then there exists a common invariant edge for all members f of F ($f|L$ is decreasing means: $x \leq y$ in L implies $fx \geq fy$, i.e., $a \geq b\,(a \in fx,\ b \in fy)$; (x, y) is an invariant edge for f means $x \in fy$ and $y \in fx$).

3.12.9. Theorem (cf. Th 3 in Dacić 1983 and Klimeš, l.c. Ths 7 and 8). Let $(L, \leq)$ be a non-empty complete lattice and $d : (L, \leq) \twoheadrightarrow P'L$ be such that for every $x \leq y$ in L and each $v \in dy$ some $u \in dx$ verifies $v \leq u$. If $\sup dx \in dx (x \in L)$, then there exists a d-fixed edge (a, b) in the sense that $a \leq b$ in $(L, \leq)$ and $a \in db, b \in da$.

3.13. *Retracts*

3.13.0. The set Inv of invariant (fixed) points could be given in advance. In this connection one has an important notion of retract R of an entity E with respect to a mapping $T : E \twoheadrightarrow R$. If $R \subset E$ and if $T : E \twoheadrightarrow R$ is such that $T|R = 1_R$ (= identity selfmapping of R), then R is called the T-retract of E. If E is a space, then one assumes that T be continuous; if E is ordered, then one assumes that T be increasing.

Several properties of a space (like connexion, compacticity, paracompacticity, fixpoint property,...) are preserved in retracts. Every closed set F in a space E is a retract of E (Borsuk). No sphere S^n is a retract of a ball K_{n+1} of dimension $n+1$ because K_{n+1} has the fixpoint property (Brouwer 1912) and S^n does not have this property. Here is a nice

3.13.1. Theorem (G. Birkhoff 1937). Let (0) $(O, \leq)$ be any ordered set. Every complete sublattice of (0) is a retract of (0).

3.13.2. Theorem. If (0) $(O \leq)$ is finite, connected and containing no crown, then a subset X of (0) is a retract of (0) if and only if there is an increasing self-mapping T of (0) such that $X = \operatorname{Inv}(T, M)$ (Duffus – Rival 1979).

Definition. A subset K of (0) is a crown if K is isomorph to $\uparrow_1^2 \uparrow_3^4$ (thus $pK = 4$) and there is no $x \in E$ such that $1, 3 \leq x \leq 2, 4$ or if pE is even > 5 then K is the union of two equinumerous disjoint antichains A, B such that every member of A has exactly two successors in B and every member of B has in A exactly two predecessors.

Polish mathematician K. Borsuk worked very much on retracts (cf. Borsuk 1967). Yugoslav mathematician Živanović Žarko [s. 1973] extended the notion of retract introducing generalized retracts: A is a generalized retract of a space containing A, if for every neighborhood $V(A)$ of A there is a continuous mapping f of the space into $V(A)$ such that $f|A = 1_A$. The notion is more general than the notion of retract, but many statements concerning retracts are holding for generalized retracts.

4. Fixpoint Equivalents of Some Mathematical Statements

In this section we shall list several fundamental notions and statements each expressible equivalently in terms of fixpoints. Our considerations are in frame of the ZF-Set Theory.

Example. In a topological space (S, closure) the closed sets are defined as fixpoints of $\operatorname{Cl}X$ ($X \in PS$), i.e., as solution of $\operatorname{Cl}X = X$ in PS. In particular we list the following statements.

4.0.0. AC (Axiom of Choice) (cf. Theorems 4.3.5, 4.4.3).

AC could be formulated in the following form: For every nonvoid system D of nonvoid disjoint sets there is a self-map $f|D$ such that $fx \in x$ ($x \in D$).

4.0.1. LO (Linear Orderability) of every set (s. 4.4.8.1 Th.)

4.0.2. χH (Alef Hypothesis) $2^{\chi_\alpha} = \chi_{\alpha+1}$ for every ordinal number α.

4.0.3. GCH (General Continuum Hypothesis). For any infinite cardinal numbers x, y, if $x \leq y \leq 2^x$, then either $x = y$ or $y = 2^x$.

4.0.4. TA (Tree Alternative). The power pT of every infinite tree T satisfies pT = length T := sup{pL : $L \subset T, L$ is a chain} or pT = width T := sup{pA : $A \subset T$, A is antichain}.

4.0.5. KA. Every ordered set contains a maximal antichain (cf. Kurepa 1952 (11), 1953 (1); pp. 61–67, Felgner 1971.).

4.0.6. MKG (Maximal Complete Graph): Every graph (G, R) contains a clique (:= maximal complete subgraph) K, i.e., such that $K \times K \subset R$ and that the conjuction $K \subset X \subset G$ and $X^2 \subset R$ implies $K = X$.

4.0.7. Remark that KA is a particular case of MKG when for any ordered set $(O, \leq)$ one considers the graph

(1) $(O; O'' \cup \text{diag}\,(O \times O))$, where $O'' := \{(x, y) \in O \times O$ and $x||y\}$; diag $(O \times O) := \{(x, x) : x \in O\}$.Then every maximal antichain A in $(O, \leq)$ satisfies $A = K \backslash \text{diag}$, $A \cup \text{diag} = K$, where K is any clique in (1).

4.1.0. Theorem (Fundamental property of ordinal numbers)

ON $(\alpha) \Rightarrow$ Ord W$(\alpha) = \alpha$, i.e., every ordinal number α is a fixpoint of the selfmapping Ord $W\,(\alpha)|$ ON; where ON denotes the class of all ordinal numbers and W $(\alpha) := \{\text{ON}\,(\beta), \beta < \alpha\} := \text{ON}\,[0, \alpha)$.

4.1.1. For cardinal numbers the situation is different. If KARD (n), i.e., if n is a cardinal, let KARD $[0, n) := \{x : \text{KARD}\,(x)$ and $x < 1\}$, then the class of all fixpoints of the selfmapping

(0) pKARD $[0, n)|$KARD is the class KARD $[0, \chi_0] \cup$ WIK, where WIK denotes the class of all weakly inaccessible cardinal numbers.

4.2. ***Invariant points of*** $Tx = 1 + x$

Obviously, in the field $R(i)$ of complex numbers one has Inv $(R(i), 1 + x) = \emptyset$. One has a different situation in classes KARD, ON.

4.2.0. Theorem. A cardinal or ordinal number n is infinite if and only if

(0) $1 + n = n$, i.e., Inv (Kard, $1 + x$) = Kard$_\infty$, Inv (ON, $1 + x$) = ON$_\infty$. It is sufficient to prove the implication $\Leftarrow$: if (0), then n is infinite. Now, let n be a cardinal number satisfying (0); let S be a set of power n and e be an object such that $e \notin S$; then $Z := \{e\} \cup S$ is a set of power $1 + n$ equinumerous, by (0), to the proper subset S. Let b be a bijection of Z into S; then, in particular, $be \subset S$; therefore, $b^k e \in S$ for every $k \in N$. We claim that for distinct $i, j \in N$ one has $b^i e \neq b^j e$. As a matter of fact, if $b^i e = b^j e$ and $i \leq j$, then acting by bijection $b^{-i} := (b^{-1})^i$ one gets

$b^{-i}(b^i e) = b^{-i}(b^j e)$, i.e., $b^{i-i}e = b^{j-i}e$ thus $e = b^{j-1}e$ and $j - i = 0$ because $b^k \in S$ for every $k \in N$ and $e \notin S$.

So we have established that S contains the infinite orbit $O(b, e)$; therefore the power of S is infinite.

The proof that every ordinal n satisfying (0) is infinite and in particular $\omega \leq n$ is simpler than the above proof for cardinals.

4.3. *Invariant points of squaring*

4.3.0. Of course, $\text{Inv}(R(i), x^2) = \{0, 1\}$. In general, for a ring $R(+, .)$ the "idempotents", i.e., solutions of $x^2 = x \in R$ play an important role. We are interested to determine all idempotents in ON and KARD respectively. For ordinal numbers the solution is simple: $\text{Inv}(\text{ON}, x^2) = \{0, 1\}$.

4.3.1. As to cardinals the equation $m^2 = m$ is satisfied not only by 0, 1 but also by pR (v. Th. A, Cantor 1878) and for every alef (LXVII p. 896 resp. [108, Hessenberg 1906 where we read (we translate it into English): "LXVII. If $\chi_\alpha \geq \chi_\beta$, then $\chi_\alpha + \chi_\beta = \chi_\alpha \chi_\beta = \chi_\alpha$. In particular, $n\chi_\alpha = \chi_\alpha^n = \chi_\alpha$ for every finite n"). The class Inv (KARD, x^2) is closely connected with AC (Choice axiom) (cf. 4.2.5).

Let us remark that in the quoted paper of Cantor 1878, end of §8, it occured for the first time the famous continuum hypothesis that pR is the immediate successor of pN.

4.3.2. Mapping $m \in \text{KARD} \twoheadrightarrow \chi(m) \in \text{Alefs}$. One knows (s.p. 229_{12-9}, Sierpiński 1928) that without the use of the axiom of choice one can prove that to every infinite cardinal number m there corresponds a least aleph $\chi(m)$ such that

(0) neither $m < \chi(m)$ nor $m > \chi(m)$.

Let us prove the following

4.3.3. Lemma. Let m be a fixed transfinite cardinal number; if m and $m + \chi(m)$ are invariant for squaring:

(1) if $m^2 = m$ and

(2) $(m + \chi(m))^2 = m + \chi(m)$,

then m is an alef.

Proof. Since for any cardinals x, y one has $(x + y)^2 = x^2 + 2xy + y^2$ this formula for $x = m, y = \chi(m)$, by (1), (2), becomes

$$m + \chi(m) = m + 2m\chi(m) + \chi(m), \text{ and therefore}$$

(3) $m\chi(m) \leq 2m\chi(m) \leq m + 2m\chi(m) + \chi(m) = m + \chi(m)$.

Since for any cardinals x, y one has $x + y \leq xy$, (3) implies

(4) $m\chi(m) = m + \chi(m)$.

Now, according to Tarski (s.L.1, Tarski 1924), if a cardinal c and an alef χ satisfy $c\chi = c + \chi$, then c, χ are comparable. Therefore (4) implies $m \leq \chi(m)$ or $\chi(m) \leq m$; thus by (0) one has $m = \chi(m)$, i.e., m is an alef.

4.3.4. Problem. Determine $\{m : \text{KARD}\,(m) \text{ and } m^2 = m\} =?$

Now, 4.3.3 Lemma implies the following

4.3.5. Theorem (= Th. II, Tarski 1924). If every infinite cardinal m satisfies $m^2 = m$, then the choice axiom is true.

Since every alef is invariant by squaring (Hessenberg 1906), the theorem 4.3.5 implies

4.3.6. Theorem. The axiom of choice AC is true if and only if each infinite cardinal m satisfies $m^2 = m$; in other words AC $\iff$ Inv (Kard$_\infty$, squaring) = Kard$_\infty$ (Hessenberg 1908 for $\Rightarrow$; Tarski 1924 for $\Leftarrow$).

There are many equivalents of AC; especially one has

4.3.7. Theorem. AC is equivalent to the following

Maximal Chain Principle: Every $(O, \leq)$ contains a branch (:= maximal linearly ordered subset), i.e., AC $\iff$ MCP (Hausdorff 1914 p. 140 for $\Rightarrow$, Birkhoff, Garrett 1948 pp. 42–43 for $\Leftarrow$).

4.4. *Branches in $(O, \leq)$. Cliques in graphs (G, R)*

4.4.0. It is very important to know some branch or the class $L_M(O, \leq)$ of all branches of a given ordered set $(O, \leq)$ and the class $L_M(G, R)$ of all cliques (= maximal complete subgraphs) of a given graph (G, R) (Reminder: A complete subgraph (clique) in $(G; R)$ is defined as any (maximal) solution X of $X^2 \subset R$). The notion of cone $(O, \leq)$ (a) $= \{x : x \in O \text{ and } x \text{ is comparable to } a\}$ relative to a given object a plays an important role; a might not belong to O.

Analogously, one defines the a-cocone of $(O, \leq)$ as the complement $O \backslash (O, \leq)(a) := \{x : x \in O \text{ and } a \| x\}$. Similarly, for graphs (G, R) and any object a one defines the a-cone as $(G, R)(a) := \{x : x \in G \text{ and } (a, x) \in R\}$, and the a-cocone as the complement $C(G, R)(a) := G \backslash (G, R)(a)$. Thus a is not in the a-cocone.

4.4.1. Theorem. Clique as a fixpoint (cf. 2:3 L., Kurepa 1976(3)). If a non-empty subset X of a graph (G, R) is a clique, i.e., if

(0) $X \in L_m(G, R)$, then

(1) $F_{ik}X = X$ where

(2) $F_{ik}X := \cap(G, R)(x), (x \in X)$; and vice versa. In other words, if (0), then X is a fixpoint of the selfmapping

(3) $F_{ik}|P'G; P'G := \{y : \emptyset \neq y \subset G)$; and conversely.

Proof. $\Rightarrow$: Claim: if $X \in L_M$, then (1), thus $(1)_1 \subset (1)_2$ and $(1)_2 \subset (1)_1$ under the condition (2). Now, if $z \in (1)_1$ then, by (2), $z \in (G, R)(x)\,(x \in X)$, thus $(z, x) \in R(x \in X)$; this means that $\{z\} \cup X$ is a complete graph containing the clique X; therefore, the clique maximality condition implies $z \in X = (1)_2$. Dually, if $y \in (1)_2 = X, X$ being a complete subgraph, one has $X^2 \subset R$, thus $(x, y) \in (G, R)(x)\,(x \in X)$ and consequently $y \in (2)_2 := TX = (1)_1$.

$\Leftarrow$: Claim: if (1) and (2) then (0), i.e., X is a clique. At first, (1) and (2) imply that X is complete because if $x, y \in X$ then by (1) $x, y \in F_{ik}X$ and by (2) $(y, x) \in R$; thus $X^2 \subset R$. It remains to prove that X is maximal. Now, if $e \in G$ and if $(e, x) \in R\,(x \in X)$, then $e \in (G, R)\,(x)\,(x \in X)$ and consequently $e \in F_{ik}X =$ (by (1)) X, thus X is maximal and complete. This finishes the proof of 4.4.1 Theorem.

4.4.2. Theorem (Branches in ordered sets as invariants points). A non-empty subset X of an ordered set $(O, \leq)$ is a branch if and only if $F_{ik}X = X$, where $F_{ik}X := \cap(O, \leq)(x)\,(x \in X \subset O)$.

Theorem 4.3.7 and Theorem 4.4.2 yield the following

4.4.3. Theorem. The choice axiom AC is equivalent to the statement that for every non-empty ordered set $(O, \leq)$ the selfmapping $F_{ik}|P'(O)$, defined by $F_{ik}X := \cap(O, \leq)(a)\,(a \in X \in P'(O) := \{Y : \emptyset \neq Y \subset O\}$ has a fixpoint.

4.4.4. Complemented graph of (G, R). Let us apply Theorem 4.4.2 to the "Complemented graph"

$(G, R)^c := (G, G \times G \backslash R \cup \mathrm{diag}\,(G \times G))$ of (G, R); let us observe that cones and cocones in $(G, R), (G, R)^c$ are related; one proves easily that $(G, R)^c(a) = \{a\} \cup C(G, R)\,(a)$ for every a (as to symbolics cf. 4.4.0). Therefore the clique version of Theorem 4.4.1 for $(G, R)^c$ yields the following anticlique version in (G, R).

4.4.5. Theorem (Anticlique as a fixed element). A non-empty subset X of a graph (G,R) is an anticlique ($\equiv$ maximal antichain), if and only if $F_{ia}X = X$ where

$$F_{ia}X := \cap(\|(G,R)(x) \cup \{x\}) \quad (x \in X).$$

If for an ordered set $(O,\leq)$ we apply Theorem 4.4.5 to the graph $(O, \|+$ diag$)$ where $\| := \{(x,y) : x,y \in O$ and neither $x \leq y$ nor $x > y\}$, one gets the following

4.4.6. Theorem (Antibranch as a fixpoint). A non-empty subset $X \subset (O,\leq)$ is an antibranch in $(O,\leq)$ if and only if X is a fixpoint for the mapping $F_{ia}|P'O$ defined by

$$F_{ia}X = \cap(\{x\} \cup C(O,\leq)(x)) \ \ (x \in X \in P'O).$$

4.4.7. Theorem (KA as a fixpoint statement). The statement KA ($\equiv$ every $(O,\leq)$ contains an antibranch) is equivalent to the statement that for every $(O,\leq) \neq \emptyset$ the selfmapping $F_{ia}|P'O$ has a fixpoint.

4.4.8. A specification of Theorem 4.4.2. Let us specify Theorem 4.4.2 for power sets (0) $(PS,\supset)$ (S is any set). Then one gets branches B in (0). Now,

4.4.8.0. **Lemma.** Each branch B in $(PS,\supset)$ allows a total order of S; in particular if for $x,y \in S$ one considers that $x <_B y$ means the existence of an $x \in B$ such that $x \notin X$ and $y \in X$, then $(S,<_B)$ is a total order and that every $X \in B$ is a right part of $(S,<_B)$ (cf. Kuratowski 1921; also Kurepa 1935 (2,3*) pp. 33–43).

The propositions 4.4.2, 4.4.8.0 imply the following

4.4.8.1. **Theorem** ($\equiv$ 2 : 1 th. in Kurepa 1976(3)). Statement

LO(S) Set S is orderable totally

is equivalent to the statement:

$(F_{ik})S$ The selfmapping $F_{ik}|P'P'S$ defined by $X \in P'P'S \twoheadrightarrow F_{ik}X := \cap(P'P'S,\supset)(a)$ $(a \in X)$ has at least one fixed point. In other words, the statement

LO Every set is totally orderable

is equivalent to the statement

F_{ik}. For every non-empty set S the mapping $F_{ik}|P'P'S$ has a fixpoint.

4.4.8.2. Let us note that AC implies F_{ik}; the converse does not hold.

4.4.8.3. **Theorem.** LO and KA $\Longleftrightarrow$ AC (Kurepa 1953(1) Th. 3.1).

4.4.8.4. **Theorem.** In full ZF-Set Theory, KA $\Longleftrightarrow$ AC (Felgner 1969). Foundation axiom is used.

4.4.8.5. In ZF° (:= ZF\Foundation axiom) AC is independent of KA (Halpern's Doctoral Thesis, Berkeley 1962; see pp. 62–66, Felgner 1971). The facts 4.4.8.3–5 are interesting in particular when one knows the following

4.4.8.6. **Theorem.** Every graph (G, R) contains a clique $\Longleftrightarrow$ AC.

It is remarkable that many special forms of $R \subset G^2$ are sufficient to imply AC. So in virtue of Theorem 4.3.7 it is sufficient that $R = \leq \cup \geq$ (comparability relation K as the union of any order relation $\leq$ and its dual $\geq$); according to Vaught 1952 it suffices to consider that $R = D$ (disjunction relation where XDY stands for $X \cap Y = \emptyset$); this is a special case of the following

4.4.8.7. **Theorem.** Let D, K, J respectively denote:

the disjunction relation $XDY \Longleftrightarrow X \cap Y = \emptyset$,

the comparability relation for sets, i.e., $XKY \Longleftrightarrow X \supset Y$ or $X \supset Y$ and

the overlapping relation $XJY := X \backslash Y \neq \emptyset \neq Y \backslash X$;

let $R \in \{$ D, nonD, J, nonJ, K, nonK$\}$; R$\neq$nonJ be fixed; if for every non-empty family G of sets the graph (G, R) contains a clique, then AC is holding (cf. Th. 3.1 in Kurepa 1952 (11)). The case R =K is Theorem 4.3.7; case R =D is due to Vaught 1952. The Theorem is not valid for R =nonJ (J. D. Halpern, Ph. D. Thesis 1961; p. 23, Rubin and J. Rubin 1963).

4.4.9. Alef Hypothesis (χH)

4.4.9.1. **Theorem** (cf. Kurepa 1972(1)) The following two statements are equivalent:

(0) χ **H**: $2^{\kappa_\alpha} = \chi_{\alpha+1}$ whenever ON (α),

(1) Alef Left Factorial Hypothesis (χLFH) $\;!\chi_\alpha = \chi_\alpha$ for every ON (α).

Proof. Put $n := \chi_\alpha, n^{+1} := \chi_{\alpha+1}$. Let us prove that (0)$\Rightarrow$(1). Now, $(1)_1 = !n =$ (Def. 6.1 in K. 1964(3))$= \Sigma_{0 \leq k < n} k! = \Sigma_0^\infty k! + \Sigma_{p\omega_0 \leq k < n} k! =$

(the first sum is χ_0; for each alef χ one has $k! = 2^\kappa$ (cf. Th. 2.2 in K 1954(4), 1954(16); 3 proofs are given) $= \chi_0 + \Sigma_{p\omega_0 \leq k<n} 2^k =$ (summand χ_0 is absorbed; apply (0))$= \Sigma_{p\omega_0 \leq k<n} k^+ = \sup_{k<n} k^+ = n = (1)_2$. We applied the implication (0)$\Rightarrow \sup_{k<n} k^+ = n$ for any alef $> \chi_0$; the implication is obvious: if $n^- < n$, the supremum is $(n^-)^+ = n$; and if $n^- = n$, then $k < k^+ < n$ and $\sup k = \sup k^+ = n$.
Proof of (1)$\Rightarrow$(0). Now, $(0)_2 = n^+ =$ (by (1) one has $!n^+ = n^+$) $=!n^+ = n! = 2^n = (0)_1$.

Theorem 4.4.9.1 should be compared to the following.

4.4.9.2. **Theorem.** The Alef hypothesis is equivalent to the equality $2^{<n} = n$ whenever n is an alef (Lemma 9; p. 194, Tarski 1930; at p. 188 one reads $a^{<b} = \Sigma_{r<b} a^r$ (Def 4) specifying at p. 188_8 that $1 \leq r < b$; thus $0 = r$ was excluded).

4.4.10. GCH as a fixpoint statement.

4.4.10.0. **Definition.** For every 2-un (a, b) of cardinal numbers let

(0) $a^{<b} := \Sigma a^r$ $(0 \leq r < b)$, i.e., r is running through the vertexless left cone Kard $(., b)$ of all cardinals $< b$ (cf. Tarski 1930 p. 188; he excluded $r = 0$).

4.4.10.1. **Theorem.** General continuum hypothesis (GCH). If $x, y \in$ KARP$_\infty$, and $x \leq y \leq 2^x$, then either $x = y$ or $y = 2^x$.

This is equivalent to the following

TAS (1) $2^{<n} = n$, whenever n is an infinite cardinal number.

Under AC, the Theorem 4.4.10.1 reduces to the Theorem 4.4.9.2. Therefore since GCH$\Rightarrow$AC, the conclusion GCH$\Rightarrow$TAS in Theorem 4.4.10.1 is true.

Proof of TAS$\Rightarrow$GCH. The proof is based on the following very interesting fact.

4.4.10.2. **Theorem** ($\equiv$Th. 1, Tarski 1954) "Statement S_1 is provable without the help of the axiom of choice".

"S_1. For every cardinal m there is a cardinal n such that

(i) $m < n$, and

(ii) the formula $m < p < n$ does not hold for any cardinal p".

In other words, each cardinal m is endowed with at least one immediate succesor – let us denote it by m^+; thus the class Succ m of all such solutions n is non-empty, in particular $m + h(m) \in$ Succ m, where $h(m)$ is the least alef which is not $\leq m$; Hartog's number hm is defined by $hm := p\{\alpha : \alpha$ is the order type of some well-orderable subset of a set M of power $m\}$. Now, let us apply TAS just for the situation in S_1; thus (1) is true. Since $m < n$, the term 2^m is a summand in the expression by which $(1)_1$ is defined by (0). Therefore $2^m \leq (1)_1 = (1)_2 = n$, thus $2^m \leq n$ and (in virtue of the Cantor's inequality $m < 2^m$) one has $m < 2^m \leq n$. Consequently, $2^m = n$ or $2^m < n$. The last inequality is excluded by the wordings of S_1. Thus $n = 2^m$ and the wordings of S_1 become the wordings of GCH for the number $x = m$. This finishes the proof of Theorem 4.4.10.1.

4.4.11. General Left Factorial Hypothesis (GLFH)

4.4.11.0. For any cardinal number n the left factorial of n is defined by $!n :\equiv \mathrm{K}n : \Sigma_{0 \leq m < n} m!$. The left factorial alef hypothesis (every alef χ satisfies $!\chi = \chi$) is equivalent to the general alef hypothesis: each alef χ satisfies $2^\chi = \chi^+$ (v. Th. 4.4.9.2). Here are the corresponding wordings in KARD.

4.4.11.1. GLFH: $!n = \mathrm{K}n = n$ whenever $\mathrm{KARD}_\infty(n)$.

4.4.11.2. SRFH (Successor Right Factorial Hypothesis): $n!$ covers n for every infinite cardinal (in the following sense).

4.4.11.3. **Definition.** Given $(O, \leq)$ and $x, y \in O; (O, \leq (x, y) := (O, \leq)$ $(y, x) := \{z; z \in O$ and $x < z < y$ or $x > z > y\}$. If $x < y$ and if $(O, \leq)(x, y) = v$, one says that y covers x or that y (resp. x) is a right (left) neighbor of x (resp. y) or that x is covered by y.

4.4.11.4. Right Factorial Hypothesis (RFH) $n! = 2^n$ for any $\mathrm{KARD}_\infty(n)$ (cf. Kurepa 1953(1) Problem 6.1; 1953(4); 1954(16) end of no. 6; 1972(1) no. 1,2).

4.4.11.5. Succesor Right Factorial Hypothesis (SRFH). If $\mathrm{KARD}_\infty n$, then $n!$ covers n.

4.4.11.6. **Theorem.** GLFH $\Rightarrow$ SRFH, i.e., whenever n is infinite cardinal, then (0) $!n = n$ implies (1) $n!$ covers n.

Proof. Let us apply the equality (0) for the number n occuring in the above Tarski's statement S_1; thus (0) holds. Now, in the expression of $!n$ occurs also the term $m!$ because $m < n$. Thus $m! \leq \mathrm{K}n = (0)_1 = (0)_2 = n$. Therefore, since (2) $m < m!$ for every infinite m (cf. Th. 4.4.11.7) one has (3) $m < m! \leq n$. By the wordings of S_1 the sign $\leq$ in (3) is prohibited to mean $<$. Thus $m! = n$, i.e., $m!$ covers n.

4.4.11.6.1. **Problem.** SRFH $\Rightarrow$ GLFH?

4.4.11.7. **Theorem.** Let n be any cardinal number; then:

(0) $2n - 2 \leq n! \leq n^n$.

(i) If $n > 2$, then

(1) $n \leq 2n - 2 \leq n!$ and $n < n!$

(ii) If $0 < n = 2n$, then n is infinite and satisfies $n < 2^n \leq n! \leq n^n$.

(iii) If $1 < n = n^2$, then n is infinite and satisfies $n < n! = 2^n = n^n$.

Proof of (0). At first, $n! \leq n^n$ because $S!$ is a part of the set ${}^S S$ of power n^n of all selfmappings of S. Further, since (0) is true for $n = 0, 1, 2$, let $n > 2$, and S be a set of power n; let 0, 1 be signs for two distinct points in S. Let $D := \{0,1\} \times S = \{0\} \times S \cup \{1\} \times S$, where $\{x\} \times S := \{(x,s) : s \in S\}$. Then

(2) $\text{p}D = 2\text{p}S = 2n$.

For every $(x, y) \in D$ let (xy) denote the permutation of S which is cyclic in $\{x, y\}$ and is the identity in $S \backslash \{x, y\}$; then obviously $(xy) \in D \backslash \{0,1\}^2$ implies $(y, x) \notin D \backslash \{0,1\}^2$; the mapping $(x, y) \in D \to (x, y) \in S!$ is a bijection of $D \backslash \{(1,0),(1,1)\}$ into $S!$, thus (0) is true.

Proof of (i). If n is finite and > 0, then (0) implies (i) because $n < 2n - 2$. If n is infinite, then so is $2n$ and obviously $2n - 2 = 2n$ and the true relation (0) becomes $2n \leq n!$; therefore (since $n \leq 2n$) $n \leq 2n \leq n!$. Consequently, if $n < 2n$, then (1) holds. If

(3) $n = 2n$, let us consider the following mapping $f|PS$ of the power set PS of S into $D!$

For any $X \in PS$ (including the cases X = empty, $X = S$) let $X' := \{0,1\} \times X$ and $fX|D$ be defined as the identity mapping in $D \backslash X'$; in X' let

$$fX(e, x) = (1 - e, x) \quad (e = 0, 1; x \in X).$$

One checks readily that $fX \in D!$ and that the mapping $f|P(S)$ is a bijection of PS into $D!$. Thus $\text{p}PS \leq \text{p}(D!) = (\text{p}D)! = (\text{by } (3), (2)) = n!$, i.e., $\text{p}PS \leq n!$ This relation jointly with Cantor's Theorem $\text{p}S < \text{p}PS = 2^{\text{p}S}$, and $\text{p}S = n$ yields $n < 2^n \leq n!$ for every infinite cardinal. The proof of (i) is done. By the way, so is for (ii) as well.

Proof of (iii). The assumption in (iii) implies that one can apply (ii); one gets the four term relation in (ii), which by raising to the n-th power yields $n^n \leq 2^{nn} \leq n!^n \leq n^{nn}$ and therefore (because $n = nn$) $n^n \leq 2^n \leq$

$n!^n \leq n^n$, thus $2^n = n^n$ and, by (ii), $2^n = n! = n^n$. This completes the proof.

4.4.11.8. **Remark.** I am acquainted with results 4.4.11.7 since 1968; since 1968 I published several papers [only a short paper 1972(1)] on finite [infinite] factorials and combinatorics. Meanwhile appeared in M. R. 50 (1975) # 9595 the rewiew, by J. E. Rudin, of my paper 1972(1) concerning the problem as to whether RFH $\Rightarrow$ AC (Problem 1.4. in the paper) one reads: "(It is still an open question whether RFH $\not\Rightarrow$ AC. Recent results of J. Dawson and P. Howard [see Howard, Notices Amer. Math. Soc. **21** (1974), A-499, Abstract 74T-E56] show that RFH is not provable in ZF. In fact they show that if n is an infinite cardinal any of the three alternatives (i) $n!$ and 2^n are incomparable. (ii) $n! < 2^n$, or (iii) $2^n < n!$ are possible in ZF)". See pp. 186–7 Dawson-Howard 1876.

4.4.12. Chain x Antichain Hypothesis for trees as a fixpoint statement.

4.4.12.0. In K. 1935 (2, 3*) general trees or ramified tables T (pseudotrees or ramified sets R) were introduced as ordered sets in which each left cone is well (linearly) ordered.

4.4.12.1. Degenerate or D-sets were defined as $(O, \leq)$ in which every cone is linearly ordered. Let $P_D(O, \leq) := \{X, X \subset O$ and X is degenerate$\}$ and $b(O, \leq) := \sup\{pX : X \in P_D(O, \leq)\}$.

Of course, these definitions are literally transferable into graphs (G, R) on substituting "linearly ordered set" by "complete subgraph". Let length $(O, \leq) := p_L(O, \leq) := \sup\{pX : X$ is a linearly ordered subset of $(O, \leq)\}$ and width $(O, \leq) := p_A(O, \leq) := \sup\{pX : X$ is without distinct comparable members and $X \subset O\}$. How are the numbers p, p_A, p_L related?

4.4.12.2. Let $\gamma(O, \leq)$ be the least ordinal number which is not embeddable into $(O, \leq)$; $\gamma(O, \leq)$ is called the rank or the ordinal height of $(O, \leq)$.

4.4.12.3. **Lemma.** If $\gamma(O, \leq)$ is finite, then

(i) $p\gamma(O, \leq) = p_L(O, \leq)$ and

(ii) $p(O, \leq) \leq p_A(O, \leq) \cdot p_L(O, \leq)$.

A corresponding majorization for infinite $(O, \leq)$ might fail; already for well-founded sets majorisation is exponential holding also for infinite binary graphs. In this way I discovered, independently, Ramsey's result — very basis of Partition Calculus (cf. K. 1937(5) "relation fondamentale" (1); 1939(2) = 1959(1), 1959(2) Th. 6.2.2): If a graph G is infinite, then

$$pG \leq x^y, \text{ where } x = \sup\{p_A G, p_L G\}, y = \inf\{p_A G, p_L G\}).$$

4.4.12.4. Henceforth, let T be a tree; then one has the following basic disjoint partition of $(T, \leq)$ into "levels" or "rows":

$$\text{(P)} \quad T = \cup R_\alpha T \quad (\alpha < \gamma T)$$

where $R_\alpha(T, \leq) := \{x : x \in T \text{ and ord } (T, \leq)(., x) = \alpha\}$. One has

$$T = \cup(T, \leq)[x, .), (x \in R_0 T).$$

Each level R_α is an antichain; the number $m(T, \leq) := \sup \mathrm{p}\, R_\alpha T$ $(\alpha < \gamma)$ is $\leq p_A(O, \leq)$. Otherwise, $p_A T$ does not depend upon mT. The disjoint partition (P) yields

$$\mathrm{p}T = \Sigma \mathrm{p} R_\alpha T \ (a < \gamma T)\,, \ \mathrm{p}T \leq mT \cdot \mathrm{p}\gamma T\,.$$

The number $\mathrm{p}\gamma T$ either equals to or covers $p_L(T, \leq)$.

4.4.12.5. **Theorem** (= Th. 1 p. 105, Kurepa 1935(2,3*)): For infinite trees T, the power $\mathrm{p}T$ either equals or covers bT.

What is the character of this alternative? My standpoint was expressed by the following quoting.

4.4.12.6. **Theorem.** "Théorème fondamental. Les hypothèses P_1, $P_2, \cdots$, P_{12} sont, logiquement, deux à deux équivalentes." (Ibidem p. 132).

"P_1: Quel que soit le tableau ramifié T, la borne supérieure bT est atteinte dans T, c'est-à-dire T contient un sous-ensemble dégénéré ayant la puissance bT (*Hypothèse ou Postulat*[7] *de ramification*);

P_2: Tout tableau ramifié infini a même puissance que l'un de ses sous-ensembles dégénérés (*Principe de réduction*);

P_3 ...

P_4: T étant un tableau ramifié infini quelconque d'ensembles, la famille T^d a même puissance que l'une de ses sous-familles *disjonctives*[10] (*Proposition fondamentale sur les tableaux ramifiés d'ensembles*;

[7] Cf. ce Postulat sur lequel est bâtie l'Analyse classique: La borne supérieure de tout ensemble borné de nombres réels est un nombre réel bien déterminé.

[10] T^d désigne la famille des ensembles distincts qu'on obtient en adjoignant à T les ensembles A–B, A, B parcourant les éléments de T (cf. ma Note des C. R., 199, 1934 p. 122).

P_5: Quel que soit l'ensemble ordonné infini E, il existe une famille *disjonctive* d'intervalles non-vides de E ayant la puissance p_1E (*Problème de la structure cellulaire d'ensembles ordonnés*);

P_6: ...

...

P_{12}: ..." (Ibidem pp. 130–131; p_1E is density number of E).

4.4.12.7. Simple consequences of the RHT (Ramification Hypothesis for Trees): $= P_1$ or of the RPT (Reduction Principle for Trees): $= P_2$ are:

4.4.12.7.1. $bT = pT$ for every infinite tree.

4.4.12.7.2. LAHT (Chain x Antichain Tree Hypothesis):

$pT \leq p_AT.p_LT$ for each tree (cf. 4.4.12.3 (ii)).

LAHT is also called Rectangle Hypothesis for Trees (ReHT) for obvious reasons when one looks on geometrical or mechanical scheme of a tree.

4.4.12.7.3. MATH (Maximal Antichain Hypothesis) Each tree T contains an antichian A such that $pA \geq pX$ for every antichain $X \subset (T, \leq)$; in other words: The antichain number p_AT is attained inside each tree T (let us remark that MATH is provable for every T unless p_AT is weakly inaccessible; cf. Kurepa 1987(2)).

4.4.12.8. In K. 1937(5) no. III the theorem 4.4.12.5 was formulated in the form $\alpha \leq n(\alpha) \leq \alpha + 1$, introducing the following

4.4.12.9. **Definition.** Mapping $n_\alpha|$ ON is defined by

$$\chi_{n(\alpha)} := \sup \{pT, T \text{ is tree and } bT \leq \chi_\alpha\}.$$

There is no restriction to require in the definition that $T \subset (C(\alpha), \leq_k); C_\alpha :=$ the class of all ξ-sequences over $\mathrm{ON}[0, \omega_\alpha), \xi$ running over ordinals $\leq \omega_{\alpha+1}; x \leq_k y := x$ is an initial section of y (cf. Kurepa 1953(12) no. 2). In such a way one has the following

4.4.12.10. **Theorem.** Chain x Antichain Hypothesis $pT \leq p_AT \cdot p_LT$ for trees is equivalent to the fixpoint equality

$n_\alpha = \alpha$ for each ordinal number α.

In particular, the equality $n_0 = 0$ is equivalent to the positive answer to the Suslin problem and is a postulate.

4.5. ***Universality of the fixpoint approach***

4.5.0. Sofar we had the opportunity to see how various mathematical statements could be equivalently worded as fixpoint statements. It is interesting that, in some sense, such an approach is feasible in each case; we have the following

4.5.1. Theorem. Given any theory X equipped with a given truth values system V of power > 1. Each statements S in X is equivalent to a fixpoint statement concerning a self-map $v_S|V$ in such a way that the decidability of S is equivalent to the existence of a unique fixpoint of v_S: the fixpoint of v_S is the truth value τS of S.

As a matter of fact, it suffices to define $v_S|V$ in such a way that $v_S(\tau S) = \tau S \in V$ and that $v_S(x) \neq x$ for each $x \in V(S) := V\backslash\{\tau S\}$. In particular, one could require that $v_S|V(S)$ be any permutation of $V(S)$ having no fixpoint, provided $\mathrm{p}V(S) > 1$. If e.g., $V = \{0, 1\}$, it suffices to define $v_S(0) = v_S(1) = 1(0)$ if S is true (false) and $v_S(0) = 1, v_S(1) = 0$ if S is indecidable.

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Đuro R. Kurepa
Matematički Institut Beograd
Zagrebačka 7
11000 Beograd
Jugoslavija

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Exact Controllability and Uniform Stabilization of Euler-Bernoulli Equations with Boundary Control Only in $\Delta w|_\Sigma$

I. Lasiecka and R. Triggiani

1. Introduction, Statement of Main Results, Literature

1.1. Introduction; Exact Controllability with Control Action Only in $\Delta w|_\Sigma$

Throughout this paper, Ω is a bounded open domain in R^n, typically $n \geq 2$, with sufficiently smooth boundary $\Gamma = \partial\Omega$. In Ω we consider the following Euler-Bernoulli mixed problem in the solution $w(t,x)$:

$$\begin{cases} w_{tt}+\Delta^2 w = 0 & \text{in } (0,T]\times\Omega; & (1.1a) \\ w(0,\cdot) = w_0;\ w_t(0,\cdot) = w_1 & \text{in } \Omega; & (1.1b) \\ w|_\Sigma = 0 & \text{in } (0,T]\times\Gamma = \Sigma; & (1.1c) \\ \Delta w|_\Sigma = u & \text{in } \Sigma, & (1.1d) \end{cases}$$

with control function u only in the boundary condition (1.1d). When (1.1c) is replaced by the non-homogeneous B.C. $w|_\Sigma = v$, regularity results in appropriate functions spaces (in fact, optimal regularity results) and corresponding exact controllability results using both boundary controls v and u were given in [Lio.1], [Lio.2], [L-T.2], [L-T.3]. The question of exact controllability with just one boundary control such as u in (1.1) is pointed out in [Lio.2, Remark 4.1] to be an open problem. In Section 2 we shall study such questions of exact controllability of (1.1) on the spaces

$$Z = \mathcal{D}(A^{1/2})\times L_2(\Omega); \qquad W = L_2(\Omega)\times[\mathcal{D}(A^{1/2})]' \qquad (1.2)$$

(the second component of W denoting the dual of $\mathcal{D}(A^{1/2})$ with respect to the $L_2(\Omega)$-topology) where A is the positive self-adjoint operator defined by

$$Af = \Delta^2 f;\ \mathcal{D}(A) = \{f \in H^4(\Omega):\ f|_\Gamma = \Delta f|_\Gamma = 0\}$$
$$\text{so that } A^{1/2}f = -\Delta f;\ \mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega). \qquad (1.3)$$

The choice of the space Z implies at the outset the homogeneous B.C. (no control action) in (1.1c). We set

$$\|x\|_{\mathcal{D}(A^\alpha)} = \|A^\alpha x\|_{L_2(\Omega)}, \text{ any real } \alpha, \qquad (1.4)$$

where, if $\alpha < 0$, by $\mathcal{D}(A^\alpha)$ we mean $\mathcal{D}(A^\alpha) = [\mathcal{D}(A^{-\alpha})]'$, the dual of $\mathcal{D}(A^{-\alpha})$ with respect to the $L_2(\Omega)$-topology. The following exact controllability result on Z for any T > 0 requires no geometrical conditions on Ω (except for smoothness of Γ).

<u>Theorem 1.1</u>. Let T > 0. Given $\{w_0, w_1\} \in Z = \mathcal{D}(A^{1/2})\times L_2(\Omega)$, there exists a suitable control $g \in L_2(0,T;H^{1/2}(\Gamma))$ such that the corresponding solution of (1.1) satisfies $w(T,\cdot) = w_t(T,\cdot) = 0$. ∎

1.2. <u>Stabilization with Control Action Only in</u> $\Delta w|_\Sigma$

The problem of <u>uniform</u> stabilization for system (1.1) with just one feedback control is markedly more difficult to solve; more importantly, the results on uniform stabilization given below in Theorems 1.3 and 1.4 require severe geometrical conditions. By contrast, the exact controllability result in Theorem 1.1, and the results of <u>strong</u> stabilization to be given in Theorem 1.2 below on the entire continuum of natural spaces $Z_\alpha = \mathcal{D}(A^{1/2+\alpha})\times\mathcal{D}(A^\alpha)$, α real, require no geometrical conditions on Ω. It should be noted that absence of geometrical conditions for strong stabilization, while typical for second order problems (in the space variable), i.e., wave problems [Lag.1], [L-T.1], [T.1], is as yet untypical for plate problems. In the present case, this is the consequence of the particular choice of boundary conditions in (1.1c-d) which produce $A^{1/2}$ as defined by the differential operator $-\Delta$ as in (1.3). For instance, if we replace the boundary conditions (1.1c-d) with $w|_{\Sigma} = u \in L_2(\Sigma)$ and $\frac{\partial w}{\partial \nu}|_{\Sigma} \equiv 0$, or else $w|_{\Sigma} = 0$ and $\frac{\partial w}{\partial \nu}|_{\Sigma} = u \in L_2(\Sigma)$, then the class of domains Ω where strong stabilization (in the appropriate spaces of optimal regularity) is presently achieved is not any larger than the class of domains Ω where uniform stabilization can be claimed [B-T.1], [O-T.1], respectively. It is the latter, not the former, that is the typical situation for plates. The technical issue behind this will be explained in Remark 2.1 below. The class of domains Ω covered by our uniform stabilization result will be singled out in Definition 1.1 below and are very restrictive. It contains spheres, or small deformations thereof, or set differences of such domains.

<u>Choices of the Feedback Operator</u>. It is justified in Section 3 that the following choice of a feedback operator on Σ:

$$\Delta w|_{\Sigma} = u = -\frac{\partial}{\partial \nu} A^{2\alpha} w_t = -G_2^* A^{1+2\alpha} w_t, \quad \alpha \text{ real}, \tag{1.5}$$

where G_2^* is the adjoint of the Green operator G_2 defined in (2.2), (2.3), below provides a reasonable <u>candidate</u> for stabilization problem of (1.1), as justified by the following results. The feedback control in (1.5), once inserted in (1.1d), gives rise to the following closed loop problem

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } (0,\infty)\times\Omega; & (1.6a) \\ w(0,\cdot) = w_0; \; w_t(0,\cdot) = w_1 & \text{in } \Omega; & (1.6b) \\ w|_{\Sigma} = 0 & \text{in } (0,\infty)\times\Gamma; & (1.6c) \\ \Delta w|_{\Sigma} = -\frac{\partial}{\partial \nu} A^{2\alpha} w_t & \text{in } (0,\infty)\times\Gamma. & (1.6d) \end{cases}$$

The abstract model for problem (1.6) is (see Section 3)

$$w_{tt} = -A[w+G_2G_2^*A^{1+2\alpha}w_t], \quad \text{or} \quad \frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix} = \mathcal{A}\begin{vmatrix} w \\ w_t \end{vmatrix}; \qquad (1.7a)$$

$$\mathcal{A} = \begin{vmatrix} 0 & I \\ -A & -AG_2G_2^*A^{1+2\alpha} \end{vmatrix}, \quad \mathcal{D}(\mathcal{A}) = \{z \in Z_\alpha : \mathcal{A}z \in Z_\alpha\}. \qquad (1.7b)$$

After the above background, we can finally state our main stabilization results for problem (1.1).

<u>Theorem 1.2</u> (well-posedness and strong stabilization on $Z_\alpha = \mathcal{D}(A^{1/2+\alpha})\times\mathcal{D}(A^\alpha))$. Consider the closed loop problem (1.6), or equivalently, (1.7). Then

(i) (well-posedness) the corresponding map $\{w_0,w_1\} \to \{w(t),w_t(t)\}$ defines a strongly continuous contraction semigroup $e^{\mathcal{A}t}$ on $Z_\alpha = \mathcal{D}(A^{1/2+\alpha})\times\mathcal{D}(A^\alpha)$, α real.

(ii) (L_2-nature of feedback operator)

$$\frac{d}{dt}\left\| e^{\mathcal{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|^2_{Z_\alpha} = -2\|G_2^*A^{1+2\alpha}w_t\|^2_{L_2(\Gamma)} = -2\left\|\frac{\partial A^{2\alpha}w_t}{\partial\nu}\right\|^2_{L_2(\Gamma)}; \qquad (1.8a)$$

$$\left\| e^{\mathcal{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|^2_{Z_\alpha} - \left\|\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|^2_{Z_\alpha} = -2\int_0^T \|G_2^*A^{1+2\alpha}w_t\|^2_{L_2(\Gamma)}dt$$
$$= -2\int_0^T \left\|\frac{\partial A^{2\alpha}}{\partial\nu} w_t\right\|^2_{L_2(\Gamma)}dt; \qquad (1.8b)$$

$$\int_{t_0}^\infty \left\|\frac{\partial}{\partial\nu} A^{1+2\alpha}w_t\right\|^2_{L_2(\Gamma)}dt = \int_{t_0}^\infty \|G_2^*A^{1+2\alpha}w_t(t)\|^2_{L_2(\Gamma)}dt \le \|\{w(t_0),w_t(t_0)\}\|^2_{Z_\alpha},$$
$$\text{for any } t_0 \ge 0. \qquad (1.8c)$$

(iii) The resolvent operator $R(\lambda,\mathcal{A})$ of the feedback generator $\mathcal{A}$ in (1.7b) is given by

$$R(\lambda,\mathcal{A}) = \begin{vmatrix} \frac{1-V^{-1}(\lambda)}{\lambda} & V^{-1}(\lambda)A^{-1} \\ -V^{-1}(\lambda) & \lambda V^{-1}(\lambda)A^{-1} \end{vmatrix} ; \tag{1.9a}$$

$$V(\lambda) = I+\lambda A^{\frac{1}{2}+\alpha}G_2G_2^*A^{\frac{1}{2}+\alpha}+\lambda^2A^{-1}, \tag{1.9b}$$

at least for Re $\lambda > 0$ and is compact on Z_α; moreover, $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$.

(iv) The resolvent operator $R(\lambda,\mathcal{A})$ is well defined and compact on Z_α on the closed half-plane Re $\lambda \geq 0$. Thus, the spectrum (i.e., point spectrum) $\sigma(\mathcal{A})$ of $\mathcal{A}$ is contained in $\{\lambda:\text{ Re } \lambda < 0\}$.

(v) (strong stabilization) We have for each fixed α real:

$$\begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} = e^{\mathcal{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \to 0 \text{ in } Z_\alpha$$

$$\text{as } t \to \infty, \ \forall\ [w_0,w_1] \in Z_\alpha. \quad \blacksquare \tag{1.10}$$

We next pass to uniform stabilization. Here we single out two values of α: $\alpha = 0$ and $\alpha = -\frac{1}{2}$, in which case we write $Z_{\alpha=0} = Z = \mathcal{D}(A^{\frac{1}{2}})\times L_2(\Omega)$; and $Z_{\alpha=-\frac{1}{2}} = W = L_2(\Omega)\times[\mathcal{D}(A^{\frac{1}{2}})]'$, as in (1.2). The class of domains Ω covered by our uniform stabilization results is singled out in the next definition.

Definition 1.1. Let Ω satisfy the following condition. There exists a vector field $h(x) \in [C^2(\overline{\Omega})]^n$ such that:

(i) h is parallel to ν (exterior unit normal) on all of Γ; i.e., $h(\sigma) = b(\sigma)\nu(\sigma)$, for $b(\sigma)$ a smooth scalar boundary function, $\sigma \in \Gamma$; $b \in H^1(\Gamma)$;

(ii) the following inequality holds,

$$\int_Q \Delta q\Big(\sum_{i=1}^{n} \nabla h_i\cdot\nabla q_{x_i}\Big)d\Omega \geq \rho\int_Q |\Delta q|^2 d\Omega, \tag{1.11}$$

where $q(x) \in \mathcal{D}(A)$, and therefore satisfies

$$q|_\Gamma \equiv 0 \text{ and } \Delta q|_\Gamma \equiv 0,$$

and $\rho > 0$ is a suitable constant, possibly depending on $h(x)$, Ω, and $q(x)$. ■

Examples of domains satisfying Definition 1.1 include n-dimensional spheres with center x_0, where $h(x) = x-x_0$ and small deformations thereof; also set differences of such domains. See Appendix C.

Theorem (1.3) (uniform stabilization on $Z = \mathcal{D}(A^{\frac{1}{2}})\times L_2(\Omega)$). Consider problem (1.6) with $\alpha = 0$ in (1.6d), i.e., with

$$\Delta w|_{\Sigma} = u = -\frac{\partial w_t}{\partial \nu}|_{\Sigma} = -G_2^* A w_t \tag{1.12}$$

in (1.6d), whose well-posedness is asserted in Theorem 1.2. Let now Ω satisfy the geometrical conditions of Definition 1.1 above. Then there exist constants M and $\delta > 0$ such that

$$\left\|\begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix}\right\|_Z = \left\| e^{\mathcal{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|_Z \le Me^{-\delta t}\left\|\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\|_Z, \quad t \ge 0. \quad ■ \tag{1.13}$$

Theorem 1.4 (uniform stabilization on $W = L_2(\Omega)\times[\mathcal{D}(A^{\frac{1}{2}})]'$). Consider problem (1.6) with $\alpha = -\frac{1}{2}$ in (1.6d), i.e., with

$$\Delta w|_{\Sigma} = u = -\frac{\partial (A^{-1}w_t)}{\partial \nu}|_{\Sigma} = -G_2^* w_t \tag{1.14}$$

in (1.6d), whose well-posedness is asserted by Theorem 1.2. Let now Ω satisfy the geometrical conditions of Definition 1.1 above. Then the same conclusion as in (1.13) holds true, with Z there replaced by W now. ■

2. Proof of Theorem 1.1: Exact Controllability

2.1. Preliminaries for Exact Controllability and Stabilization

As in [L-T.2], [L-T.3], we introduce the Green's operator G_2,

$$G_2 g_2 \equiv y; \quad G_2: \text{ continuous } H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega),$$
$$\text{[L-M.1, pp. 188-9], } s \text{ real}; \tag{2.1}$$

$$\Delta^2 y = 0 \text{ in } \Omega;\ y = 0 \text{ on } \Gamma;\ \Delta y = g_2 \text{ on } \Gamma. \tag{2.2}$$

Then, the solution at time T to the Euler-Bernoulli problem (1.1) with $w_0 = w_1 = 0$ can be written explicitly as

$$\begin{vmatrix} w(T;\ w_0 = w_1 = 0) \\ w_t(T;\ w_0 = w_1 = 0) \end{vmatrix} = \mathcal{L}_T u = \begin{vmatrix} A\int_0^T S(T-t)G_2 u(\tau)d\tau & (2.3a) \\ A\int_0^T C(T-t)G_2 u(\tau)d\tau & (2.3b) \end{vmatrix}$$

[L-T.2], [L-T.3], where C(t) is the s.c. cosine operator generated by the operator -A in (1.3) and $S(t) = \int_0^t C(\tau)d\tau$. It is expedient to introduce the Dirichlet map D [L-T.1],

$$Dv = \zeta \iff \Delta\zeta = 0 \text{ in } \Omega;\quad \zeta = v \text{ on } \Gamma; \tag{2.4a}$$

D: continuous $H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega)$, [L-M.1, pp. 188-9], s real, (2.4b)

and recall that as a consequence of the special B.C. we have the relationship [L-T.2], [L-T.3],

$$G_2 = -A^{-\frac{1}{2}}D;\quad G_2^* = -D^*A^{-\frac{1}{2}}; \tag{2.5}$$

$$(G_2 g, v)_{L_2(\Omega)} = (g, G_2^* v)_{L_2(\Gamma)};\quad (Dg, v)_{L_2(\Omega)} = (g, D^* v)_{L_2(\Gamma)};$$

$$g \in L_2(\Gamma),\ v \in L_2(\Omega). \tag{2.6}$$

We have [L-T.2], [L-T.3] via Green's second theorem

$$G_2^* A^{\frac{1}{2}} f = -D^* A f = -\frac{\partial(\Delta f)}{\partial\nu},\quad f \in \mathcal{D}(A) \tag{2.7}$$

$$G_2^* A f = -D^* A^{\frac{1}{2}} f = \frac{\partial f}{\partial\nu},\ f \in \mathcal{D}(A). \tag{2.8}$$

In the uniform stabilization problem we shall use the following properties:

$$D^* : \text{continuous } H^{-s}(\Omega) \to H^{-s+\frac{1}{2}}(\Gamma), \quad 0 \le s \le \tfrac{1}{2}, \tag{2.9}$$

which follows by duality from (2.4b), and

$$D^* D : \text{continuous } L_2(\Omega) \to H^1(\Omega), \tag{2.10}$$

which follows by applying (2.9) with s = 0 followed by (2.4b) with $s = \frac{1}{2}$.

2.2. Exact Controllability on $Z = \mathcal{D}(A^{\frac{1}{2}}) \times L_2(\Omega)$

Step 1. We need to show that the unbounded, closed operator $\mathcal{L}_T$ in (2.3) satisfies $L_2(0,T;H^{\frac{1}{2}}(\Gamma)) \supset \mathcal{D}(\mathcal{L}_T) \to$ onto Z, or equivalently that there is $C_T > 0$ such that [T-L.1, p. 235]

$$\left\| \mathcal{L}_T^* \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right\|_{L_2(0,T;H^{\frac{1}{2}}(\Gamma))} \ge C_T \left\| \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right\|_Z \tag{2.11}$$

for all $\{z_1, z_2\}$ for which the left hand side of (2.11) is finite, where $\mathcal{L}_T^*$ is the Hilbert space adjoint

$$(\mathcal{L}_T u, z)_Z = (u, \mathcal{L}_T^* z)_{L_2(0,T;H^{\frac{1}{2}}(\Gamma))} . \tag{2.12}$$

Step 2. Lemma 2.1.

(a) We have

$$\left(\Lambda^2 \mathcal{L}_T^* \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right)(t) = - \frac{\partial(\Delta\phi(t))}{\partial\nu} , \tag{2.13}$$

where

$$\Lambda : \text{isomorphism } H^r(\Gamma) \to H^{r-\frac{1}{2}}(\Gamma), \text{ self-adjoint on } L_2(\Gamma), \tag{2.14}$$

and where $\phi(t) = \phi(t, \phi_0, \phi_1)$ solves the following homogeneous problem:

$$\begin{cases} \phi_{tt}+\Delta^2\phi = 0 & \text{in } Q; \quad (2.15a) \\ \phi|_{t=T} = \phi_0,\ \phi_t|_{t=T} = \phi_1 & \text{in } \Omega; \quad (2.15b) \\ \phi|_\Sigma \equiv \Delta\phi|_\Sigma \equiv 0 & \text{in } \Sigma, \quad (2.15c) \end{cases}$$

with initial data

$$\phi_0 = A^{-½}z_2 \in \mathcal{D}(A^{½});\quad \phi_1 = -A^{½}z_1 \in L_2(\Omega). \tag{2.16}$$

(b) Inequality (2.11) is equivalent to the following inequality: There exists $C_T > 0$ such that

$$\left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|^2_{L_2(0,T;H^{-½}(\Gamma))} \geq C_T'\|\{\phi_0,\phi_1\}\|^2_{\mathcal{D}(A^{½})\times L_2(\Omega)} = C_T'E(0) \tag{2.17}$$

for all $\{\phi_0,\phi_1\} \in Z = \mathcal{D}(A^{½})\times L_2(\Omega)$ for which the left hand side of (2.17) is finite; in (2.17) we have set

$$E(t) = \|A^{½}\phi(t)\|^2_{L_2(\Omega)}+\|\phi_t(t)\|^2_{L_2(\Omega)} = \int_\Omega\{(\Delta\phi(t))^2+\phi_t^2(t)\}\,d\Omega. \tag{2.18}$$

Thus, exact controllability of (1.1) on Z over [0,T] within the class of $L_2(0,T;H^{½}(\Gamma))$-controls u is equivalent to inequality (2.17). ∎

Proof. (a) On the other hand, we have from (2.12) and (2.14) and $z = [z_1,z_2]$

$$\left(\mathcal{L}_T u, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}\right)_{\mathcal{D}(A^{½})\times L_2(\Omega)} = (u,\mathcal{L}_T^* z)_{L_2(0,T;H^{½}(\Gamma))}$$

$$= (\Lambda u,\Lambda\mathcal{L}_T^* z)_{L_2(0,T;L_2(\Gamma))} = (u,\Lambda^2\mathcal{L}_T^* z)_{L_2(0,T;L_2(\Gamma))}. \tag{2.19}$$

On the other hand, we compute from (2.3) as usual [L-T.3], [T.2],

$$\left(\mathcal{L}_T u, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}\right)_{\mathcal{D}(A^{1/2})\times L_2(\Omega)} = \left(A\int_0^T S(T-t)G_2u(t)dt, Az_1\right)_{L_2(\Omega)}$$

$$+ \left(A\int_0^T C(T-t)G_2u(t)dt, z_2\right)_{L_2(\Omega)}$$

$$= \int_0^T (u(t), G_2^*A[S(T-t)Az_1+C(T-t)z_2])_{L_2(\Gamma)}dt. \quad (2.20)$$

Thus, by comparing (2.19) with (2.20), using that C is even while S is odd, and recalling (2.7), we have

$$(\Lambda^2\mathcal{L}_T^*z)(t) = G_2^*A^{3/4}[C(t-T)A^{-1/2}z_2+S(t-T)(-A^{1/2}z_1)] = -\frac{\partial(\Delta\phi(t))}{\partial\nu}, \quad (2.21)$$

and (2.13), (2.15), (2.16) are proved.

(b) We have from (2.14), (2.13),

$$\left\|\mathcal{L}_T^*z\right\|_{L_2(0,T;H^{1/2}(\Gamma))} = \left\|\Lambda\mathcal{L}_T^*z\right\|_{L_2(0,T;L_2(\Gamma))} = \left\|\Lambda^{-1}\frac{\partial(\Delta\phi)}{\partial\nu}\right\|_{L_2(0,T;L_2(\Gamma))}$$

$$= \left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|_{L_2(0,T;H^{-1/2}(\Gamma))}. \quad (2.22)$$

Thus, (2.17) follows from (2.22), (2.11) and (see (2.16)),

$$\|\phi_0\|_{\mathcal{D}(A^{1/2})} = \|A^{1/2}\phi_0\|_{L_2(\Omega)} = \|z_2\|_{L_2(\Omega)};$$

$$\|\phi_1\|_{L_2(\Omega)} = \|A^{1/2}z_1\|_{L_2(\Omega)} = \|z_1\|_{\mathcal{D}(A^{1/2})}. \quad \blacksquare$$

<u>Step 3</u>. Thus, the key technical issue is to show that equality (2.17) holds true. This is accomplished in the next two propositions.

<u>Proposition 2.2</u>. Let T > 0 be given. With reference to problem (2.15) for ϕ, the following inequality holds true:

$$-\int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu}\frac{\partial\phi}{\partial\nu}\, h\cdot\nu\, d\Sigma \geq (T-\varepsilon)\, E(0) - \frac{c}{\varepsilon}\||\nabla\phi|\|^2_{C([0,T];L_2(\Omega))}, \qquad (2.23)$$

where $h(x) = x-x_0$, $x_0 \in R^n$, $\varepsilon > 0$ arbitrary and c is a constant. ∎

<u>Proof</u>. The proof uses the multipliers $h\cdot\nabla\phi$, $h(x) = x-x_0$; ϕ; and ϕ_t. The computations are reported in Appendices A and B where they are carried out in the case of a general vector field h(x), since such case is needed in the problem of uniform stabilization. Multiplying (2.15a) by $h\cdot\nabla\phi$ with $h(x) = x-x_0$, hence div h = dim Ω = n, one obtains

$$-\int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu}\frac{\partial\phi}{\partial\nu}\, h\cdot\nu\, d\Sigma = 2\int_Q(\Delta\phi)^2dQ+[(\phi_t,h\cdot\nabla\phi)_\Omega]_0^T+\tfrac{1}{2}[(\phi_t,\phi\text{ div } h)_\Omega]_0^T. \qquad (2.24)$$

See Appendix A. Similarly, multiplying (2.15a) by ϕ one obtains

$$\int_Q(\Delta\phi)^2dQ = \int_Q\phi_t^2dQ-[(\phi_t,\phi)_\Omega]_0^T. \qquad (2.25)$$

Inserting (2.25) into (2.24) and recalling (2.18) yields the following identity for the right hand side (R.H.S.) of (2.24):

$$\text{R.H.S. of (2.24)} = \int_0^T E(t)dt+\rho_{0T} = TE(0)+\rho_{0T}; \qquad (2.26)$$

$$\rho_{0T} = [(\tfrac{n}{2}-1)(\phi_t,\phi)_\Omega+(\phi_t,h\cdot\nabla\phi)_\Omega]_0^T. \qquad (2.27)$$

In the last step of (2.26), we have used that $E(t) \equiv E(0)$, which is proved e.g., by using the multiplier ϕ_t.

Finally, we readily obtain from (2.27) using Poincaré inequality on ϕ, (2.18) and $E(t) \equiv E(0)$:

$$\rho_{0T} \geq -\varepsilon\, E(0) - \frac{c}{\varepsilon}\||\nabla\phi|\|^2_{C([0,T];L_2(\Omega))}. \qquad (2.28)$$

By using (2.28) in (2.26), and recalling (2.24), we arrive at (2.23). ∎

__Step 4__. As to the left hand side of (2.23), we write

$$\left|\int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu}\frac{\partial\phi}{\partial\nu}\, h\cdot\nu\, d\Sigma\right| \le C_h\int_0^T \left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|_{H^{-1/2}(\Gamma)}\left\|\frac{\partial\phi}{\partial\nu}\right\|_{H^{1/2}(\Gamma)} dt$$

$$\le C_h\left\{\frac{1}{\varepsilon}\left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))} + \varepsilon\left\|\frac{\partial\phi}{\partial\nu}\right\|^2_{L_2(0,T;H^{1/2}(\Gamma))}\right\}$$

$$\le C_h\left\{\frac{1}{\varepsilon}\left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))} + \varepsilon T\, E(0)\right\}, \qquad (2.29)$$

where in the last step we have used

$$\left\|\frac{\partial\phi}{\partial\nu}\right\|_{L_2(0,T;H^{1/2}(\Gamma))} \le C\|\phi\|_{L_2(0,T;H^2(\Omega))}$$

$$\le C\|\phi\|_{L_2(0,T;\mathcal{D}(A^{1/2}))} = C\left\{\int_0^T \|A^{1/2}\phi\|^2_{L_2(\Omega)}dt\right\}^{1/2}$$

$$\le C\int_0^T E(t)dt = CT\, E(0), \qquad (2.30)$$

since $\phi|_\Sigma \equiv 0$ and since $E(t) \equiv E(0)$. Combining (2.29) with (2.24), we obtain

__Corollary 2.3__. Let $T > 0$ be given. The following inequality holds true for the solution ϕ of (2.15):

$$\frac{C_h}{\varepsilon}\left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))} + \frac{C}{\varepsilon}\,\||\nabla\phi|\|^2_{C([0,T];L_2(\Omega))}$$

$$\ge (T-\varepsilon-C_h\varepsilon)E(0). \qquad \blacksquare \qquad (2.31)$$

__Step 5__. We finally absorb lower order terms in inequality (2.30) by a compactness argument of the type used in [Lio.1-2], [Lit.1], [L-T.3], etc. in other circumstances.

<u>Proposition 3.5</u>. With reference to problem (2.15), inequality (2.30) implies that: There exists $C_T > 0$ such that

$$\||\nabla\phi|\|^2_{C([0,T];L_2(\Omega))} \le C_T\left\|\frac{\partial(\Delta\phi)}{\partial\nu}\right\|^2_{L_2(0,T;H^{-½}(\Gamma))} . \quad \blacksquare \tag{2.31}$$

<u>Proof</u>. Suppose by contradiction that there exists a sequence $\{\phi_n\}$ of solutions to problem (2.15) such that

$$\left\|\frac{\partial(\Delta\phi_n)}{\partial\nu}\right\|_{L_2(0,T;H^{-½}(\Gamma))} \to 0; \tag{2.32}$$

$$\||\nabla\phi_n|\|_{C([0,T];L_2(\Omega)} \equiv 1. \tag{2.33}$$

Then, since $\{\phi_n\}$ satisfies (2.30), we have $E_n(0) \le$ const, uniformly in n; i.e.,

$$\{\phi_{0n},\phi_{1n}\} \to \text{some } \{\tilde{\phi}_0,\tilde{\phi}_1\} \text{ weakly in } \mathcal{D}(A^{½})\times L_2(\Omega). \tag{2.34}$$

Then the function $\tilde{\phi}(t) = C(t-T)\tilde{\phi}_0+S(t-T)\tilde{\phi}_1$ satisfies

$$\{\phi_n,\phi_n'\} \to \{\tilde{\phi},\tilde{\phi}'\} \text{ in } L_\infty(0,T;\mathcal{D}(A^{½})\times L_2(\Omega)) \text{ weak star.} \tag{2.35}$$

Thus, $\{\phi_n,\phi_n'\}$ uniformly bounded in $L_\infty(0,T;\mathcal{D}(A^{½})\times L_2(\Omega))$, where $\mathcal{D}(A^{½}) = H^2(\Omega) \cap H_0^1(\Omega)$ from (1.3). Thus, by compactness [S.1],

$$\phi_n \to \tilde{\phi} \text{ strongly in } L_\infty(0,T;H_0^1(\Omega)), \tag{2.36}$$

so that (2.36) implies via (2.33),

$$\||\nabla\tilde{\phi}|\|_{C([0,T];L_2(\Omega))} \equiv 1. \tag{2.37}$$

On the other hand, $\tilde{\phi}$ solves the problem

$$\tilde{\phi}_{tt}+\Delta^2\tilde{\phi} = 0 \qquad \text{in } Q; \tag{2.38a}$$

$$\tilde{\phi}|_\Sigma = 0;\ \Delta\tilde{\phi}|_\Sigma = 0;\ \frac{\partial(\Delta\tilde{\phi})}{\partial\nu}|_\Sigma = 0 \qquad \text{in } \Sigma, \tag{2.38b}$$

the latter condition following from (2.32). Setting $\psi = \Delta\tilde{\phi} = A^{1/2}\tilde{\phi}$, we obtain the problem

$$\begin{cases} \psi_{tt}+\Delta^2\psi = 0; & (2.39a) \\ \psi|_\Sigma = 0,\ \frac{\partial\psi}{\partial\nu}|_\Sigma = 0,\ \Delta\psi|_\Sigma = 0, & (2.39b) \end{cases}$$

where the latter identity follows from $\Delta\psi = \Delta^2\tilde{\phi} = -\tilde{\phi}_{tt}$ in Q, whose restriction on Σ vanishes by the first identity in (2.41b). Then, with T > 0 arbitrary, problem (2.39) with three boundary conditions implies $\psi \equiv 0$ in Q [Lio.1-2], hence $\tilde{\phi} \equiv 0$ in Q. But this conclusion contradicts (2.37). ∎

3. The Feedback System on $Z_\alpha = \mathcal{D}(A^{1/2+\alpha})\times\mathcal{D}(A^\alpha)$ and Theorem 1.2

The Feedback System. We follow the conceptual approach of [L-T.1], [T.1] in case of the wave equation and of [B-T.1] in case of a different plate problem. The abstract differential version of problem (1.1)--which corresponds to the integral version (2.3)--is given in factor form or, respectively, in additive perturbation form by

$$w_{tt} = -A[w-G_2u] \text{ on } L_2(\Omega); \quad \text{or } w_{tt} = -Aw+AG_2u \text{ on } [\mathcal{D}(A)]'; \tag{3.1a}$$

$$\text{or } \frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}\begin{vmatrix} w \\ w_t \end{vmatrix} + \begin{vmatrix} 0 \\ AG_2u \end{vmatrix}, \tag{3.1b}$$

where A on the right of (3.1) is extended, with the same symbol, as an operator, say, $L_2(\Omega) \to [\mathcal{D}(A)]'$. For the purposes of solving the feedback stabilization problem for the dynamics (1.1) in the space $Z_\alpha = \mathcal{D}(A^{1/2+\alpha})\times\mathcal{D}(A^\alpha)$ we seek, if possible, a "feedback" operator $\mathcal{F}$ such that $u = \mathcal{F}(w_t)$ inserted in (1.1c) produces a corresponding closed loop problem which is (i) well-posed on the space Z_α; (ii) satisfies

$\mathcal{F}(w_t) \in L_2(0,\infty;L_2(\Gamma))$, and (iii) decays in the (possibly) uniform operator topology of Z_α as $t \to +\infty$. Since $\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}$ is skew-adjoint on Z_α, Eq. (3.1b) plainly suggests to take $\mathcal{F} = -G_2^*A^{1+2\alpha}$, i.e., $u = -G_2^*A^{1+2\alpha}w_t$ as a natural candidate for feedback stabilization, for this choice then makes the corresponding feedback operator introduced in (1.7b)

$$\mathcal{A} = \begin{vmatrix} 0 & I \\ -A & -AG_2G_2^*A^{1+2\alpha} \end{vmatrix} = \begin{vmatrix} 0 & I \\ -A & -A^{1/2}DD^*A^{1/2+2\alpha} \end{vmatrix}, \quad \mathcal{D}(\mathcal{A}) = \{y \in Z_\alpha : \mathcal{A}y \in Z_\alpha\} \tag{3.2}$$

dissipative on Z_α (in (3.2) we have used relation (2.5) between G_2 and D); indeed, from (1.4) and $Z_\alpha = \mathcal{D}(A^{1/2+\alpha})\times\mathcal{D}(A^\alpha)$, we obtain with $y = [y_1,y_2]$

$$\begin{aligned} \mathrm{Re}(\mathcal{A}y,y)_{Z_\alpha} &= \mathrm{Re}\left(\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} y,y\right)_{Z_\alpha} - (A^{1+2\alpha}G_2G_2^*A^{1+2\alpha}y_2,y_2)_{L_2(\Omega)} \\ &= 0-\|G_2^*A^{1+2\alpha}y_2\|^2_{L_2(\Gamma)} \le 0, \quad y \in \mathcal{D}(\mathcal{A}). \end{aligned} \tag{3.3}$$

A more explicit description of $\mathcal{D}(\mathcal{A})$ will be given in Section 4, below Remark 4.1. With the above choice for the feedback operator, and recalling (2.8), (2.5), we have explicitly (1.5), i.e.,

$$\Delta w|_\Sigma = u = -G_2^*A^{1+2\alpha}w_t = -\frac{\partial A^{2\alpha}w_t}{\partial\nu} = D^*A^{1/2+2\alpha}w_t. \tag{3.4}$$

Thus, the resulting closed loop problem, wehre (3.4) is inserted in (3.1a), takes on the form (see (2.4) and (2.5)),

$$w_{tt} = -A[w+G_2G_2^*A^{1+2\alpha}w_t] \text{ on } L_2(\Omega); \text{ or } w_{tt} = -Aw-A^{1/2}DD^*A^{1/2+2\alpha}w_t, \tag{3.5}$$

as anticipated in (1.7a) whose explicit partial differential equation version is problem (1.6).

<u>Proof of Theorem 1.2 (Sketch)</u>. We omit the details for the explicit computation of the resolvent $R(\lambda,\mathcal{A})$ in (1.9) and refer to

[L-T.1], [T.1], [B-T.1], [O-T.1], for similar computations for waves and plates. The well-posedness of the feedback problem (3.5), or (1.6.), as a s.c. contraction semigroup $e^{\mathcal{A}t}$ on Z_α, is a consequence of the Lumer-Phillips theorem, by (3.3) and the well-defined $R(\lambda,\mathcal{A})$ in (1.9) for $\lambda > 0$, from which inequality (1.8) follows at once, as usual; see the references listed above for conceptually similar situations for waves and other plate problems. It remains to show parts (iv) and (v) of Theorem 1.2. Actually, the strong stabilization in part (v) follows in the usual way, via the Nagy-Foias-Foguel decomposition of contraction semigroups, as in [L-T.1], [T.1], [B-T.1], once we prove part (iv), i.e., that there are no nonzero eigenvalues of $\mathcal{A}$ on the imaginary axis (that $0 \in \rho(\mathcal{A})$ is immediate); or, by (1.9a) that $V^{-1}(\lambda) \in \mathcal{L}(L_2(\Omega))$ for $\lambda = ir$, $r \neq 0$ real and $V(\lambda)$ as in (1.9b). For otherwise, letting $V(ir)x = 0$ and taking the $L_2(\Omega)$-inner product with x leads, as usual [L-T.1], [T.1], [B-T.1], to (*) $G_2^*A^{½+\alpha}x = 0$, or $Ax = r^2x$ and x, if nonzero, is an eigenvector $x = e_n$ of A with eigenvalue r^2, so that $e_n|_\Gamma = \Delta e_n|_\Gamma = 0$. Moreover, (*) yields both $G_2^*AA^{-½+\alpha}e_n = (r^2)^{-½+\alpha}\frac{\partial e_n}{\partial\nu}|_\Gamma = 0$ by (2.8), as well as $G_2^*A^{½}A^{-1+\alpha}e_n = -(r^2)^{-1+\alpha}\frac{\partial\Delta e_n}{\partial\nu}|_\Gamma = 0$ by (2.7). Thus, e_n satisfies all four boundary conditions and hence $x = e_n = 0$, as desired. The proof of part (iv) is complete. ■

<u>Remark 2.1</u>. Generally, for an operator A which corresponds to a fourth-order differential operator, the above argument produces the vanishing of only three boundary conditions for e_n: the two corresponding to the definition of $\mathcal{D}(A)$ and the third due to the contradiction argument which yields a relation like (*) above. In our particular set of B.C. which gives $A^{½}$ as defined by (1.3), and hence (2.7) and (2.8), the vanishing of $\frac{\partial e_n}{\partial\nu}|_\Gamma = 0$ and $\frac{\partial(\Delta e_n)}{\partial\nu}|_\Gamma = 0$ are an equivalent condition. But, in general, this is not the case; and non-trivial extra work, perhaps subject to geometrical conditions on Ω as in the case of [B-T.1], [O-T.1] where different B.C. are

considered, is needed to obtain the fourth boundary condition for e_n. This complication does not arise if instead A corresponds to a second-order differential operator, for then $\mathcal{D}(A)$ produces the vanishing of one boundary condition for e_n, while the counterpart of (*) produces the vanishing of the required second boundary condition, the total effect of which is to produce e_n. ∎

For future use, in Section 4, we note explicitly that for the case $\alpha = 0$ the closed loop problem in abstract form ((3.5) for $\alpha = 0$) becomes

$$w_{tt} = -Aw + A^{1/2}DD^*A^{1/2}w_t; \quad \mathcal{A} = \begin{vmatrix} 0 & I \\ -A & -A^{1/2}DD^*A^{1/2} \end{vmatrix}, \tag{3.6}$$

or in explicit p.d.e. version

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{on } (0,T)\times\Omega = Q; & (3.7a) \\ w(0,\cdot) = w_0 \in \mathcal{D}(A^{1/2});\ w_t(0,\cdot) = w_1 \in L_2(\Omega) & \text{on } \Omega; & (3.7b) \\ w|_\Sigma = 0 & \text{on } (0,T)\times\Gamma = \Sigma; & (3.7c) \\ \Delta w|_\Sigma = -\frac{\partial w_t}{\partial \nu}\Big|_\Sigma = D^*A^{1/2}w_t & \text{on } (0,T)\times\Gamma = \Sigma. & (3.7d) \end{cases}$$

4. Proof of Theorem 1.3: Uniform Stabilization of the Euler-Bernoulli Problem (1.1) on $Z = \mathcal{D}(A^{1/2})\times L_2(\Omega)$

4.1. Preliminaries and a Change of Variable $w \to p$

For the feedback problem (3.7) or (3.8), we define the 'energy functional' E(t) by the squared norm of the (feedback) semigroup in Theorem 1.2 for $\alpha = 0$ (see (1.4)):

$$E(t) = E(w,t) = \left\| e^{\mathcal{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z^2 = \left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_Z^2$$

$$= \|A^{1/2}w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} \le E(0) \tag{4.1}$$

by the contraction property of Theorem 1.2(i). Our main goal will be, as usual, to show that there exists a time $0 < T < \infty$ and a corresponding constant $C = C_T > 0$ such that

$$E(T) \leq C_T \int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma, \tag{4.2}$$

for then (6.1), combined with

$$E(0) = E(T) + 2\int_0^T \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma \tag{4.3}$$

(which is identity (1.8b) for $\alpha = 0$ as in the present case), yields $E(0) \geq (2+\frac{1}{C})E(T)$ and hence, as usual

$$E(T) \leq rE(0), \quad r < 1 \text{ or } \|e^{\mathcal{A}T}\|_{\mathcal{L}(Z)} < 1, \tag{4.4}$$

which implies the desired uniform (exponential) decay (1.13). (We refer to [B-T.1, Remark 3.1] for comments on the advantages and disadvantages of using criterion (4.4) over Datko's theorem.) A more explicit description of $y = [y_1, y_2] \in \mathcal{D}(\mathcal{A})$ is as follows from (3.7): (i) $y_2 \in \mathcal{D}(A^{½})$, or $A^{½}y_2 \in L_2(\Omega)$; (iii) $y_1 + A^{-½}DD^*A^{½}y_2 \in \mathcal{D}(A)$, or recalling (2.10), (1.3), and (i), we have $A^{½}y_1 \in H^1(\Omega)$. By Theorem 1.2, with $\alpha = 0$, if $\{w_0, w_1\} \in \mathcal{D}(\mathcal{A})$, then $\{w(t), w_t(t)\} \in C([0,T];\mathcal{D}(\mathcal{A}))$, and thus

$$A^{½}w(t) \in C([0,T];H^1(\Omega)); \; A^{½}w_t(t) \in C([0,T];L_2(\Omega)),$$
$$\{w_0, w_1\} \in \mathcal{D}(\mathcal{A}). \tag{4.5}$$

Adapting to present circumstances, the ideas of [L-T.1], [B-T.1], we introduce a new variable p by setting

$$p = A^{-1/2}w_t \in \begin{cases} C([0,T];\mathcal{D}(A^{1/4})) & \text{if } [w_0,w_1] \in Z; & (4.6a) \\ C([0,T];\mathcal{D}(A)) & \text{if } [w_0,w_1] \in \mathcal{D}(\mathcal{A}), & (4.6b) \end{cases}$$

see Theorem 1.2 and (4.4) respectively. Thus, by (4.6) and (3.6),

$$p_t = A^{-1/2}w_{tt}$$
$$= -A^{1/2}w - DD^*A^{1/2}w_t \in \begin{cases} L_2(0,T;H^1(\Omega)), & \text{if } [w_0,w_1] \in Z; & (4.7a) \\ C([0,T];\mathcal{D}(A^{1/2})), & \text{if } [w_0,w_1] \in \mathcal{D}(\mathcal{A}), & (4.7b) \end{cases}$$

where the regularity follows from (4.5) and (3.6), and (iii) above (4.5) respectively. Finally, by (4.5), (4.4),

$$p_{tt} = -A^{1/2}w_t - DD^*A^{1/2}w_{tt} = -Ap - DD^*A^{1/2}w_{tt}. \tag{4.8}$$

In terms of the scalar function $p(t,x)$, $x \in \Omega$, which corresponds to the vector-valued function $p(t) = p(t,\cdot)$, the abstract equation (4.8) can be rewritten explicitly as the following Euler-Bernoulli homogeneous problem:

$$\begin{cases} p_{tt} + \Delta^2 p = F & \text{on } (0,T)\times\Omega; & (4.9a) \\ p(0,\cdot) = p_0 = A^{-1/2}w_t(0) \in \mathcal{D}(\mathcal{A}) & \text{in } \Omega; & (4.9b) \\ p_t(0,\cdot) = p_1 = -A^{1/2}w(0) - DD^*A^{1/2}w_t(0) \in \mathcal{D}(A^{1/2}) & \text{in } \Omega; & (4.9c) \\ p|_\Sigma = \Delta p|_\Sigma = 0 & \text{on } (0,T)\times\Gamma = \Sigma; & (4.9d) \end{cases}$$

$$F = -DD^*A^{1/2}w_{tt} = D\frac{\partial \Delta p_t}{\partial \nu} \quad \text{on } \Sigma, \tag{4.10}$$

where (4.7b) is used in (4.9b), and where the homogeneous boundary conditions (4.9d) are a consequence of $p \in \mathcal{D}(\mathcal{A})$ from (4.6b). In our arguments in the sequel, we shall have to consider pointwise values of $p_t(t)$ in $H_0^1(\Omega)$, which make sense for actual data $[w_0,w_1] \in \mathcal{D}(\mathcal{A})$ as assumed, by (4.7b) and $\mathcal{D}(A^{1/2}) \subset \mathcal{D}(A^{1/4}) = H_0^1(\Omega)$. In the subsequent analysis we shall crucially use that the change of variables implies

$$\|w_t(t)\|^2_{L_2(\Omega)} = \|A^{1/2}p(t)\|^2_{L_2(\Omega)} = \int_\Omega (\Delta p(t))^2 d\Omega \le E(t) \le E(0); \quad (4.11)$$

$$p_t(t) = -A^{1/2}w(t) + \mathcal{O}(\|D^*A^{1/2}w_t(t)\|_{L_2(\Gamma)}); \quad (4.12)$$

$$\frac{\partial(\Delta p)}{\partial\nu}\Big|_\Sigma = D^*Ap = D^*A^{1/2}w_t = \Delta w|_\Sigma = -\frac{\partial w_t}{\partial\nu}\Big|_\Sigma, \quad (4.13)$$

where the constant in $\mathcal{O}$ is $\|D\|$. The $L_2(\Omega)$-norms $\|A^{1/2}p\|$ and $\|p_t\|$ in (4.11), (4.12) will arise in the multiplier approach used below, and this justifies the need of introducing a new variable p.

Moreover, we shall use that

$$\int_\Omega |\nabla p|^2 d\Omega \text{ equivalent to } \|A^{1/4}p\|^2 \le C\|A^{1/2}p\|^2 = C\int_\Omega (\Delta p)^2 d\Omega. \quad (4.14)$$

4.2. <u>An Identity for the p-System (4.9)</u>

<u>Proposition 4.1</u>. The following identity holds true for problem (4.9) where $[w_0,w_1] \in \mathcal{D}(\mathcal{A})$, hence $[p_0,p_1] \in \mathcal{D}(A)\times\mathcal{D}(A^{1/2})$, where $Q = (0,T)\times\Omega$; $\Sigma = (0,T)\times\Gamma$,

$$-\int_\Sigma \frac{\partial(\Delta p)}{\partial\nu}\frac{\partial p}{\partial\nu} h\cdot\nu d\Sigma = 2\int_Q \Delta p\Big(\sum_{i=1}^n \nabla h_i\cdot\nabla p_{x_i}\Big)dQ$$

$$+ \tfrac{1}{2}\int_Q p\Delta p\,\Delta(\text{div } h)dQ + \int_Q \Delta p\nabla p\cdot\nabla(\text{div } h)dQ$$

$$+ \int_Q \Delta p[\Delta h_1,\ldots,\Delta h_n]\cdot\nabla p dQ - \int_Q Fh\cdot p dQ - \tfrac{1}{2}\int_Q Fp \text{ div } h\, dQ$$

$$+ \Big[(p_t(t),h\cdot\nabla p(t))_\Omega + \tfrac{1}{2}(p_t(t),p(t)\text{div } h)_\Omega\Big]_0^T. \quad (4.15)$$

<u>Remark 4.1</u>. We note explicitly that the following identities hold true:

$$\operatorname{div}(H\nabla p) = \sum_{i=1}^{n} \nabla h_i \cdot \nabla p_{x_i} + \nabla p \cdot \nabla(\operatorname{div} h); \tag{4.16}$$

$$\operatorname{div}(H^T\nabla p) = \sum_{i=1}^{n} \nabla h_i \cdot \nabla p_{x_i} + [\Delta h_1, \ldots, \Delta h_n] \cdot \nabla p, \tag{4.17}$$

where $H = H(x)$ is the $n\times n$ matrix with (i,j)-entry given by $\frac{\partial h_i}{\partial x_j}$ as in (1.) and H^T its transpose, so that (4.16) and (4.17) imply

$$\operatorname{div}[(H+H^T)\nabla p] = 2\sum_{i=1}^{n} \nabla h_i \cdot \nabla p_{x_i} + \nabla p \cdot \nabla(\operatorname{div} h) + [\Delta h_1, \ldots, \Delta h_n] \cdot \nabla p,$$

and hence (4.15) can be rewritten as

$$\begin{aligned} -\int_\Sigma \frac{\partial(\Delta p)}{\partial\nu} \frac{\partial p}{\partial\nu} h\cdot\nu d\Sigma = & \int_Q \Delta p \operatorname{div}[(H+H^T)\nabla p] dQ \\ & + ½\int_Q p\Delta p\, \Delta(\operatorname{div} h) dQ \\ & - \int_Q Fh\cdot\nabla p dQ - ½\int_Q Fp \operatorname{div} h\, dQ \\ & - (p_1, h\cdot\nabla p_0)_\Omega - ½(p_1, p_0 \operatorname{div} h)_\Omega. \end{aligned} \tag{4.18}$$

__Proof__. The proof is carried out in Appendices A and B. (For $h(x)$ a radial field, it reduces to identities in [Lag.2], [L-L.1].) ∎

The analysis below will show *a fortiori* that the terms in identity (4.15) are well defined by establishing appropriate estimates thereof. To this end, the crucial term is the one involving $Fh\cdot\nabla p = -DD^*A^{½}w_{tt}h\cdot\nabla p$. What follows is our basic identity.

__Proposition 4.2__. Let $\{w_0, w_1\} \in \mathcal{D}(\mathcal{A})$. The following identity holds true for problem (4.9):

$$-\int_\Sigma \frac{\partial(\Delta p)}{\partial\nu}\frac{\partial p}{\partial\nu}\, h\cdot\nu d\Sigma = 2\int_Q \Delta p\Big(\sum_{i=1}^n \nabla h_i\cdot\nabla p_{x_i}\Big)dQ + \tfrac{1}{2}\int_Q p\Delta p\, \Delta(\text{div } h)dQ$$

$$+\int_Q \Delta p\nabla p\cdot\nabla(\text{div } h)dQ + \int_Q \Delta p[\Delta h_1,\dots,\Delta h_n]\cdot\nabla p dQ$$

$$-\int_0^T (DD^*A^{1/2}w_t, h\cdot\nabla p_t)_\Omega dt - \tfrac{1}{2}\int_0^T (DD^*A^{1/2}w_t, p_t \text{ div } h)_\Omega dt + \rho_{0,T}; \qquad (4.19)$$

$$\rho_{0,T} = -[(A^{1/2}w(t), h\cdot\nabla p(t))_\Omega + \tfrac{1}{2}(A^{1/2}w(t), p(t)\text{div } h)_\Omega]_0^T. \qquad (4.20)$$

__Proof__. We proceed as in Lemma B.1, Appendix B.

__Step 1__. Integrating by parts in t and recalling the term F in (4.10), we find

$$-\int_0^T (F, h\cdot\nabla p)_\Omega dt = \int_0^T (DD^*A^{1/2}w_{tt}, h\cdot\nabla p)_\Omega dt = [(DD^*A^{1/2}w_t(t), h\cdot\nabla p(t))_\Omega]_0^T$$

$$- \int_0^T (DD^*A^{1/2}w_t, h\cdot\nabla p_t)_\Omega dt; \qquad (4.21)$$

$$-\tfrac{1}{2}\int_0^T (F, p \text{ div } h)_\Omega dt = \tfrac{1}{2}\int_0^T (DD^*A^{1/2}w_{tt}, p \text{ div } h)_\Omega dt$$

$$= +\tfrac{1}{2}[(DD^*A^{1/2}w_t(t)), p(t)\text{div } h)_\Omega]_0^T - \tfrac{1}{2}\int_0^T (DD^*A^{1/2}w_t, p_t \text{div } h)_\Omega dt. \qquad (4.22)$$

__Step 2__. We now insert (4.21), (4.22) into the right hand side of (4.15) and use (4.7): $p_t + DD^*A^{1/2}w_t = -A^{1/2}w$ at $t = T$ and $t = 0$ in combining the $(\ ,\ )_\Omega$-terms. This way, (4.15) becomes (4.19). ∎

__Proposition 4.3__. Let $\{w_0, w_1\} \in \mathcal{D}(\mathcal{A})$. Assume further that Ω satisfies inequality (1.11) for some smooth vector field h(x) (but __not__ necessarily condition (i) of Definition 1.1 which requires h to be parallel to ν). With reference to the right hand side (R.H.S.) of identity (4.19), we then have for any $\varepsilon > 0$,

$$\text{R.H.S. of (4.19)} \geq (2\rho-\varepsilon)\int_Q (\Delta p)^2 dQ - C_{1h\varepsilon}\int_Q |\nabla p|^2 dQ - C_h\left[E(T)+\int_0^T\int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma\right]$$

$$- \int_0^T (DD^* A^{1/2} w_t, h\cdot\nabla p_t)_\Omega dt - 1/2\int_0^T (DD^* A^{1/2} w_t, p_t \text{ div } h)_\Omega dt, \tag{4.23}$$

where $\rho > 0$ is the constant in assumption (1.11) and where the constant $C_{1h\varepsilon}$ and C_h depend on ε and on the vector field but not on T; moreover, we have $C_{1h\varepsilon} = 0$ if h(x) is linear in x, in particular a radial field. ■

<u>Proof</u>. For the first term on the right hand side of identity (4.19), we use assumption (1.11). For the second, third, and fourth terms on the right of identity (4.19), we use (*) $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ with $a = |\Delta p|$ and b either p or else $|\nabla p|$ plus Poincaré inequality. (We note that all these three terms vanish if h(x) is linear in x.) Finally, from (4.20) we readily obtain via (4.1), (4.14), (4.11) and Poincaré inequality

$$|\rho_{0T}| \leq C_h[E(T)+E(0)] \leq C_h\left[E(T)+\int_0^T\int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma\right], \tag{4.24}$$

where in the last step we have used (4.3). The constant C_h in (4.24) depends on h, but not on T. Using (4.24) results in (4.23). ■

4.3. <u>Analysis of Terms in (4.23) Involving</u> $D(\Delta w|_\Sigma) = DD^* A^{1/2} w_t$: <u>Completion of Proof of Theorem 1.3</u>

The next proposition deals with the most demanding term in (4.23). In handling this term, we shall encounter a technical difficulty similar to the one met in [L-T.1] for the wave equation with feedback in the Dirichlet B.C., in particular [L-T.1, Lemma 3.3]. It is in overcoming this difficulty that the geometrical condition that the vector field h be parallel to ν comes into play. It is in order to emphasize the technical analogy between the present problem with the Euler-Bernoulli equation and one feedback control and the problem with the wave equation in [L-T.1] that we have reduced the original Green operator G_2 to the operator D which is the same Dirichlet map (2.4) which occurs in the wave equation problem [L-T.1].

__Theorem 4.4__. Let $\{w_0, w_1\} \in \mathcal{D}(A)$. Let the vector field h(x) be parallel to $\nu(x)$ on Γ. Then the following estimate holds true:

$$\int_0^T (DD^* A^{1/2} w_t, h\cdot\nabla p_t)_\Omega = \mathcal{O}\Big(\int_0^T \|D^* A^{1/2} w_t\|^2_{L_2(\Gamma)} dt + \int_0^T \|D^* A^{1/2} w_t\|_{L_2(\Gamma)} \|A^{1/2} w\|_{L_2(\Omega)} dt\Big) + \mathcal{O}([E(T)+E(0)]), \tag{4.24}$$

where the constants in $\mathcal{O}$ are of the form $\|D\| c_h$, in particular they do not depend on T. ■

__Proof__. The proof is given in Subsection 4.4. ■

Using now Theorem 4.4, we can complete the proof of inequality (4.2), and thus of Theorem 1.3.

__Proposition 4.5__. Let the vector field condition of Definition 1.1 holds true (so that both Proposition 4.3 and Theorem 4.4 apply). Then we have

$$-\int_\Sigma \frac{\partial(\Delta p)}{\partial\nu}\frac{\partial p}{\partial\nu} h\cdot\nu d\Sigma \geq 2(\rho-\varepsilon)\int_0^T \|A^{1/2} w\|^2_{L_2(\Omega)} + \|w_t\|^2_{L_2(\Omega)} dt - C_{1h\varepsilon} T \||\nabla p|\|^2_{C([0,T];L_2(\Omega))} - C_h E(T) + C_{h\varepsilon}\int_0^T\int_\Gamma \left[\frac{\partial w_t}{\partial\nu}\right]^2 d\Sigma. \tag{4.25}$$

Proof. Since $\Delta w|_{\Sigma} = D^{*}A^{\frac{1}{2}}w_t = -\frac{\partial w_t}{\partial \nu}\Big|_{\Sigma}$ from (3.4) with $\alpha = 0$, and recalling (4.3), we rewrite (4.24) as

$$\left|\int_0^T (DD^{*}A^{\frac{1}{2}}w_t, h\cdot\nabla p_t)_{\Omega}dt\right| \le \varepsilon\int_0^T \|A^{\frac{1}{2}}w\|^2_{L_2(\Omega)}dt + \frac{C_h}{\varepsilon}\int_0^T\int_{\Gamma}\left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma + C_h E(T). \tag{4.26}$$

Moreover, the last integral term in (4.23) can be estimated as

$$\left|\int_0^T (DD^{*}A^{\frac{1}{2}}w_t, p_t \text{ div } h)_{\Omega}\right| \le \varepsilon\int_0^T \|p_t\|^2_{L_2(\Omega)}dt + \frac{C_h}{\varepsilon}\int_0^T \|D^{*}A^{\frac{1}{2}}w_t\|^2_{L_2(\Gamma)}dt$$
$$\le \varepsilon\int_0^T \|A^{\frac{1}{2}}w\|^2_{L_2(\Omega)} + \frac{C_h}{\varepsilon}\int_0^T\int_{\Gamma}\left(\frac{\partial w_t}{\partial \nu}\right)^2 d\Sigma. \tag{4.27}$$

Inserting (4.26) and (4.27) on the right of (4.23) results in (4.25), after recalling (4.11). ∎

Corollary 4.6. Under the assumptions on Ω of Proposition 4.5, we have the inequality

$$C_h\left\{\int_{\Sigma}\left(\frac{\partial(\Delta p)}{\partial \nu}\right)^2 d\Sigma + \int_{\Sigma}\left(\frac{\partial p}{\partial \nu}\right)^2 d\Sigma\right\} + C_{1h\varepsilon}T\||\nabla p|\|^2_{C([0,T];L_2(\Omega))}$$
$$\ge [2(\rho-\varepsilon)T - C_h]E(T). \tag{4.28}$$

Proof. We recall (4.1) in the first integral term on the right of (4.25), as well as $\int_0^T E(t)dt \ge TE(0)$ by the dissipativity property (1.8a). Moreover, we recall (4.13). Thus (4.25) becomes (4.28). ∎

We next 'absorb' the lower-order terms in (4.28).

<u>Lemma 4.7</u>. Inequality (4.28) implies: there is $C_T > 0$ such that

$$\int_0^T\int_\Gamma \left(\frac{\partial p}{\partial\nu}\right)^2 d\Sigma + \| |\nabla p| \|^2_{C([0,T]L_2(\Omega))} \le C_T \int_0^T\int_\Gamma \left(\frac{\partial(\Delta p)}{\partial\nu}\right)^2 d\Sigma. \quad (4.29)$$

<u>Proof</u>. We proceed as in the proof of Proposition 3.5 in the special case $s = 0$ rather than $s = \frac{1}{2}$, with respect this time to the p-problem (4.9). We only note explicitly that when we arrive at $\frac{\partial(\Delta\tilde{p})}{\partial\nu} = 0$ on Σ (counterpart of the last identity in (2.38b)) for the limit $\tilde{p}$, we then obtain that the right hand side of the Eq. (4.9a) for the p-problem becomes $\tilde{F} = D\frac{\partial\Delta\tilde{p}_t}{\partial\nu} \equiv 0$ by (4.10). Thus the $\tilde{p}$-problem is homogeneous on the right hand side, precisely like the $\tilde{\phi}$-problem in (2.38). The rest of the proof may then follow the argument of Proposition 3.5 below (2.38), and is based on the uniqueness property of the resulting $\tilde{p}$-problem (same as the $\tilde{\phi}$-problem (2.38)) to produce a contradiction. ∎

<u>Corollary 4.8</u>. Under the assumption of Theorem 1.3, we obtain

$$\int_0^T\int_\Gamma \left(\frac{\partial w_t}{\partial\nu}\right)^2 d\Sigma \ge C_{1\rho\epsilon h}(T - C_{2\rho\epsilon h})E(T),$$

and with T sufficiently large, inequality (4.2) is proved. ∎

Thus, the proof of Theorem 1.3 is complete as soon as we prove Theorem 4.4.

4.4. <u>Proof of Theorem 4.4: h Parallel to ν on Γ</u>

We follow as a guideline the proof of [L-T.7, Proposition 3.2].

<u>Proposition 4.9</u>. We have, where we recall that $\Delta w|_\Sigma = D^* A^{\frac{1}{2}} w_t = -\frac{\partial w_t}{\partial\nu}|_\Sigma$ (from (3.4) with $\alpha = 0$ as in our present case):

$$\int_0^T (DD^* A^{\frac12} w_t, h\cdot\nabla p_t)_\Omega dt = \int_0^T (A^{\frac12} w, h\cdot\nabla(DD^* A^{\frac12} w_t))_\Omega dt$$

$$+ \mathcal{O}(\int_0^T \|D^* A^{\frac12} w_t\|^2_{L_2(\Gamma)} dt) + \mathcal{O}(\int_0^T \|D^* A^{\frac12} w_t\|_{L_2(\Gamma)} \|A^{\frac12} w\|_{L_2(\Omega)} dt), \quad (4.30)$$

where the constants in $\mathcal{O}$ are of the form $\|D\|C_h$, in particular, they do not depend on T. ■

<u>Proof</u>. Recalling p_t in (4.7), we write

$$-\int_0^T (DD^* A^{\frac12} w_t, h\cdot\nabla p_t)_\Omega dt = I_1 + I_2; \quad (4.31)$$

$$I_1 = \int_0^T (DD^* A^{\frac12} w_t, h\cdot\nabla(DD^* A^{\frac12} w_t))_\Omega dt; \quad (4.32)$$

$$I_2 = \int_0^T (DD^* A^{\frac12} w_t, h\cdot\nabla(A^{\frac12} w))_\Omega dt. \quad (4.33)$$

We first estimate I_1. We claim (see (3.4))

$$I_1 = \mathcal{O}(\int_0^T \|D^* A^{\frac12} w_t\|^2_{L_2(\Gamma)} dt). \quad (4.34)$$

In fact, to show (4.34), we use the identity (obtained from, say, (A.1) in Appendix A),

$$\int_\Omega \phi h\cdot\nabla\psi \, d\Omega = \int_\Gamma \phi\psi h\cdot\nu \, d\Gamma - \int_\Omega \psi h\cdot\nabla\phi \, d\Omega - \int_\Omega \phi\psi \,\mathrm{div}\, h \, d\Omega \quad (4.35a)$$

for, say, $\phi,\psi \in H^1(\Omega)$, which for $\psi = \phi$ specializes to

$$\int_\Omega \psi h\cdot\nabla\psi \, d\Omega = \tfrac12\int_\Gamma \psi^2 h\cdot\nu \, d\Gamma - \tfrac12\int_\Omega \psi^2 \,\mathrm{div}\, h \, d\Omega. \quad (4.35b)$$

Taking $\psi = DD^*A^{1/2}w_t$ in (4.35b), whereby $\psi|_\Gamma = D^*A^{1/2}w_t$ by (2.4a), we obtain, via (2.4b):

$$(DD^*A^{1/2}w_t, h\cdot\nabla(DD^*A^{1/2}w_t))_\Omega = \tfrac{1}{2}(D^*A^{1/2}w_t, D^*A^{1/2}w_t h\cdot\nu)_\Gamma$$

$$- \tfrac{1}{2}(DD^*A^{1/2}w_t, DD^*A^{1/2}w_t \text{ div } h)_\Omega) = \mathcal{O}(\|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)}), \quad (4.36)$$

and (4.34) follows. We next estimate I_2 in (4.33). We claim that

$$I_2 = \int_0^T (D^*A^{1/2}w_t, A^{1/2}w\ h\cdot\nu)_\Gamma dt - \int_0^T (DD^*A^{1/2}w_t, A^{1/2}w \text{ div } h)_\Omega dt$$

$$- \int_0^T (A^{1/2}w, h\cdot\nabla(DD^*A^{1/2}w_t))_\Omega dt$$

$$= -\int_0^T (A^{1/2}w, h\cdot\nabla(DD^*A^{1/2}w_t))_\Omega dt + \mathcal{O}(\int_0^T \|D^*A^{1/2}w_t\|^2_{L_2(\Gamma)} dt)$$

$$+ \mathcal{O}(\int_0^T \|D^*A^{1/2}w_t\|_{L_2(\Gamma)} \|A^{1/2}w\|_{L_2(\Omega)} dt). \quad (4.38)$$

In fact, (4.37) follows at once from (4.33) by using identity (4.35a) with $\phi = DD^*A^{1/2}w_t$, hence $\phi|_\Gamma = D^*A^{1/2}w_t$ by (2.4a) and $\psi = A^{1/2}w$. Then (4.38) follows from (4.37), by noticing that $(A^{1/2}w)|_\Gamma = (-\Delta w)|_\Gamma = \frac{\partial w_t}{\partial \nu} = D^*A^{1/2}w_t$ by (1.3), (1.1c), and (3.4) with $\alpha = 0$, via Schwarz inequality and D continuous. Finally, (4.31), (4.34), and (4.38) yield (4.30). Proposition 4.9 is proved. ■

We finally handle the first integral term at the right side of (4.30). It is this term which presents technical difficulties similar to those encountered in the wave problem with Dirichlet feedback [L-T.1]. These are overcome when the vector field h(x) is parallel to the normal ν on Γ.

<u>Lemma 4.10</u>. Let $\{w_0, w_1\} \in \mathcal{D}(A)$ and let the vector field h(x) be parallel to the normal unit vector ν on Γ, so that $h(\sigma) = b(\sigma)\nu$, $\sigma \in \Gamma$, for a smooth boundary function b. Then we have

$$(A^{\frac{1}{2}}w(t), h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t(t))_\Omega = \tfrac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial\nu} D(bD^*A^{\frac{1}{2}}w(t)), D^*A^{\frac{1}{2}}w(t)\right)_\Gamma$$

$$+ \mathcal{O}(\|D^*A^{\frac{1}{2}}w_t(t)\|_{L_2(\Gamma)}\|A^{\frac{1}{2}}w(t)\|_{L_2(\Omega)}) \quad \text{a.e. in } t. \tag{4.39}$$

<u>Proof</u>. <u>Step 1</u>. Recalling (1.3) and $w|_\Sigma = 0$ in (1.1c) and using Green's second theorem, we obtain (all inner products are in L_2, unless otherwise noted)

$$-(A^{\frac{1}{2}}w, h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t))_\Omega = (\Delta w, h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t))_\Omega = (1) + (2). \tag{4.40}$$

$$(1) = \left(\frac{\partial w}{\partial\nu}, h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t)\right)_\Gamma = -(D^*A^{\frac{1}{2}}w, h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t))_\Gamma \quad \text{(by (2.8))}; \tag{4.41}$$

$$(2) = (w, \Delta(h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t))_\Omega. \tag{4.42}$$

<u>Step 2</u>. We analyze (1). We claim that

$$(1) = -\tfrac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial\nu} D(bD^*A^{\frac{1}{2}}w), D^*A^{\frac{1}{2}}w\right) + \mathcal{O}(\|D^*A^{\frac{1}{2}}w_t\|_{L_2(\Gamma)}\|A^{\frac{1}{2}}w\|_{L_2(\Omega)})$$

$$\text{a.e. in } t. \tag{4.43}$$

To prove (4.43), we use the assumption $h(\sigma) = b(\sigma)\nu$ on the vector field and rewrite (1) from (4.41) by means of Green's second theorem recalling that $Dg|_\Gamma = g$ by (2.4a):

$$(1) = -\left(D^*A^{\frac{1}{2}}w, b\frac{\partial}{\partial\nu}(DD^*A^{\frac{1}{2}}w_t)\right)_\Gamma = -\left(bD^*A^{\frac{1}{2}}w, \frac{\partial}{\partial\nu}(DD^*A^{\frac{1}{2}}w_t)\right)_\Gamma \tag{4.44}$$

$$= -\left(\frac{\partial}{\partial\nu} D(bD^*A^{\frac{1}{2}}w), D^*A^{\frac{1}{2}}w_t\right)_\Gamma, \tag{4.45}$$

where cancellations occur because of the definition of D in (2.4a). Next, we compute by (4.44), (4.45),

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial \nu} D(bD^*A^{1/2}w), D^*A^{1/2}w\right]_\Gamma = \left[\frac{\partial}{\partial \nu} D(bD^*A^{1/2}w_t), D^*A^{1/2}w\right]_\Gamma$$

$$+ (D^*A^{1/2}w, b\frac{\partial}{\partial \nu}(DD^*A^{1/2}w_t))_\Gamma. \quad (4.46)$$

Using $\Delta(\beta\gamma) = \beta\Delta\gamma+\gamma\Delta\beta+2\nabla\beta\cdot\nabla\gamma = 2\nabla(Db)\cdot\nabla(Dg)$, if $\beta = Db$, and $\gamma = Dg$ for some vector $g \in L_2(\Gamma)$, we can readily verify that $D(bg) = (Db)(Dg)-\chi$, hence

$$\text{on } \Gamma\text{:}\ \frac{\partial D(bg)}{\partial \nu} = b\frac{\partial (Dg)}{\partial \nu} + g\frac{\partial (Db)}{\partial \nu} - \frac{\partial \chi}{\partial \nu}, \quad (4.47)$$

where χ satisfies

$$\Delta\chi = 2\nabla(Db)\cdot\nabla(Dg) \text{ in } \Omega; \quad \chi = 0 \text{ on } \Gamma;$$

$$\text{or } \chi = 2A^{-1}[\nabla(Db)\cdot\nabla(Dg)]. \quad (4.48)$$

We now specialize (4.47) to the case of our interest where $g = D^*A^{1/2}w_t \in L_2(\Gamma)$ a.e. in t by (3.6). Thus, the right hand side (R.H.S.) of (4.46) becomes by (4.47),

$$\text{R.H.S. of (4.46)} = (b\frac{\partial}{\partial \nu}(DD^*A^{1/2}w_t, D^*A^{1/2}w)_\Gamma + (D^*A^{1/2}w,\ b\frac{\partial}{\partial \nu}(DD^*A^{1/2}w_t))_\Gamma$$

$$+ (D^*A^{1/2}w_t\frac{\partial (Db)}{\partial \nu},\ D^*A^{1/2}w)_\Gamma - (\frac{\partial \chi}{\partial \nu},\ D^*A^{1/2}w)_\Gamma \quad (4.49)$$

$$= 2(b\frac{\partial}{\partial \nu}(DD^*A^{1/2}w_t, D^*A^{1/2}w)_\Gamma$$

$$+ \mathcal{O}(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\|A^{1/2}w\|_{L_2(\Omega)}), \quad (4.50)$$

since with $g = D^*A^{1/2}w_t \in L_2(\Gamma)$ a.e., $\nabla(Dg) \in H^{-1/2-\epsilon}(\Omega)$ [L-M.1, p. 85], we obtain $\chi \in H^{3/2-\epsilon}(\Omega)$ by (4.48), hence $\frac{\partial \chi}{\partial \nu} \in H^{-\epsilon}(\Gamma)$ [K.1, Theorem 3.8.1]; on the other hand, $D^*A^{1/2}w \in H^{1/2}(\Gamma)$ by (2.9) with $s = 0$, and thus

$$(\frac{\partial \chi}{\partial \nu},\ D^*A^{1/2}w)_\Gamma = \mathcal{O}(\|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\|A^{1/2}w\|_{L_2(\Omega)}) \text{ a.e. in t}, \quad (4.51)$$

which completes the proof of the step from (4.49) to (4.50). (The validity of (4.48) can be proved also by the use of Green's second theorem followed by identity (4.35a).) Then (4.46) and (4.50), along with (4.44), yield (4.43) as desired.

Step 3. We analyze (2). We claim that

$$(2) = (w,\Delta(h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t)))_\Omega = \mathcal{O}(\|D^*A^{\frac{1}{2}}w_t\|_{L_2(\Gamma)}\|A^{\frac{1}{2}}w\|_{L_2(\Omega)})$$

a.e. in t. (4.52)

This follows by writing

$$(2) = (A^{-\frac{1}{2}}A^{\frac{1}{2}}w,\Delta(h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t)))_\Omega, \tag{4.53}$$

with $A^{\frac{1}{2}}w \in L_2(\Omega)$, $D^*A^{\frac{1}{2}}w_t \in L_2(\Gamma)$ a.e. in t, and proceeding as in the proof of [L-T.1, Lemma 3.3] from (A.5) to (A.15) in Appendix A of this reference.

Step 4. Using identity (4.40), and the estimates (4.43) and (4.52), we obtain (4.39). ∎

The proof of Lemma 4.10 is complete. ∎

Lemma 4.10 allows us to complete the estimate of the first integral term on the right of (4.30), hence of the desired integral term of Proposition 4.9.

Corollary 4.11. Under the assumptions of Lemma 4.10, we have

$$\int_0^T (A^{\frac{1}{2}}w,h\cdot\nabla(DD^*A^{\frac{1}{2}}w_t))_\Omega dt = \mathcal{O}\Big(\int_0^T \|D^*A^{\frac{1}{2}}w_t\|_{L_2(\Gamma)}\|A^{\frac{1}{2}}w\|_{L_2(\Omega)}dt\Big)$$

$$+ \mathcal{O}(E(T)+E(0)), \tag{4.54}$$

where the constants in $\mathcal{O}$ depend on $\|D\|$, b, but not on T. ∎

Proof. From (4.39) by integration by parts in t:

$$\int_0^T (A^{1/2}w(t), h\cdot\nabla(DD^*A^{1/2}w_t(t)))_\Omega = +\tfrac{1}{2}\left[(\frac{\partial}{\partial\nu} D(bD^*A^{1/2}w(t), D^*A^{1/2}w(t))_\Gamma\right]_0^T + \mathcal{O}(\int_0^T \|D^*A^{1/2}w_t\|_{L_2(\Gamma)}\|A^{1/2}w\|_{L_2(\Omega)}dt). \quad (4.55)$$

Now $A^{1/2}w(t) \in L_2(\Omega)$ implies $D^*A^{1/2}w(t) \in H^{1/2}(\Gamma)$ by (2.9) with $s = 0$ and with b smooth, we have that $D(bD^*A^{1/2}w(t)) \in H^1(\Omega)$ by (2.4b), and it solves the Laplace equation. Therefore $\frac{\partial}{\partial\nu} D(bD^*A^{1/2}w(t)) \in H^{-1/2}(\Gamma)$ [K.1, Theorem 3.8.1, p. 71 and ff]. Thus

$$|(\frac{\partial}{\partial\nu} D(bD^*A^{1/2}w(t), D^*A^{1/2}w(t))_\Gamma| \le \|\frac{\partial}{\partial\nu} D(bD^*A^{1/2}w(t)\|_{H^{-1/2}(\Gamma)} \|D^*A^{1/2}w(t)\|_{H^{1/2}(\Gamma)}$$

$$\le C\|A^{1/2}w(t)\|^2_{L_2(\Omega)} \le CE(t). \quad (4.56)$$

Thus (4.56) used in (4.55) yields (4.54). ∎

To complete the proof of Theorem 4.4, we combine (4.54) with (4.30), thus obtaining (4.24). ∎

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Appendix A: Proof of Identity (4.15) of Proposition 4.1; and of Identity (2.24)

For future reference to exact controllability/uniform stabilization problems for Euler-Bernoulli equations with boundary conditions possibly different from (1.1c-d), we shall first derive a general identity, (A.9) below, for p only solution of Eq. (4.9a) in terms of an arbitrary smooth vector field $h(x) = h_1(x),\dots,h_n(x)]$ in, say, $C^2(\bar{\Omega})$. Only subsequently, we shall specialize such an identity (A.9) to p which also satisfies the boundary conditions (4.9d).

Identity for p Solution of (4.9a). We use the multiplier $h\cdot\nabla p$. We shall repeatedly invoke the identity

$$\int_\Omega h\cdot\nabla\psi \, d\Omega = \int_\Gamma \psi\cdot h\nu \, d\Gamma - \int_\Omega \psi \operatorname{div} h \, d\Omega \tag{A.1}$$

obtained from $\operatorname{div}(\psi h) = h\cdot\nabla\psi + \psi \operatorname{div} h$ and the divergence theorem.

Term $p_{tt}h\cdot\nabla p$. Integrating by parts in t yields, after setting throughout $Q = (0,T)\times\Omega$; $\Sigma = (0,T)\times\Gamma$:

$$\int_Q p_{tt}h\cdot\nabla p \, dt\, d\Omega = [\int_\Omega p_t h\cdot\nabla p \, d\Omega]_0^T - \int_Q p_t h\cdot\nabla p_t d\Omega. \tag{A.2}$$

Using (A.1) with h there replaced by $p_t h$ now, and with $\psi = p_t$ yields readily

$$\int_\Omega p_t h\cdot\nabla p_t d\Omega = ½\int_\Gamma p_t^2 h\cdot\nu \, d\Gamma - ½\int_\Omega p_t^2 \operatorname{div} h \, d\Omega. \tag{A.3}$$

Thus, by using (A.3) into (A.2), we obtain

$$\int_Q p_{tt}h\cdot\nabla p \, dQ = -½\int_\Sigma p_t^2 h\cdot\nu \, d\Sigma + [(p_t(t),h\cdot\nabla p(t))_\Omega]_0^T$$

$$+ ½\int_Q p_t^2 \operatorname{div} h \, dQ. \tag{A.4}$$

<u>Term</u> $\Delta^2 p h\cdot\nabla p$. By Green's second theorem,

$$\int_\Omega \Delta^2 p(h\cdot\nabla p)d\Omega = \int_\Omega \Delta p\ \Delta(h\cdot\nabla p)d\Omega + \int_\Gamma \frac{\partial(\Delta p)}{\partial\nu}\ h\cdot\nabla p\ d\Gamma - \int_\Gamma \Delta p\ \frac{\partial(h\cdot\nabla p)}{\partial\nu}\ d\Gamma. \quad (A.5)$$

Using the identity

$$\Delta(h\cdot\nabla p) = 2\sum_{i=1}^{n} \nabla h_i\cdot\nabla p_{x_i} + h\cdot\nabla(\Delta p) + [\Delta h_1,\ldots,\Delta h_n]\cdot\nabla p, \quad (A.6)$$

and invoking (A.1) for the second term of (A.6) with h there replaced by (Δp) h now and with $\psi = \Delta p$, we obtain

$$\int_\Omega \Delta p\ \Delta(h\cdot\nabla p)d\Omega = 2\int_\Omega \Delta p\left(\sum_{i=1}^{n} \nabla h_i\cdot\nabla p_{x_i}\right)d\Omega + \tfrac{1}{2}\int_\Gamma (\Delta p)^2 h\cdot\nu\ d\Gamma - \tfrac{1}{2}\int_\Omega (\Delta p)^2 \mathrm{div}\ h\ d\Omega + \int_\Omega \Delta p[\Delta h_1,\ldots,\Delta h_n]\cdot\nabla p\ d\Omega. \quad (A.7)$$

Using (A.7) and (A.5) yields finally

$$\int_\Omega \Delta^2 p(h\cdot\nabla p)d\Omega = \tfrac{1}{2}\int_\Gamma (\Delta p)^2 h\cdot\nu\ d\Gamma - \int_\Gamma \Delta p\ \frac{\partial(h\cdot\nabla p)}{\partial\nu}\ d\Gamma + 2\int_\Omega \Delta p\left(\sum_{i=1}^{n} \nabla h_i\cdot\nabla p_{x_i}\right)d\Omega - \tfrac{1}{2}\int_\Omega (\Delta p)^2 \mathrm{div}\ h\ d\Omega + \int_\Omega \Delta p[\Delta h_1,\ldots,\Delta h_n]\cdot\nabla p\ d\Omega. \quad (A.8)$$

<u>Combining the Above Terms</u>. Summing up (A.4) and the term obtained from (A.8) after integrating in time, we obtain by use of (4.9a):

$$\int_\Sigma \Delta p \frac{\partial(h\cdot\nabla p)}{\partial\nu}\, d\Sigma - \int_\Sigma \frac{\partial(\Delta p)}{\partial\nu}(h\cdot\nabla p)d\Sigma - \tfrac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu\, d\Sigma + \tfrac{1}{2}\int_\Sigma p_t^2 h\cdot\nu\, d\Sigma$$

$$= 2\int_Q \Delta p\Big(\sum_{i=1}^n \nabla h_i\cdot\nabla p_{x_i}\Big)dQ + \tfrac{1}{2}\int_Q [p_t^2-(\Delta p)^2]\text{div } h\, dQ + [(p_t(t),h\cdot\nabla p(t))_\Omega]_0^T$$

$$+ \int_Q \Delta p[\Delta h_1,\dots,\Delta h_n]\cdot\nabla p\, dQ - \int_Q Fh\cdot\nabla p\, dQ. \tag{A.9}$$

The second integral on the right of (A.9) is evaluated in the subsequent Appendix B, in (B.4). Using this result, we finally obtain the identity

$$\int_\Sigma \Delta p \frac{\partial(h\cdot\nabla p)}{\partial\nu}\, d\Sigma - \int_\Sigma \frac{\partial(\Delta p)}{\partial\nu}(h\cdot\nabla p)d\Sigma - \tfrac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu\, d\Sigma + \tfrac{1}{2}\int_\Sigma p_t^2 h\cdot\nu\, d\Sigma$$

$$- \int_\Sigma \frac{\partial(\Delta p)}{\partial\nu}\, p \text{ div } h\, d\Sigma + \tfrac{1}{2}\int_\Sigma \Delta p \frac{\partial(p \text{ div } h)}{\partial\nu}\, d\Sigma$$

$$= 2\int_Q \Delta p\Big(\sum_{i=1}^n \nabla h_i\cdot\nabla p_{x_i}\Big)dQ + \tfrac{1}{2}\int_Q p\Delta p\, \Delta(\text{div } h)dQ + \int_Q \Delta p\, \nabla p\cdot\nabla(\text{div } h)dQ$$

$$+ \int_Q \Delta p[\Delta h_1,\dots,\Delta h_n]\cdot\nabla p\, dQ - \int_Q Fh\cdot\nabla p\, dQ$$

$$+ \tfrac{1}{2}[(p_t(t),p(t)\text{div } h)_\Omega]_0^T + [(p_t(t),h\cdot\nabla p(t))_\Omega]_0^T$$

$$- \tfrac{1}{2}\int_Q Fp \text{ div } h\, dQ - \int_Q Fh\cdot\nabla p\, dQ. \tag{A.10}$$

Specialization of (A.10) to p solution of problem (4.9). Using $p|_\Sigma = \Delta p|_\Sigma = 0$, see (4.9d), hence $p_t|_\Sigma = 0$ and $h\cdot\nabla p = \frac{\partial p}{\partial\nu} h\cdot n$ in (A.10), we obtain (4.15) as desired. Moreover, setting $F = 0$ and $h(x) = x-x_0$ yields (2.24). ■

Appendix B: An Identity for the Integral of $[p_t^2-(\Delta p)^2]$

Lemma B.1. (a) For p solution of problem (4.9) we have the following identity where $Q = (0,T)\times\Omega$,

$$\int_Q [p_t^2-(\Delta p)^2]\operatorname{div} h \, dQ = \int_Q p\Delta p \, \Delta(\operatorname{div} h)dQ + 2\int_Q \Delta p \, \nabla p\cdot\nabla(\operatorname{div} h)dQ$$

$$- \int_0^T (DD^* A^{\frac{1}{2}} w_t, p_t \operatorname{div} h)_\Omega dt + [(A^{\frac{1}{2}} w(t), p(t)\operatorname{div} h)_\Omega]_0^T. \quad (B.1)$$

(b) In particular

$$\int_Q [p_t^2-(\Delta p)^2]dQ = -\int_0^T (DD^* A^{\frac{1}{2}} w_t, p_t)_\Omega dt + [(A^{\frac{1}{2}} w(t), p(t))_\Omega]_0^T \quad (B.2)$$

$$= \mathcal{O}(E(t)) + \mathcal{O}\Big(\int_0^T \|D^* A^{\frac{1}{2}} w_t\|_{L_2(\Gamma)} \|A^{\frac{1}{2}} w\|_{L_2(\Omega)} dt\Big). \quad (B.3)$$

Step 1. We shall show that for p solution of (4.9a) we have the identity

$$\int_Q [p_t^2-(\Delta p)^2]\operatorname{div} h \, dQ = \int_\Sigma \frac{\partial(\Delta p)}{\partial\nu} p \operatorname{div} h \, d\Sigma - \int_\Sigma \Delta p \, \frac{\partial(p \operatorname{div} h)}{\partial\nu} \, d\Sigma$$

$$+ \int_Q p\Delta p \, \Delta(\operatorname{div} h)dQ + 2\int_Q \Delta p\nabla p\cdot\nabla(\operatorname{div} h)dQ$$

$$- \int_Q Fp \operatorname{div} h \, dQ + [(p_t(t), p(t)\operatorname{div} h)_\Omega]_0^T. \quad (B.4)$$

To this end, we shall use the multiplier p div h.

Term $p_{tt}p$ div h. Integrating by parts in t yeilds

$$\int_Q p_{tt} p \operatorname{div} h \, dQ = \int_Q p_t^2 \operatorname{div} h \, dQ + [(p_t(t), p(t)\operatorname{div} h)_\Omega]_0^T. \quad (B.5)$$

Term $\Delta^2 p$ p div h. Using the identity

$$\Delta(p \text{ div } h) = \Delta p \text{ div } h + p\Delta(\text{div } h) + 2\nabla(\text{div } h)\cdot\nabla p, \qquad (B.6)$$

as well as Green's second theorem, we obtain

$$\int_\Omega \Delta^2 p\, p \text{ div } h\, d\Omega = \int_\Omega (\Delta p)^2 \text{div } h\, d\Omega + \int_\Omega \Delta p\, p\, \Delta(\text{div } h) d\Omega$$

$$+ 2\int_\Omega \Delta p(\nabla p\cdot\nabla(\text{div } h))d\Omega$$

$$+ \int_\Gamma \frac{\partial(\Delta p)}{\partial\nu} p \text{ div } h\, d\Gamma - \int_\Gamma \Delta p \frac{\partial(p \text{ div } h)}{\partial\nu} d\Gamma. \qquad (B.7)$$

Summing up (B.5) and the term obtained from (B.7) after integration in time, we obtain (B.4) by the use of (4.9a).

<u>Step 2</u>. Next, we integrate by parts in t after recalling the term F in (4.10),

$$-\int_0^T (F, p \text{ div } h)_\Omega dt = \int_0^T (DD^* A^{1/2} w_{tt}, p \text{ div } h)_\Omega dt$$

$$= [(DD^* A^{1/2} w_t(t), p(t)\text{div } h)_\Omega]_0^T$$

$$- \int_0^T (DD^* A^{1/2} w_t, p_t \text{ div } h)_\Omega dt. \qquad (B.8)$$

<u>Step 3</u>. Using (B.8) and (4.7): $p_t + DD^* A^{1/2} w_t = -A^{1/2} w$, we obtain

$$-\int_0^T (F, p \text{ div } h)_\Omega dt + [(p_t(t), p(t)\text{div } h)_\Omega]_0^T$$

$$= -[(A^{1/2} w(t), p(t)\text{div } h)_\Omega]_0^T - \int_0^T (DD^* A^{1/2} w_t, p_t \text{ div } h)_\Omega dt. \qquad (B.9)$$

Then, using the B.C. (4.9d) in (B.4) and inserting (B.9) in (B.4) yields (B.1) as desired. Part (a) is proved. For part (b), Eq. (B.2), we simply take div h = 1 in the above argument, i.e., we

The constant k in (C.8) depends on $\sup|h_{ij}|$, $i \neq j$; on the constant d in (C.2); on the constant c in (C.6). Thus, if these quantities are sufficiently small with respect to $m > 0$, we may obtain $m-k = \rho > 0$ as desired. This situation occurs in particular for a linear field $h_i(x) = a_i(x_i - x_{0,i})$, with constant a_i positive such that $\sup|a_i - m| = d$ is sufficiently small for some $m > 0$.

I. Lasiecka and R. Triggiani
Department of Applied Mathematics
Thornton Hall
University of Virginia
Charlottesville, VA 22903 USA

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ON A CLASS OF FUNCTIONAL EQUATIONS CHARACTERIZING THE SINE FUNCTION

Stefania Paganoni Marzegalli *

ABSTRACT. In this paper we show that an analytic function f, defined in a neighbourhood of the origin, is a solution of the following class of functional equations

$$f(x)\sum_{h=1}^{n} f[(2h-1)x] = (f(nx))^2 \qquad , \qquad n \geq 2$$

if and only if it has one of the following forms

$$f(x) = \lambda x \qquad \text{or} \qquad f(x) = \lambda \sin(\gamma x) \qquad (\lambda \in \mathbf{C}\ ,\ \gamma \in \mathbf{C} \setminus \{0\})\ .$$

1. Introduction

It is well known that there are functional equations in several variables which characterize the trigonometric functions and, in particular, the sine function (see Ref. 1–2 for a rich Bibliography).

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The aim of this paper is to characterize the sine function by means of a class of functional equations in a single variable.

Let us consider the following class of functional equations

$$f(x)\sum_{h=1}^{n} f[(2h-1)x] = (f(nx))^2 \qquad , \qquad n \geq 2 \qquad\qquad (*)_n$$

where the unknown function f is a complex variable function defined in a neighbourhood of the origin.

Denote by H_n the class of the solutions of $(*)_n$ for a fixed n and denote by H the class of the common solutions of $(*)_n$ for all $n \geq 2$.

It's easy to prove the following statements:

i) If $f \in H_n$ $[f \in H]$, then $\lambda f \in H_n$ $[\lambda f \in H]$ for every $\lambda \in \mathbf{C}$.

ii) If $f(kx) = kf(x)$ for every $k \in \mathbf{N}$ then $f \in H_n$ $[f \in H]$.
(Therefore if f is an additive function then $f \in H_n$ $[f \in H]$).

iii) The functions $f(x) = \lambda|x|$, $f(x) = \lambda x$, $f(x) = \lambda\sin(\gamma x)$ belong to the class H_n and to the class H .
(To prove that $f(x) = \sin(\gamma x)$ belongs to the class H it is sufficient to put $w = \exp(i\gamma x)$ and to write $\sin(\gamma x) = \frac{1}{2i}(w - w^{-1})$. Then the first member of $(*)_n$ is

$$-\frac{1}{4}(w-w^{-1})\sum_{h=1}^{n}\left(w^{2h-1} - w^{-(2h-1)}\right) =$$

$$= -\frac{1}{4}(w-w^{-1})\left\{w\frac{1-w^{2n}}{1-w^2} - w^{-1}\frac{1-w^{-2n}}{1-w^{-2}}\right\} = -\frac{1}{4}w^{-2n}(w^{2n}-1)^2$$

and it is equal to the second member of $(*)_n$.)

The equations $(*)_n$ and $(*)_m$ are not equivalent if $n \neq m$, that is $H_n \neq H_m$. For the sake of brevity we show this property in the particular case of $n = 2, m = 3$ by the following two examples.

<u>Example 1</u>. Consider the function $\varphi : \mathbf{C} \to \mathbf{C}$ defined in the following way:

$$\begin{cases} \varphi(x) = 0 & , \quad \text{if } x \notin \mathbf{N} \\ \varphi(x) = 2^{\alpha_1}3^{\alpha_2} & , \quad \text{if } x \in \mathbf{N} \text{ and } x = 2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k} . \end{cases}$$

It is easy to verify that $\varphi \in H_2$ but $\varphi \notin H_3$.

Example 2. Consider the function $\varphi : \mathbf{C} \to \mathbf{C}$ defined in the following way:

$$\begin{cases} \varphi(x) = 0 & , \quad \text{if } x \notin \mathbf{N} \\ \varphi(x) = 3^{\alpha_2}5^{\alpha_3} & , \quad \text{if } x \in \mathbf{N} \text{ and } x = 2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k} . \end{cases}$$

It is easy to verify that $\varphi \notin H_2$ but $\varphi \in H_3$.

2. Analytic Solutions

If we look for the solutions of $(*)_n$ in the class of the analytic functions defined in a neighbourhood of the origin, we are in a different situation. Namely we can prove the following

Theorem 1. *Let f be an analytic function defined in a neighbourhood of the origin and $n \geq 2$. f belongs to H_n if and only if f has one of the following forms :*

i) $f(x) = \lambda x$, $\lambda \in \mathbf{C}$

ii) $f(x) = \lambda \sin(\gamma x)$, $\lambda \in \mathbf{C}$, $\gamma \in \mathbf{C}\setminus\{0\}$.

Proof. We have already shown that $f(x) = \lambda x$ and $f(x) = \lambda \sin(\gamma x)$ belong to H_n . Now let us consider an analytic function defined in a neighbourhood of the origin. Then $f(x) = \sum_{k=0}^{+\infty} a_k x^k$ and $(*)_n$ becomes

$$\sum_{m=0}^{+\infty} \left\{ \sum_{k=0}^{m} a_k a_{m-k} \left(\sum_{h=1}^{n} (2h-1)^k - n^m \right) \right\} x^m = 0 . \tag{1}$$

So (1) is fulfilled if and only if for every $m \geq 0$ the following relation holds :

$$\sum_{k=0}^{m} a_k a_{m-k} \left(\sum_{h=1}^{n} (2h-1)^k - n^m \right) = 0 . \tag{2}$$

If we put $m = 0$ in (2) we have $a_0^2(n-1) = 0$ and therefore $a_0 = 0$. If we consider $m = 1$ in (2) we get

$$\sum_{k=0}^{1} a_k a_{1-k} \left(\sum_{h=1}^{n} (2h-1)^k - n \right) = 0 \tag{3}$$

and, since $a_0 = 0$, (3) is fulfilled by every a_1 . We get the same result if we put $m = 2$ in (2) and remember that $\sum_{h=1}^{n}(2h-1) = n^2$.

Now we prove that $a_1 \neq 0$. Assume on the contrary $a_1 = 0$ and let $r\ (>1)$ be the least integer p for which $a_p \neq 0$. Then, if we define

$$\gamma_n(2r) := \sum_{h=1}^{n}(2h-1)^r - n^{2r} \quad , \tag{4}$$

by (2) with $m = 2r$ it follows $\gamma_n(2r) = 0$ and this is impossible because

$$\gamma_p(2r) < 0 \qquad \text{for all } r > 1 \text{ and } p \geq 2\ . \tag{5}$$

(We may prove (5) by induction on p . For $p = 2$ the property is obvious for all $r > 1$. Now if we suppose $\sum_{h=1}^{p}(2h-1)^r < p^{2r}$ for all $r > 1$ we have

$$\sum_{h=1}^{p+1}(2h-1)^r = \sum_{h=1}^{p}(2h-1)^r + (2p+1)^r < p^{2r} + (2p+1)^r$$

and so it is sufficient to show that $p^{2r} + (2p+1)^r < (p+1)^{2r}$ for all $r > 1$, that is

$$\begin{aligned}\sum_{k=0}^{2r}\binom{2r}{k}p^k - p^{2r} - \sum_{k=0}^{r}\binom{r}{k}2^k p^k &= \\ = \sum_{k=r+1}^{2r-1}\binom{2r}{k}p^k + \sum_{k=1}^{r}\left\{\binom{2r}{k} - \binom{r}{k}2^k\right\}p^k &> 0.\end{aligned} \tag{6}$$

(6) is true since, for every $k \in \{1, \cdots, r\}$,

$$\binom{2r}{k} - \binom{r}{k}2^k = \frac{2^k}{k!}\{r(r-\frac{1}{2})\cdots(r-\frac{k}{2}+\frac{1}{2}) - r(r-1)\cdots(r-k+1)\} > 0\ .$$

(5) is so proved).

Therefore from now on we assume, without loss of generality, $a_1 = 1$, that is we look for a *normalized solution* f of $(*)_n$.

Now if we put $m = 3$ in (2) we get

$$\sum_{k=0}^{3} a_k a_{3-k}\left(\sum_{h=1}^{n}(2h-1)^k - n^3\right) = 0\ .$$

As $a_0 = 0$, $a_1 = 1$, we have

$$a_2 \left(\sum_{h=1}^{n} (2h-1) + \sum_{h=1}^{n} (2h-1)^2 - 2n^3 \right) = 0$$

and if we remember that $\sum_{h=1}^{n} (2h-1)^2 = \frac{1}{3} n(4n^2-1)$ we deduce

$$a_2 \left[\frac{n}{3} (n-1)(2n-1) \right] = 0 \ .$$

Therefore $a_2 = 0$. Similarly if we put $m = 4$ in (2) we get

$$\sum_{k=0}^{4} a_k a_{4-k} \left(\sum_{h=1}^{n} (2h-1)^k - n^4 \right) = 0 \ . \tag{7}$$

As $a_0 = a_2 = 0$ and $a_1 = 1$ we obtain

$$a_3 \left(\sum_{h=1}^{n} (2h-1) + \sum_{h=1}^{n} (2h-1)^3 - 2n^4 \right) = 0$$

and, since $\sum_{h=1}^{n} (2h-1)^3 = n^2(2n^2-1)$, (7) is fulfilled by every $a_3 \in \mathbf{C}$.

Till now we have proved that every analytic and normalized solution of $(*)_n$ defined in a neighbourhood of the origin has the form $f(x) = \sum_{k=0}^{+\infty} a_k x^k$ with $a_0 = a_2 = 0$, $a_1 = 1$, $a_3 = \alpha \in \mathbf{C}$. If we prove that the coefficients $a_m, m \geq 4$, are functions of a_k with $k < m$ then the family of all the analytic normalized solutions of $(*)_n$ depends only on the arbitrary complex parameter α . Since we already know a one–parameter family of analytic normalized solutions of $(*)_n$, namely $f(x) = \frac{1}{\alpha} \sin(\alpha x)$ if $\alpha \neq 0$ and $f(x) = x$ if $\alpha = 0$, the families have to coincide.

If order to get informations on a_m , $m \geq 4$ we have to consider the coefficient of x^{m+1} in (1). By (2) we have

$$\sum_{k=0}^{m+1} \left\{ a_k a_{m+1-k} \left(\sum_{h=1}^{n} (2h-1)^k - n^{m+1} \right) \right\} = 0 \ . \tag{8}$$

and the proof is complete if we show that the coefficient $B_n(m)$ of a_m in (8) is different from zero. We have:

$$B_n(m) = \sum_{h=1}^{n} (2h-1) + \sum_{h=1}^{n} (2h-1)^m - 2n^{m+1} \ . \tag{9}$$

By induction on m we prove that $B_n(m) \neq 0$ for all $m \geq 4$ and $n \geq 2$. Indeed, since

$$B_n(4) = n^2 + \frac{n}{15}(12n^2 - 7)(4n^2 - 1) - 2n^5 = \frac{n}{15}(18n^4 - 40n^2 + 15n + 7)$$

it is easy to see that $B_n(4) > 0$ for all $n \geq 2$. Now, assuming $B_n(m-1) = \sum_{h=1}^{n}(2h-1)^{m-1} - 2n^m + n^2 > 0$, we have to prove $B_n(m) > 0$. But

$$B_n(m) = B_n(m-1) + \sum_{h=1}^{n}(2h-1)^m - \sum_{h=1}^{n}(2h-1)^{m-1} - 2n^{m+1} + 2n^m .$$

So it is sufficient to show that, for all $n \geq 2$,

$$\sum_{h=1}^{n}(2h-1)^m - \sum_{h=1}^{n}(2h-1)^{m-1} > 2n^m(n-1) . \tag{10}$$

We prove (10) by induction on n, for every fixed m. (10) is clearly true for $n = 2$. Assume (10) true for $n = p$; then

$$\begin{aligned}&\sum_{h=1}^{p+1}(2h-1)^m - \sum_{h=1}^{p+1}(2h-1)^{m-1} = \\ &\qquad = \sum_{h=1}^{p}(2h-1)^m - \sum_{h=1}^{p}(2h-1)^{m-1} + (2p+1)^m - (2p+1)^{m-1} > \\ &\qquad > 2p^m(p-1) + (2p+1)^{m-1}2p\end{aligned}$$

and so, in order to prove (10) for $n = p+1$ it is sufficient to show that

$$p^{m-1}(p-1) + (2p+1)^{m-1} > (p+1)^m$$

that is

$$p^m - p^{m-1} + \sum_{k=0}^{m-1}\binom{m-1}{k}2^k p^k > \sum_{k=0}^{m}\binom{m}{k}p^k$$

or

$$\{2^{m-1} - (1+m)\}p^{m-1} + \sum_{k=1}^{m-2}\left\{\binom{m-1}{k}2^k - \binom{m}{k}\right\}p^k > 0 .$$

But $2^{m-1} > 1 + m$ for every $m \geq 4$ and

$$\binom{m-1}{k}2^k - \binom{m}{k} = \frac{(m-1)\cdots(m-k+1)}{k!}\{2^k(m-k) - m\} > 0$$

since $2^k(m-k) > m$ for every $k = 1, \cdots, m-2$, $m \geq 4$. So (10) holds for every $n \geq 2$. Therefore all the analytic solutions of $(*)_n$ defined in a neighbourhood of the origin are of the form described in the Theorem 1.

The following Corollary is an obvious consequence of Theorem 1.

Corollary 1. *If f is an analytic function defined in a neighbourhood of the origin and $n \geq 2$, then*

$$f \in H_n \qquad \textit{if and only if} \qquad f \in H .$$

3. Some Generalizations

Let $N > n \geq 1$ and consider the following class of functional equations

$$f(x) \sum_{h=n+1}^{N} f[(2h-1)x] = (f(Nx))^2 - (f(nx))^2 . \qquad (*)_{N,n}$$

Obviously $(*)_{N,1}$ is equal to $(*)_N$. So in this paragraph we consider $N > n \geq 2$. Denote by $K_{N,n}$ the class of the solutions of $(*)_{N,n}$ for fixed N and n and by K the class of the common solutions of $(*)_{N,n}$ for all pair N, n with $N > n \geq 2$. Obviously $K_{N,n} \supset H_N \cap H_n$; on the other hand the following example shows that there exist functions that belong neither to H_N nor to H_n but belong to $K_{N,n}$.

Example 3. Let $N = 7, n = 3$; consider the function $\varphi : \mathbf{C} \to \mathbf{C}$ defined in the following way :

$$\varphi(x) = \begin{cases} 0 & \text{if } x \notin \mathbf{N} \\ 3^{2\alpha_2} 7^{\alpha_3} 11^{\alpha_4} 13^{\alpha_5} & \text{if } x \in \mathbf{N} ,\ n = 2^{\alpha_1} 3^{\alpha_2} \cdots p_k^{\alpha_k} . \end{cases}$$

It is easy to verify that φ belongs neither to H_7 nor to H_3 but it belongs to $(*)_{7,3}$.

As in paragraph 2 we are looking for the analytic solutions of $(*)_{N,n}$, defined in a neighbourhood of the origin.

Theorem 2. *Let f be an analytic function defined in a neighbourhood of the origin and $N > n \geq 2$. $f \in K_{N,n}$ if and only if it has one of the following forms :*

i) $f(x) = \lambda x$, $\lambda \in \mathbf{C}$

ii) $f(x) = \lambda \sin(\gamma x)$, $\lambda \in \mathbf{C}$, $\gamma \in \mathbf{C}\backslash\{0\}$.

Proof. Obviously, if f has the form i) or ii) it belongs to $K_{N,n}$. Now consider an analytic function given by $f(x) = \sum_{k=0}^{+\infty} a_k x^k$. Then $(*)_{N,n}$ becomes

$$\sum_{m=0}^{+\infty} \left\{ \sum_{k=0}^{m} a_k a_{m-k} \left(\sum_{h=n+1}^{N} (2h-1)^k - N^m + n^m \right) \right\} x^m = 0$$

and so f is a solution of $(*)_{N,n}$ if and only if, for every $m \geq 0$,

$$\sum_{k=0}^{m} a_k a_{m-k} \left(\sum_{h=n+1}^{N} (2h-1)^k - N^m + n^m \right) = 0 . \qquad (11)$$

If we put $m = 0$ in (11) we have $a_0 = 0$. Moreover if $m = 2$ (11) holds for every complex number a_1 .

Now we prove, as in Theorem 1, $a_1 \neq 0$. Assume on the contrary $a_1 = 0$ and let r (> 1) be the least integer p for which $a_p \neq 0$. Then, by (11) with $m = 2r$, it follows

$$\delta_{N,n}(2r) := \sum_{h=n+1}^{N} (2h-1)^r - N^{2r} + n^{2r} = 0$$

By (4) we have $\delta_{N,n}(2r) = \gamma_N(2r) - \gamma_n(2r)$. But $\gamma_p(2r)$ is strictly decreasing with respect to p . Indeed

$$\gamma_{p+1}(2r)-\gamma_p(2r)=(2p+1)^r-(p+1)^{2r}+p^{2r}=$$
$$=\sum_{k=1}^{r}\left\{\binom{r}{k}2^k-\binom{2r}{k}\right\}p^k-\sum_{k=r+1}^{2r-1}\binom{2r}{k}p^k .$$

and by (6) this difference is always negative. So $\delta_{N,n}(2r)<0$ and this is impossible.

Therefore from now on we assume, without loss of generality, $a_1=1$. If we put $m=3$ in (11), as $a_0=0$, $a_1=1$, we have

$$a_2\left\{\sum_{h=n+1}^{N}(2h-1)+\sum_{h=n+1}^{N}(2h-1)^2-2N^3+2n^3\right\}=0$$

that is $\quad a_2(N-n)\varphi_{N,n}(2)=0 \quad$ where

$$\varphi_{N,n}(2):=\frac{2}{3}N^2+\frac{2}{3}n^2+\frac{2}{3}nN-N-n+\frac{1}{3}>\frac{2}{3}N^2-\frac{2}{3}N+3>0 \quad , \quad N>n\geq 2 .$$

Therefore $a_2=0$.

Moreover if $m=4$ (11) holds with every complex number a_3 . The proof is complete if, as in Theorem 1, we prove that the coefficients a_m , $m\geq 4$, are functions of a_k with $k<m$. Therefore we consider

$$\sum_{k=0}^{m+1}a_k a_{m+1-k}\left\{\sum_{h=n+1}^{N}(2h-1)^k-N^{m+1}+n^{m+1}\right\}=0 . \tag{12}$$

In (12) the coefficient $C_{N,n}$ of a_m is

$$C_{N,n}(m)=\sum_{h=n+1}^{N}(2h-1)+\sum_{h=n+1}^{N}(2h-1)^m-2N^{m+1}+2n^{m+1}=B_N(m)-B_n(m)$$

where $B_h(m)$ is given by (9). We have to prove $C_{N,n}\neq 0$. It is sufficient to show that $B_h(m)$ is strictly increasing with h .

$$B_{h+1}(m)-B_h(m)=2h+1+(2h+1)^m-2(h+1)^{m+1}+2h^{m+1}=$$
$$=\sum_{k=2}^{m}\left\{\binom{m}{k}2^k-2\binom{m+1}{k}\right\}h^k>0 \quad , \quad m\geq 4$$

because, for everey $k = 2, \cdots, m$,

$$\binom{m}{k}2^{k-1} - \binom{m+1}{k} = \frac{1}{k!}m(m-1)\cdots(m-k+2)\{(m-k+1)2^{k-1} - (m+1)\} > 0$$

(it is elementary to show that $(m-k+1)2^{k-1} > m+1$ for every $k = 2, \cdots, m$).

So the normalized family of analytic solutions is a one–parameter family and Theorem 2 is proved.

From Theorem 2 we have immediately

Corollary 2. *If f is an analytic function defined in a beighbourhood of the origin and $N > n \geq 2$, then*

$$f \in K_{N,n} \quad \text{if and only if} \quad f \in K .$$

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Stefania Paganoni Marzegalli
Dipartimento di Matematica
Università degli Studi di Milano
Via C. Saldini 50
20133 , Milano
Italia

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A GENERALIZATION OF HÖLDER'S AND MINKOWSKI'S INEQUALITIES AND CONJUGATE FUNCTIONS

Janusz Matkowski

A function h: $(0,\infty)\rightarrow \mathbb{R}$ is convex iff for every positive x_1, x_2, y_1, y_2:

$$h\left(\frac{x_1+x_2}{y_1+y_2}\right)(y_1+y_2)\leq h\left(\frac{x_1}{y_1}\right)y_1+h\left(\frac{x_2}{y_2}\right)y_2\,.$$

We show that this is a generalization of Hölder's and Minkowski's inequalities. This inequality establishes a strict relation between h and $h^*(t):=h(\frac{1}{t})t$, $(t>0)$, which is said to be a conjugate of h. In particular h satisfies this inequality iff h^* does; $(h^*)^*=h$; and the inequality is symmetric iff $h^*=h$ i.e., iff h is selfconjugate. Several examples of selfconjugate functions are given.

An integral version of the basic inequality is also considered.

1. Introduction

Hölder's and Minkowski's inequalities, owing to their extreme importance, have already got several proofs and generalizations (cf. G. H. Hardy, J. E. Littlewood, G. Pólya [1] and D. S. Mitrinović [2]). In the first part of this paper we present a simple inequality which is equivalent to convexity of a function: $h:(0,\infty)\rightarrow \mathbb{R}$ and which contains the discrete Hölder's and Minkowski's inequalities as very special cases. In Sec. 2 for $h:(0,\infty)\rightarrow\mathbb{R}$ we define a "conjugate function" h^*. It turns out that the basic inequality, which, in general, is not "symmetric" with respect to the occuring variables, establishes a strict relation between them. Namely, h satisfies this inequality if and only if h^* does. Moreover $(h^*)^*=h$ and, the inequality is "symmetric" iff $h^*=h$ i.e., iff h is selfconjugate. We give several examples of such functions.

In the third section we give an integral version of the basic inequality from which we obtain the integral Hölder's and Minkowski's inequalities as well as some accompanying ones. In Sec. 4 we present an n dimensional generalization of the basic inequality.

It is worth to emphasize here that the basic inequality is quite elementary and obvious (it requires no proof). Using this inequality we obtain "one line proof" of Minkowski's inequality without any refering to Hölder's inequality. In our opinion this fact is of some value in the didactic point of view.

2. A Characterization of a Convex Function Defined on $(0, \infty)$ a Generalized Discrete Hölder's and Minkowski's Inequality

The following theorem is the fundamental result of this paper.

Theorem 1. A function $h : (0, \infty) \to \mathbb{R}$ is convex iff for every $x_1, x_2, y_1, y_2 > 0$

$$h\left(\frac{x_1 + x_2}{y_1 + y_2}\right)(y_1 + y_2) \le h\left(\frac{x_1}{y_1}\right) y_1 + h\left(\frac{x_2}{y_2}\right) y_2 . \tag{1}$$

A function h is concave iff reversed inequality holds.

(This is obvious. But for the sake of completeness one can give the following reasoning. Suppose that h is convex. Then, for positive x_1, x_2, y_1, y_2 we have

$$h\left(\frac{x_1}{y_1}\right) y_1 + h\left(\frac{x_2}{y_2}\right) y_2 = \left[\frac{y_1}{y_1 + y_2} h\left(\frac{x_1}{y_1}\right) + \frac{y_2}{y_1 + y_2} h\left(\frac{x_2}{y_2}\right)\right](y_1 + y_2)$$

$$\le h\left(\frac{y_1}{y_1 + y_2}\frac{x_1}{y_1} + \frac{y_2}{y_1 + y_2}\frac{x_2}{y_2}\right)(y_1 + y_2) = h\left(\frac{x_1 + x_2}{y_1 + y_2}\right)(y_1 + y_2) .$$

To prove the converse implication it is enough to put in (1): $y_1 = \lambda \in (0, 1); y_2 = 1 - \lambda; x_1 = \lambda x; x_2 = (1 - \lambda) y$; where $x, y \in (0, \infty)$.)

From Theorem 1, by induction, we obtain

Corollary 1. A function $h : (0, \infty) \to \mathbb{R}$ is convex iff for every positive integer k and for all positive $x_1, \ldots, x_k; y_1, \ldots, y_k$:

$$h\left(\frac{x_1 + \ldots + x_k}{y_1 + \ldots + y_k}\right)(y_1 + \ldots + y_k) \le h\left(\frac{x_1}{y_1}\right) y_1 + \ldots + h\left(\frac{x_k}{y_k}\right) y_k .$$

A function h is concave iff the above inequality is reversed.

Remark 1. (A proof of Minkowski's inequality) The function $h(t) = (t^p + 1)^{1/p}, (t > 0)$, is convex for $p \geq 1$ and concave for $p < 1, p \neq 0$. Applying Corollary 1 for $p \geq 1$ we obtain

$$\left[\left(\frac{x_1 + \ldots + x_k}{y_1 + \ldots + y_k}\right)^p + 1\right]^{1/p} (y_1 + \ldots + y_k) \leq$$
$$\left[\left(\frac{x_1}{y_1}\right)^p + 1\right]^{1/p} y_1 + \ldots + \left[\left(\frac{x_k}{y_k}\right)^p + 1\right]^{1/p} y_k$$

which is the discrete Minkowski's inequality. For $p < 1, p \neq 0$, we get the converse inequality.

Remark 2. (A proof of Hölder's inequality) Take p and q such that $1/p + 1/q = 1$. For $p > 1$ the function $h(t) = t^{1/p}, (t > 0)$, is concave therefore, by Corollary 1,

$$x_1^{1/p} y_1^{1/q} + \ldots + x_k^{1/p} y_k^{1/q} \leq (x_1 + \ldots + x_k)^{1/p} (y_1 + \ldots + y_k)^{1/q}.$$

Replacing here x_i by x_i^p and y_i by y_i^q we obtain the discrete Hölder's inequality. For $p < 1, p \neq 0$, the inequality is reversed.

2. Conjugate Functions

Let $h : (0, \infty \rightarrow \mathbb{R}$ be an arbitrary function. The function $h^* : (0, \infty) \rightarrow \mathbb{R}$ defined by the formula

$$h^*(t) := h\left(\frac{1}{t}\right) t, \ (t > 0),$$

is said to be a *conjugate* of h.

It follows from Theorem 1 that there is a strong connection between h and h^*. Namely, we have the following

Theorem 2. Suppose that $h : (0, \infty) \rightarrow \mathbb{R}$. Then

1. $(h^*)^* = h$;

2. h satisfies inequality (1) if and only if h^* does (an application of h^* to (1) interchanges the positions of x_i and y_i in this inequality);

3. h is convex (concave) iff h^* is convex (concave);

4. if $h(t) = t^{1/p}$ then $h^*(t) = t^{1/q}$ where $1/p + 1/q = 1$;

5. if $h(t) = (t^p + 1)^{1/p}$ then $h^* = h$.

Proof. Properties 1, 4 and 5 follow from the definition of h. Writing inequality (1) for h we have

$$h\left(\frac{y_1 + y_2}{x_1 + x_2}\right)(x_1 + x_2) \leq h\left(\frac{y_1}{x_1}\right)x_1 + h\left(\frac{y_2}{x_2}\right)x_2 .$$

Interchanging here the positions x_i and $y_i, i = 1, 2$, we get inequality (1). It shows that if h^* satisfies inequality (1) then so does the function h. The converse implication is now a consequence of property 1. Property 3 follows from 2 and Theorem 1.

The expressions on the left and right hand side of inequality (1), in general, are not symmetric with respect to the occuring variables (in other words the function of two variables s and t given by the formula: $(s,t) \to h(s/t)t$ is not symmetric with respect to s and t). From Theorem 2 it follows that we have symmetry here if and only if h is *selfconjugate* i.e., iff $h^* = h$.

Examples.

1. A power function $h(t) = t^p$ is selfconjugate if and only if $p = 1/2$. Moreover $h(t) = t^{1/2}$ is concave.

2. $h(t) = (t^p + 1)^{1/p}$ is selfconjugate for every $p \neq 0$; it is convex for $p \geq 1$ and concave for $p < 1, p \neq 0$.

3. $h(t) = \frac{t}{t+1}$ is selfconjugate and concave. Applying Theorem 1 (or Corollary 1) we obtain the following inequality

$$\frac{x_1 y_1}{x_1 + y_1} + \ldots + \frac{x_k y_k}{x_k + y_k} \leq \frac{(x_1 + \ldots + x_k)(y_1 + \ldots + y_k)}{(x_1 + \ldots + x_k) + (y_1 + \ldots + y_k)}$$

for every positive integer k and positive $x_1, \ldots, x_k; y_1, \ldots, y_k$.

4. For every positive integer k the function

$$h(t) = \frac{t + t^2 + \ldots + t^k}{1 + t + t^2 + \ldots + t^k}$$

is selfconjugate and concave.

5. For every positive integer k the function

$$h(t)=\sum_{i=1}^{k} c_i(t^{r_i}+t^{s_i})$$

where $c_i>0; r_i+s_i=1, (i=1,\dots,k)$, is selfconjugate; it is concave if r_i and s_i are positive for $i=1,\dots,k$.

Because of the above-mentioned symmetry, inequality (1) seems to be especially interesting for selfconjugate functions.

Remark 3. According to the definition, h is selfconjugate iff it satisfies the functional equation $h(t)=h(1/t)t, t>0$. Let us note that every function defined on (0,1] or [1,∞) can be uniquely extended onto $(0,\infty)$ to a solution of this functional equation.

3. An Integral Analogue of The Fundamental Inequality

For a measure space $(\Omega,\sum,\mu)$ we denote by $S_+=S_+(\Omega,\sum,\mu)$ the set of all μ-integrable step functions $x:\Omega\to(0,\infty)$. We write χ_A for the characteristic function of a set A.

Theorem 3. Let $(\Omega,\sum,\mu)$ be a measure space such that $0<\mu(\Omega)<\infty$. If $h:(0,\infty)\to\mathbb{R}$ is convex then

$$h\left(\frac{\int_\Omega x d\mu}{\int_\Omega y d\mu}\right)\int_\Omega y d\mu \le \int_\Omega \left[h\circ\left(\frac{x}{y}\right)\right] y d\mu\,, x,y\in S_+\,. \tag{2}$$

If h is concave then the reversed inequality holds.

Proof. For arbitrary $x,y\in S_+$ there exist a positive integer k and disjoint sets $A_1,\dots,A_k\in\sum$ such that

$$x=x_1\chi_{A_1}+\dots+x_k\chi_{A_k}\,, y=y_1\chi_{A_1}+\dots+y_k\chi_{A_k}$$

for some positive $x_1,\dots,x_k; y_1,\dots,y_k$. Replacing in Corollary 1 x_i by x_ia_i and y_i by y_ia_i where $a_i:=(A_i), i=1,\dots,k$, we get

$$h\left(\frac{x_1a_1+\dots+x_ka_k}{y_1a_1+\dots+y_ka_k}\right)(y_1a_1+\dots+y_ka_k)\le h\left(\frac{x_1}{y_1}\right)$$

$$y_1a_1+\dots+h\left(\frac{x_k}{y_k}\right)y_ka_k$$

which completes the proof of (2).

Remark 4. Assuming in Theorem 3 that the measure space $(\Omega, \sum, \mu)$ is nontrivial, i.e., there exists a set $A \in \sum$ such that $0 < \mu(A) < \mu(\Omega)$, one can easily prove that h is convex if and only if inequality (2) holds.

Remark 5. (A proof of integral Minkowski's inequality) The function $h(t) = (t^{1/p} + 1)^p, t > 0$, is concave for $p > 1$ and convex for $p < 1, p \neq 0$. Applying Theorem 3 with this function h we obtain for $p > 1$ the integral version of Minkowski's inequality and for $p < 1, p \neq 0$, the reversed, but only for measure space $(\Omega, \sum, \mu)$ such that $\mu(\Omega) < \infty$ and for $x, y \in S_+$. The general inequality immediately follows from Lebesgue monotone convergence theorem.

Remark 6. (A proof of integral Holder's inequality) Applying Theorem 3 with $h(t) = t^{1/p}, t > 0$, we obtain for $p > 1$:

$$\int_\Omega xy\,d\mu \leq \left(\int_\Omega x^p\,d\mu\right)^{1/p} \left(\int_\Omega y^q\,d\mu\right)^{1/q}, x, y \in S_+ ; q := \frac{p}{p-1},$$

and for $p < 1, p \neq 0$, the reversed inequality.

Remark 7. Taking in Theorem 3 $y = \chi_\Omega$ we get

$$h\left(\frac{\int_\Omega x\,d\mu}{\mu(\Omega)}\right) \mu(\Omega) \leq \int_\Omega h \circ x\,d\mu$$

for every convex function $h : (0, \infty) \to \mathbb{R}$ and for every $x \in S_+$. In the case $\mu(\Omega) = 1$ this is the well known Jensen inequality.

Example. Applying Theorem 3 with the concave and selfconjugate function $h(t) = \frac{t}{t+1}$ we obtain the following inequality

$$\int_\Omega \frac{xy}{x+y}\,d\mu \quad \frac{(\int_\Omega x\,d\mu)(\int_\Omega y\,d\mu)}{\int_\Omega x\,d\mu + \int_\Omega y\,d\mu}, x, y \in S_+ .$$

4. A Finite Dimensional Generalization of Fundamental Inequality

Our next result generalizes Theorem 1, Corollary 1 and Theorem 3 (cf. also Remark 4).

Theorem 4. Let $n \geq 2$ be a positive integer; let $h : (0,\infty)^{n-1} \to \mathbb{R}$ and suppose that $(\Omega, \sum, \mu)$ is a measure space satisfying condition $0 < \mu(\Omega) < \infty$ and such that there exists a set $A \in \sum$ such that $0 < \mu(A) < \mu(\Omega)$. Then the following conditions are equivalent:

(i) h is convex;

(ii) for all positive $x_i, y_i (i = 1, \dots, n)$,

$$h\left(\frac{x_1 + y_1}{x_n + y_n}, \dots, \frac{x_{n-1} + y_{n-1}}{x_n + y_n}\right)(x_n + y_n) \leq$$
$$h\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n + h\left(\frac{y_1}{y_n}, \dots, \frac{y_{n-1}}{y_n}\right) y_n ;$$

(iii) for every positive integer k and for all positive $x_{ij}, (i = 1, \dots, k;$ $j = 1, \dots, n)$:

$$h\left(\frac{\sum_{j=1}^{k} x_{1j}}{\sum_{j=1}^{k} x_{nj}}, \dots, \frac{\sum_{j=1}^{k} x_{n-1,j}}{\sum_{j=1}^{k} x_{nj}}\right) \sum_{j=1}^{k} x_{nj} \leq \sum_{j=1}^{k} h\left(\frac{x_{1j}}{x_{nj}}, \dots, \frac{x_{n-1,j}}{x_{nj}}\right) x_{nj} ;$$

(iv) for every $\bar{x}_1, \dots, \bar{x}_n \in S_+(\Omega, \sum, \mu)$:

$$h\left(\frac{\int_\Omega \bar{x}_1 d\mu}{\int_\Omega \bar{x}_n d\mu}, \dots, \frac{\int_\Omega \bar{x}_{n-1} d\mu}{\int_\Omega \bar{x}_n d\mu}\right) \int_\Omega \bar{x}_n d\mu \leq \int_\Omega h\left(\frac{\bar{x}_1}{\bar{x}_n}, \dots, \frac{\bar{x}_{n-1}}{\bar{x}_n}\right) x_n d\mu .$$

For a concave function all these inequalities are reversed.

Proof. It is obvious that (i) and (ii) are equivalent. Inequality (iii) follows from (ii) by induction on k. Repeating the argument used in the proof of Theorem 3 we can easily show that (iii) implies (iv). Finally put $B := \Omega \backslash A; a := \mu(A)$ and $b := \mu(B)$. Setting in inequality (iv) the function $\bar{x}_1, \dots, \bar{x}_n \in S_+$ given by

$$\bar{x}_i := x_i \chi_A + y_i \chi_B \, , \ x_i, y_i > 0 \, , \ (i = 1, \dots, n) \, ,$$

we obtain the inequality

$$h\left(\frac{x_1 a + y_1 b}{x_n a + y_n b}, \dots, \frac{x_{n-1} a + y_{n-1} b}{x_n a + y_n b}\right)(x_n a + y_n b) \le$$
$$h\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n a + h\left(\frac{y_1}{y_n}, \dots, \frac{y_{n-1}}{y_n}\right) y_n b .$$

Replacing here $x_i a$ by x_i and $y_i b$ by y_i for $i = 1, \dots, n$, we obtain inequality (ii). This completes the proof.

Remark 8. Let $r_1, \dots, r_n$ be positive and such that $r_1 + \dots + r_n = 1$. Since the function $h : (0,\infty)^{n-1} \to \mathbb{R}$ defined by the formula

$$h(t_1, \dots, t_{n-1}) := t_1^{r_1} \dots t_{n-1}^{r_{n-1}}$$

is concave, we have from Theorem 4:

$$x_1^{r_1} \dots x_n^{r_n} + y_1^{r_1} \dots y_n^{r_n} \le (x_1 + y_1)^{r_1} \dots (x_n + y_n)^{r_n}$$

for all positive $x_1, \dots, x_n; y_1, \dots, y_n$. This is a generalization of Hölder's inequality given in [1], p. 21.

Remark 9. Theorem 4 provides us a simple criterion of subadditivity of a function $f : (0,\infty)^n \to \mathbb{R}$. Moreover it allows us to give an interesting characterization of a symmetric norm in the linear space $\mathbb{R}^n$. The relevant results will be published elsewhere.

5. Final Remark

Let $(\Omega, \sum, \mu)$ be a measure space and let $\varphi : [0,\infty) \to [0,\infty)$ be a bijection such that $\varphi(0) = 0$. One can easily check that the functional $\mathbb{P}_\varphi : S_+(\Omega, \sum, \mu) \to [0,\infty)$ given by the formula

$$\mathbb{P}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ x d\mu\right)$$

is well defined. It is worth to mention here that inequalities (1) and (ii) have appeared, quite unexpectedly, in the course of the proof of the following converse of Hölder's inequality.

Theorem. Let $(\Omega, \sum, \mu)$ be a measure space, let $A, B \in \sum$ be sets such that $0 < \mu(A) < 1 < \mu(B) < \infty$ and suppose that φ and ψ are bijections of $[0,\infty)$ such that $\varphi(0)=\psi(0)=0$. If

$$\int_\Omega xy\,d\mu \le \mathbb{P}_\varphi(x)\mathbb{P}_\psi(y)\,,\; x,y\in S_+\,,$$

then φ and ψ are conjugate power functions i.e., there exist $p>1, q>1$ such that $p^{-1}+q^{-1}=1$ and $\varphi(t)=\varphi(1)t^p, \psi(t)=\psi(1)t^q$.

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Janusz Matkowski
Department of Mathematics
Technical University
43-300 Bielsko-Biała
Poland

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INTEGRATION AND THE FUNDAMENTAL THEORY OF ORDINARY DIFFERENTIAL EQUATIONS : A HISTORICAL SKETCH

Jean Mawhin

Abstract. Constantin Caratheodory is famous for having introduced the Lebesgue integral in the fundamental theory of ordinary differential equations : the concepts of a Caratheodory function and of solutions in the sense of Caratheodory are now classical. This paper shows the evolution of the basic existence theorem for the Cauchy problem related to ordinary differential equations from the pioneering contributions of Cauchy to the present time.

1. Introduction

The influence of ordinary differential equations in the creation and progress of the differential and integral calculus has been important and constant. The aim of this short essay is to show the basic interaction of the concept of integral and that of solution of the Cauchy problem for ordinary differential equations, from the pioneering work of Cauchy to contemporary researchs. One of the milestones of this development will be the important contribution of CONSTANTIN CARATHEODORY, who injected in the fundamental theory of ordinary differential equations the concepts and techniques of the Lebesgue integral. For other surveys of the evolution of the basic theory of ordinary differential equations, see **[13,18,27,39,41,45,49,53,54,55,59,60]**.

2. Euler, Cauchy and Lipschitz

The pre-Eulerian period in ordinary differential equations has seen a flowering of ingenious tricks in trying to reduce to quadratures the obtention of explicit solutions of many particular ordinary differential equations and, as noticed by PAINLEVE [44] "The wave stopped when all what was integrable, in natural philosophy problems, was integrated". The *Institutiones Calculi Integrali* of EULER (1768) [17] constitute the master piece of this period but also the fundamental link to the next one. Realizing that even the simplest differential equation

(1) $$y'(x) = f(x)$$

cannot always be integrated in finite terms, EULER, in Chapter 7 of the first Section of Volume 1, returns, to obtain an approximate solution of (1), to the old idea of approximating y(x) by a finite sum through a partition of [a,x] through the points

$$a = a_0 < a_1 < \ldots < a_{m-1} < a_m = x$$

and approximating y(x) by the expression

(2) $$y(a) + \Sigma_{1 \leq j \leq m} f(a_{j-1})(a_j - a_{j-1}).$$

He applies the same idea, in Chapter 7 of the second Section of Volume 1, to the approximate integration of a first order ordinary differential equation

(3) $$y'(x) = f(x,y(x))$$

by proposing the following approximate solution

(4) $$y(a) + \Sigma_{1 \leq j \leq m} f(a_{j-1}, y_{j-1})(a_j - a_{j-1}),$$

where the y_j are defined recursively by the relations

$$y_0 = y(a),$$

(5)

$$y_j = y_{j-1} + f(a_{j-1}, y_{j-1})(a_j - a_{j-1}), \quad (1 \leq j \leq m-1).$$

The similarity between formulas (4) and (2) is clear, the only difference being the implicit character of (4), due to the recursive definition of the y_j. EULER does not worry about the convergence of expressions (4) or (2) to the exact solution of the problem (whose existence is not questionned). But he gives judicious advices of how to obtain satisfactory approximations by choosing suitably the partitions of [a,x] in what is called to-day the *Euler's polygonal method* in the numerical integration of ordinary differential equations.

CAUCHY had read Euler's Institutiones and the updated version given by LACROIX in his monumental *Traité du calcul différentiel et du calcul intégral* of 1797-1798 [**33,14**]. He will apply to the integral and to ordinary differential equations the bright idea that he already introduced in his study of continuous and differentiable functions : to use the limit concept to transform known approximation schemes in existence proofs. Like BOLZANO and GAUSS, but still more systematically, CAUCHY is concerned with the question of the existence of the mathematical notions. In front of the impossibility of finding, in general, explicit solutions of a differential equation, CAUCHY defines and solves, under rather general conditions, the problem of their existence. His philosophy, well summarized in a note written in 1842 [**9**], consists, in the theory of integration, in placing the concept and the study of the definite integral before that of the indefinite integral, and, in differential equations, in setting the *Cauchy problem*, i.e. y_0 being given, find a solution y of (3) such that

(6) $$y(a) = y_0 .$$

instead of looking first for a "general solution" of the differential equation.

Recall also that CAUCHY proved the existence of the definite integral of f over [a,x], for a continuous function f, by showing that (2) has a limit when the mesh

$$\max_{1 \leq j \leq m}(a_j - a_{j-1}) \tag{7}$$

tends to zero. He applies successfully the same procedure to the problem (3)-(6) for f and $D_y f$ continuous and bounded, by going to the limit in the approximate expressions (4) and (5) where y(a) is replaced by y_0.

The recent discovery by GILLAIN [7] of unpublished printed material of the *Résumé du Cours de calcul infinitésimal* de CAUCHY at the Ecole Polytechnique [6] has definitely shown the deep unity of thinking of CAUCHY in his approach of the integral calculus. We can conclude, with DOBROWOLSKY [13] that "one of the main reasons which have led Cauchy to create his first method (for the fundamental theorem on ordinary differential equations) is the one which motivated him in rethinking analysis in general".

It is interesting to notice that in 1835, in his *Mémoire sur l'intégration des équations différentielles* [8], CAUCHY will use his "calcul des limites" (i.e. the method of majorating functions) to "transform into a completely rigorous theory the method of integrating an arbitrary system of differential equations through series". The basic tool in this memoir is another famous creation of Cauchy, namely the theory of integration of holomorphic functions along a path of the complex plane. Of course, this method only works when the right-hand member of the equation is itself holomorphic. A similar result will be obtained independently by WEIERSTRASS [58] in 1842.

Therefore, together with its revolutionnary character, CAUCHY's contribution shares the other characteristic of outstanding contributions : to find its roots in the work of predecessors. Being based upon Euler's polygonal method, Cauchy's first method can be linked to the Leibnizian tradition of calculus, although his second method justifies the powerful method of integration through power series introduced by NEWTON.

Being apparently unaware of Cauchy's contribution, LIPSCHITZ [**35**] will reproduce in 1868 Cauchy's first method, under slightly weaker conditions. He assumes only that f is continuous and such that

(8) $$|f(x,y) - f(x,z)| \leq L|y-z|$$

in the neighbourhood of the point (a,y_0). This is what is now called a *Lipschitz condition*, and it already appears implicitely in CAUCHY's work as a consequence of the continuity of D_yf and another Cauchy's famous tool, the mean value theorem [**15,20**]. LIPSCHITZ expresses clearly the link between his approach and Cauchy's integral when he writes : "In the case where the function f does not contain the variable y, the function remaining uniform and continuous with respect to x, our analysis shows that the integral from a to x of f is well defined and that the derivative of this function, with respect to the upper extremity of the interval of integration, is equal to f(x)".

3. Riemann, Volterra, Peano, de La Vallée Poussin

Everybody knows how RIEMANN, in his famous memoir of 1857 *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe* [**50**], will create the first *integration theory* by obtaining a characterization of the class of functions f for which the *Riemann sums* (already considered by CAUCHY !)

$$\Sigma_{1\leq j\leq m} f(x_j)(a_j-a_{j-1})$$

converge to an unique value whenever (7) goes to zero, independently of the choice of the x_j in $[a_{j-1},a_j]$ $(1\leq j\leq m)$. It is the case, as shown by CAUCHY, when f is continuous, but it still happens for some discontinuous functions. There is no trace, in RIEMANN's work, of an extension of his ideas to the fundamental theory of ordinary differential equations. The brevity of his life may be the reason of it.

LIPSCHITZ, in the above quoted paper [35], seems to be the first one to mention RIEMANN's work in a memoir devoted to ordinary differential equations. He makes it only in his conclusion, in the following way : "Riemann's posthumous memoir on the representation of a function by a trigonometrical series has emphasized the fact that the existence of the definite integral holds under a condition more general than continuity", and he recalls Riemann's definition. He then concludes in the following way : "But it seems to me that those conditions do not imply that the derivative of this integral, with respect to the extremity of the interval of integration, is equal to f(x); it is the reason why I have thought necessary to keep the condition of continuity of the function f(x) for the study of the integration of the ordinary differential equations

$$y'(x) = f(x)." \tag{9}$$

Indeed, the Riemann integral had destroyed the full reciprocity between the operations of differentiation and of indefinite integration which holds in Cauchy's theory for continuous functions. In Riemann's frame, the equivalence between the differential form (9) of the equation and its integral form

$$y(x) = a + \int_{[a,x]} f(s)ds, \tag{10}$$

is lost when f is Riemann-integrable without being continuous. And it is interesting to notice that LIPSCHITZ refuses to make the first step toward the concept of *generalized solution* of (9), namely a function y satisfying (10).

This step was boldly made in 1881 by VOLTERRA, when he was still a student (of DINI) at Pisa., who incorporated Riemann ideas in the Cauchy problem in his memoir *Sui principii del calcolo integrale* [56]. VOLTERRA, who does not seem to be aware of DARBOUX paper of 1875 on discontinuous functions [10], introduces independently the *lower and upper integrals* of a bounded function over [a,x] as respective limits of the sums

(11) $$\Sigma_{1\leq j\leq m} m_j(a_j-a_{j-1}) \text{ and } \Sigma_{1\leq j\leq m} M_j(a_j-a_{j-1}),$$

when (7) goes to zero, where

$$m_j = \inf \{f(x) : x \in [a_{j-1},a_j]\} \text{ and } M_j = \sup \{f(x) : x \in [a_{j-1},a_j]\}.$$

VOLTERRA also solves a problem raised by DINI by giving the first example of a bounded derivative which is not Riemann integrable, which confirms the complete dissymetry between the operations of differentiation and indefinite integration in the frame of Riemann-integrable functions. Section III of Volterra's memoir is devoted to the Cauchy problem for ordinary differential equations. Again, VOLTERRA seems to be unaware of CAUCHY's first approach when he writes : "The first existence proof for the integral of an ordinary differential equation passing through one point is due to Cauchy. Briot and Bouquet have proved Cauchy's theorem in a very simple way; their method imposes various restrictions to the differential equations, which consist in conditions under which the integrals can be developed in Taylor series. Independently of those considerations, Lipschitz and Houël have given a proof of the existence of the integrals of ordinary differential equations in which the argument is similar to that used in the proof of the existence of Riemann definite integrals. Following the same argument, but applying the method used here in the proof of Riemann's theorem, one is led to somewhat more general results".

Indeed, if M denotes the supremum of |f| over a neighbourhood C of (a,y_0), VOLTERRA considers the sums (11) where, now, m_j and M_j are defined recursively by the formulas

$$m_j = \inf \{f(x,y) : (x,y) \in R_j\} , M_j = \sup \{f(x,y) : (x,y) \in R_j\},$$

with

$$R_j = [a_{j-1},a_j] \times [y_0 + \Sigma_{1\leq k\leq j-1} m_k(a_k-a_{k-1}) - M(a_j-a_{j-1}), y_0 + \Sigma_{1\leq k\leq j-1} M_k(a_k-a_{k-1}) + M(a_j-a_{j-1})]$$

($1\leq j\leq m$), where we make sure that the R_j stay in the neighbourhood C, by taking $|x-a|$ small enough. VOLTERRA then shows that, like in Riemann integration theory, the necessary and sufficient condition in order that (4) has a limit y(x) whenever (7) goes to zero is that

$$\Sigma_{1\leq j\leq m}(a_j-a_{j-1})D_j \to 0 \tag{12}$$

whenever (7) goes to zero, where D_j denotes the oscillation of f over R_j ($1\leq j\leq m$). When it is the case, VOLTERRA shows that the function y is continuous, the function f(.,y(.)) is Riemann integrable over [a,x] and the function y satisfies the integral equation

$$y(x) = y_0 + \int_{[a,x]} f(s,y(x))ds \ . \tag{13}$$

It remains then to find conditions over f in order that condition (12) holds and to discuss the relation between the solutions of (13) and those of the corresponding Cauchy problem. VOLTERRA first shows that condition (12) necessary implies the *uniqueness of the solution* of (13) and that each solution of (13) is a classical solution of the Cauchy problem when f is continuous over C. He then proves that (12) holds when f is continuous and satisfies the Lipschitz condition (8) over C. More generally, he shows also that condition (12) holds if the above conditions upon f hold on C except on a subset which can be covered by an at most countable family of rectangles with sides parallel to the axes and such that the sum of the lengths of their sides parallel to Ox is arbitrary small. In this case, the function y satisfies the differential equation (3) at each point x for which f is continuous at (x,y(x)).

VOLTERRA then raises the important question of the *solvability of the Cauchy problem* (3)-(6) *when f is only continuous* on C. He gives a positive answer under the supplementary assumption that f(s,.) is monotone for each s. This last restriction will be dropped by PEANO in 1886 [**47**]. The same author had already defined, in 1883, [**46**] the Riemann integral independently of any limit concept by observing that the lower and upper integrals could be respectively defined as the supremum and infimum of the expressions (11) for all finite partitions $a = a_0 < a_1 < ... < a_{m-1} < a_m < x$ of [a,x]. In his paper [**47**], PEANO applies the same idea to problem (3)-(6) with f continuous by showing that the infimum V (resp. supremum U) of the function v (resp. u) such that

$$v(a) = u(a) = y_0$$

and

$$v'(s) > f(s,v(s)) \ , \ (\text{resp. } u'(s) < f(s,u(s))$$

for $a \leq s \leq x$, provide solutions of (3)-(6) and that any other solution y of the problem satisfies, on this interval, the inequality

$$u(s) \leq y(s) \leq v(s),$$

introducing in this way the concepts of *maximal and minimal solutions*. Like LIPSCHITZ and VOLTERRA, PEANO does not seem to be aware of CAUCHY's contribution based upon the Euler's polygons (he only quotes Cauchy's result for f holomorphic and its simplification by BRIOT and BOUQUET), but he refers to LIPSCHITZ's memoir [**35**], to the treatises on analysis by HOUEL [**23**] and GILBERT [**19**], and, of course, to VOLTERRA's paper [**56**]. IN 1898, OSGOOD [**43**] will give another proof of Peano's result, under the sames assumptions, by using an approach more reminiscent of Volterra's one. In contrast to the methods of CAUCHY, LIPSCHITZ AND VOLTERRA, thoses of PEANO in [**47**] and OSGOOD do not extend to the case of systems of differential equations, and the corresponding existence of at least one solution for the Cauchy's problem for a continuous right-hand side will be proved in 1890 by the same PEANO [**48**], by combining the approximation method of Euler-Cauchy with a theorem of ASCOLI and ARZELA. His proof will be simplified by DE LA VALLEE POUSSIN [**11**], MIE [**40**] and ARZELA [**3,4**]. One shall notice that PEANO never tried to weaken the continuity condition of f with respect to x.

Apparently unaware of VOLTERRA'S memoir [**56**], but well informed about the contributions of GILBERT [**19**] and DARBOUX memoir [**10**], DE LA VALLEE POUSSIN obtains, in his *Mémoire sur l'intégration des équations différentielles* of1893 [**12**], results very similar to those of VOLTERRA by a closely related method. His motivation is expressed very clearly in the introduction of his memoir : "The present work has been inspired by the study of the memoir on discontinuous functions of M. Darboux and the note that M. C. Jordan has added to the third volume of his Traité d'analyse. Our aim is to extend, whenever it is possible, to ordinary differential equations, the concept of integrability introduced by Riemann for the special case of quadratures. In the same way that one can integrate discontinuous functions, we have intended to show that one can integrate differential equations containing such functions". Like VOLTERRA and in contrast to LIPSCHITZ, DE LA VALLEE POUSSIN will not hesitate to call "integral of equation (3) a function y which satisfies, for each x, the relation (13), and hence to consider solutions which are not differentiable everywhere. Among the aspects which complete Volterra's work, let us mention the proof of the *continuous dependence of the solution with respect to* y_0 when condition (12) holds, and the obtention of an interesting condition upon f, a forerunner of *Caratheodory condition,* in order that (12) holds, namely the Riemann integrability of f(.,y) over [a,x] for each fixed y and the existence and continuity with respect to y of $D_y f$. For example, DE LA VALLEE POUSSIN proposes as an application of his theory the differential equation

$$y'(x) = \Sigma_{0\le j\le n} X_j(x) y^j(x)$$

where the functions X_j are Riemann-integrable over [a,x]. Let us observe finally that in an historical appendix written upon request of P. MANSION, DE LA VALLEE POUSSIN makes a short comparaison between his results and those of PEANO's paper [**47**]. It is not known if this gave to DE LA VALLEE POUSSIN the opportunity to discover the existence and the content of VOLTERRA's memoir.

5. Lebesgue, Caratheodory and Kurzweil

Everybody knows the immense progress that the *Lebesgue integral* made possible in analysis and how LEBESGUE himself used it in the study of Fourier series and in the calculus of variations. LEBESGUE's thesis *Intégrale, longueur, aire* of 1902 [**34**] contains only a few lines about the possible consequences of his new integration theory in the theory of ordinary differential equations : "(the new integral) allows indeed to solve the fundamental problem of the differential calculus in all cases where the derivative is bounded and, consequently, it allows to integrate ordinary differential equations which can be reduced to quadratures. For example, f(x) being an arbitrary bounded function, we shall be able to recognize if the equation

$$y' + ax = f(x)$$

has solutions, and, if it is the case, to find them.". He adds, in a footnote : "This remark leads to interesting problems. For example, f(x) and g(x) being bounded, are all the solutions of the equation

$$y' + f(x)y = g(x)$$

contained in the classical formula

$$y(x) = \exp(-\int f(x)dx).\int g(x) \exp(\int f(x)dx)dx \text{ ?"}.$$

It was left to CARATHEODORY in his famous *Vorlesungen über reelle Funktionen* [5] to incorporate Lebesgue integral into the fundamental theory of ordinary differential equations. His conditions correspond in a way, in this new setting, to the synthesis of those of DE LA VALLEE POUSSIN and PEANO, if we observe that he assumes f(.,y) to be measurable over [a,x] for each fixed y, f(s,.) continuous for almost each s in [a,x] and that

$$|f(s,y)| \le F(s)$$

over C (almost everywhere in s) for some Lebesgue integrable function F over [a,x] (*Caratheodory conditions*). A *solution* of (3)-(6) *in the sense of Caratheodory* will be a solution of the integral equation (13) and will satisfy therefore the differential equation (3) almost everywhere on [a,x]. In CARATHEODORY's approach, the fundamental tool to go from approximate solutions to exact ones is the LEBESGUE's dominated convergence theorem, after the extraction of a convergent subsequence with the use of the ASCOLI-ARZELA's theorem.

There has been systematic studies, due to NEMICKII, VAINBERG, KRASNOSEL'SKII, LADYZENSZKII, RUTICKII and others on the structure of *Nemickii operators* defined on various spaces of functions as mappingsof the type

$$x(.) \to f(.,x(.))$$

when the function f satisfies the Caratheodory conditions (see [**1,28,29**] for references). An axiomatic definition of *Caratheodory operators* has been introduced by KARTAK [**24,25,26**] in order to extend Caratheodory theory, and VRKOC [**57**] has shown that they can always be associated to a unique Caratheodory function. In 1955, AQUARO [**2**] has proposed an extension of the fundamental theory where the Caratheodory conditions are replaced by the assumption of Lebesgue integrability of $f(.,y(.))$ for all continuous y and the equiabsolute continuity of the family of its indefinite integrals OPIAL **[42]** has shown in 1960 that Aquaro's conditions are indeed equivalent to Caratheodory ones.

The conditions and the method of CARATHEODORY will also serve as a model for the obtention, by KARTAK **[24,25,26]** and MANOUGIAN **[36]** of a fundamental theory for the Cauchy problem in the frame of the DENJOY-PERRON's extension of Lebesgue integral. See also the book **[16]**.

Equation (13) can also be used to introduce an interesting extension of the concept of ordinary differential equation motivated by the fact that important properties of the solution, and in particular its continuous dependence with respect to a parameter, can be expressed uniquely in terms of the application F defined by

$$F(x,y) = \int_{[a,x]} f(s,y)ds \tag{14}$$

rather than in terms of f itself. To introduce this extension, let y be a solution of (13) and (y_m) a sequence of piecewise constants functions

$$y_m(s) = y(s_j),\ a_{j-1} \leq s < a_j, \tag{15}$$

where

$$a_{j-1} \leq s_j \leq a_j\ ,\ 1 \leq j \leq m,\ \ a = a_0 < a_1 < \ldots < a_{m-1} < a_m = x\ , \tag{16}$$

such that (y_m) converges uniformly on $[a,x]$ to y. From Caratheodory conditions, it follows that

$$f(s,y_m(s)) \rightarrow f(s,y(s))$$

for a.e. s in $[a,x]$ and then

$$\int_{[a,x]} f(s,y_m(s))ds \rightarrow \int_{[a,x]} f(s,y(s))ds$$

when $m \to \infty$, by Lebesgue dominated convergence theorem. On the other hand,

$$\int_{[a,x]} f(s,y_m(s))ds = \Sigma_{1\leq j\leq m}\int_{[a,x]} f(s,y(s_j))ds =$$

$$\Sigma_{1\leq j\leq m}[F(a_j,y(s_j))-F(a_{j-1},y(s_j))],$$

where F is defined by (14). In other words, y can be obtained as a limit of the expressions

(17) $$y_0 + \Sigma_{1\leq j\leq m}[F(a_j,y(s_j)) - F(a_{j-1},y(s_j))]$$

when m tends to infinity, i.e. when the partition of [a,x] defined by (15) and (16) gets finer and finer. The right-hand member of (15) is similar to Riemann sums associated to the a_j and s_j. This observation, that can be found in the monograph [**51**], had led KURZWEIL [**30**] to introduce in 1957 the following generalization of the Riemann sums. If U maps [a,x] x [a,x] into R and if

$$A = \{a_0,s_1,a_1,s_2,...,a_{m-1},s_m,a_m\}$$

where the a_j and s_j verify (16), is given, KURZWEIL introduces the *generalized Riemann sum*

(18) $$S(U,A) = \Sigma_{1\leq j\leq m}[U(a_j,s_j) - U(a_{j-1},s_j)],$$

which reduces to the usual Riemann sum when

(19) $$U(a,s) = f(s)a,$$

with f mapping [a,x] into R. It is immediately seen that the right-hand member of (17) is the generalized Riemann sum associated to the function U defined by

$$U(a,s) = F(a,y(s)).$$

It is therefore natural to define the integral

$$\int_{[a,x]} DU(a,s)$$

associated to U as a suitable limit of the sums (18), and the second main contribution of KURZWEIL consists in modifying the filter of the partitions on which the limit is computed to get a sufficiently general integral which reduces, when U is given by (19), to the DENJOY-PERRON integral (see e.g. [**22,30,31**]).J will be the *Kurzweil integral* of DU over [a,x] if for each positive ε one can find a positive function δ on [a,x] such that

$$|S(U,A) - J| \leq \varepsilon$$

for each *δ-fine partition* A of [a,x], i.e. each partition A satisfying

$$s_j - \delta(s_j) \leq a_{j-1} \leq s_j \leq a_j \leq s_j + \delta(s_j), \ (1 \leq j \leq m).$$

Riemann-type integrals correspond to the restriction to constant functions δ in the definition.This modification was independently introduced a few years later by HENSTOCK [**21**] and has had, in integration theory, important developments that we cannot describe here (see e.g. [**22,31,37,38,**]).

Starting now from an arbitrary function F from [a,x] x R into R, KURZWEIL defined a *solution of the Cauchy problem for the generalized differential equation*

(20) $$y'(x) = DF(x,y(x)), \ y(a) = y_0,$$

as a function y which is solution of the integral equation

(21) $$y(x) = y_0 + \int_{[a,x]} DF(a,y(s)),$$

where the integral in the right-hand member is a Kurzweil-Henstock integral as defined above. One can see that the differential notation (20) is purely a symbolic one, as the solution of (21) will not necessarily be a differentiable function (not even necessarily a continuous function). Of course, like above, it will be necessary to find explicit conditions upon F which insure the existence of the integral in (21) and determine the regularity properties of the solution. One can consult, in this respect, the monographs [**51**] and [**52**] and their references, where it is shown in particular that the generalized differential equations contain as special case not only the Caratheodory situation, but also measure differential equations and differential equations with impulses. When $F(x,y) = A(x)y$ for some function A with bounded variation, the solutions of (20) are nothing but those of the integral equation

$$y(x) = y_0 + \int_{[a,x]} y(s)dA(s),$$

where the integral in the right-hand member is a Perron-Stieltjes integral.

In his recent book *Lectures on the Theory of Integration* [**22**], HENSTOCK has used the Kurzweil-Henstock integral, with the usual Riemann sums associated to (19), i.e. the Denjoy-Perron integral, together with a generalized convergence theorem valid for this integral, to propose an extension of the fundamental theory of ordinary differential equations. He replaces the Caratheodory conditions by the following ones : $f(s,.)$ is continuous for a.e. s in $[a,x]$, $f(.,y)$ is integrable for each y , and for some compact set S in R, some positive function δ on $[a,x]$, all δ-fine partitions A of $[a,x]$ and all functions w on $[a,x]$, one has

$$\Sigma_{1 \leq j \leq m} f(s_j, w(s_j))(a_j - a_{j-1}) \in S.$$

Very recently, KURZWEIL and SCHWABIK [**32**] has proved that the above conditions imply that f is necessarily of the form

$$f(x,y) = g(x) + h(x,y)$$

with g Perron-Denjoy-integrable and h satisfying the Caratheodory conditions. Hence, a change of variables can reduce this situation to Caratheodory's one.

Those recent examples show that the interaction between integration theories and the fundamental theory of ordinary differential equations continue to be a fruitful source of inspiration for the mathematicians.

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Jean Mawhin
Université de Louvain
Institut mathématique
B-1348 Louvain-la-Neuve, Belgium

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SOME BOUNDARY VALUE PROBLEMS FOR A PARTIAL DIFFERENTIAL EQUATION OF NON-INTEGER ORDER.

Marek W. Michalski

Abstract. The paper concerns a Goursat-like problem and a generalized Cauchy problem for some nonlinear partial differential equation of non-integer order. Assuming the *Caratheodory* conditions for the nonlinear part of the equation, we reduce the said problems to integro-functional equations and then prove the existence of the global solutions by using the Schauder fixed point theorem.

1. Derivatives of Arbitrary Order

In this paper, which extends earlier researches of J. Conlan (cf. [1]) and the present author (cf. [8]) we deal with boundary value problems containing the derivatives of non-integer order. Therefore, we first quote the definition and some basic properties of such derivatives (cf. [3]).

$$D_x^{-\alpha}D_y^{-\beta}f(x,y) = \int_0^x d\xi\int_0^y (x-\xi)^{\alpha-1}(y-\eta)^{\beta-1}f(\xi,\eta)d\eta/(\Gamma(\alpha)\Gamma(\beta))$$

$$D_x^{-\alpha}f(x,y) = \int_0^x (x-\xi)^{\alpha-1}f(\xi,y)d\xi/\Gamma(\alpha)$$

and

$$D_y^{-\beta}f(x,y) = \int_0^y (y-\eta)^{\beta-1}f(x,\eta)d\eta/\Gamma(\beta),$$

respectively, a. e. on Ω. Thus, the derivative $D_x^{-\alpha}D_y^{-\beta}$ $(\alpha,\beta \geq 0)$ can be treated as a transform of the space of integrable functions into itself. One can also prove (cf. [8]) that this transform is completely continuous in case when $\alpha,\beta > 0$.

Further properties of the derivative of non-integer order can be found in [1], [8] and [9].

2. <u>The Goursat-like Problem</u>

In what follows, we assume that the numbers α and β satisfy the inequalities $0 < \alpha,\beta \leq 1$.

Let $g:[0,A] \longrightarrow [0,B]$, $h:[0,B] \longrightarrow [0,A]$, $G:(0,A) \longrightarrow \mathbb{R}$, and $H:(0,B) \longrightarrow \mathbb{R}$ be given functions[1], (x_0,y_0) an arbitrarily fixed point of $\bar{\Omega}$ and c_0 a given number.

We deal with the following partial differential equation

(1) $$D_x^{\alpha}D_y^{\beta}u(x,y) = F(x,y,\{D_x^{\gamma}D_y^{\lambda}u(x,y)\}) \quad (x,y) \in \Omega,$$

where $\{D_x^{\gamma}D_y^{\lambda}u\}$ denotes the finite sequence of all derivatives $D_x^{\gamma}D_y^{\lambda}u$ such that $\gamma \leq \alpha$; $\lambda \leq \beta$; $\gamma + \lambda < \alpha + \beta$ (the total number of these derivatives will be denoted by m)

---

[1] The curves of equations $y = g(x)$ and $x = h(y)$ will be denoted by l_1 and l_2, respectively.

Let Ω be the rectangle $\Omega := (0,A)\times(0,B)$, where $0 < A,B < \infty$, and $f:\Omega \longrightarrow \mathbb{R}$ a Lebesgue integrable function. In what follows, α and β are real numbers ($\alpha,\beta \in \mathbb{R}$) and p and q positive integers ($p,q \in \mathbb{N}$) such that $\alpha \le p$; $\beta \le q$. The derivative $D_x^\alpha D_y^\beta f$ is defined by the following equality

$$D_x^\alpha D_y^\beta f(x,y) = \begin{cases} D_x D_y \int_0^x d\xi \int_0^y (x-\xi)^{-\alpha}(y-\eta)^{-\beta} f(\xi,\eta)d\eta/(\Gamma(1-\alpha)\Gamma(1-\beta)) & \text{(a)} \\ \qquad \text{for } \alpha,\beta \le 0 & \\ D_x^p D_y^q (D_x^{\alpha-p} D_y^{\beta-q} f(x,y)) \quad \text{for } \alpha > 0 \text{ or } \beta > 0 & \text{(b)} \end{cases}$$

where Γ is the Euler gamma function, D_x^p and D_y^q denote the classical partial derivatives and $D_x = D_x^1$, $D_y = D_y^1$ (in case (b) we additionally assume that the function $D_x^{\alpha-p} D_y^{\beta-q} f$ is differentiable p times with respect to x and q times with respect to y).

Let us note that the constants p and q above can be chosen arbitrarily (so that $\alpha \le p$, $\beta \le q$), which easily results from the following proposition (cf. [3])

<u>Proposition 1.</u> *If* $v:\Omega \longrightarrow \mathbb{R}$ *is measurable with respect to* y, *possesses the derivative* $D_x v \in L(\Omega)$ *and satisfies the inequality*

(*) $\qquad |v(x,y)| \le M(y)$

with $M \in L(0,B)$, *then the formula*

(**) $$\left(\int_0^x v(x,\eta)d\eta\right)' = \int_0^x D_x v(x,\eta)d\eta + v(x,x)$$

holds good a. e. on (0,A).

Basing on Proposition 1 and the Fubini theorem, we can assert that for $\alpha = p$, $\beta = q$ ($p,q \in \mathbb{N}$) the following relations

$$D_x^\alpha D_y^\beta f(x,y) = D_x^{p+1} D_y^{q+1} \int_0^x d\xi \int_0^y f(\xi,\eta)d\eta = D_x^p D_y^q f(x,y)$$

are valid a. e. on Ω. Moreover, for $\alpha,\beta > 0$ the derivatives $D_x^{-\alpha} D_y^{-\beta}$, $D_x^{-\alpha}$ and $D_y^{-\beta}$ are integrable and satisfy the equalities.

By a *solution* of equation (1) in Ω we mean a function $u:\Omega \longrightarrow \mathbb{R}$ which possesses an integrable derivative $D_x^{\alpha}D_y^{\beta}u$ [2)] and which satisfies (1) a. e. in Ω.

We study the Goursat-like problem (G) which consists in finding a solution of equation (1) in Ω satisfying the conditions

(2) $$D_x^{\alpha}D_y^{\beta-1}u(x,g(x)) = G(x);\ D_x^{\alpha-1}D_y^{\beta}u(h(y),y) = H(y);$$

(3) $$D_x^{\alpha-1}D_y^{\beta-1}u(x_0,y_0) = c_0.$$

Let us note that the above problem was considered by Z. Szmydt (cf. [7]) in the case $\alpha = \beta = 1$ as a generalization of the classical Goursat problem.

We assume the following

I. The function $F:\Omega\times\mathbb{R}^m \longrightarrow \mathbb{R}$ satisfies the *Caratheodory* conditions (cf. [5], def. 12.2) and the inequality

(4) $$|F(x,y,\{z_{\gamma\lambda}\})| \le K(x,y) + \sum_{\gamma,\lambda}\sum_{l=1}^{m_{\gamma\lambda}} K_{\gamma\lambda l}(x,y)|z_{\gamma\lambda}|^{x_{\gamma\lambda l}}$$

($z_{\gamma\lambda} \in \mathbb{R}$) holds true a. e. on Ω, where $m_{\gamma\lambda} \in \mathbb{N}$, $0 < x_{\gamma\lambda l} \le 1$ are given numbers and $K, K_{\gamma\lambda l}:\Omega \longrightarrow \mathbb{R}_+$ given functions of class $L(\Omega)$, and $L^{1/(1-x_{\gamma\lambda l})}(\Omega)$ respectively.

II. The functions g and h are continuous.

III. The functions G and H are integrable.

---

2) As a result $D_x^{\alpha-1}D_y^{\beta}u$ and $D_x^{\alpha}D_y^{\beta-1}u$ are absolutely continuous with regard to x and y, respectively, and $D_x^{\alpha-1}D_y^{\beta-1}u$ is absolutely continuous with respect to both x and y.

3. Solutions of the Goursat-like problem

Let us denote $s = D_x^{\alpha} D_y^{\beta} u$. One can prove that if u is a solution of equation (1) in Ω, then there are integrable functions $\varphi:(0,A) \longrightarrow \mathbb{R}$ and $\psi:(0,B) \longrightarrow \mathbb{R}$, and a constant $c \in \mathbb{R}$, such that

(5) $$u(x,y) = u_0(x,y) + cx^{\alpha-1}y^{\beta-1}/(\Gamma(\alpha)\Gamma(\beta)) + x^{\alpha-1}\psi^{(-\beta)}(y)/\Gamma(\alpha) + \varphi^{(-\alpha)}(x)y^{\beta-1}/\Gamma(\beta) + D_x^{-\alpha}D_y^{-\beta}s(x,y)$$

($\varphi^{(-\alpha)}(x) := D_x^{-\alpha}\varphi(x)$; $\psi^{(-\beta)}(y) := D_y^{-\beta}\psi(y)$) with

(6) $$u_0(x,y) = \left(c_0 - \int_0^{x_0} G(\xi)d\xi - \int_0^{y_0} H(\eta)d\eta\right)x^{\alpha-1}y^{\beta-1}/(\Gamma(\alpha)\Gamma(\beta)) + x^{\alpha-1}H^{(-\beta)}(y)/\Gamma(\alpha) + G^{(-\alpha)}(x)y^{\beta-1}/\Gamma(\beta).$$

One can also observe that the function u_0 satisfies the conditions (2) and (3), and the homogeneous equation (1).

Conversely, if u is given by formula (5) with some integrable functions $\varphi:(0,A) \longrightarrow \mathbb{R}$ and $\psi:(0,B) \longrightarrow \mathbb{R}$, and a constant $c \in \mathbb{R}$, then u is a solution of equation (1) in Ω.

Imposing on function u (cf. (5)) conditions (2) and (3), we obtain

(7) $$\varphi(x) = -\int_0^{g(x)} s(x,\eta)d\eta; \qquad \psi(y) = -\int_0^{h(y)} s(\xi,y)d\xi$$

and

(8) $$c = \int_0^{x_0} d\xi \int_{y_0}^{g(\xi)} s(\xi,\eta)d\eta + \int_0^{y_0} d\eta \int_0^{h(\eta)} s(\xi,\eta)d\xi.$$

Let us observe that the following inequalities

$$(9)\qquad \int_0^A |\varphi(x)|dx \le ||s||;\ \int_0^B |\psi(y)|dy \le ||s||; \qquad |c| \le 2||s||$$

hold good, where $||\cdot||$ is the norm in the space $L(\Omega)$. Assuming that $\alpha,\beta \ge 0$, and using the first two of relations (9), we get the estimates

$$(10)\qquad \int_0^A |\varphi^{(-\alpha)}(x)|dx \le A^{\alpha}/\Gamma(1+\alpha)||s||; \qquad \int_0^B |\psi^{(-\beta)}(y)|dy \le B^{\beta}/\Gamma(1+\beta)||s||$$

Denote

$$(11)\qquad L_{\gamma\lambda}s(x,y) = cx^{\alpha-\gamma-1}y^{\beta-\lambda-1}/(\Gamma(\alpha-\gamma)\Gamma(\beta-\lambda)) +$$

$$x^{\alpha-\gamma-1}\psi^{(\lambda-\beta)}(y)/\Gamma(\alpha-\gamma) + \varphi^{(\gamma-\alpha)}(x)y^{\beta-\lambda-1}/\Gamma(\beta-\lambda) + D_x^{\gamma-\alpha}D_y^{\lambda-\beta}s(x,y)$$

with c, φ and ψ (depending on s) being given by formulae (7) and (8), respectively. It is easily observed that the Problem (G) is equivalent to the following integro-functional equation

$$(12)\qquad s(x,y) = F(x,y,\{D_x^{\gamma}D_y^{\lambda}u_0(x,y)+L_{\gamma\lambda}s(x,y)\})$$

$((x,y) \in \Omega)$.

In the sequel we will show that equation (12) has at least one solution and that the set of its solutions is bounded in the space $L(\Omega)$. To this end we consider on $L(\Omega)$ the transformation T

$$(13)\qquad Ts(x,y) := F(x,y,\{D_x^{\gamma}D_y^{\lambda}u_0(x,y)+L_{\gamma\lambda}s(x,y)\}).$$

One can observe that T is a composition of three transforms: the linear $L_{\gamma\lambda}$, a translation and a substitution operator.

The relation (resulting from (7) - (11))

$$(14)\qquad ||L_{\gamma\lambda}s|| \le 5A^{\alpha-\gamma}B^{\beta-\lambda}/(\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\lambda))||s||$$

implies that $L_{\gamma\lambda}$ is continuous mapping of $L(\Omega)$ into itself. By a standard argument, based on the Riesz theorem on compactness (cf. [4], p.166 and [5], Th. 4.20.1) one can prove the validity of

PROPOSITION 2. *If $\gamma < \alpha$, $\lambda < \beta$, then the transformation* $L_{\gamma\lambda}: L(\Omega) \longrightarrow L(\Omega)$ *is completely continuous.*

Due to Assumptions I and III the substitution operator (cf. [6], Th. 12.10) and the translation are continuous mappings of $L(\Omega)$ into itself, whence, and by the continuity of $L_{\gamma\lambda}$, the transformation T is continuous. Moreover, by Proposition 2, it is completely continuous in case if $\gamma < \alpha$, $\lambda < \beta$.

Let us observe that in the said case Proposition 2 implies the complete continuity of the transformation defined by the right hand side of equation (5), where c, φ and ψ are given by formulae (7) and (8), respectively.

Denote $h = (h_x, h_y)$ and $t_h s(x,y) = s(x+h_x, y+h_y)$. We will use the following definition:

A set $Z \subset L(\Omega)$ is *uniformly continuous in average* if and only if $\|t_h s - s\| \longrightarrow 0$, uniformly with respect to $s \in Z$, when $h \longrightarrow 0$.

Now, let us consider the set

$$B_\rho = \{s \in L(\Omega) : \|s\| \le \rho_K + \rho\},$$

where $\rho_K := \|K\|$ and ρ is a positive number, and its continuous in average subset Z_ρ. By the Riesz theorem, Z_ρ is relatively compact in $L(\Omega)$.

Let $s \in L(\Omega)$. In virtue of inequality (4), the estimate

$$(15) \qquad |Ts(x,y)| \le K(x,y) + $$

$$+ \sum_{\gamma,\lambda} \sum_{l=1}^{m_{\gamma\lambda}} K_{\gamma\lambda l}(x,y)\left(|D_x^\gamma D_y^\lambda u_0(x,y)|^{\varkappa_{\gamma\lambda l}} + |L_{\gamma\lambda} s(x,y)|^{\varkappa_{\gamma\lambda l}}\right)$$

holds true.

Bearing in mind the Hölder inequality and basing on relations (14) and (15), we get

$$\|Ts\| \le \rho_K + \sum_{\gamma,\lambda} \sum_{l=1}^{m_{\gamma\lambda}} \|K_{\gamma\lambda l}^{1/(1-x_{\gamma\lambda l})}\|^{1-x_{\gamma\lambda l}} \Big(\|D_x^{\gamma} D_y^{\lambda} u_0\|^{x_{\gamma\lambda l}} +$$

$$+ (A^{\alpha-\gamma} B^{\beta-\lambda}/(\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\lambda)\|s\|)^{x_{\gamma\lambda l}}\Big),$$

whence and by the inequality

$$\|D_x^{\gamma} D_y^{\lambda} u_0\| \le \text{const } A^{\alpha-\gamma} B^{\beta-\lambda},$$

we obtain

(16) $$\|Ts\| \le \rho_K + \text{const } \max(A,B)\ (1 + \|s\|^{x}),$$

where const is a positive constant independent of s and $x := \max_{\gamma,\lambda,l}(x_{\gamma\lambda l})$.

Evidently, a sufficient condition for the inclusion $T(B_\rho) \subset B_\rho$ is

(17) $$\text{const } \max(A,B)\ (1 + (\rho_K + \rho)^{x}) \le \rho.$$

Let us distinguish two cases: a) $x < 1$, b) $x = 1$.

In case a), inequality (17) is satisfied provided that ρ is chosen sufficiently large.

In case b), the said inequality holds good if diamΩ is sufficiently small.

We can assert that by the continuity of T, the set $T(\bar{Z}_\rho)$ is a compact subset of $L(\Omega)$ and hence it is a closure of a certain set which is uniformly continuous in average.

Thus, the inclusion $T(\bar{Z}_\rho) \subset \bar{Z}_\rho$ holds true.

By the Schauder fixed point theorem (cf. [4], p. 57 and [5]. Th. 3.6.1) equation (12) has an integrable solution. Moreover, if s is a solution to equation (12) then, due to relation (16), the inequality $\|s\| \le \rho_0$ holds, where ρ_0 fulfils condition (17).

Aforegoing considerations establish

THEOREM 1. *If Assumptions* I-III *are satisfied, then Problem* (G) *has a global solution* [3] *in the case* $x < 1$ *and a local one in the case* $x = 1$ *(cf. the discussion subsequent to* (17)).

4. Extention of the local solutions

It is known from the former Section that Problem (G) possesses a local solution in the case $x = 1$. In this section we are going to extend this solution to obtain a global one.

First of all we consider the problem (G_0), that is the problem (G) in which

$$x = 1,\ \lambda < \beta\ ,\ y_0 = 0;\ g \equiv 0 \tag{18}$$ [4]

(it is clear that (G_0) is a counterpart of the Picard problem).

Let us equip the space $L(\Omega)$ with the norm

$$\|s\|_\tau = \iint_\Omega |s(x,y)| \exp(-\tau y)dxdy, \tag{19}$$

where τ is a positive number.

Using the Hölder inequality, we obtain

$$\begin{aligned} \iint_\Omega K_{\gamma\lambda 1}(x,y)|L_{\gamma\lambda}s(x,y)|^{x_{\gamma\lambda 1}}\exp(-\tau y)dxdy &\le \\ \le \left\|K_{\gamma\lambda 1}^{1/(1-x_{\gamma\lambda 1})}\right\|_\tau^{1-x_{\gamma\lambda 1}} \left|\iint_\Omega L_{\gamma\lambda}s(x,y)\exp(-\tau y)dxdy\right|^{x_{\gamma\lambda 1}}&. \end{aligned} \tag{20}$$

By direct calculation, one can show that the inequalities

$$\left.\begin{aligned} &\iint_\Omega \exp(-\tau y)x^{\alpha-\gamma-1}|\varphi^{(\lambda-\beta)}(y)|dxdy/\Gamma(\alpha-\gamma) \\ &\iint_\Omega \exp(-\tau y)D_x^{\gamma-\alpha}D_y^{\lambda-\beta}|s(x,y)|dxdy \end{aligned}\right\} \le \tag{21}$$

---

[3] The said solution is continuous in case when $\alpha = \beta = 1$.

[4] As a result $\varphi \equiv 0$ and $c = 0$.

$$\le A^{\alpha-\gamma}/\Gamma(1+\alpha-\gamma)\tau^{\lambda-\beta}||s||_{\tau}$$

hold true.

Finally, we have

$$(22)\qquad ||Ts||_{\tau} \le \rho_K + \text{const} \sum_{\gamma,\lambda} \sum_{l=1}^{m_{\gamma\lambda}} \left(1 + (\tau^{\lambda-\beta}||s||_{\tau})^{\varkappa_{\gamma\lambda l}}\right).$$

Hence, for τ and ρ sufficiently large to fulfil the relation

$$(23)\qquad \text{const} \sum_{\gamma,\lambda} \sum_{l=1}^{m_{\gamma\lambda}} \left(1 + (\tau^{\lambda-\beta}(\rho_K + \rho))^{\varkappa_{\gamma\lambda l}}\right) \le \rho,$$

T continuously maps the compact set $\bar{Z}_{\rho,\tau}$ (i. e. $\bar{Z}_{\rho}$ with $||\cdot||$ replaced by $||\cdot||_{\tau}$) into itself.

As a result we can formulate

PROPOSITION 3. *If Assumptions* I-III *and* (18) *are satisfied, then Problem* (G_0) *has a global solution.*

REMARK 1. The thesis of Proposition 3 is valid if condition (18) is replaced by

$$(18')\qquad \varkappa = 1;\ \gamma < \alpha;\ x_0 = 0;\ h \equiv 0.$$

REMARK 2. Since the right-hand sides of estimates (21) do not depend on B, one can show that if $\Omega = (0,A)\times(0,\infty)$ then equation (12) has a solution s in the class of measurable functions such that $||s||_{\tau} < \infty$ (the parameters τ in (19) and ρ in the definition of B_{ρ} are chosen so that inequality (23) is satisfied).

REMARK 3. (the characteristic problem). It is known from paper [8] that under Assumptions I-III, and the additional assumptions $g \equiv 0$; $h \equiv 0$, Problem (G) has a solution.

We assume that $g(0) = h(0) = 0$ and the curves l_1 and l_2 do not intersect each other in $\Omega \setminus \{(A,B)\}$, $\gamma < \alpha$, $\lambda < \beta$ and $x_0 = y_0 = 0$.

It can be noticed (cf. relations (5) and (17)) that there exists a sufficiently small number $\delta > 0$ such that Problem (G) has a solution, say u_1, in the set $(0,\delta)^2 \subset \Omega$.

Now, we will use Proposition 3 to extend the local solution (cf. [3]) of Problem (G).To this end we assume that $g(\delta) < \delta$ (in the opposite case $h(\delta) < \delta$) and define $a := \max\{x \in [\delta,A]: g(x) \le \delta\}$.

We seek a function $u:(0,a)\times(0,\delta) \longrightarrow \mathbb{R}$, such that $u = u_1$ in $(0,\delta)^2$, which is a solution of equation (1) in $(\delta,a)\times(0,\delta)$ and satisfies the conditions

$$D_x^{\alpha}D_y^{\beta-1}u(x,g(x)) = G(x); \; D_x^{\alpha-1}D_y^{\beta}u(\delta,y) = D_x^{\alpha-1}D_y^{\beta}u_1(\delta,y); \tag{24}$$

$$D_x^{\alpha-1}D_y^{\beta-1}u(\delta,0) = D_x^{\alpha-1}D_y^{\beta-1}u_1(\delta,0) \tag{25}$$

$(x \in (\delta,a);\ y \in (0,\delta))$.

It can be shown by an argument analogous to that in the proof of Proposition 3 that there is a solution, say u_2, of the above problem.

Set $b := \max\{y \in [\delta,B]: h(y) \le a\}$. Similarly as above, we search for a function $u:(0,a)\times(0,b) \longrightarrow \mathbb{R}$, such that $u = u_2$ in $(0,a)\times(0,\delta)$, being a solution of equation (1) in $(0,a)\times(\delta,b)$ and satisfying the conditions

$$D_x^{\alpha}D_y^{\beta-1}u(x,\delta) = D_x^{\alpha}D_y^{\beta-1}u_2(x,\delta) \; ; \; D_x^{\alpha-1}D_y^{\beta}u(h(y),y) = H(y); \tag{26}$$

$$D_x^{\alpha-1}D_y^{\beta-1}u(0,\delta) = D_x^{\alpha-1}D_y^{\beta-1}u_2(0,\delta). \tag{27}$$

$(x \in (0,a);\ y \in (\delta,b))$.

We denote by u_3 a solution of the above problem. It is easily seen that u_3 is a solution to equation (1) in $(0,a)\times(0,b)$ and satisfies the condition (2) for $x \in (0,a)$; $y \in (0,b)$ and condition (3).

Continuing this process, we can extend a local solution of Problem (G) to obtain a global one.

The above-obtained results can be gathered in

<u>Theorem 2.</u> *If Assumptions* I–III *(with* $x = 1$*), as well as those formulated above in the present Section, are satisfied then Problem* (G) *has a global solution.*

5. Generalized Cauchy Problem

Let us keep in force Assumptions I-III, and assume additionally that h is absolutely continuous. The above-presented method allows to examine the following generalized Cauchy problem (C): Find a solution of equation (1) in Ω satisfying the conditions

$$(28)\qquad \begin{aligned} &D_x^{\alpha}D_y^{\beta-1}u(x,g(x)) = G(x);\\ &D_x^{\alpha-1}D_y^{\beta-1}u(h(y),y) = c + \int_0^y H(\eta)d\eta. \end{aligned}$$

One can show that if the function $h'G\circ h$ is integrable, then

$$(29)\qquad u_c(x,y) = cx^{\alpha-1}y^{\beta-1}/(\Gamma(\alpha)\Gamma(\beta)) + G^{(-\alpha)}(x)y^{\beta-1}/\Gamma(\beta) +$$

$$+ x^{\alpha-1}H^{(-\beta)}(y)/\Gamma(\alpha) - x^{\alpha-1}\int_0^y(y-\eta)^{\beta-1}h'(\eta)G(h(\eta))d\eta/(\Gamma(\alpha)\Gamma(\beta))$$

is a solution of the homogeneous equation (1) and satisfies the conditions (28).

We seek a solution of Problem (C) in the form

$$(30)\qquad u(x,y) = u_c(x,y) + x^{\alpha-1}\psi^{(-\beta)}(y)/\Gamma(\alpha) +$$

$$+ \varphi^{(-\alpha)}(x)y^{\beta-1}/\Gamma(\beta) + D_x^{-\alpha}D_y^{-\beta}s(x,y),$$

where φ and ψ are integrable functions. Imposing on the above function u the conditions (28), we get

$$(31)\qquad \begin{aligned} &\varphi(x) = -\int_0^{g(x)} s(x,\eta)d\eta;\\ &\psi(y) = -h'(y)\int_{g(h(y))}^{y} s(h(\eta),\eta)d\eta - \int_0^{h(y)} s(\xi,y)d\xi. \end{aligned}$$

Repeating the argument from Sections 3 and 4 we obtain

THEOREM 3. *Assume that:* 1^0 *Hypotheses* I-III *are satisfied;*
2^0 *the functions* h *and* $h'G\circ h$ *are absolutely continuous and integrable, respectively,*
3^0 *in case* $x \le 1$, *the curves* l_1 *and* l_2 *do not intersect each other in* $\Omega \setminus \{(A,B)\}$ *and the condition* $g(0) = h(0) = 0$ *is satisfied.*

Under these assumptions Problem (C) *has a solution.*

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Dr. Marek W. Michalski
Insitute of Mathematics
Warsaw University of Technology
PL-00-661, Warsaw

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ON THE COMPLEX ANALYSIS METHODS FOR SOME CLASSES OF PARTIAL DIFFERENTIAL EQUATIONS

L. G. Mikhailov

The development and applications of complex analysis methods in application to partial differential equations have a long history starting from B. Riemann probably. Its great development in the USSR took place since 1940 to 1970 in the works by Investigations and Science Organizations Activity of M. A. Lavrentiev, N. J. Mushelishvili [1], J. N. Vecua [2], F. D. Gakhov [3] and others. In the recent time similar International Science Organization Activity was displayed with the participation of R. P. Gilbert (USA), W. Wendland and E. Meister (FRG), W. Tutschke (DDR) and others.

The review of some of the author's results in the direction denoted in the title will be given in this paper, see [4]–[14].

1. Generalized Cauchy-Riemann System with Singular Points [4]

We shall consider complex-valued functions of two real variables and in addition to the customary designation $f(x,y)$ we shall use the notation $f(z)$, where $z = x + iy$. If $f(x,y) = f(z) \in C^1(D)$ the formulae $\partial_{\bar z} f = \frac{1}{2}(\partial_x f + i\partial_y f), \partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f)$ define the formal complex derivatives with respect to $\bar z$ and z. Let RA and A denote classes of functions $f(x,y)$ analytic in (x,y) and $f(z)$ analytic in z, respectively. If $f(x,y) \in C^1$ then

$$f(x,y) = \sum_{k,j=0}^{\infty} f_{kj} x^k y^j = \sum_{k,j=0}^{\infty} \tilde{f}_{kj} z^k \bar{z}^j = f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \tilde{f}(z,\bar{z}).$$

In order to find $\partial_{\bar{z}} f$ it is necessary to differentiate $\tilde{f}$ with respect to $\bar{z}$ considering z to be constant. If $\partial_{\bar{z}} f = 0$, then f independent to $\bar{z}$ and $f = f(z)$. On the other hand if

$$W = W(z) = u + iv \in C^1(D)\,, \text{then}\, \partial_{\bar{z}} W = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]$$

and Cauchy-Riemann conditions become $\partial_{\bar{z}} W = 0$. Relative to the operation $\partial_{\bar{z}}$ the integral $Tf = -\frac{1}{\pi}\int_D\int \frac{f(\zeta)}{\zeta - z} ds (\zeta = \xi + i\eta, ds = d\xi d\eta)$ possesses an important property $\partial_{\bar{z}} Tf = f(z)$ for all points within D (and $\partial_{\bar{z}} Tf = 0$ for exterior points), where $\partial_{\bar{z}}$ is understood in the conventional sense, if $f(z) \in \mathcal{H}(D)$, and in generalized sense of S. L. Sobolev, if $f(z) \in L(D)$. In addition to the well-known classes of functions and Banach Spaces $C(D)$ and $L^p(D)$ let $M(D)$ denote a class of bounded functions and $\mathcal{H} \equiv \text{Lip}\alpha$ denotes a class of functions, for which a Hölder-condition or Lipschitz-α condition is available.

The operator Tf is linear and completely continuous from $M(D)$ and $C(D)$ into $C(D)$ and from $L^p(D), p > 2$, into $C(D)$ as well; in the latter case the function $Tf \in \text{Lip}\alpha, \alpha = \frac{p-2}{p}$. The integral defines the primitive of $f(z)$ in respect to $\bar{z}$, the set of all primitives is given by the formula $W(z) = \Phi(z) + Tf$, where $\Phi(z)$ is arbitrary analytic function. If $f(z) \in L^p(D), p > 2$, and $W(z) \in C(D + \Gamma)$, then $\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{W(t)}{t-z} dt$; if $f(z) \in RA$, then its primitive is bounded by conventional integration with respect to $\bar{z}$ $F(z,\bar{z}) = \int f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) d\bar{z}$ and after this we will have the formula $-\frac{1}{\pi}\int_D\int \frac{f(\zeta)}{\zeta - z} ds = F(z,\bar{z}) - \frac{1}{2\pi i}\int_\Gamma \frac{F(t,\bar{t})}{t-z} dt$. All formulas remain valid if $f(z)$ has isolated singular points, exactly the same way in fundamental singular integral $\omega(z) = -\frac{1}{\pi}\int_D\int \frac{a(\zeta)ds}{|\zeta|(\zeta - z)}$. If $a(z) \in L^p(D), p > 2$, then $\omega(z) = o(|z|^{-\frac{2}{p}}), z \to 0$; if $a(z) = o(|z|^{-\beta}), 0 < \beta < 1$, then $\omega(z) = o(|z|^{-\beta})$; if $a(z) \in M(D)$ and $\overline{\lim\limits_{z\to 0}} |a(z)| = \mu$, then $|\omega(z)| \le (2\mu + \varepsilon)\ln\frac{1}{|z|} + N_\varepsilon$ and if $a(z) \in C(D)$, then $\omega(z) = o(\ln\frac{1}{|z|}), z \to 0$.

We shall consider the system

$$\partial_{\bar{z}} W = \frac{a(z)}{|z|} W + \frac{b(z)}{|z|} \overline{W}\,, \tag{1.1}$$

where $a(z), b(z)$ are bounded in D, the domain D is finite and $z = 0$ is its inner point. Every solution of (I.1) admits the representation by the formula

$$W(z) = \varphi(z) \exp \Omega(z), \Omega(z) = -\frac{1}{\pi} \int_D \int \frac{a(\zeta) + b(\zeta) \frac{\overline{W(\zeta)}}{W(\zeta)}}{|\zeta|(\zeta - z)} ds \qquad (1.2)$$

and $\varphi(z)$ is an analytic function (corresponding to $W(z)$). Utilizing the notation $\mu = \overline{\lim_{z\to 0}} |a(z)| + \overline{\lim_{z\to 0}} |b(z)|$ we have

$$-(2\mu + \varepsilon) \ln \frac{1}{|z|} - N_\varepsilon \le |\Omega(z)| \le (2\mu + \varepsilon) \ln \frac{1}{|z|} + N_\varepsilon , \qquad (1.3)$$

where ε is an arbitrary small number and N_ε is a constant that may $\to \infty$ when $\varepsilon \to 0$. If $K_1(\varepsilon), K_2(\varepsilon)$ are analogous constants, we have

$$K_1(\varepsilon)|z|^{2\mu+\varepsilon} \le |\exp \Omega(z)| \le K_2(\varepsilon)|z|^{-(2\mu+\varepsilon)} . \qquad (1.4)$$

Examples of singular equations $\partial_{\bar{z}} W = \lambda \cdot \frac{a(z)}{z} W$ with respect to formula (1.2)

1. $a(z) = \frac{1}{\ln \frac{1}{|z|^2}}, W(z) = \varphi(z)(\ln \frac{1}{|z|^2})^\lambda$;
2. $a(z) = 1, W(z) = \varphi(z)|z|^{2\lambda}$;
3. $a(z) = \frac{\alpha}{2} \cdot \frac{1}{|z|^\alpha}, W(z) = \varphi(z) \cdot \exp(-\frac{\lambda}{|z|^\alpha})$.

Violation of the Carlemann and Liouville theorems

The Carlemann theorem that the solutions may have only zeroes of finite order and that the zeroes are discrete is a fundamental one for the theory of generalized analytic functions. From the estimation (1.4) follows that if $a(z) \in M(D)$, the $\exp \Omega(z)$ and $W(z) = \varphi(z) \exp \Omega(z)$ as well may have zeroes of finite order only. If in example 3 we put $\varphi(z) = \exp(\frac{1}{z})$, then for $W(z) = \exp(\frac{1}{z}) \exp(-\frac{\lambda}{|z|^\alpha})$ and $\alpha > 1$ the singular point $z = 0$ is a limit point of zeroes. Thus the Carlemann theorem may be violated if the singularities are higher than the first order.

In the regular case no non-zero generalized analytic function exists which is continuously continuable through boundary Γ into an analytic function which will vanish at $z = 0$ (Liouville theorem). Let us consider example 2. Putting $\varphi(z) = z^{-n}$, we obtain $W(z) = z^{-n} \cdot |z|^{2\lambda}$ and if $2\text{Re}\,\lambda > n$, then $W(z)$ is continuous everywhere in D, including $z = 0$, and is continuously continuable through Γ into the function cz^{-n}, which is analytic and vanishes for $z = \infty$. Selecting λ we obtain equtions, which have any previously assigned number of functions violating the Liouville theorem.

Fundamental theorems. Integrating (1.1) with respect to $\bar{z}$, we obtain the equivalent integral equation

$$W(z) = \Phi(z) - \frac{1}{\pi} \int_D \int \frac{a(\zeta)W(\zeta) + b(\zeta)\overline{W(\zeta)}}{|\zeta|(\zeta - z)} ds\,, \qquad (1.5)$$

where $\Phi(z)$ is an arbitrary analytic function. The singular integral equations of this new type have been investigated in [4]. If $W(z) = |z|^{-\beta} \cdot W_0(z)$, where $W_0(z) \in M(D), C(D), \ldots$, then $W(z) \in M_\beta(D), C_\beta(D), \ldots$ and $\|W\|_\beta = \|W_0\|$. Thus $M_\beta, C_\beta, \ldots$ are Banach spaces isometrical to $M, C, \ldots$. As it was shown in [4] for $0 < \beta < 1$ the integral operator in (1.5) is linear, but not completely continuous in $M_\beta, C_\beta, \ldots$ and contrary to a regular case and may have a non-zero eigen-functions.

Theorem 1.1. The Liouville theorem for (1.1) is violated by those and only those functions which are solutions of homogeneous equation (I.5). Their number is $\leq [\mu + \beta]$ in M_β, C_β and $\leq [\mu]$ in M, C, where $[\mu]$ is integer part of the number μ.

Let

$$K = \sup_D |a(z)| + \sup_D |b(z)|, q(\beta) = \frac{1}{\pi} \int_{|\zeta| \leq \infty} \int \frac{ds}{|\zeta|^{1+\beta} |\zeta - 1|}\,.$$

Theorem 1.2. For some $\beta, 0 < \beta < 1$, let one of the following two conditions be satisfied:

1) $a(z), b(z)$ are bounded and $K \cdot q(\beta) < 1$;

2) $a(z), b(z)$ are bounded in D and continuous at the point $z = 0$ and $\mu \cdot q(\beta) < 1$.

Then all the solutions for (I.1) of class $M_\beta(D)$ are expressed by means of the formula

$$W(z) = \Phi(z) + \int_D\int [\Gamma_1(z,\zeta)\Phi(\zeta) + \Gamma_2(z,\zeta)\overline{\Phi(\zeta)}]ds \qquad (1.6)$$

in terms of analytic function $\Phi(z)$, where the correspondence between $W(z)$ and $\Phi(z)$ is mutually one-to-one.

The conversion of the representation formula (1.2).

Using notation $v(z) = \frac{W(z)}{\varphi(z)} = \exp\Omega(z)$, we obtain a differential equation $\partial_{\bar z} v = \frac{a(z)}{|z|}v + \frac{b(z)}{|z|}\cdot\frac{\overline{\varphi(z)}}{\varphi(z)}\bar v$. Applying a formula of type (1.6) and denoting the resolvents by $\Gamma_1^\varphi(z,\zeta), \Gamma_2^\varphi(z,\zeta)$, we will obtain:

Theorem 1.3. Let one of the conditions:

1) $a(z), b(z)$ be bounded and $K\cdot q(1/2) < 1$, or

2) $a(z), b(z)$ be bounded in D and continuous at the point $z = 0$ and $\mu\cdot q(1/2) < 1$ be satisfied.

Then the formula (1.2) and the inverse formula

$$W(z) = \varphi(z)\left\{1 + \int_D\int [\Gamma_1^\varphi(z,\zeta) + \Gamma_2^\varphi(z,\zeta)]ds\right\}$$

establish a mutually one-to-one correspondence between $W(z)$ and $\varphi(z)$ in the class of functions with isolated singularities.

$$K_1(\varepsilon)\cdot|z|^{2\mu+\epsilon} \le \left|\frac{W(z)}{\varphi(z)}\right| \le K_2(\varepsilon)\cdot|z|^{-(2\mu+\epsilon)}\left(\mu < \frac{1}{2}\right).$$

The theorems 1.1, 1.2, 1.3 permit us to expand on the singular case the whole of the theory of J. N. Vecua from regular case [2].

These results have been obtained by the author in 1958–1963. Later by the author and his co-workers in Dushanbe the other methods have been developed as well: the method, based on separation of variables with more exact studying of the model equation; the method used in connection (1.1) with Partial Differential Equations of the second order of ellyptic type and others. In this paper we have no possibility to give full observation of all investigations made in Dushanbe. It must be mentioned that many papers are written in Alma-Ata and Tbilisi as well. But it must be said that the central problem on the correspondence between $W(z)$ and $\varphi(z)$ remains open to-day, in general case.

2. Generalized Analytic Functions in Many Variables, [5]–[7]

The functions named in the title are solutions of the system

$$\partial_{\bar{z}_k} W = a_k(z)\overline{W} + b_k(z)W + c_k(z)\,, k = 1, \dots, n\,, \qquad (2.1)$$

where $z = (z_1, \dots, z_n), z_k = x_k + iy_k, 2\partial_{\bar{z}_k} = \partial_{x_k} + i\partial_{y_k} a_k, b_k, c_k$ are known and $W = W(z) = W(z_1, \dots, z_n)$ are unknown functions of class $C^2(D), D$ is a polycylindrical domain.

1) $a_k = b_k = 0$ (inhomogeneous Cauchy-Riemann system). Let all the necessary conditions $\partial_{\bar{z}_k} c_j = \partial_{\bar{z}_j} c_k$ be available. Then the formula

$$W(z) = T_1C_1 + S_1T_2C_2 + \dots + S_1 \dots S_{n-1}T_nC_n (\equiv R[c_1, \dots, c_n]) \quad (2.2)$$

where $S_kW = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{W(z_1,\dots,t_k,\dots,z_n)}{t_k - z_k} dt_k$,
$T_kW \equiv -\frac{1}{\pi}\int_{D_k}\int \frac{W(z_1,\dots,\sigma_k,\dots,z_n)}{\sigma_k - z_k} ds_k$ gives a particular solution. As far as (2.2) was received by composition of one-dimentional formula mentioned above, then besides (2.2), by transposition of indices many analogous formulas may be formed.

2) All $a_k = 0$. After cross-differentiations we will have many relations on b_k, c_k, which are sufficient for constructing such functions $\omega(z) = R[b_1, \dots, b_n]$ and $U(z) = R[\exp(-\omega)c_1, \dots, \exp(-\omega)c_n]$, therefore the formula $W(z) = [\Phi(z) + U(z)]\exp\Omega(z)$ gives a general solution, $\Phi(z)$ is an arbitrary analytic function.

3) The general case. Let $a_p \neq 0$. The first series of cross-differentiations leads to the relations of type

$$\partial_{z_j} W = \sigma_{jp}\partial_{zp} W + q_{jp}\overline{W} + h_{jp}W + f_{jp}\,, \qquad (2.3)$$

where $a_p\overline{\sigma_{jp}} = a_j, a_p\overline{q_{jp}} = \partial_{\bar{z}_j} b_p - \partial_{\bar{z}_p} b_j$. The second series of cross-differentiations into (2.3) leads to the relations

$$\partial_{\bar{z}_k}\sigma_{jp} \cdot \partial_{z_p} W + q_{jp} \cdot \sigma_{jp} \cdot \overline{\partial_{z_p} W} = \dots \qquad (2.4)$$

(on the right side are members without derivatives). Having even one non-zero equality (2.4), we obtain a relation of type $\partial_{z_p} W = \lambda_p\overline{W} + \mu_p W + v_p$, substituting it in (2.3) gives n similar relations; together with equation (2.1) they form a total differentials system which can have no more than manifold of solutions with finite number of arbitrary constants — (let us name it *trivial manifold of solutions*). For existence of a non-trivial manifold of solutions it is necessary that all the coefficients of (2.4) are equal to zero:

$$\partial_{\bar{z}_k}\sigma_{jp} = 0\,, q_{jp} = 0 \text{ or } \partial_{\bar{z}_j} b_p = \partial_{\bar{z}_p} b_j\,.$$

If $\omega = R[b_1, \ldots, b_n]$ then substitution of $W = \exp(-\omega)\cdot V$ reduces (2.1) to a canonical form

$$\partial_{\bar{z}_k} W = a_k(z)\overline{W} + c_k(z)\,, k = 1, \ldots, n\,. \tag{2.5}$$

Let us consider the first and second series of cross-differentiations in (2.5) and by the method mentioned above many previous and new equalities, necessary for the existence of non-trivial manifold solutions, may be obtained. In particular, if we consider the holomorphic system $\partial_{\bar{z}_k}\chi = \sigma_{k_1}\cdot\partial_{z_1}\chi\,, k = 2, \ldots, n$, it be totally integrable and if $\chi(z)$ is its solution satisfying initial condition $\chi(z_1, 0, \ldots, 0) = z_1$, then changing variables $\zeta_1 = \chi(z_1, \ldots, z_n), \zeta_k = z_k, k = 2, \ldots, n$, we transform (2.5) to the system $\sigma_{\bar{\zeta}_1} W = b(\zeta)\overline{W} + d_1, \partial_{\bar{\zeta}_k} W = d_k, k = 2, \ldots, n$. After the next similar transformation we will deduce $d_k = 0, k = 2, \ldots, n$, and $b(\zeta)$ must have a form $b(\zeta) = \frac{\overline{f(\zeta)}}{f(\zeta)}\cdot d(\zeta_1)$, where $f(\zeta)$ is an analytic function on ζ. After substitution $W = f\cdot\psi$ we shall obtain finally $\partial_{\bar{\zeta}_1}\psi = \alpha(\zeta_1)\overline{\psi} + h(\zeta_1)$.

Theorem 2. Let in the system (2.1) $a_p \neq 0$ and all conditions necessary for existence of non-trivial manifold of solutions are available. Then by changes of variables, the system (2.1) reduces to the single equation relative to a function of one variable.

From this basic position many results are followed and, in particular, representation formulas of the first and second kind similar (1.2) and (1.6) respectively.

3. Systems with Arbitrary Complex Operators, [8]–[10]

Let

$$P_j W \equiv \sum_{k=1}^{2n} a_k^j(x)\left(\frac{\partial W}{\partial x_k}\right) = 0\,, j = 1, \ldots, n\,. \tag{3.1}$$

$x = (x_1, \dots, x_n)$ is real and $a_k^j(x)$ are complex given functions. Let commutators $[P_j, P_k]$ be linear combinators of $P_1, \dots, P_n$. As it can be judged from lectures of Nirenberg(*), if $a_k^j(x) \in C^\infty$ then there exist local coordinates in which (3.1) takes the form of the Cauchy-Riemann system. We can add that here naturally arises the question of studying equations with operators $P_j W$ on the left side and with general linear terms on the right side:

$$P_j W = \sum_{k=1}^{2n} a_k^j(x) \cdot \left(\frac{\partial W}{\partial x_k}\right) = a_j(x)\overline{W} + b_j(x)W + c_j(x), j = 1, \dots, n. \quad (3.2)$$

From the main theorem of (*) it follows that in the local coordinates indicated above it takes a form of a generalized Cauchy-Riemann system type of (2.1).

In this second half of Sec. 3 we shall obtain formulas for representing solutions in terms of Analytic Functions (A. F.), first of all, for one complex analytic equation

$$LW \equiv \sum_{k=1}^{n} a_k(x)\partial_{x_k} W = 0 \quad (3.3)$$

and then for system m equations of this type with arbitrary number of variables, even or odd and arbitrary number of equations $m, 1 < m < n$. Let in (3.3) $a_k(x)$ be complex-valued given functions and W be a desired one in some polydisc neighbourhood of the origin. We exclude the cases when all $a_k(x)$ are real or L and $\bar{L}$ (or L^1 and L^2, where $L^1 + iL^2 = L$, are linearly dependent. Moreover we also exclude various singular cases, assuming, for example, that $a_1(x) \neq 0$ (then dividing by it we may assume that $a_1(x) \equiv 1$). We shall use analytic continuation method (in many variables) [8]. This method allows us to obtain concrete formulas very useful for practical computations.

Theorem 3.1. If $a_k(x) \in RA$, then the manifold of solutions of (3.3) in RA is given by the formula $W = \Phi[A_2(x), \dots, A_n(x)]$, where Φ is A. F. on complex variables $A_2, \dots, A_n$ and $A_k(x)$ are the construction in real

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(*)L. Nirenberg, *Lectures on Linear Partial Differential Equations*, Conf. Board Math. Sci Regional Conf. Ser. Math. No. 17, Amer. Math. Soc. Providence, R. I, 1973.

means of the left sides of the first integrals of the complex analytic system of ordinary differential equations $\frac{dz_k}{dz_1} = a_k(z_1, \dots, z_n), k = 2, \dots, n$.

If we change variables according to the formulas $\zeta_1 = z_1, \zeta_k = A_k(z_1, \dots, z_n), k = 2, \dots, n$, and denote the inverse transformation by $z_k = \alpha_k(\zeta_1, \dots, \zeta_n), k = 2, \dots, n$, then we obtain $\mathcal{L}W \equiv \sum_{k=1}^{n} a_k(z)\partial_{z_k} W = \partial_{\zeta_1} W$ and for inhomogeneous equation $LW = f(x), f(x) \in RA$ by the same method we obtain a formula for a particular solution,

$$W_0(x) = F(x) = \int_0^{x_1} f\{t, \alpha_2(x)[t, A_2(x), \dots, A_n(x)], \dots, \\ \alpha_n[t, A_2(x), \dots, A_n(x)]\} dt\,.$$

For the more general equation $LW = b(x)W + f(x)$ we obtain the representation formula

$$W(x) = \exp \omega(x)\{\Phi[A_2(x), \dots, A_n(x)] + W_0(x)\}\,,$$

where $\omega(x)$ and $W_0(x)$ are particular solutions of the equations

$$LW = b, LW_0 = \exp(-\omega) \cdot f\,.$$

The equation with the most general right hand side has the form $LW = \alpha(x)\overline{W} + \beta(x)W + \gamma(x)$, but the term $\beta(x)W$ can be eliminated.

Theorem 3.2. Suppose that $\alpha(x), \beta(x), \gamma(x)$ are complex and belong to RA. Then the manifold of all its solutions in RA is given by $W(x) = \Phi[A_2(x), \dots, A_n(x)] + \Gamma_1\Phi + \Gamma_2\overline{\Phi} + W_0(x)$, where Φ is an arbitrary A. F., $W_0 = F + \Gamma_1 F + \Gamma_2 \overline{F}$ and Γ_1, Γ_2 are the resolvents of the integral equation.

Proceeding to the system

$$\Lambda^k W \equiv \sum_{j=1}^{n} \alpha_j^k(x) \left(\frac{\partial W}{\partial x_j}\right) = 0\,, k = 1, \dots, m, 1 < m < n\,,$$

we assume that the $\alpha_j^k(x)$ are complex and belong to RA, the operators $\Lambda^1, \dots, \Lambda^m$ are linearly independent, and their commutators $\equiv 0$. By solving the system algebraically for $\partial_{x_1} W, \dots, \partial_{x_m} W$, we transform it into the form

$$L^k W \equiv \partial_{x_k} W + \sum_{j=m+1}^{n} a_j^k(x) \left(\frac{\partial W}{\partial x_j}\right) = 0\,, k = 1, \dots, m\,. \tag{3.4}$$

Carrying out analytic continuation (by exchanging x_k on Z_k), we arrive at the system to which a theory is available that is well known for real systems. For the first of equations it is necessary to find a system of the first integrals and take their left sides as new independent variables. The first equation may be transformed to the form $\partial_{\zeta_1} W = 0$ and the variable ζ_1 will be missing in all the remaining equations. We then proceed in a similar way with a newly obtained system. After m steps we arrive at the assertion that there exists a homeomorphic analytic change of variables such that $L^1 W = 0, \ldots, L^m W = 0$ can be transformed into $\partial_{\zeta_1} W = 0, \ldots, \partial_{\zeta_m} W = 0$.

Theorem 3.3. Suppose that in the system (3.4) $a_j^k(x) \in RA$ and complex, the operators $L^1, \ldots, L^m$ are linearly independent and their commutators are identically zero. Then the manifold of all solutions of (3.4) in RA is given by $W = \Phi[A_{m+1}(x), \ldots, A_n(x)]$, where Φ is an arbitrary A. F.

For the same operators $L^k W$ we consider the corresponding inhomogeneous system $L^k W = f_k, k = 1, \ldots, m$, where $f_k(x) \in RA$. Commutation of it leads to the necessary compatibility conditions $L^k f_j = L^j f_k$. If they are satisfied, a particular solution of the inhomogeneous system is given by a concrete formula. For more general system $L^k W = b_k(x) W + f_k(x), k = 1, \ldots, m$ there may be prescribed the necessary and sufficient conditions for compatibility; if they are satisfied, all solutions are given by the formula

$$W(x) = \exp \omega(x) \{\Phi[A_2(x), \ldots, A_n(x)] + V(x)\} .$$

4. Non-linear Systems [11]–[14]

The well-known overdetermined system

$$\partial_x u = p(x, y; u), \partial_y u = q(x, y; u) \tag{4.1}$$

is named a total integrable if the condition necessary for compatibility

$$p_y + q \cdot p_u = q_x + p \cdot q_u \tag{4.2}$$

will be satisfied identically. If we have for (4.1) the initial data condition $[u]_{x=x_0, y=y_0} = u_0$, then this problem is equivalent to the next chain of integral equations [11]:

$$\left.\begin{array}{c} u(x, y) = v(y) + \int_{x_0}^{x} P[t, y; u(t, y)] dt \, , \\ v(y) = u_0 + \int_{y_0}^{y} q[\tau, y; v(\tau)] d\tau \, . \end{array}\right\} \tag{4.3}$$

If we consider a complex system

$$\partial_{\bar{z}} W = p[z,\zeta;W], \partial_{\bar{\zeta}} W = q[z,\zeta;W], \tag{4.4}$$

where p, q are analytic on W and R-analytic on z, ζ, then by analytic continuation method (with exchange $\bar{z}$ and $\bar{\zeta}$ on new and independent variables s and σ) we will come to the system

$$\partial_s W = p[s,z;\sigma,\zeta;W], \partial_\sigma W = q[s,z;\sigma,\zeta;W]. \tag{4.5}$$

A chain of complex integral equations may be written analogously (4.3).

Theorem 4.1. Let in system (4.4) the functions p, q be analytic on W and R-analytic on z, ζ; let the condition necessary for compatibility

$$p_{\bar{\zeta}} + q \cdot p_W = q_{\bar{z}} + p \cdot q_W (\equiv h) \tag{4.6}$$

be satisfied identically, and some conditions of smallness be satisfied too. Then mutually one-to-one correspondence between the solutions of (4.4) and analytic functions $\Phi(z,\zeta)$ exists.

If $p, q \in C^2$ on z, ζ the new integral representation formula must be constructed first:

$$W(z,\zeta) = \Phi(z,\zeta) + T_{\bar{z}} p + T_{\bar{\zeta}} q - T_{\bar{z}} T_{\bar{\zeta}} h, \tag{4.7}$$

where $T_{\bar{z}}$ and $T_{\bar{\zeta}}$ are operators of type Tf (see Sec. 1) on the first or second variables of the function in variables (z,ζ). Integral equation (4.7) (respectively to W) is equivalent to the overdetermined system (4.4), but it is very difficult to establish that each solution of (4.7) is differentiable and according to this fact for (4.4) the theorem of mutual one-to-one correspondence will have been established as well.

In recent papers of the author [10]–[14] the similar results are extended to the overdetermined systems with arbitrary number of independent complex variables:

$$\partial_{\bar{z}_k} W = p_k[z_1,\ldots,z_n;W], k = 1,\ldots,n,$$

where $p_k(z, W), z = (z_1, \dots, z_n) \in C^{n-1}$ on z and analytic on W and all the conditions necessary for compatibility are available identically, the theorem of mutual one-to-one correspondence between W and A. F. $\Phi(z), z = (z_1, \dots, z_n)$, is obtained.

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Leonid Mikhailov
734063, Dushanbe
Aini-Str., 299
USSR

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INEQUALITIES CONNECTED WITH TRIGONOMETRIC SUMS

G. V. Milovanović and Th. M. Rassias

In this survey paper we consider inequalities connected with trigonometric sums. In the first part we give several classical results which lead to inequalities of Fejér, Jackson, Gronwall, Young, Rogosinski and Szegõ, and their extensions. In the second part we start with Turán's inequalities and study positivity and monotonicity of some classes of trigonometric sums and certain classes of orthogonal polynomial sums.

1. CLASSICAL RESULTS

1.1. Preliminaries

In this paper we consider various inequalities including trigonometric sums of the form

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \tag{1.1.1}$$

Several applications of these results can be given in the Fourier analysis. A very special role is played by the following sum

$$S_n(x) = \sum_{k=1}^{n} \frac{1}{k} \sin kx, \tag{1.1.2}$$

which represents the nth partial sum of the Fourier series

$$\frac{1}{2}(\pi - x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin kx \qquad (0 < x < 2\pi). \tag{1.1.3}$$

The above expansion, as well as the following two expansions are due to L. Euler

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx = \frac{1}{2}x \qquad (|x| < \pi), \tag{1.1.4}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos(2k+1)x = \begin{cases} \pi/4 & (|x| < \pi/2), \\ -\pi/4 & (\pi/2 < |x| < \pi). \end{cases} \tag{1.1.5}$$

The expansions (1.1.3) and (1.1.4) go back to the year 1755, and (1.1.5) to 1772 (cf. Burkhardt [1, pp. 857, 858, and 933] and E. Hewitt and R. E. Hewitt [1]). These expansions were investigated by the following mathematicians: H. Wilbraham studied (1.1.5) in the year 1848 and H. S. Carslaw in 1917; J. W. Gibbs studied (1.1.4) in 1899 and Kneser in 1905; Kneser also studied (1.1.3) in 1905 as well as Fejér in 1910, Jackson in 1911, and Gronwall in 1912. The partial sums of these series are continuous functions, while the sums of the series are functions with discontinuities.

It is well known that the Fourier series for a given periodic function f does not converge uniformly to $f(x)$ on an interval where f has a discontinuity. The nature of the deviation of the partial sums from $f(x)$ on such intervals is known as the *Gibbs phenomenon*, or the *Gibbs-Wilbraham phenomenon* (cf. an exellent survey paper by E. Hewitt and R. E. Hewitt [1]).

The modern theory of Fourier series started with Fejér's celebrated theorem that the Fourier series of a continuous function is uniformly Cesàro summable to the function (cf. Zygmund [1]). The proof of this fact was based upon the following

$$\sum_{k=0}^{n} \sin(k+\tfrac{1}{2})x = \frac{1-\cos(n+1)x}{2\sin\frac{x}{2}} = \frac{\sin^2(n+1)\frac{x}{2}}{\sin\frac{x}{2}} \geq 0$$

for $0 \leq x \leq 2\pi$. Fejér used this inequality to prove that

$$\frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kx = \frac{1}{2(n+1)} \left(\frac{\sin(n+1)\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 \geq 0.$$

Also, Fejér [3] used the inequality

$$|S_n(x)| < M \qquad (\forall x \in \mathbb{R} \text{ and } \forall n \in \mathbb{N}), \tag{1.1.6}$$

where $M \leq 3.6$, to estimate the Lebesgue constant and study divergence properties of Fourier series. The existence of a constant M was first proved by Kneser [1], without giving any numerical estimate of M. Fejér [4] proved that M can take the value $\pi/2 + 1 \approx 2.57$. However, Fejér had conjectured (see Fejér [5]) that the maximum of $|S_n(x)|$ increases as n tends to infinity and this limit is equal to

$$\mathrm{Si}(\pi) = \int_0^{\pi} \frac{\sin x}{x}\, dx = 1.8519370\ldots,$$

so that $M = 1.8519\ldots$ gives the *best constant*. Jackson [1] and Gronwall [1] verified Fejér conjecture.

The term *Gibbs's phenomenon* (for the convergence of $S_n(x)$ to $(\pi - x)/2$ in $(0,\pi)$) is usually attached to the fact that

$$\lim_{n\to\infty} S_n\Big(\frac{\pi}{n+1}\Big) = \mathrm{Si}(\pi) = \Big(\frac{\pi}{2}\Big)\cdot 1.1789797\ldots > \frac{\pi}{2}.$$

Fejer, also stated another conjecture on the positivity of the sum (1.1.2) on $(0,\pi)$, i.e. that

$$S_n(x) > 0 \qquad \text{if} \qquad (0 < x < \pi). \tag{1.1.7}$$

The above conjecture influenced several mathematicians who tried to verify it. This conjecture was answered positively in various ways. It was also generalized by many mathematicians. Fejer [8], [15] himself gave two different proofs of his conjecture.

These inequalities can be considered as inequalities for sums of special orthogonal polynomials. There are many inequalities for sums of special, but more general orthogonal polynomials. In particular, the polynomial inequality that de Branges [1] used in his proof of the well-known Bieberbach conjecture [1] is equivalent to the positivity of the sum of certain Jacobi polynomials given by Askey and Gasper [1] (see also Askey [7], Askey and Gasper [2], and Gasper [5]).

1.2. Fejér-Gronwall-Jackson's, Young's and Related Inequalities

Let $S_n(x)$ be given by (1.1.2). Since $S_n(0) = S_n(\pi) = 0$ and $S_n(x) = S_n(2\pi + x) = -S_n(-x)$, to describe completely the behavior of S_n, it is enough only to know its behavior in the open interval $(0,\pi)$.

The first proofs of Fejér's conjecture

$$S_n(x) > 0 \qquad (0 < x < \pi) \tag{1.2.1}$$

were published by Jackson [1] and Gronwall [1] in 1911 and 1912, respectively. Fejér [16] stated that the proof of Gronwall was communicated to him in a letter of October 22, 1910 that he received from Gronwall himself. Fejér also stated that Jackson's proof was communicated to him in a letter of December 19, 1910 that Jackson sent to him.

Jackson [1] proved the following conjectures by Fejér:

1° *The function $x \mapsto S_n(x)$ has a maximum at the point $x = x_n = \pi/(n+1)$;*

2° $S_n(x_n) > S_{n-1}(x_{n-1})$;

3° $\lim_{n\to\infty} S_n(x_n) = \int_0^\pi (\sin x/x)\,dx = 1.8519370\ldots < \pi/2 + 1.$

Gronwall [1] proved that the function $x \mapsto S_n(x)$ has maxima in $(0, \pi)$ at the points $x_m^{(n)} = (2m+1)\pi/(n+1)$ $(m = 0, 1, \ldots, N)$, and minima at $y_m^{(n)} = 2m\pi/n$ $(m = 1, \ldots, N)$, where $N = N(n) = [(n-1)/2]$. This follows immediately from

$$\frac{d}{dx} S_n(x) = \sum_{k=1}^{n} \cos kx = \frac{\sin \frac{nx}{2} \cos \frac{(n+1)x}{2}}{\sin \frac{x}{2}} = 0.$$

We note that

$$x_0^{(n)} < y_1^{(n)} < x_1^{(n)} < \cdots < y_m^{(n)} < x_m^{(n)} < \cdots < y_N^{(n)} < x_N^{(n)}.$$

Furthermore, Gronwall proved also the following results concerning the behavior of $S_n(x)$ at its extrema:

Theorem 1.2.1. *Let $m = 0, 1, \ldots, N$. Then*

$$S_{n+1}\left(\tfrac{2m+1}{n+2}\pi\right) > S_n\left(\tfrac{2m+1}{n+1}\pi\right).$$

If $m = 1, \ldots, N$, we have

$$S_{n+1}\left(\tfrac{2m}{n+1}\pi\right) > S_n\left(\tfrac{2m}{n}\pi\right).$$

PROOF. Since $\frac{2m+1}{n+2}\pi < \frac{2m+1}{n+1}\pi < \frac{2m+2}{n+1}\pi$, i.e., $x_m^{(n+1)} < x_m^{(n)} < y_{m+1}^{(n+1)}$, we conclude that the function S_{n+1} is decreasing in this interval $\left(\frac{2m+1}{n+2}\pi, \frac{2m+2}{n+1}\pi\right)$. Thus, we have

$$S_{n+1}(x_m^{(n+1)}) > S_{n+1}(x_m^{(n)}) = S_n(x_m^{(n)}) + \tfrac{1}{n+1}\sin((n+1)x_m^{(n)}),$$

i.e.,

$$S_{n+1}\left(\tfrac{2m+1}{n+2}\pi\right) > S_n\left(\tfrac{2m+1}{n+1}\pi\right).$$

Similarly, from $\frac{2m-1}{n+1}\pi < \frac{2m}{n+1}\pi < \frac{2m}{n}\pi$, i.e., $x_{m-1}^{(n)} < y_m^{(n+1)} < y_m^{(n)}$, we have

$$S_{n+1}(y_m^{(n+1)}) = S_n(y_m^{(n+1)}) + \frac{1}{n+1}\sin((n+1)y_m^{(n+1)}) > S_n(y_m^{(n)}).$$

This is the second inequality in Theorem 1.2.1. □

Theorem 1.2.2. *The inequality $S_n(x) > 0$ holds for all $n \in \mathbb{N}$ and all $x \in (0, \pi)$.*

PROOF. First, we estimate S_n at its minima $2m\pi/n$, where $2m\pi/n < \pi$, i.e., $2m + 1 \leq n$. By Theorem 1.2.1 we have

$$S_n\left(\frac{2m}{n}\pi\right) > S_{n-1}\left(\frac{2m}{n-1}\pi\right) > \cdots > S_{2m+1}\left(\frac{2m}{2m+1}\pi\right),$$

i.e.,

$$S_n\left(\frac{2m}{n}\pi\right) > \sum_{k=1}^{2m+1} \frac{1}{k} \sin\left(k\frac{2m\pi}{2m+1}\right)$$
$$= \sum_{k=1}^{2m+1} \frac{1}{k} \sin\left[k\left(\pi - \frac{\pi}{2m+1}\right)\right]$$
$$= \sum_{k=1}^{2m+1} \frac{(-1)^{k-1}}{k} \sin\frac{k\pi}{2m+1}.$$

Since the function $x \mapsto \sin x/x$ is decreasing in $[0,\pi]$, the last alternating series is positive, and thus $S_n(x) > 0$, when $x \in (0,\pi)$.

Theorem 1.2.3. *For two successive maxima of* S_n*, we have*

$$S_n\left(\frac{2m+1}{n+1}\pi\right) > S_n\left(\frac{2m+3}{n+1}\pi\right) \qquad (m = 0, 1, \ldots, N-1).$$

PROOF. After some elementary trigonometric manipulations, we find

$$S_n'(x) - S_n'(x + \tfrac{\pi}{n+1}) = \frac{\sin(\frac{\pi}{n+1})\sin(n+1)x}{\cos(\frac{\pi}{n+1}) - \cos(\frac{\pi}{n+1} + x)}.$$

Set (Gronwall [**1**])

$$\psi_\pm(t) = S_n\left(\frac{(2m+2)\pi \pm t}{n+1}\right)$$

and

$$w(t) = \psi_-(t) - \psi_-(t-2\pi) - \psi_+(t) + \psi_+(t+2\pi).$$

Then

$$\frac{dw(t)}{dt} = \frac{\sin\frac{\pi}{n+1}\sin t}{n+1}\left\{\frac{1}{\cos(\frac{\pi}{n+1}) - \cos\frac{(2m+3)\pi-t}{n+1}} - \frac{1}{\cos(\frac{\pi}{n+1}) - \cos\frac{(2m+3)\pi+t}{n+1}}\right\}.$$

If $0 < t < \pi$ and $0 < \frac{2m+3}{n+1} < 1$, we have

$$\cos(\tfrac{\pi}{n+1}) > \cos\tfrac{(2m+3)\pi-t}{n+1} > \cos\tfrac{(2m+3)\pi+t}{n+1},$$

and then $w'(t) > 0$. Therefore the function $t \mapsto w(t)$ is strictly increasing in $[0,\pi]$. Since $w(0) = 0$, we conclude that $w(\pi) > 0$, i.e.,

$$S_n\left(\frac{2m+1}{n+1}\pi\right) - 2S_n\left(\frac{2m+3}{n+1}\pi\right) + S_n\left(\frac{2m+5}{n+1}\pi\right) > 0. \qquad (1.2.2)$$

This inequality holds for $2m+3 < n+1$, i.e., for $2m+3 \le n$ (if n is odd) or $2m+3 \le n-1$ (if n is even).

Let $(2m+3)\pi/(n+1)$ be the last maximum of S_n in $(0,\pi)$. Then $m = N-1$ ($N = [(n-1)/2]$) and $(2m+5)\pi/(n+1) = \pi + x_0$, where $x_0 \in (0,\pi)$. Since

$$S_n(\pi + x_0) = S_n(x_0 - \pi) = -S_n(\pi - x_0)$$

and $S_n(\pi - x_0) > 0$ (by Theorem 1.2.2), on the basis of (1.2.2) we claim that

$$S_n\left(\frac{2m+1}{n+1}\pi\right) > S_n\left(\frac{2m+3}{n+1}\pi\right) \tag{1.2.3}$$

for $m = N-1$. By making use of induction and using (1.2.2), we obtain a proof of (1.2.3) for $m = N-2, \ldots, 1, 0$. □

We give the following result in the form presented in the paper of E. Hewitt and R. E. Hewitt [1].

Theorem 1.2.4. *For every $m \in \mathbb{N}$, the sequence $\left\{S_n\left(\frac{2m-1}{n+1}\pi\right)\right\}_{n=1}^{\infty}$ is ultimately increasing and has limit* Si$((2m-1)\pi)$. *The sequence $\left\{S_n\left(\frac{2m}{n}\pi\right)\right\}_{n=1}^{\infty}$ is ultimately increasing and has limit* Si$(2m\pi)$.

From this theorem, for the first maximum ($m=1$), we obtain (1.1.7). Gronwall [1] found further over and under shoots in the convergence (also see E. Hewitt and R. E. Hewitt [1]):

Theorem 1.2.5. *For $n \le 42$, the minimum values of S_n in the interval $(0,\pi)$ form a decreasing sequence. For $n \ge 43$, there is an integer m_0 such that*

$$S_n\left(\tfrac{2m}{n}\pi\right) < S_n\left(\tfrac{2m+2}{n}\pi\right) \qquad (m = 1, \ldots, m_0 - 1)$$

and

$$S_n\left(\tfrac{2m}{n}\pi\right) > S_n\left(\tfrac{2m+2}{n}\pi\right) \qquad (m = m_0, m_0+1, \ldots, N-1),$$

where $N = \left[\frac{n-1}{2}\right]$. The number m_0 is $\left[\frac{\sqrt{2n}}{2\pi}\right]$ or $\left[\frac{\sqrt{2n}}{2\pi}\right] + 1$. Also, the asymptotic equality

$$S_n\left(\frac{2m_0\pi}{n}\right) = \frac{\pi}{2} - \frac{2}{\sqrt{2n}} + O\left(\frac{1}{n}\right)$$

holds.

It seems that the shortest proof of Fejér's inequality (1.2.1) has been given by Landau [1]. In the following we give his inductive proof:

Suppose that $n > 1$ and $S_{n-1}(x) > 0$ for $0 < x < \pi$. Let t be any extremum point of the function $x \mapsto S_n(x)$ in the interval $(0, \pi)$. Then from the equality

$$0 = 2\sin\frac{t}{2}S'_n(t) = 2\sin\frac{t}{2}\sum_{k=1}^{n}\cos kt = \sin\left(n+\tfrac{1}{2}\right)t - \sin\tfrac{t}{2}$$

it follows

$$\sin nt = \sin\left(n+\tfrac{1}{2}\right)t\,\cos\tfrac{t}{2} - \cos\left(n+\tfrac{1}{2}\right)t\,\sin\tfrac{t}{2} = \begin{Bmatrix} 0, \text{ or} \\ \sin t \end{Bmatrix} \geq 0,$$

i.e.,

$$S_n(t) = S_{n-1}(t) + \frac{1}{n}\sin nt \geq S_{n-1}(t) > 0.$$

Thus, the function $x \mapsto S_n(x)$ does not have nonpositive minimum on $(0, \pi)$, such that $S_n(x) > 0$ for all $x \in (0, \pi)$.

Fejér [3] also proved the inequality

$$\left|\sum_{k=1}^{n}\frac{1}{k}\sin(2k-1)x\right| < 2 + 3M \qquad (x \in \mathbb{R},\ n \in \mathbb{N}), \tag{1.2.4}$$

where the constant M is the same as in the inequality (1.1.6).

Lenz [1] estimated trigonometric sums of the form

$$N_n(x) = \sum_{k=1}^{n} b_k \sin(2k-1)x \quad \text{and} \quad P_n(x) = \sum_{k=1}^{n} c_k \sin 2kx,$$

under the conditions

$$0 \leq b_k \leq \frac{B}{k} \ \ (1 \leq k \leq n), \qquad 0 \leq b_k - b_{k+1} \leq \frac{B}{k^2} \ \ (1 \leq k \leq n)$$

and

$$0 \leq c_k \leq \frac{C}{k} \ \ (1 \leq k \leq n), \qquad 0 \leq c_k - c_{k+1} \leq \frac{C}{k(k+1)} \ \ (1 \leq k \leq n),$$

respectively, where B and C are constants. Then Lenz obtained for all real values of x and all values of $n \in \mathbb{N}$ the following inequalities

$$|N_n(x)| \leq \frac{b_1}{\sin(b_1/2B)} \qquad \text{and} \qquad |P_n(x)| \leq \frac{c_1}{\sin(c_1/2C)}.$$

In the following we state a few special cases of the above inequalities:

1° If $b_k = 1/k$ and $B = 1$ then

$$\Big|\sum_{k=1}^{n} \frac{1}{k} \sin(2k-1)x \Big| \le \frac{1}{\sin(1/2)} < 2.1,$$

which gives a stronger estimate than (1.2.4);

2° If $b_k = 1/(k+1)$ and $B = 1$ then

$$\Big|\sum_{k=1}^{n} \frac{1}{k+1} \sin(2k-1)x \Big| \le \frac{1}{2\sin(1/4)} < 2.03;$$

3° If $c_k = 1/k$ and $C = 1$ then

$$|S_n(2x)| = \Big|\sum_{k=1}^{n} \frac{1}{k} \sin 2kx \Big| \le \frac{1}{\sin(1/2)} < 2.1,$$

where $S_n(x)$ is given by (1.1.2).

A better estimate was obtained by Bohr [1]. Bohr proved that

$$|S_n(x)| < 2.$$

It has been mentioned before that the best constant is $\mathrm{Si}(\pi) = 1.8519\ldots$.

We will consider now an analogous classical result for the cosinus sum

$$C_n(x) = \sum_{k=1}^{n} \frac{1}{k} \cos kx \qquad (0 \le x \le \pi), \tag{1.2.5}$$

which represents the nth partial sum of the Fourier series

$$\log\left(2\sin\frac{x}{2}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \cos kx \qquad (0 < x \le \pi).$$

The greatest value of $C_n(x)$ is attained at the origin. Since

$$C_n'(x) = -\sum_{k=1}^{n} \sin kx = -\frac{1}{2}\csc\frac{x}{2}\left(\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}\right),$$

i.e.,

$$C_n'(x) = -\csc\frac{x}{2}\sin\frac{nx}{2}\sin\frac{(n+1)x}{2},$$

it follows that the maxima of $C_n(x)$ occur at $x = 2m\pi/n$ ($m = 0, 1, \ldots, [n/2]$), and the minima occur at $x = 2m\pi/(n+1)$ ($m = 1, \ldots, [(n+1)/2]$).

For $\lambda < m$ it follows that

$$C_n\Big(\frac{2m\pi}{n+1}\Big) - C_n\Big(\frac{2\lambda\pi}{n+1}\Big) = \int_{2\lambda\pi/(n+1)}^{2m\pi/(n+1)} C_n'(x)\,dx$$

$$= -\frac{2}{n+1}\int_{\lambda\pi}^{m\pi}\Big(\sin x\cot\frac{x}{n+1} - \cos x\Big)\sin x\,dx$$

$$= -\frac{2}{n+1}\int_{\lambda\pi}^{m\pi}\sin^2 x\cot\frac{x}{n+1}\,dx,$$

which is negative, because $m \le (n+1)/2$ and the cotangent in the integral is positive. Thus the minima of $C_n(x)$ form a decreasing sequence in the interval $[0,\pi]$. The smallest value of $C_n(x)$ is attained at $\frac{2\pi}{n+1}\big[\frac{n+1}{2}\big]$. Therefore, if n is odd, the least value of $C_n(x)$ is $C_n(\pi)$, while if n is even, it is then $C_n\big(\pi - \frac{\pi}{n+1}\big)$. Using this fact, Young [1] has proved the following result:

Theorem 1.2.6. *The cosinus polynomial* (1.2.5) *satisfies the inequality*

$$C_n(x) > -1 \qquad (0 \le x \le \pi). \tag{1.2.6}$$

PROOF. We consider two cases:

Case 1. If n is odd, then

$$C_n(\pi) = -1 + \frac{1}{2} - \cdots - \frac{1}{n} > -1.$$

Case 2. If n is even, then for $p = \pi/(n+1)$ we have

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) = C_n(\pi - p) = \sum_{k=1}^{n}\frac{(-1)^k}{k}\cos kp,$$

i.e.,

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) = -\Big(1 - \frac{1}{n}\Big)\cos p + \Big(\frac{1}{2} - \frac{1}{n-1}\Big)\cos 2p$$

$$- \cdots + (-1)^{n/2}\Big(\frac{2}{n} - \frac{1}{(n/2)+1}\Big)\cos\frac{np}{2}.$$

From

$$1 - \frac{1}{n} > \frac{1}{2} - \frac{1}{n-1} > \frac{1}{3} - \frac{1}{n-2} > \cdots > \frac{2}{n} - \frac{1}{(n/2)+1}$$

and

$$\cos p \ge \cos 2p \ge \cdots \ge \cos\frac{np}{2}$$

it follows that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) > -\Big(1 - \frac{1}{n}\Big)\cos p > -1.$$

Therefore, the inequality (1.2.6) is valid. $\square$

Young [1] also proved the following result:

Theorem 1.2.7. *The inequality*

$$C_n(x) \le 5 + \frac{1}{2}\log\frac{1}{2(1-\cos x)}$$

holds.

Nikonov [1] (see also Pak [1, p. 132]) obtained the following results for $C_n(x)$:

1° *All maxima of $C_n(x)$ for $x \in (\pi/2, 3\pi/2)$ are negative;*

2° *The maxima of $C_n(x)$ for $x \in (0,\pi)$ form a monotone decreasing sequence;*

3° *All maxima of $C_n(x)$ for $x \in (0,\pi/3)$ are positive;*

4° *$C_n(x)$ has a zero in $(\pi/3, \pi/2)$;*

5° *The minima of $C_n(x)$ for $x \in (\pi/3,\pi/2)$ are negative and for a given k, the k-th minimum increases as n increases.*

In 1925 Fejér [6] obtained the following three results about the nonnegativity of trigonometric sums:

Theorem 1.2.8. *Let the sequence $\lambda_0, \lambda_1, \dots, \lambda_n, \lambda_{n+1}$ be nonnegative, monotonically nonincreasing and convex, i.e.,*

$$\begin{cases} \lambda_0 \ge \lambda_1 \ge \cdots \ge \lambda_n \ge \lambda_{n+1} = 0, \\ \Delta^2\lambda_k = \lambda_{k+2} - 2\lambda_{k+1} + \lambda_k \ge 0 \quad (k = 0, 1, \dots, n-1). \end{cases} \tag{1.2.7}$$

Then

$$\Lambda_n(x) = \frac{\lambda_0}{2} + \sum_{k=1}^{n} \lambda_k \cos kx \ge 0 \tag{1.2.8}$$

for all values of x.

PROOF. Let

$$c_0 = \frac{1}{2}, \quad c_k = \frac{1}{2} + \cos x + \cdots + \cos kx \quad (0 < k \le n),$$
$$\sigma_k = c_0 + c_1 + \cdots + c_k \quad (0 \le k \le n).$$

Then

$$\sigma_k = \frac{1}{2}\left(\frac{\sin((k+1)x/2)}{\sin(x/2)}\right)^2 \ge 0. \tag{1.2.9}$$

From

$$\frac{1}{2} = c_0 = \sigma_0, \quad \cos x = c_1 - c_0 = \sigma_1 - 2\sigma_0,$$

$$\cos kx = c_k - c_{k-1} = \sigma_k - 2\sigma_{k-1} + \sigma_{k-2} = \Delta^2\sigma_{k-2},$$

we obtain

$$\Lambda_n(x) = \sum_{k=0}^{n-2} \Delta^2 \lambda_k \sigma_k + (\lambda_{n-1} - 2\lambda_n)\sigma_{n-1} + \lambda_n \sigma_n. \tag{1.2.10}$$

The last equality, because of conditions (1.2.7) and (1.2.9), implies inequality (1.2.8). □

The function $D_k(x) = c_k$ is known as the Dirichlet kernel, and

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) = \frac{\sigma_n}{n+1} \tag{1.2.11}$$

as the Fejér kernel. We can see that

$$D_k(x) = \frac{\sin(k+1/2)x}{2\sin(x/2)}$$

and

$$F_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \frac{n-k+1}{n+1} \cos kx.$$

Remark 1.2.1. The conditions (1.2.7) can be replaced by

$$\lambda_0 - \lambda_1 \geq \lambda_1 - \lambda_2 \geq \cdots \geq \lambda_{n-1} - \lambda_n \geq \lambda_n \geq \lambda_{n+1} = 0.$$

Remark 1.2.2. If we set $x = \arccos t$ $(-1 \leq t \leq 1)$, the inequality (1.2.8) reduces to

$$\frac{1}{2}\lambda_0 T_0(t) + \lambda_1 T_1(t) + \cdots + \lambda_n T_n(t) \geq 0,$$

where T_k is the Chebyshev polynomial of degree k.

Theorem 1.2.9. *For all $n \in \mathbb{N}$ and $0 \leq r \leq 1/2$, the inequality*

$$\frac{1}{2} + r\cos x + \cdots + r^n \cos nx \geq 0 \qquad (0 \leq x \leq 2\pi)$$

holds.

Theorem 1.2.10. *Let $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$. Then the inequality*

$$a_0 P_0(t) + a_1 P_1(t) + \cdots + a_n P_n(t) \geq 0 \qquad (-1 \leq t \leq 1)$$

holds, where P_k is the Legendre polynomial of degree k.

In the special case, when $a_0 = a_1 = \cdots = a_n = 1$, the above theorem was proved by Fejér [1] in 1908 (see, also [2]). In fact, using Mehler formula

$$P_k(\cos x) = \frac{2}{\pi} \int_x^{\pi} \frac{\sin(2k+1)\frac{\theta}{2}}{\sqrt{2(\cos x - \cos\theta)}}\, d\theta, \quad k = 0, 1, \ldots,$$

Fejér obtained the following representation

$$U_n(x) = \sum_{k=0}^{n} P_k(\cos x) = \frac{2}{\pi}\int_x^{\pi} \frac{\sin^2 \frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}\sqrt{2(\cos x - \cos\theta)}}\, d\theta\,,$$

wherefrom $U_n(x) \geq 0$, when $0 \leq x \leq \pi$. The general case follows from $a_k = 1$ by summation by parts

$$\sum_{k=0}^{n} a_k P_k(t) = \sum_{k=0}^{n-1}(a_k - a_{k+1})U_k(x) + a_n U_n(x).$$

Also, Fejér [1] proved the following inequality

$$U_n(x) \leq \frac{2}{\left(\sin\dfrac{x}{2}\right)^{3/2}} \qquad (0 < x \leq \pi).$$

However, if $0 < \alpha \leq x \leq \pi$, then

$$U_n(x) \leq \frac{2}{\left(\sin\dfrac{\alpha}{2}\right)^{3/2}}.$$

Remark 1.2.3. From the proof of Theorem 1.2.8 the following Fejér's inequality can be obtained

$$\sigma_n = \frac{n+1}{2} + n\cos x + \cdots + \cos nx \geq 0 \qquad (0 \leq x \leq 2\pi).$$

In fact this inequality follows from (1.2.8) if we set $\lambda_0 = n+1$, $\lambda_1 = n$, $\lambda_2 = n-1, \ldots, \lambda_n = 1$.

When $\lambda_0 = 2$, $\lambda_k = 1/k$ $(k = 1, \ldots, n)$ the conditions of Theorem 1.2.8 are not satisfied for $n > 2$. Namely, the following inequality

$$\Delta^2\lambda_{n-1} = 0 - 2\lambda_n + \lambda_{n-1} \geq 0$$

does not hold. However, in this case, the inequality (1.2.8), i.e.,

$$1 + \sum_{k=1}^{n} \frac{1}{k}\cos kx \geq 0$$

holds, because it is Young's inequality (1.2.6).

Fejér [6, § 3] has also proved the following inequalities

$$\begin{aligned} \sigma_n(x) &= n\sin x + (n-1)\sin 2x + \cdots + \sin nx \geq 0, \\ h_n(x) &= \sin x + \sin 2x + \cdots + \sin(n-1)x + \frac{\sin nx}{2} \geq 0, \end{aligned} \tag{1.2.12}$$

which hold for all values of $n \in \mathbb{N}$ and $0 \leq x \leq \pi$.

Fejér [8] considered the generalized sum of the form

$$Q_n(x) = q_1 \sin x + q_2 \sin 2x + \cdots + q_{n-1} \sin(n-1)x + q_n \frac{\sin nx}{2}.$$

Theorem 1.2.11. *Let $q_1, q_2, \ldots, q_n$ be a positive, monotone decreasing and convex sequence, then the polynomial $Q_n(x)$ is nonnegative in $[0, \pi]$, i.e.,*

$$Q_n(x) \geq 0 \qquad (0 \leq x \leq \pi). \tag{1.2.13}$$

PROOF. Let

$$\begin{aligned} s_k &= s_k(x) = \sin x + \sin 2x + \cdots + \sin kx, \\ \sigma_k &= \sigma_k(x) = s_1 + s_2 + \cdots + s_k, \\ h_k &= h_k(x) = \sin x + \sin 2x + \cdots + \sin(k-1)x + \frac{\sin kx}{2}. \end{aligned}$$

According to the identity

$$\sum_{k=1}^{n} q_k \sin kx = \sum_{k=1}^{n-2} \Delta^2 q_k \sigma_k + (q_{n-1} - q_n)\sigma_{n-1} + q_n h_n + q_n \frac{\sin nx}{2} \tag{1.2.14}$$

we conclude that (1.2.13) holds, since the following inequalities

$$\sigma_k = \sigma_k(x) \geq 0 \quad (k = 1, \ldots, n-1), \quad h_n = h_n(x) \geq 0 \quad (0 \leq x \leq \pi)$$

are satisfied on the basis of (1.2.12). □

By making use of this result, Fejér gave one proof of his conjecture (1.2.1), which was stated in 1910. In fact by setting $q_k = 1/k$ $(k = 1, \ldots, n)$, Fejér [8] obtained from (1.2.14), for $n \geq 3$, the following inequality

$$S_n(x) \geq \left(\frac{1}{3} - 2 \cdot \frac{1}{2} + 1\right)\sigma_1 + \frac{1}{n} \cdot \frac{\sin nx}{2},$$

i.e.,

$$S_n(x) \geq \frac{1}{3} \sin x + \frac{1}{2n} \sin nx = U_n(x). \tag{1.2.15}$$

The trigonometric polynomial $U_n(x)$ in (1.2.15) is positive for $\pi/n \leq x \leq \pi - \pi/n$, when $n \geq 3$, which follows from the following inequalities

$$\frac{1}{3} \sin x \geq \frac{1}{3} \sin \frac{\pi}{n} > \frac{1}{3} \cdot \frac{2}{\pi} \cdot \frac{\pi}{n} = \frac{4/3}{2n} > \frac{|\sin nx|}{2n}.$$

Thus, $S_n(x) > 0$ for $\pi/n \leq x \leq \pi - \pi/n$ and $n \geq 3$.

For $0 < x < \pi/n$, we have that $0 < kx < k\pi/n \leq \pi$, $1 \leq k \leq n$, so that each term in $S_n(x)$ is positive and $S_n(x) > 0$.

For $\pi - \pi/n < x < \pi$, we put $x = \pi - t$, so $0 < t < \pi/n$. Then we have

$$S_n(x) = S_n(\pi - t) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin kt}{k},$$

i.e.,

$$S_n(x) = t \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin kt}{kt}.$$

Since the function $z \mapsto \sin z/z$ is positive and decreasing in $(0, \pi)$, the last alternating series is positive, and thus $S_n(x) > 0$ for $\pi - \pi/n < x < \pi$ and $n \geq 3$.

For $n = 1$ and $n = 2$ the inequality (1.2.1) is evident.

For $q_k = n - k + 1$ the inequality (1.2.13) reduces to the first inequality in (1.2.12). This inequality was first proved by F. Lukács (cf. Fejér [8]), who transformed $\sigma_n(x)$ to

$$\begin{aligned}\sigma_n(x) &= (n+1)(\sin x + \cdots + \sin nx) - (\sin x + 2\sin 2x + \ldots + n \sin nx) \\ &= \frac{(n+1)\sin x - \sin(n+1)x}{4\sin^2(x/2)},\end{aligned}$$

from which it follows that $\sigma_n(x) > 0$, when $0 < x < \pi$. It is evident that $\sigma_n(0) = \sigma_n(\pi) = 0$.

If we set $q_k = \binom{n+m-k}{m}$, where $m \geq 1$, then the inequality (1.2.13) reduces to the inequality

$$\sum_{k=1}^{n} \binom{n+m-k}{m} \sin kx > 0 \qquad (0 < x < \pi), \tag{1.2.16}$$

which was proved by induction (Turán [1]). It can be proved (see Fejér [15]) that the following inequality

$$\sum_{k=1}^{n} \binom{n+m-k}{m} \frac{\sin ka \sin kx}{k} > 0 \qquad (0 < a < \pi,\ 0 < x < \pi)$$

holds.

If for the geometric series $1 + z + z^2 + \cdots$ we define the sums of the order m by means of

$$\begin{aligned}s_{n,m}(z) &= s_{0,m-1}(z) + s_{1,m-1}(z) + \cdots + s_{n,m-1}(z), \\ s_{k,0}(z) &= 1 + z + \cdots + z^k,\end{aligned}$$

then we have

$$s_{n,m}(z)=\sum_{k=0}^{n}\binom{n+m-k}{m}z^k.$$

Setting $z=e^{ix}$ we obtain

$$s_{n,m}(e^{ix})=X_{n,m}(x)+iY_{n,m}(x),$$

where

$$X_{n,m}(x)=\sum_{k=0}^{n}\binom{n+m-k}{m}\cos kx \quad \text{and} \quad Y_{n,m}(x)=\sum_{k=0}^{n}\binom{n+m-k}{m}\sin kx.$$

We can remark that the inequality (1.2.16) can be written in the form $Y_{n,m}(x)>0$ $(0<x<\pi)$, where $m\geq 1$.

Szegö [5] proved the following result:

Theorem 1.2.12. *Let* γ *be defined by* $\sin^2(\gamma/2)=0.7$ $(\pi/2\leq\gamma<\pi)$. *Then the following inequalities*

$$Y_{n,2}(x)=\sum_{k=0}^{n}\binom{n+2-k}{2}\sin kx>0 \qquad (0<x<\pi),$$

$$-X'_{n,2}(x)=\sum_{k=0}^{n}\binom{n+2-k}{2}k\sin kx>0 \qquad (0<x\leq\gamma),$$

$$Y'_{n,2}(x)=\sum_{k=0}^{n}\binom{n+2-k}{2}k\cos kx<0 \qquad (\pi/2<x\leq\pi),$$

$$X_{n,2}(x)-X_{n,2}(y)=\sum_{k=0}^{n}\binom{n+2-k}{2}(\cos kx-\cos ky)>0$$

$$(0\leq x\leq\pi/2;\ \gamma\leq y\leq\pi)$$

hold.

Considering the inequality $-X'_{n,2}>0$, Schweitzer [1] proved that instead of γ one can put $2\pi/3$ and that this bound is best possible. Another way to state this inequality is given by Askey and Fitch [4], using absolutely monotonic functions. In Section 2.5 we will consider such problems.

By a geometric interpretation of the sum

$$\sum_{k=0}^{n}a_ke^{ikx} \qquad (a_k\geq a_{k+1}),$$

Tomić [1] gave a method which can be applied to determine bounds of trigonometric polynomials and series. At almost the same time Hyltén-Cavallius [1] used similar geometric methods, proving that

$$C_n(x) \le -\log\sin\frac{x}{2} + \frac{1}{2}(\pi - x) \qquad (0 \le x \le \pi)$$

and

$$0 < S_n(x) < \pi - x \qquad (0 < x < \pi),$$

where the sums $C_n(x)$ and $S_n(x)$ are given by (1.2.5) and (1.1.2) respectively.

Tomić [1] illustrated applications of his geometric method by proving Fejér's theorem 1.2.8, as well as proving of the inequality (see Fejér [15])

$$\sum_{k=0}^{n} a_k \sin\left(k + \frac{1}{2}\right) x > 0 \qquad (0 < x < 2\pi),$$

where $a_0 \ge a_1 \ge \cdots \ge a_n > 0$.

Using the same geometric method Karamata and Tomić [1] proved the following results (see also Tomić's thesis [2]):

Theorem 1.2.13. *Let $a_{k-1} \ge a_k$ $(k = 1, \ldots, n)$ and*

$$S_{\alpha,\beta}(x) = \sum_{k=0}^{n} a_k \sin(\alpha k + \beta)x,$$

where α and β are arbitrary real numbers. Then

$$-a_0 \sin^2\left(\beta - \frac{\alpha}{2}\right)\frac{x}{2} \le \sin\frac{\alpha x}{2} S_{\alpha,\beta}(x) \le a_0 \cos^2\left(\beta - \frac{\alpha}{2}\right)\frac{x}{2} \quad (0 < x < \pi).$$

Theorem 1.2.14. *Let $a_{k-1} \ge a_k \ge 0$ $(k = 1, \ldots, n)$, then*

$$0 \le \sum_{k=0}^{n} a_k \sin\left(k + \frac{1}{2}\right) x \le a_0 \csc\frac{x}{2} \qquad (0 < x < 2\pi).$$

If $a_{k-1} \ge a_k \ge 0$ $(k = m+1, \ldots, n)$, then

$$-a_m \sin^2 mx \le \sin x \sum_{k=m}^{n} a_k \sin(2k+1)x \le a_m \cos^2 mx \qquad (0 < x < \pi).$$

Theorem 1.2.15. *Suppose that the sequence* $\{\lambda_k\}_{k=0}^{n+1}$ *satisfies the hypotheses of Theorem* 1.2.8, *then*

$$0 \le \Lambda_n(x) \le \frac{\lambda_0 - \lambda_1}{2} \csc^2 \frac{x}{2} \qquad (0 < x < 2\pi),$$

where $\Lambda_n(x)$ *is given by* (1.2.10).

Theorem 1.2.16. *Let* $\lambda_m \ge \lambda_{m+1} \ge \cdots \ge \lambda_n \ge \lambda_{n+1} = \lambda_{n+2} = 0$ *and* $\Delta^2\lambda_k = \lambda_{k+2} - 2\lambda_{k+1} + \lambda_k \ge 0$ $(k = m, \dots, n)$, *then*

$$w_m(x) \le \sum_{k=m}^{n} \lambda_k \cos kx \le W_m(x) \qquad (0 < x < 2\pi),$$

where

$$2w_m(2x) = \lambda_m \frac{\sin^2(m-1)x}{\sin^2 x} - \lambda_{m-1} \frac{\sin^2 mx}{\sin^2 x},$$
$$2W_m(2x) = \lambda_{m-1} \frac{\sin^2 mx}{\sin^2 x} - \lambda_m \frac{\cos^2(m-1)x}{\sin^2 x}.$$

Similarly to Fejér's theorem 1.2.11 Tomić [3] (see also [2]) obtained the following results:

Theorem 1.2.17. *Let* $q_k \ge 0$ $(k = 1, \dots, n)$,

$$q_1 - q_2 \ge q_2 - q_3 \ge \cdots \ge q_{n-1} - q_n \ge 0, \tag{1.2.17}$$

and

$$mq_{2m} \le \sum_{k=1}^{m-1} k\Delta^2 q_{2k-1} + m(q_{2m-1} - q_{2m}) \qquad (m = 1, \dots, [n/2]).$$

Then

$$\overline{Q}_n(x) = \sum_{k=1}^{n} q_k \sin kx \ge 0 \qquad (0 \le x \le \pi). \tag{1.2.18}$$

Theorem 1.2.18. *The polynomial* (1.2.18) *is positive in* $(0, \pi/2)$ *if the inequalities* (1.2.17) *hold and* $q_{n+1} \le q_n/2$.

Tomić [2] also proved the following result:

Theorem 1.2.19. *Let the sequence $\{a_k\}_{k=1}^{\infty}$ be four times monotonic, i.e.,*

$$\binom{k}{0}a_n - \binom{k}{1}a_{n+1} + \cdots + (-1)^k\binom{k}{k}a_{n+k} \geq 0 \qquad (k = 1, 2, 3, 4).$$

Then, for $0 < x < \pi$,

$$\sum_{k=1}^{\infty} a_k \sin kx \leq \frac{a_1}{2}\cot\frac{x}{2}.$$

In a recent paper, Steinig [1] considered sine polynomials with real coefficients of the form (1.2.18) and proved the following result:

Theorem 1.2.20. *Let*

$$q_k \geq 2\sum_{i=1}^{n-k}(-1)^{i+1}q_{k+i} \quad (i = 1, \dots, n-1) \quad \text{and} \quad q_n > 0. \tag{1.2.19}$$

Then

$$\overline{Q}_n(x) \geq 0 \qquad (0 < x < \pi), \tag{1.2.20}$$

where $\overline{Q}_n(x)$ is given by (1.2.18). Inequality (1.2.20) is strict on $(0, \pi)$ if only if $(k_1, \dots, k_s, n) \leq 2$, where $k_1, \dots, k_s$ are those k $(1 \leq k \leq n-1)$ for which strict inequality holds in (1.2.19), and $(k_1, \dots, k_s, n)$ is their greatest common divisor with n.

It is easy to show that Fejér's theorem 1.2.11 is a corollary of Theorem 1.2.20. Examples show that Fejér's conditions are more restrictive than those in (1.2.19). For instance, Theorem 1.2.20 implies the positivity of

$$4\sin x + 3\sin 2x + (2 - \varepsilon)\sin 3x + \frac{1}{2}\sin 4x$$

for $0 < x < \pi$ and $0 \leq \varepsilon \leq 1/2$, but Theorem 1.2.11 applies only to the case $\varepsilon = 0$. Steinig [1] also showed that his theorem is equivalent to Theorem 1.2.8.

Some extensions of the above inequalities to two variables were considered by Koschmieder [1]. Namely, he studied estimates for sums of the type

$$s_n(x, y) = \sum_{k=1}^{n}\frac{\varphi_k(x)\varphi_k(y)}{\lambda_k} \qquad (x, y \in (0, \pi)), \tag{1.2.21}$$

when

(a) $\lambda_k = k^2\pi^2, \quad \varphi_k(x) = \sqrt{2}\sin kx;$

(b) $\lambda_k = \left(k - \frac{1}{2}\right)^2\pi^2, \quad \varphi_k(x) = \sqrt{2}\sin(k - \frac{1}{2})x;$

(c) $\lambda_k = k^2\pi^2, \quad \varphi_k(x) = \sqrt{2}\cos kx.$

Theorem 1.2.21. *The following inequalities hold:*

$$\frac{2}{\pi^2}\sum_{k=1}^{n}\frac{\sin kx \sin ky}{k^2} > 0 \qquad (0 < x < \pi,\ 0 < y < \pi), \tag{1.2.22}$$

$$0 < \frac{2}{\pi^2}\sum_{k=1}^{n}\frac{\sin(k-\frac{1}{2})x \sin(k-\frac{1}{2})y}{(k-\frac{1}{2})^2} < 1 \qquad (0 < x \le \pi,\ 0 < y \le \pi),$$

$$-\frac{1}{6} < \frac{2}{\pi^2}\sum_{k=1}^{2n}\frac{\cos kx \cos ky}{k^2} < \frac{1}{3}\ .$$

The inequality (1.2.22) follows from the Fejér-Gronwall-Jackson's inequality (see Fejér [**17**] and [**14**]).

Fejér [**12**] also stated the following more general result:

Theorem 1.2.22. *Let* $\sum\limits_{k=1}^{\infty} |\lambda_k| < \infty$. *If*

$$f(x) = \sum_{k=1}^{\infty}\lambda_k \sin kx \ge 0 \qquad (0 \le x \le \pi), \tag{1.2.23}$$

then

$$g(x,y) = \sum_{k=1}^{\infty}\frac{\lambda_k}{k}\sin kx \sin ky \ge 0 \qquad (0 \le x, y \le \pi), \tag{1.2.24}$$

and conversely.

Proof. Let $x, y \in [0,\pi]$ and let $a = |x-y|$ and $b = \min(x+y, 2\pi-(x+y))$. Then $a, b \in [0,\pi]$.

Since

$$g(x,y) = \frac{1}{2}\int_a^b f(t)\,dt,$$

(1.2.23) implies (1.2.24). In the other direction we have

$$f(x) = \lim_{y\to 0}\frac{g(x,y)}{\sin y}. \quad \square$$

Similarly, we can prove:

Theorem 1.2.23. *Let* $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. *If*

$$f(x) = \sum_{k=1}^{\infty} \lambda_k \sin\left(k - \tfrac{1}{2}\right) x \geq 0 \qquad (0 \leq x \leq \pi),$$

then

$$g(x,y) = \sum_{k=1}^{\infty} \frac{\lambda_k}{k - \frac{1}{2}} \sin\left(k - \tfrac{1}{2}\right) x \sin\left(k - \tfrac{1}{2}\right) y \geq 0 \qquad (0 \leq x, y \leq \pi),$$

and conversely.

Using Theorem 1.2.22, Askey and Fitch [2] proved that the inequality

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\sin \pi k x}{k}\right)^{2r} \geq 0 \qquad (r = 1, 2, \ldots)$$

holds for all real x. This inequality was stated as a problem by Lyness and Moler [1]. Askey and J. Fitsch [1] gave the following generalization of the above result:

Theorem 1.2.24. *Let* $0 < x_i < \pi$ *and* $N = 1, \ldots, n$, *where* $n = 1, 2, \ldots$, *then*

$$\sum_{k=1}^{N} \prod_{i=1}^{n} \frac{\sin k x_i}{k} > 0.$$

If $0 < x_i < \pi$ *and* $n = 3, 4, \ldots$, *then*

$$\sum_{k=1}^{\infty} k \prod_{i=1}^{n} \frac{\sin k x_i}{k} > 0.$$

Koschmieder [1] considered also inequalities for $s_n'(x,y) = \frac{\partial}{\partial x} s_n(x,y)$, where $s_n(x,y)$ is given by (1.2.21). For example, in the case (a), he proved that

$$s_n'(x,y) > 0 \qquad (0 < y - x < \pi,\ 0 < y + x < \pi)$$

and

$$s_n'(x,y) < 0 \qquad (0 < x - y < \pi,\ 1 < x + y < 2\pi)$$

and, in the case (b), that

$$s_n'(x,y) > 0 \qquad (0 \leq x < y < \pi).$$

1.3. Inequalities of Rogosinski and Szegö and Their Extensions

Let K be a class of all series

$$f(z) = \sum_{k=0}^{\infty} c_k z^n,$$

which converge in $|z| \le 1$ and satisfy $|f(z)| \le 1$ for $|z| \le 1$.

Rogosinski and Szegö [1] considered the bounds for the following partial sums

$$s_n(z) = \sum_{k=0}^{n} c_k z^k \qquad (n = 0, 1, \dots),$$

using three methods for estimating the upper bound of the $|L_n|$, where

$$L_n = \sum_{k=0}^{n} \lambda_k c_k$$

and the coefficients $\lambda_0, \lambda_1, \dots, \lambda_n$ satisfy certain conditions.

These methods are based on the following facts:

1° Let $\lambda_n \neq 0$ and

$$\sqrt{\lambda_n + \lambda_{n-1} z + \cdots + \lambda_0 z^n} = \sum_{k=0}^{\infty} \mu_k z^k,$$

then

$$|L_n| \le |\mu_0|^2 + |\mu_1|^2 + \cdots + |\mu_n|^2.$$

2° For $n \ge 2$ the following inequality

$$|L_n| \le \sum_{k=0}^{n-2} (k+1) |\lambda_k - 2\lambda_{k+1} + \lambda_{k+2}| + n|\lambda_{n-1} - 2\lambda_n| + (n+1)|\lambda_n|$$

holds. If the conditions (λ_k real)

$$\begin{aligned} &\lambda_k - 2\lambda_{k+1} + \lambda_{k+2} \ge 0 \qquad (k = 0, 1, \dots, n-2), \\ &\lambda_{n-1} - 2\lambda_n \ge 0, \qquad \lambda_n \ge 0 \end{aligned} \tag{1.3.1}$$

hold, then we have that

$$|L_n| \le \lambda_0, \tag{1.3.2}$$

with equality if and only if $f(z) = \varepsilon$, $|\varepsilon| = 1$.

3° Let λ_k be real numbers. Then the inequality (1.3.2) holds if the cosine polynomial

$$\Lambda_n(x) = \frac{\lambda_0}{2} + \lambda_1 \cos x + \cdots + \lambda_n \cos nx \tag{1.3.3}$$

is negative. This fact follows directly from the following equality

$$L_n = \lim_{r\to 1} \frac{1}{\pi} \int_0^{2\pi} f\left(re^{ix}\right) \Lambda_n(x)\, dx.$$

Using 2° Rogosinski and Szegö [1] considered bounds for the absolute value of the sum

$$\gamma_n(z) = \alpha s_n(ze^{a/n}) + \beta s_n(ze^{b/n}) \qquad (\alpha, \beta \neq 0,\ \alpha \neq \beta,\ n \geq 1),$$

where α, β, a, b are real or complex constants. Namely, if $f \in K$, they proved that the relation

$$\alpha e^a + \beta e^b = 0$$

represents a condition necessary and sufficient for the $|\gamma_n(z)|$ to be uniformly bounded for all $n \in \mathsf{N}$ and all $|z| \leq 1$.

The following cases are specially interesting:

1. *Case when $\alpha = 0$ and $b < 0$.*

Theorem 1.3.1. *Let $|z| \leq 1$. Then*

$$|\alpha s_n(z) + \beta s_n(ze^{b/n})| \leq |\alpha + \beta| \qquad (b < 0,\ \alpha + \beta e^b = 0).$$

For $\alpha = -1$ and $\beta = e^{-b}$, this result reduces to:

Corollary 1.3.2. *Let $|z| \leq 1$. Then*

$$\left| \frac{s_n(\rho z) - \rho^n s_n(z)}{1 - \rho^n} \right| \leq 1 \qquad (0 \leq \rho < 1;\ n \in \mathsf{N}).$$

2. *Case when a and b are purely imaginary.*

Theorem 1.3.3. *Let $f \in K$. Then the inequality*

$$\left| s_n(ze^{i\pi/2n}) + s_n(ze^{-i\pi/2n}) \right| \leq M_n \qquad (|z| \leq 1)$$

holds, where

$$M_n = 2 \sin\frac{\pi}{2n} \sum_{k=0}^{n-1} P_k^2 \left(\cos\frac{\pi}{2n}\right)$$

and P_k is the Legendre polynomial of degree k.

Some bounds for M_n follow:

$$M_n \leq 2n \sin \frac{\pi}{2n} < \pi,$$

$$M_n < \lim_{n\to\infty} M_n = 2 \int_0^{\pi/2} J_0(x)^2\, dx = 2.155\ldots,$$

where J_0 is the Bessel function of the order zero.

Regarding to nonnegativity of the polynomial (1.3.3), Rogosinski and Szegö considered a special cosine polynomial

$$C_n^{(1)}(x) = \frac{1}{2} + \frac{\cos x}{2} + \frac{\cos 2x}{3} + \cdots + \frac{\cos nx}{n+1}, \tag{1.3.4}$$

which is similar to Young's polynomial (1.2.5).

Theorem 1.3.4. *For all real x and all $n \in \mathbb{N}$ the inequality*

$$C_n^{(1)}(x) \geq 0$$

holds.

PROOF. For $n = 0, 1, 2$ we find easily

$$C_0^{(1)}(x) = \frac{1}{2}, \qquad C_1^{(1)}(x) = \frac{1}{2} + \frac{\cos x}{2} \geq 0,$$

$$C_2^{(1)}(x) = \frac{1}{2} + \frac{\cos x}{2} + \frac{\cos 2x}{3} = \frac{2}{3}\left(\cos x + \frac{3}{8}\right)^2 + \frac{7}{96} \geq \frac{7}{96}.$$

Now, we consider the case when $n \geq 3$. Evidently, $C_n^{(1)}(0) > 0$. Because of 2π-periodicity of the sum $C_n^{(1)}(x)$ and $C_n^{(1)}(-x) = C_n^{(1)}(x)$, it is enough to consider only the case when $x \in (0, \pi]$.

Using the formulae

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin \dfrac{(2n+1)x}{2}}{2 \sin \dfrac{x}{2}}$$

and

$$\sum_{k=0}^{n} \frac{\sin \frac{(2k+1)x}{2}}{2 \sin \frac{x}{2}} = \frac{\sin^2 (n+1)\frac{x}{2}}{2 \sin^2 \frac{x}{2}},$$

we obtain the identity

$$2\sin^2\frac{x}{2}C_n^{(1)}(x)=\sum_{k=0}^{n-2}\frac{2\sin^2\frac{(k+1)x}{2}}{(k+1)(k+2)(k+3)}+\frac{\sin^2\frac{nx}{2}}{n(n+1)} \\ +\frac{\sin^2\frac{(n+1)x}{2}-\sin^2\frac{nx}{2}}{n+1}. \tag{1.3.5}$$

Since

$$\sin^2\frac{(n+1)x}{2}-\sin^2\frac{nx}{2}=\sin\frac{(2n+1)x}{2}\sin\frac{x}{2},$$

we conclude that $C_n^{(1)}(x)\geq 0$, when $(2n+1)x\leq 2\pi$, i.e., when the condition

$$n+1\leq\frac{\pi}{x}+\frac{1}{2}=\frac{2\pi+x}{2x}=P \tag{1.3.6}$$

is satisfied.

On the other hand, according to the above property, we have

$$2\sin^2\frac{x}{2}C_n^{(1)}(x)\geq\frac{1}{3}\sin^2\frac{x}{2}+\frac{1}{12}\sin^2 x-\frac{1}{n+1}\sin\frac{x}{2},$$

which also means that $C_n^{(1)}(x)\geq 0$, when the condition

$$n+1\geq\frac{\sin\frac{x}{2}}{\frac{1}{3}\sin^2\frac{x}{2}+\frac{1}{12}\sin^2 x}=\frac{6}{\sin\frac{x}{2}(3+\cos x)}=Q \tag{1.3.7}$$

is fulfilled.

To prove the statement, we will divide the interval $(0,\pi]$ in two intervals $(0,\pi/3]$ and $[\pi/3,\pi]$.

At first, we suppose that $0<x\leq\pi/3$. Then we can prove that $P\geq Q$, i.e.,

$$\sin\frac{x}{2}(3+\cos x)\geq\frac{12x}{2\pi+x}. \tag{1.3.8}$$

Since

$$\cos x\geq 1-\frac{1}{2}x^2 \qquad\text{and}\qquad \frac{\sin(x/2)}{x/2}\geq\frac{\sin(\pi/6)}{\pi/6}=\frac{3}{\pi},$$

in order to prove (1.3.8) it is enough to verify that $(2\pi+x)(4-x^2/2)\geq 8\pi$, i.e., $x^2/2+\pi x\leq 4$. Since $0<x\leq\pi/3$, the last inequality is indeed valid, because of

$$\frac{x^2}{2}+\pi x\leq\frac{\pi^2}{18}+\frac{\pi^2}{3}=\frac{7\pi^2}{18}<\frac{70}{18}<4.$$

Thus, we have proved that $P \geq Q$. Then the inequality $C_n^{(1)}(x) \geq 0$ holds, because at least one of the conditions (1.3.6) or (1.3.7) is satisfied.

Finally, consider the case when $x \in [\pi/3, \pi]$. Setting $t = \sin(x/2)$, we have that $1/2 \leq t \leq 1$. Since

$$t \mapsto g(t) = \sin\frac{x}{2}(3 + \cos x) = 2t(2 - t^2)$$

is a concave function on $[1/2, 1]$ ($g''(t) = -12t < 0$), we conclude that

$$g(t) \geq \min\bigl(g(1/2), g(1)\bigr) = g(1/2) = 7/4$$

and $Q \leq 24/7 < 4$, which means that the condition (1.3.7) is fulfilled for every $n \geq 1$. □

Tomić [4] improved the above mentioned result of Rogosinski and Szegö. Namely, he proved the following result:

Theorem 1.3.5. *For all real numbers x and all $n \geq 2$ there is a constant $K > 0$ (independing on x and n) such that the inequality*

$$C_n^{(1)}(x) > K > \frac{1}{168} \qquad (n \geq 2)$$

holds.

Remark. Using the Tomić's idea, the lower bound of the constant K can be replaced by $1/73$. Furthermore, if we take $n \geq 4$, the constant K can be improved so that $K > 1/67$.

PROOF. From the proof of Theorem 1.3.4 we see that $C_2^{(1)}(x) \geq 7/96$. Because of that, we suppose that $n \geq 3$, and, of course, $0 < x \leq \pi$. Notice that all the terms on the right side in identity (1.3.5) are positive in the interval $(0, 2\pi/(2n+1))$. Taking only the first term ($k = 0$), we obtain the following inequality

$$C_n^{(1)}(x) > \frac{1}{6} \qquad \left(0 < x < \frac{2}{2n+1}\right).$$

On the other hand, from the identity (1.3.5) it follows that

$$C_n^{(1)}(x) \geq \frac{1}{6} + \frac{1}{6}\cos^2\frac{x}{2} - \frac{1}{2(n+1)\sin(x/2)} = \varphi(x).$$

We consider now the interval $[2\pi/(2n+1), \pi]$. By setting $t = \sin(x/2)$, we have

$$\varphi(x) = \frac{1}{3} - \frac{1}{6}t^2 - \frac{1}{2(n+1)t} = \phi(t),$$

where $\sin(\pi/(2n+1)) \leq t \leq 1$. Since the function $t \mapsto \phi(t)$ is concave ($\phi''(t) < 0$) on that interval, we have that

$$\varphi(x) \geq \min\left(\phi\left(\sin\frac{\pi}{2n+1}\right), \phi(1)\right).$$

Evidently, the right side of this inequality depends on n. Notice that

$$\phi(1) = \frac{1}{6} - \frac{1}{2(n+1)} \geq \frac{1}{24} \quad (n \geq 3).$$

In order to determine $\inf_{n\geq 3} \phi\left(\sin\frac{\pi}{2n+1}\right)$, it is enough to investigate the function

$$\theta \mapsto g(\theta) = \frac{1}{3} - \frac{1}{6}\sin^2\theta - \frac{\theta}{(\pi+\theta)\sin\theta}$$

for $\theta \in (0, \pi/7]$.

Since $g(\theta) \geq g(\pi/7) = 1/72.14\ldots$ ($\theta \in (0, \pi/7]$), according to the above we conclude that we can take the value $1/73$ for K.

If we suppose that $n \geq 4$, the constant K can be improved. Namely, than we have

$$\inf_{n\geq 4} \phi\left(\sin\frac{\pi}{2n+1}\right) = \lim_{\theta\to 0} g(\theta) = \frac{1}{3} - \frac{1}{\pi} > \frac{1}{67}. \quad \square$$

Using the same method, Tomić [4] found a better bound for $n > 1$ in Young's inequality (1.2.6). He proved that

$$1 + \frac{\cos x}{1} + \frac{\cos 2x}{2} + \cdots + \frac{\cos nx}{n} > K', \qquad n = 2, 3, \ldots,$$

where K' is a positive constant independing on n and x. One can take $K' = 1/20$.

The following result is a corollary of Theorem 1.3.4:

Corollary 1.3.6. *The cosine polynomial*

$$\overline{\Lambda}_n(x) = \frac{\lambda_0}{2} + \frac{\lambda_1 \cos x}{2} + \frac{\lambda_2 \cos 2x}{3} + \cdots + \frac{\lambda_n \cos nx}{n+1},$$

with nonnegative and nonincreasing coefficients λ_k ($k = 0, 1, \ldots, n$), is nonnegative.

This result follows from

$$\overline{\Lambda}_n(x) = \sum_{k=0}^{n-1} C_k^{(1)}(x)(\lambda_k - \lambda_{k+1}) + C_n^{(1)}(x)\lambda_n \geq 0.$$

Rogosinski and Szegö [1] considered a more general cosine sum than (1.3.4), i.e.,

$$C_n^{(\alpha)}(x) = \frac{1}{1+\alpha} + \frac{\cos x}{1+\alpha} + \frac{\cos 2x}{2+\alpha} + \cdots + \frac{\cos nx}{n+\alpha}. \tag{1.3.9}$$

Notice that

$$C_n^{(\alpha)}(x) \geq 0 \tag{1.3.10}$$

for $\alpha = 1$ (Theorem 1.3.4) and $\alpha = 0$ (Young's inequality).

Putting $\lambda_0 = 2/(1+\alpha)$, $\lambda_k = (k+1)/(k+\alpha)$ $(k = 1, \ldots, n)$, we see that $\lambda_k \geq \lambda_{k+1} > 0$ $(k = 0, 1, \ldots, n-1)$, for $-1 < \alpha \leq 1$. So, the inequality (1.3.10) holds for such values of α. Rogosinski and Szegö showed that there is a number A, $1 \leq A \leq 2(1+\sqrt{2})$, such that the polynomials $C_n^{(\alpha)}(x)$ $(n = 0, 1, \ldots)$ are nonnegative for $-1 < \alpha \leq A$, while this is not the case for $\alpha > A$.

Gasper [1] determined this constant A. Namely, he proved the following result:

Theorem 1.3.7. *Let A be the positive root of the equation*

$$9\alpha^7 + 55\alpha^6 - 14\alpha^5 - 948\alpha^4 - 3247\alpha^3 - 5013\alpha^2 - 3780\alpha - 1134 = 0. \tag{1.3.11}$$

If $-1 < \alpha \leq A$, then $C_n^{(\alpha)}(x) \geq 0$ $(n = 0, 1, \ldots)$. However, if $\alpha > A$ then $C_n^{(\alpha)}(x) < 0$ for some x,

An elementary computation yields

$$A = 4.5678018\ldots\ . \tag{1.3.12}$$

Theorem 1.3.7 ia an immediate consequence of (1.3.12) and the following three lemmas:

Lemma 1.3.8. *Let $\alpha > \beta > -1$. If $C_k^{(\alpha)}(x) \geq 0$ $(k = 0, 1, \ldots, n)$, then $C_n^{(\beta)}(x) \geq 0$.*

PROOF. Sum by parts. □

Lemma 1.3.9. *Let A be defined as in Theorem (1.3.7). If $-1 < \alpha \leq A$, then $C_n^{(\alpha)}(x) \geq 0$ $(n = 0, 1, 2, 3)$. However, if $\alpha > A$ then $C_n^{(\alpha)}(x) < 0$ for some x.*

PROOF. Clearly

$$C_0^{(\alpha)}(x) = \frac{1}{1+\alpha} > 0, \quad C_1^{(\alpha)}(x) = \frac{1+\cos x}{1+\alpha} \geq 0,$$

and

$$C_2^{(\alpha)}(x) = \frac{2}{2+\alpha}\left(\left(\cos x + \frac{2+\alpha}{4+4\alpha}\right)^2 + \frac{4+4\alpha-\alpha^2}{16(1+\alpha)^2}\right) \geq 0$$

for $-1 < \alpha \leq 2(1+\sqrt{2}) = 4.8284\ldots$.

Putting $t = \cos x$, we have

$$\frac{3+\alpha}{4}C_3^{(\alpha)}(x) = t^3 + \frac{3+\alpha}{4+2\alpha}t^2 - \frac{\alpha}{2+2\alpha}t + \frac{3+\alpha}{4(1+\alpha)(2+\alpha)}$$
$$= t^3 + pt^2 + qt + r = f(t;\alpha).$$

The polynomial $f(t;\alpha)$ has at least two equal real zeros if and only if $\Delta = 0$, where

$$\Delta = 27b^2 + 4a^3, \quad a = \frac{1}{3}(3q - p^2), \quad b = \frac{1}{27}(2p^3 - 9pq + 27r).$$

Since Δ is a (strictly) positive multiple of

$$-9\alpha^7 - 55\alpha^6 + 14\alpha^5 + 948\alpha^4 + 3247\alpha^3 + 5013\alpha^2 + 3780\alpha + 1134,$$

the equality (1.3.11) holds if only if $\Delta = 0$. Denoting the (unique) positive root of (1.3.11) by A, we find that $\Delta < 0$ for $\alpha > A$. Hence $f(t;\alpha)$ $(\alpha > A)$ has a relative minimum in $(-1,1)$, it follows that if $\alpha > A$ then $f(t;\alpha) < 0$ for some point in $(-1,1)$. Also, since $F(-1;A) > 0$, $f(1;A) > 0$, and $f(t;A)$ is tangent to the t-axis, $f(t;A)$ has a zero in $(-1,1)$ and is nonnegative in $[-1,1]$. Interpreting these statements in terms of $C_3^{(\alpha)}(x)$ and applying Lemma 1.3.8, we get Lemma 1.3.9. □

Lemma 1.3.10. *Let $n \geq 4$ and $\alpha = 4.57$. Then $C_n^{(\alpha)}(x) > 0$.*

This Lemma was proved by Gasper [1], spliting the interval $(0,\pi]$ in a special way and using the methods similar to ones used in the proof of Theorem 1.3.4.

Also, Gasper [1] gives the following extensions of Theorem 1.3.7:

Theorem 1.3.11. *Let $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$. Then*

$$\frac{a_0}{1+A} + \frac{a_1\cos x}{1+A} + \frac{a_2\cos 2x}{2+A} + \cdots + \frac{a_n\cos nx}{n+A} \geq 0,$$

where A is given as in Theorem 1.3.7.

Theorem 1.3.12. *Let $-1 < \alpha \leq A$. Then*

$$\frac{1}{1+\alpha} + \sum_{k=1}^{n}\frac{1}{k+\alpha}\prod_{j=1}^{m}\cos(kx_j) \geq 0.$$

2. POSITIVITY AND MONOTONICITY OF SOME SUMS

2.1. Turán's Inequalities

In [3] Turán has proved the following results:

Theorem 2.1.1. *If the real numbers $b_1, \dots, b_n$ are not all zero and*

$$f(x) = \sum_{k=1}^{n} b_k \sin(2k-1)x \geq 0 \qquad (0 < x < \pi), \tag{2.1.1}$$

then we have for the same n

$$g(x) = \sum_{k=1}^{n} \frac{b_k}{k} \sin kx > 0 \qquad (0 < x < \pi). \tag{2.1.2}$$

Remark 2.1.1. For $b_k = 1$ ($k = 1, \dots, n$), the inequality (2.1.2) reduces to Fejér-Gronwall-Jackson's inequality (1.2.1). This exhibits (1.2.1) as a consequence of the basic inequality

$$\sum_{k=1}^{n} \sin(2k-1)x \geq 0 \qquad (0 < x < \pi).$$

Theorem 2.1.2. *If the numbers $a_0, a_1, \dots, a_n$ are not all zero and*

$$\sum_{k=0}^{n} a_k = 0 \tag{2.1.3}$$

and

$$A(x) = \sum_{k=0}^{n} a_k \cos kx \geq 0 \qquad (0 \leq x \leq 2\pi), \tag{2.1.4}$$

then

$$B(x) = \sum_{k=1}^{n} \frac{a_0 + a_1 + \cdots + a_{k-1}}{k} \sin kx > 0 \qquad (0 < x < \pi).$$

Remark 2.1.2. If we put $a_0 = 1$, $a_1 = a_2 = \cdots = a_{n-1} = 0$, $a_n = -1$, i.e., $A(x) = 1 - \cos nx \geq 0$, we get again (1.2.1).

Theorem 2.1.3. *If a_k ($k = 0, 1, \dots, n$) are real,*

$$\sum_{k=0}^{n} a_k = 0 \qquad \text{and} \qquad \Big| \sum_{k=0}^{n} a_k \cos kx \Big| \leq M \qquad (0 \leq x \leq 2\pi),$$

then for $0 < x < \pi$ we have

$$\Big| \sum_{k=1}^{n} \frac{a_0 + a_1 + \cdots + a_{k-1}}{k} \sin kx \Big| \leq M \frac{\pi - x}{2}.$$

Remark 2.1.3. Putting $a_0 = 1$, $a_1 = a_2 = \cdots = a_{n-1} = 0$, $a_n = -1$, from this result we obtain the inequality

$$\sum_{k=1}^{n} \frac{1}{k} \sin kx < 2 \sum_{k=1}^{n} \frac{1}{k} \sin kx = \pi - x \qquad (0 < x < \pi),$$

which was proved by Turán [2].

In order to deduce Theorem 2.1.1 from Theorem 2.1.2, Turán expresses (2.1.1) in the form

$$b_1 \sin \frac{x}{2} + b_2 \sin \frac{3x}{2} + \cdots + b_n \sin \frac{(2n-1)x}{2} \geq 0$$

which is valid for $0 < x < 2\pi$. Multiplying by $2\sin(x/2)$, he obtains, again for $0 < x < 2\pi$,

$$b_1(1 - \cos x) + b_2(\cos x - \cos 2x) + \cdots + b_n(\cos(n-1)x - \cos nx) \geq 0,$$

i.e.,

$$b_1 + (b_2 - b_1)\cos x + (b_3 - b_2)\cos 2x + \cdots + (b_n - b_{n-1})\cos(n-1)x - b_n \cos nx \geq 0.$$

Thus (2.1.3) and (2.1.4) are satisfied for

$$a_0 = b_1,\ a_1 = b_2 - b_1,\ \ldots,\ a_{n-1} = b_n - b_{n-1},\ a_n = -b_n.$$

Hence, Theorem 2.1.1 follows as an application of Theorem 2.1.2.

In order to prove Theorem 2.1.2, Turán considered the function

$$F(z) = \sum_{k=0}^{n} a_k z^k.$$

Then by hypothesis

$$\frac{F(z)}{1-z} = \sum_{k=0}^{n} (a_0 + a_1 + \cdots + a_{k-1}) z^{k-1},$$

Turán showed that

$$\sum_{k=1}^{n} \frac{a_0 + a_1 + \cdots + a_{k-1}}{k} \sin kx = \int_0^{(\pi - x)/2} \mathrm{Re}\{F(1 - \rho e^{-i\alpha})\}\, d\alpha, \qquad (2.1.5)$$

where $\rho = |1 - z| = |1 - e^{ix}|$ $(0 < x < \pi)$. Using (2.1.4) this gives Theorem 2.1.2.

In the case of Fejér's sum (1.1.2), the representation (2.1.5) takes the form

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k} = \int_0^{(\pi - x)/2} \mathrm{Re}\{1 - (1 - \rho e^{-i\alpha})^n\}\, d\alpha$$

found by Turán [2]. From this, we have

$$S_n(x) \geq \int_0^{(\pi-x)/2} \left\{1 - |1-\rho e^{-i\alpha}|^n\right\} d\alpha,$$

and, since $|1-\rho e^{-i\alpha}| \leq 1$, we get further that

$$S_n(x) \geq \int_0^{(\pi-x)/2} \left\{1 - |1-\rho e^{-i\alpha}|^2\right\} d\alpha \qquad (n \geq 2).$$

Using

$$\begin{aligned}\int_0^{(\pi-x)/2} (-\rho^2 + 2\rho\cos\alpha)\, d\alpha &= \left(2\cos\frac{x}{2} - \rho\frac{\pi-x}{2}\right)\rho \\ &= 4\sin^2\frac{x}{2}\left(\tan\frac{\pi-x}{2} - \frac{\pi-x}{2}\right)\end{aligned}$$

we obtain a simple positive minorant to Fejér's polynomial $S_n(x)$, i.e., the inequality

$$\sum_{k=1}^{n} \frac{\sin kx}{k} > 4\sin^2\frac{x}{2}\left(\tan\frac{\pi-x}{2} - \frac{\pi-x}{2}\right) \qquad (n \geq 2,\ 0 < x < \pi).$$

The representation (2.1.5) gives immediately Theorem 2.1.3.

In [2] Hyltén-Cavallius studied the trigonometrical kernel

$$P(x,t) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \cdot \frac{\sin(k-\frac{1}{2})t}{2\sin\frac{1}{2}t}, \tag{2.1.6}$$

i.e.,

$$P(x,t) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \left(\frac{1}{2} + \cos t + \cdots + \cos(k-1)t\right).$$

The series (2.1.6) converges in the domain $0 \leq x \leq \pi$, $-\pi \leq t \leq \pi$ except on the three segments where $0 < x < \pi$ and $t = 0$ or $|t| = x$. The formulae for $0 < x < 2\pi$

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\log\left(2\sin\frac{x}{2}\right) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2},$$

give the following explicit expression for $P(x,t)$:

$$4P(x,t) = \begin{cases} \cot\dfrac{t}{2}\log|T(x,t)| + x & (0 < x < t < \pi), \\[2mm] \cot\dfrac{t}{2}\log|T(x,t)| + x - \pi & (0 < t < x < \pi), \end{cases} \tag{2.1.7}$$

where $T(x,t) = (\tan(t/2) + \tan(x/2))\,/\,(\tan(t/2) - \tan(x/2))$. Using this formula, Hyltén-Cavallius has obtained:

Theorem 2.1.4. *$P(x,t)$ is positive when $0 < x < \pi$, $-\pi < t < \pi$ and $|t| \neq x$, $t \neq 0$.*

PROOF. We can suppose $0 < t < \pi$. In (2.1.7), however, the expression $T(x,t)$ is always greater than 1 and the assertion is proved for $0 < x < t < \pi$. For $0 < t < x < \pi$ we use the inequality

$$\log \frac{1+u}{1-u} = 2\left(u + \frac{1}{3}u^3 + \dots\right) > 2u \qquad (0 < u < 1).$$

Then we get

$$4P(x,t) > 2\cot\frac{t}{2} \cdot \frac{\tan\frac{t}{2}}{\tan\frac{x}{2}} + x - \pi = 2\left(\tan\frac{\pi - x}{2} - \frac{\pi - x}{2}\right) > 0$$

and thus the theorem is proved. □

By a partial summation we can express $P(x,t)$ in the form

$$P(x,t) = \frac{1}{2}r_0(x) + \sum_{k=1}^{\infty} r_k(x)\cos kt,$$

where

$$r_k(x) = \sum_{m=k+1}^{\infty} \frac{\sin mx}{m} = \int_x^{\pi} \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt.$$

Similarly, for the trigonometrical kernel

$$\begin{aligned} Q(x,t) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k}(\sin t + \sin 2t + \dots + \sin kt) \\ &= \sum_{k=1}^{\infty} \frac{\sin kx}{k} \cdot \frac{\cos\frac{1}{2}t - \cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t} \end{aligned}$$

we obtain the following expression

$$4Q(x,t) = \begin{cases} \log|T(x,t)| + \pi\cot\frac{t}{2} & (0 < x < t < \pi), \\ \log|T(x,t)| & (0 < t < x < \pi). \end{cases}$$

Theorem 2.1.5. *$Q(x,t)$ is positive when $0 < x < \pi$, $0 < t < \pi$ and $t \neq x$.*

Using Theorem 2.1.4, Hyltén-Cavallius [2] has given proofs of Turán's theorems 2.1.1 and 2.1.2. For example, for Theorem 2.1.2 he obtains

$$\begin{aligned} 0 < \frac{1}{\pi}\int_{-\pi}^{\pi} A(t)P(x,t)\,dt &= a_0r_0 + a_1r_1 + \dots + a_nr_n \\ &= \sum_{k=1}^{\infty} \frac{\sin kx}{k}\left(\sum_{m=0}^{\min(k-1,n)} a_m\right) = B(x), \end{aligned}$$

and the assertion is proved.

We conclude with two proofs of Turán's theorem 2.1.1.

PROOF (Hylten-Cavallius [2]). We have

$$\begin{aligned} g(x) &= \sum_{k=1}^{n} \frac{b_k}{k} \sin kx = \sum_{k=1}^{n} \frac{\sin kx}{k} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(2k-1)t \, dt \\ &= \sum_{k=1}^{\infty} \frac{\sin kx}{k} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(2k-1)t \, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{k=1}^{\infty} \frac{\sin kx}{k} \sin(2k-1)t \right) dt, \end{aligned}$$

i.e.,

$$g(x) = \frac{2}{\pi} \int_{-\pi}^{\pi} f(t) P(x, 2t) \sin t \, dt > 0 \qquad (0 < x < \pi). \quad \square$$

PROOF (Askey, Fitch, Gasper [1]). A simple computation shows that

$$\frac{d}{dt} \left(\frac{\sin \alpha t}{\alpha (\sin t)^{\alpha}} \right) = -\frac{\sin(\alpha - 1)t}{(\sin t)^{\alpha+1}}.$$

Letting $\alpha = 2k$ and $t = x/2$ we see that

$$\frac{\sin kx}{k} = 2 \int_{x/2}^{\pi/2} \left(\frac{\sin x/2}{\sin t} \right)^{2k} \frac{\sin(2k-1)t}{\sin t} \, dt.$$

Thus

$$\sum_{k=1}^{n} \frac{b_k}{k} \sin kx = 2 \int_{x/2}^{\pi/2} \sum_{k=1}^{n} b_k \left(\frac{\sin x/2}{\sin t} \right)^{2k} \frac{\sin(2k-1)t}{\sin t} \, dt.$$

But $\sum_{k=1}^{n} b_k r^{2k-1} \sin(2k-1)t > 0$ $(0 < r < 1)$ if $\sum_{k=1}^{n} b_k \sin(2k-1)t \geq 0$ and not all b_k are zero. $\square$

2.2. Positivity of Some Classes of Trigonometric Sums

In 1958 Vietoris [1] published a dramatic improvement of the inequalities of Fejér-Gronwall-Jackson and Young ((1.2.1) and (1.2.6)). Namely, he proved:

Theorem 2.2.1. *If* $a_0 \geq a_1 \geq \cdots \geq a_n > 0$ *and* $(2k)a_{2k} \leq (2k-1)a_{2k-1}$ $(k \geq 1)$, *then*

$$s_n(x) = \sum_{k=1}^{n} a_k \sin kx > 0 \qquad (0 < x < \pi) \tag{2.2.1}$$

and

$$t_n(x) = \sum_{k=0}^{n} a_k \cos kx > 0 \qquad (0 < x < \pi). \tag{2.2.2}$$

Vietoris observed that (2.2.1) and (2.2.2) follow from the corresponding assertions for the special case in which $a_k = c_k$, where

$$c_0 = c_1 = 1, \quad c_{2k} = c_{2k+1} = \frac{2k-1}{2k} c_{2k-1} \quad (k \geq 1),$$

i.e.,

$$c_{2k} = c_{2k+1} = 2^{-2k} \binom{2k}{k} \qquad (k \geq 0). \tag{2.2.3}$$

This is the extreme case of equality in the inequalities for the numbers a_k.

Theorem 2.2.2. *If c_k are given by* (2.2.3), *then*

$$\sigma_n(x) = \sum_{k=1}^{n} c_k \sin kx > 0 \qquad (0 < x < \pi) \tag{2.2.4}$$

and

$$\tau_n(x) = \sum_{k=0}^{n} c_k \cos kx > 0 \qquad (0 < x < \pi). \tag{2.2.5}$$

These two theorems are equivalent (see Vietoris [1]). Theorem 2.2.2 is obviously a special case of the first theorem. On the other hand, Theorem 2.2.1 follows from Theorem 2.2.2. For $d_0 \geq d_1 \geq \cdots \geq d_n > 0$, a summation by parts shows that

$$\sum_{k=1}^{n} c_k d_k \sin kx > 0 \qquad (0 < x < \pi)$$

and

$$\sum_{k=0}^{n} c_k d_k \cos kx > 0 \qquad (0 < x < \pi).$$

Letting $a_k = c_k d_k$ $(0 \leq k \leq n)$ Theorem 2.2.1 follows.

Remark 2.2.1. It is of interest to note that c_k has order of magnitude $k^{-1/2}$ as apposed to the order of magnitude k^{-1} for the coefficients in the earlier inequalities (1.2.1), (1.2.6), and (1.3.10).

Remark 2.2.2. This paper of Vietoris was unknown up to the appearance of D. S. Mitrinović's book [1], where this result has been treated in p. 255. Later, Askey and Steinig [1] have also performed a valuable service in drawing attention to Vietoris's theorem (see, also, a paper of Brown and E. Hewitt [1]). In a recent paper about these inequalities, Askey [6] writes: "Times had changed enough so that Mathematical Reviews gave up trying to get a review of [7]* after it was sent back unreviewed by three people".

Askey and Steinig [1] gave an alternative version of Vietoris's proof of Theorem 2.2.2. They use some of Vietoris' ideas, but many of the difficulties of his proof they replace by easier arguments. For the proof they need the three following lemmas:

Lemma 2.2.3. *Let* $m \geq 1$. *Then* $\binom{2m}{m} < 2^{2m}(\pi m)^{-1/2}$.

PROOF. Let $\gamma_m = m^{1/2}2^{-2m}\binom{2m}{m}$. Then $\gamma_m < \gamma_{m+1}$ for $m \geq 1$; and by Stirling formula, $\gamma_m \to \pi^{-1/2}$ as $m \to \infty$. □

Lemma 2.2.4. *Let the sequence* $\{c_k\}_{k=0}^{\infty}$ *be defined by* (2.2.3). *Then for* $0 < x < \pi$,

$$\sum_{k=1}^{\infty} c_k \sin kx = \sum_{k=0}^{\infty} c_k \cos kx = \left(\frac{1}{2}\cot\frac{x}{2}\right)^{1/2}. \qquad (2.2.6)$$

PROOF. For $|z| \leq 1$, $z \neq 1$, we have $(1-z)^{-1/2} = \sum_{k=0}^{\infty} c_{2k} z^k$. Since $c_{2k} = c_{2k+1}$, it follows that

$$(1+z)(1-z^2)^{-1/2} = \sum_{k=0}^{\infty} c_k z^k$$

for $|z| \leq 1$, $z \neq \pm 1$. On setting $z = e^{ix}$ $(0 < x < \pi)$ and separating real and imaginary parts, we get (2.2.6). □

Lemma 2.2.5. *Let* $P_r(x) = \sum_{k=0}^{r} b_k e^{ikx}$, *where* $b_0 \geq b_1 \geq \cdots \geq b_r > 0$. *Then for* $n \geq m \geq 0$ *we have*

$$|P_n(x) - P_m(x)| \leq \frac{b_{m+1}}{\sin(x/2)} \qquad (0 < x < 2\pi). \qquad (2.2.7)$$

PROOF. Sum by parts and use the standard estimate

$$\left|\sum_{k=0}^{n} e^{ikx}\right| \leq \frac{1}{\sin(x/2)}. \quad \square$$

Now we give Askey–Steinig's proof of Theorem 2.2.2.

PROOF OF (2.2.4). We may assume that $n \geq 2$. Different arguments are needed for each of the intervals $0 < x \leq \pi/n$, $\pi/n < x < \pi - \pi/n$ and $\pi - \pi/n \leq x < \pi$.

In the first case, all terms in the sum are nonnegative, and the first is strictly positive.

* [7] is the Vietoris's paper in references of Askey's paper [6].

For $\pi - \pi/n \le x < \pi$, we set $x = \pi - y$, so that $0 < y \le \pi/n$. If n is even, i.e., $n = 2m$, we have

$$\begin{aligned}\sigma_n(x) &= \sum_{k=1}^{2m}(-1)^{k-1}c_k \sin ky \\ &= \sum_{k=1}^{m}(c_{2k-1}\sin(2k-1)y - c_{2k}\sin 2ky) \\ &= \sum_{k=1}^{m}(2k-1)c_{2k-1}\left(\frac{\sin(2k-1)y}{2k-1} - \frac{\sin 2ky}{2k}\right).\end{aligned}$$

The last sum has positive terms since $t \mapsto \sin t/t$ is decreasing function on $(0,\pi]$ and $2ky \le 2my = ny \le \pi$. And if n is odd there is an extra term, $c_n \sin ny$, which is positive for $0 < y < \pi/n$.

If $n \ge 3$, we must still consider the interval $\pi/n < x < \pi - \pi/n$. There we have $\sin x > \sin(\pi/n) \ge (\pi/n)(1 - \pi^2/6n^2)$. Now by Lemmas 2.2.4 and 2.2.5 we obtain

$$\sigma_n \ge \left(\frac{1}{2}\cot\frac{x}{2}\right)^{1/2} - \frac{c_{n+1}}{\sin(x/2)}.$$

Hence, for $\pi/n < x < \pi - \pi/n$, we find

$$2\sin\frac{x}{2}\sigma_n(x) \ge \left(\frac{\pi}{n}\left(1 - \frac{\pi^2}{6n^2}\right)\right)^{1/2} - 2c_{n+1}. \tag{2.2.8}$$

The first term on the right hand side of (2.2.8) is decreasing in n for $n \ge 3$, and $c_{2m} = c_{2m+1}$ for $m \ge 0$. Hence, the right hand side of (2.2.8) is positive for $n = 2m - 1$, if it is positive for $n = 2m$. And for $n = 2m$ it follows from Lemma 2.2.3 that the right hand side of (2.2.8) is at least equal to $(2\pi m)^{-1/2}\{\pi(1-\pi^2/24m^2)^{1/2} - 2\sqrt{2}\} > 0$ $(m \ge 2)$. Therefore $\sigma_n(x) > 0$ for $\pi/n < x < \pi - \pi/n$.
□

PROOF OF (2.2.5). The result is obvious for $n = 0$ and $n = 1$, and an elementary computation shows that $\tau_2(x) = \cos^2 x + \cos x + 1/2 > 0$. We can therefore assume $n \ge 3$.

Firstly, we observe that $\tau_n(x) > 0$ for $0 < x \le \pi/n$ since

$$\frac{d\tau_n}{dx} = -\sum_{k=1}^{n} kc_k \sin kx < 0 \qquad (0 < x < \pi/n)$$

and

$$\tau_n(\pi/n) = \sum_{k=0}^{[n/2]}(c_k - c_{n-k})\cos\frac{k\pi}{n} > 0.$$

Secondly, we show that $\tau_n(x) > 0$ for $\pi - \pi/(n+1) < x < \pi$. We set $y = \pi - x$, and write

$$\tau_n(x) = \sum_{k=0}^{[(n-1)/2]} c_{2k}\big(\cos 2ky - \cos(2k+1)y\big) + \delta_n,$$

where $\delta_n = 0$ if $n = 2m-1$ and $\delta_n = c_{2m}\cos 2my$ if $n = 2m$. When $\delta_n = 0$, the monotonicity of $\cos x$ $(0 \le x \le \pi)$ shows that $\tau_n(x) > 0$ for $0 < y < \pi/n$. When $n = 2m$, we have

$$\begin{aligned}\tau_n(x) &\ge c_{2m}(1 - \cos y + \cos 2y - \cos 3y + \cdots + \cos 2my)\\ &= c_{2m}(1 + \cos x + \cos 2x + \cos 3x + \cdots + \cos 2mx)\\ &= c_{2m}\frac{\sin(m+1/2)x\cos mx}{\sin(x/2)} = c_{2m}\frac{\cos(m+1/2)y\cos my}{\cos(y/2)}.\end{aligned}$$

It follows that $\tau_n(x) > 0$ for $0 < (m+1/2)y < \pi/2$, i.e., $0 < y < \pi/(n+1)$.

Lastly, we consider the interval $\pi/(n+1) \le x \le \pi - \pi/(n+1)$ for $n \ge 3$. The same argument as for $\sigma_n(x)$ on $\pi/n < x < \pi - \pi/n$ shows that it is enough if

$$\left(\frac{\pi}{n+1}\left(1 - \frac{\pi^2}{6(n+1)^2}\right)\right)^{1/2} - 2c_{n+1} > 0.$$

Here, again, it suffices to consider even values of n, say $n = 2m$. Computation shows that this inequality holds for $n = 4$ and 6. For $m \ge 4$, the stronger inequality

$$\left(\frac{\pi}{2m+1}\left(1 - \frac{\pi^2}{6(2m+1)^2}\right)\right)^{1/2} - \frac{2}{\sqrt{\pi m}} > 0$$

holds, since it holds for $m = 4$ and since its left hand side, when multiplied by $\sqrt{m}$, is an increasing function of m. □

Three corollaries of Theorem 2.2.1 are given also in Askey and Steinig [1].

Corollary 2.2.6. *Let* $(2k-1)A_{k-1} \ge 2kA_k > 0$ *for* $k \ge 1$, *and* $0 < x < 2\pi$. *Then*

$$\sum_{k=0}^{n} A_k \sin(k+\tfrac{1}{4})x > 0 \quad \textit{and} \quad \sum_{k=0}^{n} A_k \cos(k+\tfrac{1}{4})x > 0.$$

Corollary 2.2.7. *Let* $A_1, \ldots, A_n$ *satisfy the conditions of Corollary* 2.2.6. *If* $0 \le \nu \le 1/4$ *and* $0 < x < 2\pi$, *or* $-1/4 \le \nu \le 1/4$ *and* $0 < x < \pi$, *then*

$$\sum_{k=0}^{n} A_k \cos(k+\nu)x > 0.$$

Corollary 2.2.8. *Let $A_1, \ldots, A_n$ satisfy the conditions of Corollary* 2.2.6. *If* $1/4 \le \nu \le 1/2$ *and* $0 < x < 2\pi$, *or* $1/4 \le \nu \le 3/4$ *and* $0 < x < \pi$, *then*

$$\sum_{k=0}^{n} A_k \sin(k+\nu)x \ge 0.$$

Combining the above results with an argument due to Szegö, Askey and Steinig [1] gave bounds for the zeros of a wide class of trigonometric polynomials.

We consider the trigonometric polynomials

$$p(x) = \lambda_0 \cos nx + \lambda_1 \cos(n-1)x + \cdots + \lambda_{n-1} \cos x + \lambda_n \tag{2.2.9}$$

and

$$q(x) = \lambda_0 \cos\left(n+\tfrac{1}{2}\right)x + \lambda_1 \cos\left(n-\tfrac{1}{2}\right)x + \cdots + \lambda_n \cos \tfrac{1}{2}\,x, \tag{2.2.10}$$

and their conjugate functions

$$\tilde{p}(x) = \lambda_0 \sin nx + \lambda_1 \sin(n-1)x + \cdots + \lambda_{n-1} \sin x \tag{2.2.11}$$

and

$$\tilde{q}(x) = \lambda_0 \sin\left(n+\tfrac{1}{2}\right)x + \lambda_1 \sin\left(n-\tfrac{1}{2}\right)x + \cdots + \lambda_n \sin \tfrac{1}{2}\,x, \tag{2.2.12}$$

respectively.

First, we give the theorem of Pólya–Szegö (see Szegö [6, pp. 134–135]):

Theorem 2.2.9. *Under the conditions* $\lambda_0 > \lambda_1 \ge \cdots \ge \lambda_n \ge 0$, *the polynomials* $p(x)$ *and* $q(x)$, *given by* (2.2.9) *and* (2.2.10), *respectively, have only real and simple zeros. There is, respectively, exactly one zero in each of intervals*

$$\frac{k-1/2}{n+1/2}\,\pi < x < \frac{k+1/2}{n+1/2}\,\pi \quad \text{and} \quad \frac{k-1/2}{n+1}\,\pi < x < \frac{k+1/2}{n+1}\,\pi, \tag{2.2.13}$$

where $k = 1, 2, \ldots, 2n$, *and* $k = 1, 2, \ldots, 2n+1$, *respectively.*

Proof. The first part of the statement was proved by Pólya [1], using the principle of argument. Szegö [3] used the classical Fejér inequality

$$\sum_{k=0}^{n} \sin\left(k+\tfrac{1}{2}\right)x = \frac{1-\cos(n+1)x}{2\sin(x/2)} \ge 0 \quad (0 < x < 2\pi) \tag{2.2.14}$$

to prove the estimates (2.2.13).

According to the summation by parts and using (2.2.14), we find that

$$W(x) = \operatorname{Im}\left\{e^{-i(n+1/2)x}(p(x) + i\tilde{p}(x))\right\} = \operatorname{Im}\left\{e^{-i(n+1)x}(q(x) + i\tilde{q}(x))\right\}$$

and

$$-W(x) = \sum_{k=0}^{n} \lambda_k \sin\left(k + \tfrac{1}{2}\right)x > 0 \qquad (0 < x < 2\pi).$$

Therefore,

$$p(x)\sin\left(n + \tfrac{1}{2}\right)x - \tilde{p}(x)\cos\left(n + \tfrac{1}{2}\right)x > 0,$$
$$q(x)\sin(n+1)x - \tilde{q}(x)\cos(n+1)x > 0,$$

for $0 < x < 2\pi$, whence

$$\operatorname{sgn} p\left(\frac{k-1/2}{n+1/2}\pi\right) = \operatorname{sgn} q\left(\frac{k-1/2}{n+1}\pi\right) = (-1)^{k+1}.$$

This shows the existence of at least one zero in each of the intervals in (2.2.13). On the other hand, the polynomials $p(x)$ and $q(x)$ cannot have more than $2n$ and $2n+1$ zeros in $[0, 2\pi]$, respectively. □

Similar results about zeros of the polynomials $\tilde{p}(x)$ and $\tilde{q}(x)$, defined by (2.2.11) and (2.2.12), respectively, can be given. Also, some improvements of bounds can be obtained for additional restrictions of the coefficients λ_k (see Szegö [3]).

Askey and Steing [1] proved the following stronger result:

Theorem 2.2.10. *If*

$$(2k-1)\lambda_{k-1} \geq 2k\lambda_k > 0, \qquad k \geq 1, \tag{2.2.15}$$

then

$$\frac{k-1/2}{n+1/4}\pi < s_k < \frac{k}{n+1/4}\pi, \qquad k = 1, \ldots, n, \tag{2.2.16}$$

$$\frac{k}{n+1/4}\pi < t_k < \frac{k+1/2}{n+1/4}\pi, \qquad k = 1, \ldots, n-1, \tag{2.2.17}$$

where s_k and t_k denote the zeros of the polynomials $p(x)$ and $\tilde{p}(x)$ in $(0, \pi)$, respectively.

PROOF. Multiplying $p(x) + i\tilde{p}(x) = \sum_{k=0}^{n} \lambda_k e^{i(n-k)x}$ by $e^{-i(n+1/4)x}$ and using Corollary 2.2.6, we find that

$$p(x)\cos\left(n + \tfrac{1}{4}\right)x + \tilde{p}(x)\sin\left(n + \tfrac{1}{4}\right)x = \sum_{k=0}^{n} \lambda_k \cos\left(k + \tfrac{1}{4}\right)x > 0,$$

and

$$p(x)\sin\left(n+\tfrac{1}{4}\right)x-\tilde{p}(x)\cos\left(n+\tfrac{1}{4}\right)x=\sum_{k=0}^{n}\lambda_k\sin\left(k+\tfrac{1}{4}\right)x>0,$$

for $0<x<2\pi$. Putting $x=k\pi/(n+1/4)$ and $x=(k+1/2)\pi/(n+1/4)$ in the above inequalities, respectively, we obtain

$$(-1)^k p(k\pi/(n+1/4))>0,\quad k=0,1,\ldots,n,$$
$$(-1)^k p((k+1/2)\pi/(n+1/4))>0,\quad k=0,1,\ldots,n-1,$$

which imply (2.2.16). Similarly, we prove (2.2.17). □

The other zeros of $p(x)$ are at $x=2m\pi\pm s_k$ and those of $\tilde{p}(x)$, at $x=2m\pi\pm t_k$ and at $x=m\pi$ $(m=0,\pm1,\pm2,\ldots)$.

Recently, for the sequence

$$d_{2k}=d_{2k+1}=\frac{k!}{\left(\frac{3}{2}\right)_k},\tag{2.2.18}$$

Brown and E. Hewitt [1] have proved the following inequalities:

$$d_0+d_1\cos x+d_2\cos 2x+\cdots+d_n\cos nx>0\qquad(0\le x<\pi),\tag{2.2.19}$$

$$d_1\sin x+d_2\sin 2x+\cdots+d_{2m+1}\sin(2m+1)x>0\qquad(0<x<\pi),\tag{2.2.20}$$

$$d_1\sin x+d_2\sin 2x+\cdots+d_{2m}\sin 2mx>0\qquad\left(0<x<\pi-\frac{\pi}{2m}\right).\tag{2.2.21}$$

Using a summation by parts it can be stated (Brown and E. Hewitt[1]):

Theorem 2.2.11. *Suppose that $\{a_k\}_{k=0}^{\infty}$ is a nonincreasing sequence of non-negative real numbers such that $a_0>0$ and*

$$a_{2k}\le\frac{2k}{2k+1}a_{2k-1}\qquad(k=1,2,\ldots).\tag{2.2.22}$$

Then, for all positive integers n, we have

$$a_0+a_1\cos x+a_2\cos 2x+\cdots+a_n\cos nx>0\quad(0\le x<\pi).$$

The sequence (2.2.18) is the extreme case of equality in the inequalities (2.2.22).

Let $\delta_k=d_{2k}$. Using (2.2.19) for $n:=2n+1$ and (2.2.20) for $m:=n$, we have that

$$\sum_{k=0}^{n}\delta_k(\cos 2kx+\cos(2k+1)x)=2\cos\frac{x}{2}\sum_{k=0}^{n}\delta_k\cos(2k+\tfrac{1}{2})x>0$$

and

$$\sum_{k=0}^{n}\delta_k(\sin 2kx+\sin(2k+1)x)=2\cos\frac{x}{2}\sum_{k=0}^{n}\delta_k\sin(2k+\tfrac{1}{2})x>0,$$

for $0<x<\pi$.

According to the above, Brown and Hewitt [1] gave:

Theorem 2.2.12. *Suppose that $\{b_k\}_{k=0}^{\infty}$ is a sequence of nonnegative real numbers such that*

$$b_k \le \frac{2k}{2k+1} b_{k-1} \qquad (k = 1, 2, \dots).$$

Then we have

$$\sum_{k=0}^{n} b_k \sin(k + \tfrac{1}{4})x > 0 \quad \textit{and} \quad \sum_{k=0}^{n} b_k \cos(k + \tfrac{1}{4})x > 0$$

for all positive integers n and $0 < x < 2\pi$.

Suppose that $\{a_k\}_{k=1}^{\infty}$ is a nonincreasing sequence of nonnegative real numbers, such that $a_1 > 0$ and $2ka_{2k-1} \ge (2k+1)a_{2k} = (2k+\alpha)a_{2k+1}$ for $k = 1, 2, \dots$, where $\alpha = 1$ or $\alpha = 2$. Brown and Wilson [1] considered the inequalities

$$a_1 \sin x + a_2 \sin 2x + \cdots + a_{n-1} \sin(n-1)x + a_n \log 2 \sin nx > 0 \quad (0 < x < \pi)$$

and

$$a_1 \sin x + a_2 \sin 2x + \cdots + a_{n-1} \sin(n-1)x + \frac{3}{4} a_n \sin nx > 0 \quad (0 < x < \pi),$$

for $\alpha = 2$ and $\alpha = 1$, respectively.

They have made the following remarks:

Remark 2.2.3. The constant $\log 2$ in the first inequality is the best possible. It can be proved that the derivative of

$$\sum_{k=1}^{n-1} \frac{\sin kx}{k+1} + \alpha \frac{\sin nx}{n+1}$$

is positive when $x = \pi$ for $\alpha > \log 2$, when n is even and sufficiently large. Then the sum will take negative values when x is close to π.

Also, the constant $3/4$ in the second inequality is the best possible. Note that

$$d_1 \sin x + \frac{3}{4} d_2 \sin 2x = (1 + \cos x) \sin x,$$

where d_1 and $d_2 = 2/3$.

2.3. Positivity of Some Orthogonal Polynomial Sums

Fejér [14] proved the following result:

Theorem 2.3.1. *For $0 < \lambda \le 1/2$, we have*

$$\sum_{k=0}^{n} C_k^{\lambda}(t) > 0 \qquad (-1 < t < 1), \tag{2.3.1}$$

where $C_k^\lambda(t)$ are Gegenbauer ultraspherical polynomials defined by the generating function

$$(1-2tz+z^2)^{-\lambda}=\sum_{k=0}^{\infty}C_k^\lambda(t)z^k.$$

In other words, the power series coefficients of the function

$$z\mapsto(1-z)^{-1}(1-2tz+z^2)^{-\lambda}$$

are positive if $0<\lambda\le 1/2$.

Szegö [4] proved that the Fejér inequality (2.3.1) holds for $-1/2<\lambda<0$. This inequality fails to hold for $\lambda>1/2$.

In the case where $\lambda>1/2$, Feldheim [1]* proved the following result:

Theorem 2.3.2. *For $\lambda\ge 1/2$ the inequality*

$$\sum_{k=0}^{n}\frac{C_k^\lambda(t)}{C_k^\lambda(1)}=\sum_{k=0}^{n}\frac{k!}{(2\lambda)_k}C_k^\lambda(t)>0\qquad(-1<t<1)\tag{2.3.2}$$

holds.

PROOF. For two ultraspherical polynomials of equal degrees but of different parameters, Feldheim [1] obtained the following relation:

$$C_k^\lambda(\cos x)=\frac{2\Gamma(\lambda+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})\Gamma(\lambda-\nu)}\cdot\frac{(2\lambda)_k}{(2\nu)_k}\int_0^{\pi/2}\sin^{2\nu}y\cos^{2\lambda-2\nu-1}y\,u^kC_k^\nu\left(\frac{\cos x}{u}\right)dy,$$

where $u=(1-\sin^2x\cos^2y)^{1/2}$, $\lambda>\nu>-1/2$, $\lambda\ne0$, $\nu\ne0$, $0\le x\le\pi$.

We conclude from this that

$$\sum_{k=0}^{n}\frac{(2\nu)_k}{(2\lambda)_k}C_k^\lambda(t)>0\qquad\left(\lambda\ge\nu,\ -\frac{1}{2}<\nu\le\frac{1}{2},\ \nu\ne0,\ -1<t<1\right),$$

since the sequence $\{u^k\}_{k=0}^n$ is decreasing, so that by (2.3.1)

$$\sum_{k=0}^{n}u^kC_k^\nu\left(\frac{\cos x}{u}\right)>0.$$

---

* This paper was translated and edited by Szegö. In a footnote he writes: "The main part of this note is a slightly modified version of a letter of the young and able Hungarian mathematician Ervin Feldheim, dated March 12, 1944, a few months before he became the victim of the terror of the Nazis. The letter was addressed to Fejér and found in his posthumous papers by Turán".

In particular for $\nu = 1/2$, $(2\nu)_k = k!$, we obtain (2.3.2), where $\lambda \geq 1/2$. □

Remark 2.3.1. In the case $\lambda = 1/2$ we obtain the Legendre polynomials and (2.3.2) reduces to

$$\sum_{k=0}^{n} P_k(t) > 0 \qquad (-1 < t < 1). \tag{2.3.3}$$

This is a result of Fejér (see Fejér [1], [2] and Theorem 1.2.10).

The case $\lambda = 1$ is particularly interesting since

$$C_k^1(\cos x) = \frac{\sin(k+1)x}{\sin x} \qquad (0 < x < \pi),$$

so that (2.3.2) yields the classical Fejér-Gronwall-Jackson's inequality

$$S_{n+1}(x) = \sum_{k=1}^{n+1} \frac{\sin kx}{k} > 0 \qquad (0 < x < \pi). \tag{2.3.4}$$

Another generalization of (2.3.4) is the following:

$$\sum_{k=0}^{n} \frac{C_k^\lambda(t)}{k+1} > 0 \qquad (-1 < t < 1;\ -1/2 < \lambda \leq 1), \tag{2.3.5}$$

which follows from (2.3.2) for $1/2 \leq \lambda \leq 1$, since the sequence $\left\{\binom{k+2\lambda-1}{k}\frac{1}{k+1}\right\}_{k=0}^{n}$ is decreasing for these values of λ. On the other hand, the inequality (2.3.5) follows from (2.3.1) for $-1/2 < \lambda \leq 1/2$ since $\left\{\frac{1}{k+1}\right\}_{k=0}^{n}$ is a decreasing sequence.

Using some asymptotic estimates, Kogbetliantz [1] proved that

$$\sum_{k=0}^{n} \frac{(\gamma+1)_{n-k}}{(n-k)!} \cdot \frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{(\alpha+\beta+1)k!} \cdot \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} \geq 0, \tag{2.3.6}$$

when $-1 \leq t \leq 1$, for $\alpha = \beta > -1/2$ and $\gamma = 2\alpha + 2$. Here, $P_k^{(\alpha,\beta)}(t)$ are Jacobi polynomials defined by the Rodrigues formula

$$(1-t)^\alpha(1+t)^\beta P_k^{(\alpha,\beta)}(t) = \frac{(-1)^k}{2^k k!} \cdot \frac{d^k}{dx^k}\left((1-t)^{k+\alpha}(1+t)^{k+\beta}\right),$$

or by the hypergeometric representation

$$P_k^{(\alpha,\beta)}(t) = \frac{(\alpha+1)_k}{k!}\, {}_2F_1\left(-k, k+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right), \tag{2.3.7}$$

where the generalized hypergeometric series is defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} \cdot \frac{t^k}{k!}.$$

Actually, Kogbetliantz [1] proved the following equivalent inequality

$$\sum_{k=0}^{n} \frac{(2\lambda+2)_{n-k}(k+\lambda)}{(n-k)!\lambda} C_k^{\lambda}(t) \geq 0 \quad (-1 \leq t \leq 1,\ \lambda > 0). \tag{2.3.8}$$

We note

$$C_k^{\lambda}(t) = \frac{(2\lambda)_k}{(\lambda+\frac{1}{2})_k} P_k^{(\alpha,\alpha)}(t), \qquad \alpha = \lambda - 1/2.$$

The limit case of (2.3.8), when $\lambda \to 0$, gives the Fejér kernel $F_n(x)$, $t = \cos x$, defined by (1.2.11).

Using the inequality (2.3.3), Fejér [2] proved a special case of (2.3.8), when $\lambda = 1/2$, i.e.,

$$\sum_{k=0}^{n} \frac{(3)_{n-k}}{(n-k)!}(2k+1)P_k(t) \geq 0 \qquad (-1 \leq t \leq 1),$$

where P_k is Legendre polynomial of degree k.

For $\lambda = 1$, the inequality (2.3.8) reduces to the following inequality

$$\sum_{k=0}^{n} \frac{(4)_{n-k}}{(n-k)!}(k+1)\sin(k+1)x \geq 0 \qquad (0 \leq x \leq \pi),$$

because of $C_k^1(t) = \sin((k+1)x)/\sin x$, $t = \cos x$. The sign $\geq$ in this inequality can be replaced by $>$, if $0 < x < \pi$. This inequality was proved by Fejér [12].

Also, Fejér [13] proved (2.3.6) when $\alpha = -\beta = 1/2$, $\gamma = 2$, i.e.,

$$\sum_{k=0}^{n} \frac{(3)_{n-k}}{(n-k)!}\left(k+\tfrac{1}{2}\right)\sin\left(k+\tfrac{1}{2}\right)x > 0 \qquad (0 \leq x \leq \pi). \tag{2.3.9}$$

We note that

$$\frac{P_n^{(1/2,-1/2)}(t)}{P_n^{(-1/2,1/2)}(1)} = \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin(x/2)}, \quad t = \cos x.$$

A weaker result than (2.3.9), i.e.,

$$\sum_{k=0}^{n} \frac{(3)_{n-k}}{(n-k)!}(k+1)\sin\left(k+\tfrac{1}{2}\right)x > 0 \qquad (0 < x < \pi),$$

was proved by Robertson [2]. He used it to prove a theorem on univalent functions.

Ruscheweyh [1] proved the following result:

Theorem 2.3.3. *Let $\lambda \geq m/2$, $m \in \mathbb{N}$. Let $a_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$, satisfy*

$$1 = a_0 \geq a_1 \geq \cdots \geq a_n \geq 0.$$

Then for $-1 < t < 1$ we have

$$\sum_{k=0}^{n} a_k \frac{C_{km}^{\lambda}(t)}{C_{km}^{\lambda}(1)} z^k \neq 0, \qquad |z| \leq 1. \tag{2.3.10}$$

The case $\lambda = 1/2$, $a_k = 1$ $(k = 0, 1, \ldots, n)$ reduces to the well-known Szegö result [2].

Lewis [1] extended the case $a_k = 1$ to

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} z^k \neq 0, \qquad |z| < 1,$$

for $-1 \leq t \leq 1$ when $0 \leq \lambda \leq \alpha + \beta$ and $\beta \geq \alpha$.

The following result is contained in Theorem 2.3.3.

Corollary 2.3.4. *Let $\lambda \geq m/2$ and $m \in \mathbb{N}$. Then for $-1 < t < 1$ we have*

$$\sum_{k=0}^{n} \frac{C_{km}^{\lambda}(t)}{C_{km}^{\lambda}(1)} > 0.$$

For $m = 1$, this result reduces to Theorem 2.3.2.

2.4. Completely Monotonic Functions

Definition 2.4.1. *If $x \mapsto f(x)$, $x > 0$, is the Laplace transform of a nonnegative measure, i.e.,*

$$f(x) = \int_0^{\infty} e^{-xt}\, d\mu(t), \qquad d\mu(t) \geq 0, \tag{2.4.1}$$

then f is called completely monotonic on $(0, \infty)$.

An equivalence of (2.4.1) with

$$(-1)^n \frac{d^n}{dx^n} f(x) \geq 0, \qquad x > 0,\ n = 0, 1, \ldots,$$

is given by the Hausdorff-Bernstein-Widder theorem (see Widder [1]).

For example, the function $x \mapsto x^{-\alpha}$, $\alpha > 0$, is completely monotonic since

$$x^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-xt} t^{\alpha-1}\, dt, \qquad \alpha > 0.$$

Alternatively, for $x > 0$ and $n = 0, 1, \ldots$,

$$(-1)^n \frac{d^n}{dx^n} \left(x^{-\alpha}\right) = (\alpha)_n x^{-(\alpha+n)} > 0, \qquad \alpha > 0.$$

Askey and Pollard [1] proved the following result:

Theorem 2.4.1. *If λ is a real number, then $x \mapsto x^{-2|\lambda|}(x^2+1)^{-\lambda}$ are completely monotonic functions for $x > 0$, i.e.,*

$$\frac{1}{x^{2|\lambda|}(x^2+1)^{\lambda}} = \int_0^{\infty} e^{-xt}\, d\mu_\lambda(t), \qquad x > 0, \quad d\mu_\lambda(t) \geq 0.$$

This theorem follows immediately from the following theorem of Schoenberg [1] (see Askey and Pollard [1]):

Theorem 2.4.2. *A function $x \mapsto f(x)$, $x \geq 0$, with $f(0) = 1$ has the property that $x \mapsto f(x)^\lambda$ is completely monotonic for $x \geq 0$ and all $\lambda > 0$ if and only if*

$$f(x) = \exp\left(-\int_0^x g(t)\, dt\right),$$

where $t \mapsto g(t)$ is a completely monotonic function.

It is enough to identify $g(x)$ as a completely monotonic function for $f(x) = f_\varepsilon(x) = 1/x^2(x^2+1)^\varepsilon$, $\varepsilon = \pm 1$. Indeed, we have

$$g(x) = -\frac{d}{dx}(\log f(x)) = 2\left(\frac{1}{x} + \varepsilon\,\frac{x}{x^2+1}\right),$$

i.e.,

$$g(x) = 2\int_0^{\infty} e^{-xt}(1 + \varepsilon \cos t)\, dt.$$

Now, consider another example. Using

$$x(x^2+1)^{-\alpha-3/2} = \frac{2^{-\alpha-1}\Gamma(1/2)}{\Gamma(\alpha+3/2)} \int_0^{\infty} e^{-xt} t^{\alpha+1} J_\alpha(t)\, dt, \quad \alpha > -1,$$

where J_α is the Bessel function of the first kind, defined by

$$J_\alpha(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+\alpha}}{n!\Gamma(n+\alpha+1)},$$

we have (see Gasper [2])

$$\int_0^{\infty} e^{-xt}\left(\int_0^t (t-\xi)^{\alpha+3/2}\xi^{\alpha+1} J_\alpha(\xi)\, d\xi\right) dt = \frac{2^{\alpha+1}\Gamma(\alpha+3/2)\Gamma(\alpha+5/2)}{\Gamma(1/2)x^{\alpha+3/2}\,(x^2+1)^{\alpha+3/2}},$$

and thus we conclude that the following inequality

$$\int_0^t (t-\xi)^{\alpha+3/2}\xi^{\alpha+1} J_\alpha(\xi)\, d\xi \geq 0, \qquad t > 0, \tag{2.4.2}$$

is equivalent to the complete monotonicity of the function

$$x \mapsto \frac{1}{x^{\alpha+3/2}\,(x^2+1)^{\alpha+3/2}}.$$

In the case when $\alpha = -1/2$, the complete monotonicity is a consequence of the formula

$$\frac{1}{x(x^2+1)} = \int_0^\infty e^{-xt}(1-\cos t)\,dt.$$

Using the above property and the fact that the product of completely monotonic functions is completely monotonic, Askey [4] proved:

Theorem 2.4.3. *For $k = 1, 2, \ldots$, the function*

$$x \mapsto \frac{1}{x^k(x^2+1)^k}$$

is completely monotonic.

This suggest that the function

$$x \mapsto \frac{1}{x^c(x^2+1)^c} \tag{2.4.3}$$

is also completely monotonic for $c \geq 1$, i.e. that the inequality (2.4.2) holds for $\alpha \geq -1/2$. Here $c = \alpha + 3/2$.

Fields and Ismail ([1], [2]) proved this by applying an asymptotic argument of Darboux type to an integral representation for a ${}_1F_2$. They first proved (2.4.2) for $1/2 \leq \alpha \leq 1/2$ and then used the multiplicative property of completely monotonic functions to prove this result for $\alpha > 1/2$.

Gasper [3] found another proof of (2.4.2). By an integration by parts,

$$\int_0^t (t-\xi)^{\alpha+3/2}\xi^{\alpha+1}J_\alpha(\xi)\,d\xi = \left(\alpha+\frac{3}{2}\right)\int_0^t (t-\xi)^{\alpha+1/2}\xi^{\alpha+1}J_{\alpha+1}(\xi)\,d\xi,$$

Gasper considered this problem in the equivalent form

$$\int_0^t (t-\xi)^{\alpha-1/2}\xi^\alpha J_\alpha(\xi)\,d\xi \geq 0, \quad \alpha \geq 1/2,\, t > 0,$$

where c in (2.4.3) is equal to $\alpha + 1/2$.

Expanding this integral as a sum of squares of Bessel functions with nonnegative coefficients, Gasper obtained

$$\int_0^t (t-\xi)^{\alpha-1/2}\xi^\alpha J_\alpha(\xi)\,d\xi = At^{\alpha+1/2}\sum_{n=0}^\infty a_n J_{n+\alpha}^2(t/2),$$

where

$$A = \frac{2^{3\alpha}\Gamma(\alpha+1/2)\Gamma(2\alpha+1)\Gamma(\alpha+1)}{\Gamma(3\alpha+3/2)}$$

and

$$a_n = \frac{((2\alpha+1)/4)_n((2\alpha-1)/4)_n}{((6\alpha+3)/4)_n((6\alpha+5)/4)_n} \cdot \frac{(2\alpha+1)_n}{n!} \cdot \frac{2n+2\alpha}{n+2\alpha}, \quad n \geq 0.$$

Thus, (2.4.4) is true.

Gasper [**3**] extended this case to

$$\int_0^t (t-\xi)^{\lambda-1/2}\xi^\lambda J_\alpha(\xi)\,d\xi \geq 0, \quad 1/2 \leq \lambda \leq \alpha,\ \alpha \geq 1/2,\ t > 0.$$

A more general inequality

$$\int_0^t (t-\xi)^{\alpha+2\mu-1/2}\xi^{\alpha+\mu} J_\alpha(\xi)\,d\xi \geq 0, \qquad t > 0, \tag{2.4.5}$$

for $0 \leq \mu \leq 1$, $\alpha + \mu \geq 1/2$, was conjectured by Gasper [**3**].

In [**4**] Gasper proved this conjecture:

Theorem 2.4.4. *If* $0 \leq \mu \leq 1$ *and* $\alpha+\mu \geq 1/2$, *then inequality* (2.4.3) *holds. The equality occurs when* $\mu = 0$, $\alpha = -1/2$ *or* $\mu = 1$, $\alpha = -1/2$.

2.5. Absolutely Monotonic Functions

In this section we will consider the absolute monotonicity of functions. A function is absolutely monotonic if its power series has nonnegative coefficients.

As we mentioned in Section 2.3, the statement of Theorem 2.3.1 can be expressed as the power series

$$\frac{1}{(1-z)(1-2tz+z^2)^\lambda} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} C_k^\lambda(t)\right) z^n \qquad (-1 < t < 1),$$

with positive coefficients for $0 < \lambda \leq 1/2$.

One extension of this result was proved by Askey and Pollard [**1**].

Theorem 2.5.1. *The function* $z \mapsto \varphi(z) = (1-z)^{-2\lambda}(1-2tz+z^2)^{-\lambda}$ *has positive power series coefficients for* $-1 < t < 1$, $\lambda > 0$.

PROOF. Letting $t = \cos x$ and

$$g(z) = \log\varphi(z) = -2\log(1-z) - \log(1-2z\cos x + z^2),$$

we have

$$\begin{aligned} g'(z) &= \frac{2}{1-z} + \frac{e^{ix}}{1-ze^{ix}} + \frac{e^{-ix}}{1-ze^{-ix}} \\ &= \sum_{n=0}^{\infty} \left(2 + e^{i(n+1)x} + e^{-i(n+1)x}\right) z^n \\ &= \sum_{n=0}^{\infty} \left(4\cos^2 \frac{(n+1)x}{2}\right) z^n \end{aligned}$$

i.e., $z \mapsto g'(z)$ is an absolutely monotonic function. Since $g(0) = 0$, we conclude that $z \mapsto g(z)$ is also absolutely monotonic, and hence so is

$$\varphi(z) = e^{\lambda g(z)} = \sum_{n=0}^{\infty} \frac{\lambda^n g(z)^n}{n!},$$

for $\lambda > 0$. □

This theorem is equivalent to

$$\sum_{k=0}^{n} \frac{(2\lambda)_{n-k}}{(n-k)!} C_k^{\lambda}(t) > 0 \qquad (-1 < t < 1,\ \lambda > 0). \tag{2.5.1}$$

As in Theorem 2.4.1, there is a second result of this type. They can be stated together as:

Theorem 2.5.1′. *If λ is a real number then the function $z \mapsto (1-z)^{-2|\lambda|}(1-2tz+z^2)^{-\lambda}$ is absolutely monotonic for $-1 \le t \le 1$.*

When $\lambda = 2$, there is a different extension of this theorem (Askey [2]):

Theorem 2.5.2. *The function $z \mapsto (1-zt)(1-z)^{-3}(1-2zt+z^2)^{-2}$ is absolutely monotonic for $-1 \le t \le 1$.*

This theorem can be considered as a consequence of other extensions of Theorem 2.5.1 when $\lambda = 2$, but these extensions are only partial extensions, since these do not hold for $-1 \le t \le 1$, but only for part of this interval. The first result was proved by Schweitzer [1] (see Theorem 1.2.12). Another way to state his result is the following (Askey and Fitch [4]):

Theorem 2.5.3. *The function $z \mapsto (1+z)(1-z)^{-2}(1-2zt+z^2)^{-2}$ is absolutely monotonic for $-1/2 \le t \le 1$.*

Askey and Fitch [4] proved also the following similar result:

Theorem 2.5.4. *The function* $z \mapsto (1-z)^{-2}(1-2zt+z^2)^{-2}$ *is absolutely monotonic for* $0 \le t \le 1$.

Putting $t = \cos x$, we can see that this result is equivalent to the inequality

$$\sum_{k=0}^{n}\left(\cos\frac{1}{2}x - \cos\left(k+\frac{3}{2}\right)x\right)\left(\cos\frac{1}{2}x - \cos\left(n-k+\frac{3}{2}\right)x\right) \ge 0 \quad (0 < x \le \pi/2).$$

Since

$$\frac{1-z\cos x}{(1-z)^3(1-2zt+z^2)^2} = \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \frac{(3)_{n-k}}{(n-k)!}\cdot\frac{(3)_k}{k!}\cdot\frac{\sin(k+1)x}{(k+1)\sin x}\right),$$

Askey [2] obtained the following result

$$\sum_{k=0}^{n} \frac{(3)_{n-k}}{(n-k)!}\cdot\frac{(3)_k}{k!}\cdot\frac{\sin(k+1)x}{k+1} \ge 0 \qquad (0 \le x \le \pi), \tag{2.5.2}$$

which is equivalent to Theorem 2.5.2.

Also, Askey [2] proved the following result:

Theorem 2.5.5. *The function* $z \mapsto (1-zt)^2(1-z)^{-2}(1-2zt+z^2)^{-2}$ *is absolutely monotonic for* $-1 \le t \le 0$.

PROOF OF THEOREM 2.5.2. Let $0 \le t \le 1$. Then

$$\frac{1-zt}{(1-z)^3(1-2zt+z^2)^2} = \frac{1-zt}{1-z}\cdot\frac{1}{(1-z)^2(1-2zt+z^2)^2}.$$

The second factor on the right is absolutely monotonic by Theorem 2.5.4, with $t = \cos x$, and the first factor is also absolutely monotonic, since it is

$$\frac{1-zt}{1-z} = 1 + \sum_{n=1}^{\infty}(1-t)z^n.$$

Since the product of two absolutely monotonic functions is absolutely monotonic, we see that Theorem 2.5.2 holds for $0 \le t \le 1$.

Let now $-1 \le t \le 0$ and

$$\frac{1-zt}{(1-z)^3(1-2zt+z^2)^2} = \frac{1}{(1-zt)(1-z)}\cdot\frac{(1-zt)^2}{(1-z)^2(1-2zt+z^2)^2}.$$

By Theorem 2.5.5, the second factor on the right side in the above equality is absolutely monotonic and the first factor is

$$\frac{1}{(1-zt)(1-z)} = \sum_{n=0}^{\infty}\left(\frac{1-t^{n+1}}{1-t}\right)z^n,$$

and so it is absolutely monotonic. □

Proof of Theorem 2.5.5 is simple except when t is close to zero (see Askey [3]).

The sum (2.3.8) has the relatively simple generating function, namely,

$$\sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \frac{(2\lambda+2)_{n-k}(k+\lambda)}{(n-k)!\lambda} C_k^{\lambda}(x) \right) = G_\lambda(z,t),$$

where

$$G_\lambda(z,t) = \frac{1-z^2}{(1-z)^{2\lambda+2}(1-2tz+z^2)^{\lambda+1}}. \tag{2.5.3}$$

The inequality (2.3.8) can be interpreted in terms of absolutely monotonic functions.

Theorem 2.5.6. *The functions $z \mapsto G_\lambda(z,t)$, defined by* (2.5.3) *are absolutely monotonic for $-1 \le t \le 1$, $\lambda > -1/2$.*

PROOF. We will use the statement of Theorem 2.5.1. Since

$$G_\lambda(z,t) = G_0(z,t)\varphi(z)$$

and $z \mapsto \varphi(z)$ is absolutely monotonic, it is enough to prove that

$$G_0(z,t) = \frac{1+z}{(1-z)(1-2zt+z^2)}$$

is also absolutely monotonic.

Letting $t = \cos x$, we have

$$\frac{1+z}{(1-2zt+z^2)} = (1+z)\sum_{k=0}^{\infty} C_k^1(t)z^k = 2\sum_{k=0}^{\infty} D_k(x)z^k,$$

because of

$$C_k^1(\cos x) + C_{k-1}^1(\cos x) = \frac{\sin(k+1)x}{\sin x} + \frac{\sin kx}{\sin x} = \frac{\sin(k+1/2)x}{\sin x} = 2D_k(x).$$

Using Fejér kernel (1.2.11), we obtain

$$G_0(z,t) = \frac{2}{1-z}\sum_{k=0}^{\infty} D_k(x)z^k = 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} D_k(x)\right) z^n,$$

i.e.,

$$G_0(z,t) = 2(n+1)\sum_{n=0}^{\infty} F_n(x)z^n,$$

which concludes the proof. □

This shows that Kogbetliantz's asymptotic argument can be replaced by a very easy proof.

In [1] Askey and Gasper found several absolutely monotonic functions:

Theorem 2.5.7. *If $\alpha > 0$ or $\alpha < 0$, then the functions*

$$z \mapsto (1-z)^{-|\alpha|}\left(1 \pm z + (1-2zt+z^2)^{1/2}\right)^{\alpha}$$

have positive power series coefficients for $-1 < t < 1$ when they are expanded in power series in z.

Theorem 2.5.8. *The functions*

$$z \mapsto (1-z)^{-|\alpha|}\left(1 - tz + (1-2zt+z^2)^{1/2}\right)^{\alpha}$$

and

$$z \mapsto (1-z)^{-|\alpha|}\left(z - t + (1-2zt+z^2)^{1/2}\right)^{\alpha}$$

are absolutely monotonic for $-1 \le t < 1$, α a real number.

One stronger result is the following:

Theorem 2.5.9. *For $\alpha > 0$ the functions*

$$z \mapsto (1-z)^{-3\alpha/4}\left(1 - z + (1-2zt+z^2)^{1/2}\right)^{-\alpha}$$

are absolutely monotonic for $-1 \le t \le 1$.

Theorem 2.5.10. *The functions*

$$f_\lambda(z) = \frac{\operatorname{Im}(1-ze^{ix})^{-\lambda}}{\lambda z(1-z)^{\lambda+1}} \qquad (-1 < \lambda \le 1,\ \lambda \ne 0)$$

and

$$f_0(z) = \lim_{\lambda\to 0} f_\lambda(z) = \frac{1}{z(1-z)} \arctan \frac{z\sin x}{1 - z\cos x}$$

have positive power series coefficients for $0 < x < \pi$.

Askey and Gasper [1] obtained the following formulas:

1° For $\lambda \ne 0$, we have

$$\begin{aligned} f_\lambda(z) &= \frac{1}{\lambda z(1-z)^{\lambda+1}} \sum_{k=0}^{\infty} z^{k+1} \frac{(\lambda)_{k+1}}{(k+1)!} \sin(k+1)x \\ &= \sum_{k=0}^{\infty} z^k \frac{(\lambda+1)_k}{(k+1)!} \sin(k+1)x \sum_{n=k}^{\infty} \frac{(\lambda+1)_{n-k}}{(n-k)!} z^{n-k} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{\sin(k+1)x}{k+1} \right) z^n. \end{aligned}$$

Using the limit case $\lambda \to 0$ we conclude that

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{\sin(k+1)x}{k+1} > 0 \qquad (0 < x < \pi) \tag{2.5.4}$$

for every λ such that $-1 < \lambda \leq 1$.

2° Express $f_\lambda(z)$ in the form

$$f_\lambda(z) = \frac{1}{\lambda z(1-z)} \operatorname{Im}\left[(1-z)(1-ze^{ix})\right]^{-\lambda}.$$

If $t = ze^{ix/2}$ then

$$\begin{aligned} f_\lambda(z) &= \frac{1}{\lambda z(1-z)} \operatorname{Im}\left(1 - 2t\cos\frac{x}{2} + t^2\right)^{-\lambda} \\ &= \frac{1}{\lambda z(1-z)} \operatorname{Im} \sum_{n=0}^{\infty} C_n^\lambda\left(\cos\frac{x}{2}\right) t^n \\ &= \frac{1}{\lambda(1-z)} \operatorname{Im} \sum_{n=0}^{\infty} C_n^\lambda\left(\cos\frac{x}{2}\right) e^{inx/2} z^{n-1} \\ &= \frac{1}{\lambda(1-z)} \operatorname{Im} \sum_{n=0}^{\infty} C_n^\lambda\left(\cos\frac{x}{2}\right) e^{inx/2} z^{n-1} \\ &= \frac{1}{\lambda(1-z)} \sum_{n=0}^{\infty} \left(C_{n+1}^\lambda\left(\cos\frac{x}{2}\right) \sin\frac{(n+1)x}{2}\right) z^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{\lambda} \sum_{k=0}^{n} C_{k+1}^\lambda\left(\cos\frac{x}{2}\right) \sin\frac{(k+1)x}{2}\right) z^n. \end{aligned}$$

Hence

$$\frac{1}{\lambda} \sum_{k=0}^{n} C_{k+1}^\lambda\left(\cos\frac{x}{2}\right) \sin\frac{(k+1)x}{2} > 0 \qquad (0 < x < \pi),$$

for $-1 < \lambda \leq 1$. As λ approaches 0 this inequality reduces to Fejér-Gronwall-Jackson inequality (1.2.1).

Remark 2.5.1. This method is due to Turán [3].

The inequality (2.5.4) is also valid when $1 \leq \lambda \leq 2$. This follows from a combination of results of Askey [3] and Bustoz and Savage [1].

Bustoz and Ismail [1] established the same inequality for $2 < \lambda \leq 4$ when $\pi/3 \leq x < \pi$.

2.6. Monotonicity of Some Trigonometric Sums

Askey and Steinig [2] proved the following result on monotonicity of the trigonometric sum

$$T_n(x) = \frac{S_n(x)}{\sin(x/2)} \qquad (0 < x < \pi), \tag{2.6.1}$$

where $S_n(x)$ is the Fejér sum given by (1.1.2).

Theorem 2.6.1. *For any positive integer n, we have*

$$\frac{d}{dx} T_n(x) < 0 \qquad (0 < x < \pi), \tag{2.6.2}$$

where $T_n(x)$ is given by (2.6.1).

For $n = 1$ as well as for $n = 2$, with $0 < x < \pi$, we have

$$T_1'(x) = -\sin\frac{x}{2} < 0 \qquad \text{and} \qquad T_2'(x) = -6\sin\frac{x}{2}\cos^2\frac{x}{2} < 0,$$

respectively.

The proof of Askey and Steinig uses the following facts:

1° Since $S_n(x) > 0$ $(0 < x < \pi)$,

$$\frac{dS_n(x)}{dx} = \sum_{k=1}^{n} \cos kx = \sin\frac{nx}{2}\cos\frac{(n+1)x}{2} \Big/ \sin\frac{x}{2},$$

and

$$\sin\left(n + \frac{1}{2}\right)x - \sin\frac{x}{2} = 2\sin\frac{nx}{2}\cos\frac{(n+1)x}{2},$$

the inequality (2.6.2) is equivalent to

$$g_n(x) > 0 \qquad (0 < x < \pi), \tag{2.6.3}$$

where

$$g_n(x) = S_n(x)\cos\frac{x}{2} + \sin\frac{x}{2} - \sin\left(n + \frac{1}{2}\right)x.$$

2° Using (1.1.3), the function g_n can also be written in the form

$$g_n(x) = \frac{1}{2}(\pi - x)\cos\frac{x}{2} + \sin\frac{x}{2} - \sin\left(n + \frac{1}{2}\right)x - \cos\frac{x}{2}\sum_{k=n+1}^{\infty} \frac{\sin kx}{x}.$$

Then, a summation by parts and the classical inequality

$$\left| \sum_{k=M}^{N} \sin kx \right| \le \left(\sin\frac{x}{2}\right)^{-1} \qquad (0 < x < \pi),$$

yield

$$g_n(x) \ge \frac{\pi - x}{2}\cos\frac{x}{2} + \sin\frac{x}{2} - 1 - \frac{1}{n+1}\cot\frac{x}{2} \qquad (0 < x < \pi).$$

3° For $n \ge 3$ the inequality (1.2.15), i.e.,

$$S_n(x) \ge \frac{1}{3}\sin x + \frac{1}{2n}\sin nx \qquad (0 < x < \pi)$$

holds.

4° The inequality (2.6.3) follows from the well-known Fejér-Gronwall-Jackson inequality (1.2.1), i.e., $S_n(x) > 0 \quad (0 < x < \pi)$, on those subintervals of $(0, \pi)$, where

$$\sin\frac{x}{2} - \sin\left(n + \frac{1}{2}\right)x \ge 0.$$

This is the case if $x < \pi$ and

$$\frac{(2m+1)\pi}{n+1} \le x \le \frac{2(m+1)\pi}{n},$$

for some m, $0 \le m \le [(n+1)/2]$.

Using the above mentioned facts and considering separately three intervals $(0, 2\pi/n)$, $[2\pi/n, 2\pi/3]$, and $(2\pi/3, \pi)$, Askey and Steinig [2] gave the proof of Theorem 2.6.1.

The inequality (2.6.2) is equivalent to a special case ($\lambda = 0$, $\alpha = 3/2$, $\beta = -1/2$) of the following inequality

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} > 0 \quad (-1 < t \le 1), \tag{2.6.4}$$

which was conjectured by Askey and Gasper [1], for $0 \le \lambda \le \alpha + \beta$, $\beta \ge -1/2$, except when $\lambda = 0$, $\alpha = -\beta = 1/2$, when the sum is nonnegative and there are cases of equality. It was shown that this conjecture holds for $\beta \ge \alpha$, for $|\beta| \le \alpha \le \beta + 1$, for $0 \le \lambda \le \beta$, and for some other special cases. More details about (2.6.4) will be given in the next section.

Using (2.6.4), with $\alpha = 3/2$, $\beta = -1/2$, $0 \le \lambda \le 1$, Gasper [4] proved a more general result than (2.6.2). Namely, he showed that for $0 \le \lambda \le 1$, the inequality

$$\frac{d}{dx}\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{\sin(k+1)x}{(k+1)\sin(x/2)} < 0 \qquad (0 < x < \pi)$$

is true. It is stronger than (2.5.4).

2.7. Positivity of Some Jacobi Polynomial Sums

In this section we will give several general results on positivity sums, which include Jacobi orthogonal polynomials. The basic results on this area belong to Askey [**1–7**], Askey and Gasper [**1–2**] and Gasper [**2–5**].

Feldheim's result given in Theorem 2.3.2 is

Theorem 2.7.1. *If*

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\alpha,\alpha)}(t)}{P_k^{(\alpha,\alpha)}(1)} \geq 0 \quad (-1 \leq t \leq 1,\ \alpha > -1),$$

then

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\beta,\beta)}(\xi)}{P_k^{(\beta,\beta)}(1)} \geq 0 \quad (-1 \leq \xi \leq 1,\ \beta > \alpha).$$

Askey [**1**] gave another result of this type:

Theorem 2.7.2. *If*

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} \geq 0 \quad (-1 \leq t \leq 1,\ \alpha,\beta > -1),$$

then

$$\sum_{k=0}^{n} a_k \frac{P_k^{(\alpha-\mu,\beta+\nu)}(\xi)}{P_k^{(\beta+\nu,\alpha-\mu)}(1)} \geq 0 \quad (-1 \leq \xi \leq 1,\ 0 \leq \mu \leq \nu).$$

As an application Askey [**1**] showed the following result:

Theorem 2.7.3. *If* $\alpha \geq \beta \geq -1/2$ *and*

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} \geq 0,$$

then

$$\sum_{k=0}^{n} a_k r^k P_k^{(\alpha,\beta)}(t) P_k^{(\alpha,\beta)}(\xi) \geq 0 \qquad (-1 \leq t,\xi \leq 1), \tag{2.7.1}$$

for $0 \leq r \leq 1/(\alpha+\beta+3)$, *where*

$$a_k = \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}.$$

The inequality (2.7.1) *fails for* $n = 1,\ t = -1,\ \xi = 1$ *if* $r > 1/(\alpha+\beta+3)$.

Later, conditions were given so that the conditional assumption is true. The case $\alpha = \beta = 0$ was considered by Szegö [**1**].

As an extension of Feldheim's result (2.3.2), the most useful one is

Theorem 2.7.4. *The inequality*

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} > 0 \qquad (-1 < t \le 1) \tag{2.7.2}$$

holds when (α, β) *satisfies*

$$\alpha + \beta \ge -2,\ \beta \ge 0 \qquad \text{or} \qquad \alpha + \beta \ge 0,\ \beta \ge -\frac{1}{2},$$

except when $\alpha = -2$, $\beta = 0$ *and* $n = 1$, *or* $t = 1$ *and* $n \ge 2$, *or when* $\alpha = 1/2$, $\beta = -1/2$.

This was obtained in steps, first some easy cases by Askey [1], then the case $\beta \ge 0$ by Askey and Gasper [1], and finally the hardest case $\beta \ge -1/2$ by Gasper [4]. The essential cases are $\alpha \ge -2$, $\beta = 0$ and $\alpha \ge 1/2$, $\beta = -1/2$. The remaining cases follow from Bateman's integral formula ([1])

$$\frac{P_k^{(\alpha-\mu,\beta+\mu)}(\xi)}{P_k^{(\beta+\mu,\alpha-\mu)}(1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^{\xi} \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} \cdot \frac{(1+t)^{\beta}}{(1+\xi)^{\beta+\mu}} (\xi - t)^{\mu-1}\, dt.$$

Askey and Gasper [1] considered three cases when $\beta = -1/2$ in the inequality (2.7.2). If $\alpha = -\beta = 1/2$ one has the inequality of Fejeér

$$\sum_{k=0}^{n} \sin\left(k + \frac{1}{2}\right) x = \frac{1 - \cos(n+1)x}{2\sin(x/2)} \ge 0 \qquad (0 < x < \pi).$$

If $\alpha = 3/2, \beta = -1/2$ and $-1 < t < 1$, the inequality (2.7.2) is equivalent to an inequality of the form (2.6.2). If $\alpha = 5/2, \beta = -1/2$, then the corresponding inequality is equivalent to

$$(n+1)\frac{\sin(n-1)x}{\sin x} - (n-1)\frac{\sin(n+1)x}{\sin x} \le (3 + \cos x)\left(n - \frac{\sin nx}{\sin x}\right)$$

for $0 < x < \pi$. This is stronger than an inequality of Robertson [1], in which on the right side we have factor 4 instead of $3 + \cos x$.

The remaining cases $\alpha > 1/2$, $\beta = -1/2$ were proved by Gasper [4].

Remark 2.7.1. Inequality (2.7.2) does not hold for $\beta < -1/2$ or for $\alpha + \beta < -2$.

Since the case $\beta = 0$ has been the most useful so far, an outline of the argument of Askey and Gasper [1] follows:

Using the hypergeometric representation (2.3.7) they obtained

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(t) = \sum_{k=0}^{n} \frac{(\alpha+1)_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j (k+\alpha+1)_j}{j!(\alpha+1)_j} \left(\frac{1-t}{2}\right)^j$$

$$= \sum_{j=0}^{n} \frac{(\alpha+1)_{2j}}{j!(\alpha+1)_j} \left(\frac{t-1}{2}\right)^j \sum_{k=0}^{n-j} \frac{(2j+\alpha+1)_k}{k!}.$$

Applying $(2a)_{2j} = 2^{2j}(a)_j(a+1/2)_j$ and $\sum_{k=0}^{n}(a)_k/k! = (a+1)_n/n!$, it follows that

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(t) = \frac{(\alpha+2)_n}{n!}\,{}_3F_2\left(-n, n+\alpha+2, \frac{\alpha+1}{2};\ \alpha+1, \frac{\alpha+3}{2};\ \frac{1-t}{2}\right).$$

Using Euler's beta integral, a formula of Gegenbauer which expresses an ultraspherical polynomial as a sum of another ultraspherical polynomial with positive coefficients when the parameters inside the sum is lower than the other one, and Clausen's formula expressing a special ${}_3F_2$ as the square of a ${}_2F_1$, they obtained

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(t) = \sum_{j=0}^{[n/2]} A_j(n,\alpha)\left(C_{n-2j}^{(\alpha+1)/2}\left(\left(\tfrac{1+t}{2}\right)^{1/2}\right)\right)^2,$$

where

$$A_j(n,\alpha) = \frac{\left(\frac{1}{2}\right)_j\left(\frac{\alpha+2}{2}\right)_{n-j}\left(\frac{\alpha+3}{2}\right)_{n-2j}(n-2j)!}{j!\left(\frac{\alpha+3}{2}\right)_{n-j}\left(\frac{\alpha+1}{2}\right)_{n-2j}(\alpha+1)_{n-2j}}$$

and $C_k^{\lambda}(t)$ is the ultraspherical polynomial. Using this identity Askey and Gasper [1] proved:

Theorem 2.7.5. *If $\alpha \geq -2$, then the inequality*

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(t) \geq 0 \qquad (-1 < t \leq 1)$$

holds. The equality is achieved only when $\alpha = -2$ and either $n = 1$ or $t = 1$, $n \geq 1$. However, if $\alpha < -2$ then $1 + P_1^{(\alpha,0)}(t) = (\alpha+2)(1+t)/2 < 0$, if $t > -1$.

Much to their surprise (see Askey [8]), this inequality for $\alpha = 2, 4, \ldots$ was the final step in L. de Branges's remarkable proof of the Bieberbach conjecture, and even the stronger conjectures of Robertson, and Lebedev and Milin.

Remark 2.7.2. Let S denote the class of functions of the form

$$f(z) = z + c_2z^2 + c_3z^3 + \cdots,$$

which are analytic and univalent in the unit disk $|z| < 1$. In 1916, Bieberbach [1] conjectured that if $f \in S$, then

$$|c_n| \leq n \qquad (n = 2, 3, \ldots),$$

with equality holding only for rotations of the *Köbe function*

$$k(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + \cdots.$$

This conjecture was proved for $n = 2$ by Bieberbach in his paper [1], using the area principle which had just been proved by Gronwall [2]. The inequality for $n = 2$ led to sharp forms of Köbe's distortion and covering theorems. Löwner [1] introduced a representation of slit mappings in terms of a differential equation. The convergence theorem of Carathéodory proves that the slit mappings are dense in S. Löwner using Carathéodory's method verified Bieberbach conjecture for $n = 3$. For $n = 4$ the conjecture was proved by Garabedian and Schiffer [1], for $n = 5$ by Pederson and Schiffer [1], and for $n = 6$ by Pederson [1] and Ozawa [1]. L. de Branges [1], building an earlier idea of Löwner and introducing new ideas of his own, succeeded in reducing this conjecture to an inequality equivalent to that given above.

The inequality (2.5.1) can be stated for Jacobi sums. It is

$$\sum_{k=0}^{n} \frac{(2\gamma+1)_{n-k}}{(n-k)!} \cdot \frac{(2\gamma+1)_k}{k!} \cdot \frac{P_k^{(\gamma,\gamma)}(t)}{P_k^{(\gamma,\gamma)}(1)} > 0 \quad (-1 < t \le 1;\, \gamma > -1/2).$$

Bateman's integral suggests consideration of the sum

$$\sum_{k=0}^{n} \frac{(\alpha+\beta+1)_{n-k}}{(n-k)!} \cdot \frac{(\alpha+\beta+1)_k}{k!} \cdot \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)}. \tag{2.7.3}$$

More generally, consider

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} > 0 \quad (-1 < t \le 1), \tag{2.7.4}$$

which reduces to (2.7.2) for $\lambda = 0$ and to (2.7.3) for $\lambda = \alpha + \beta$.

This inequality was conjectured by Askey and Gasper [1] for $0 \le \lambda \le \alpha + \beta$, $\beta \ge -1/2$. In [1] they proved that if $\beta \ge \alpha$ and $-1 < \lambda \le \alpha + \beta$ then their conjecture holds, as well as it holds when $|\beta| \le \alpha \le \beta + 1$.

If $\alpha = \beta = 1/2$, $-1 < t < 1$, the inequality (2.7.4) reduces to Fejér-Gronwall-Jackson inequality (1.2.1), i.e.,

$$\sum_{k=0}^{n} \frac{\sin(k+1)x}{k+1} > 0 \qquad (0 < x < \pi)$$

when $\lambda = 0$, and to Lukács' inequality (cf. Fejér [8])

$$\sum_{k=0}^{n} (n+1-k)\sin(k+1)x > 0 \qquad (0 < x < \pi)$$

when $\lambda = 1$. If λ is between -1 and 1 one has to (2.5.4).

The inequality (2.7.4) can be extended to a large set of (α, β, λ). So, Gasper [4] proved the following results:

Theorem 2.7.6. *If $0 \le \lambda \le \alpha + \beta$, $\beta \ge -1/2$, then*

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\beta,\alpha)}(1)} \ge 0 \quad (-1 \le t \le 1), \tag{2.7.5}$$

and the only cases of equality occur when $t = -1$ for n odd and when $\lambda = 0$, $\alpha = -\beta = 1/2$.

Theorem 2.7.7. *Let $\beta \geq \alpha$, $\beta \geq \lambda > -1$ and $2\beta \geq \lambda \geq \beta - \alpha - 2$. Then inequality* (2.7.5) *holds and the only cases of equality occur when $t = -1$ for n odd, when $\alpha = -2$, $\beta = \lambda = 0$, $n = 1$, and when $\alpha = -2$, $\beta = \lambda = 0$, $t = 1$, $n \geq 1$.*

Theorem 2.7.8. *Let $\alpha > -1$, $\lambda > \max(-1, \beta - \alpha - 1)$, and either $-1 < \beta < -1/2$ or $-1 < \beta < 1/2$, $\lambda = \alpha + \beta + 1 > 0$. Then inequality* (2.7.5) *fails to hold, and the integral $\int_0^t (t-\xi)^\lambda \xi^{\lambda-\beta} J_\alpha(\xi)\, d\xi$, $t > 0$, changes sign infinitely often as $t \to \infty$.*

Gasper [4] also obtained similar inequalities for sums of Laguerre polynomials using

$$\lim_{\alpha \to \infty} P_k^{(\alpha,\beta)}(-1 + 2t/\alpha) = (-1)^k L_k^\beta(t)$$

and the fact that $P_k^{(\beta,\alpha)}(1) = L_k^\beta(0)$, where $L_k^\beta(t)$ is the generalized Laguerre polynomial. Namely, Gasper obtained the inequality

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \cdot \frac{(\lambda+1)_k}{k!} \cdot \frac{(-1)^k L_k^\beta(t)}{L_k^\beta(0)} > 0 \qquad (2.7.5)$$

for $t \geq 0$, which holds when $\beta \geq \lambda \geq -1/2$. This inequality is a limit case of (2.7.4).

Also, in [4] Gasper connected the inequality (2.7.4) with completly monotonic functions (cf. Section 2.4).

The reader who is interested in these results should first read Chapters 1, 8 and 9 in Askey [5], and then read Gasper [4].

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Gradimir V. Milovanović
Faculty of Electronic Engineering
Department of Mathematics
University of Niš
P. O. Box 73, 18000 Niš
YUGOSLAVIA

Themistocles M. Rassias
University of La Verne
Department of Mathematics
P. O. Box 51105
Kifissia, Athens
GREECE 145 10

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ORDERED GROUPS, COMMUTING MATRICES AND ITERATIONS OF FUNCTIONS IN TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS

František Neuman

ABSTRACT

This paper describes a topological structure of a certain group of iterations of functions. This problem arose in the transformation theory of differential equations. By contrast to analytic methods having mostly been used in this area , an algebraic approach is dominated here.

Key words: Ordered groups, matrices, iterations of functions, transformation of differential equations.

AMS classification: 06F15, 15A27, 26A18, 34C20

1. INTRODUCTION

The structure of an Ehresmann groupoid is given by stationary groups of its elements taken by one from each of its connected components. Each of these components is a Brandt groupoid and stationary groups of its elements are always conjugate. Linear differential equations of the nth order, $n \geq 2$, as objects and global transformations of them as morphisms form an Ehresmann groupoid. Each of its connected component is a set of globally equivalent equations. The stationary group of a linear differential equation is the set of all global transformations of this equation into itself, the group operation being a composition of these transformations. Characterization of these stationary groups can be reduced to description of certain groups of bijections of open intervals of reals onto themselves, see [5], [6] and also [8]. The present note shows how technically complicated proof in [7] can be substantially simplified by using purely algebraic approach involving linearly ordered group and commuting matrices.

2. NOTATION, DEFINITIONS AND SOME PRELIMINARY RESULTS

Let F denote a group of all functions $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(t) = \operatorname{Arctan} \frac{a \tan t + b}{c \tan t + d}, \tag{1}$$

$a,b,c,d \in \mathbf{R}$, $|ad - bc| = 1$, where Arctan for a particular t denotes this branch of $t \mapsto \arctan t + k$ that makes functions from F continuous. Each element f of F is a real analytic bijection of $\mathbf{R}$ onto $\mathbf{R}$, $f'(t) > 0$ on $\mathbf{R}$ exactly when $ad - bc = 1$. In accordance with O. Borůvka [1] we call the group F fundamental. This group is not such a special one as it may seem to be. In fact, it is (locally) conjugate to the three-parameter homographic group

$$x \mapsto \frac{ax + b}{cx + d}$$

that is, up to conjugacy, the most general Lie group transforming $\mathbf{R}$ onto $\mathbf{R}$ having finite number of parameters, see e.g. [3]. In this sense for this type of groups the fundamental group is a general representation which, in addition, has this nice property that it is real analytic on the whole $\mathbf{R}$.

Consider also the following groups whose elements are some functions from the fundamental group F or their restrictions to an open interval of reals:

F_1: all increasing elements of F, i.e. those with ad - - bc = 1;

F_2: $f : (0,\infty) \to (0,\infty)$,

$f(t) = \text{Arctan} \dfrac{a \tan t}{b \tan t + 1/a}$, $a \in (0,\infty)$, $b \in \mathbf{R}$;

F_{3m}: for each posivite integer m

$f : (0,m\pi) \to (0,m\pi)$,

$f(t) = \text{Arctan} \dfrac{a \tan t}{b \tan t + 1/a}$, $a \in (0,\infty)$, $b \in \mathbf{R}$;

F_{4m}: for each positive integer m

$f : (0,m\pi - \pi/2) \to (0,m\pi - \pi/2)$,

$f(t) = \text{Arctan}\,(a \tan t)$, $a \in (0,\infty)$.

Let F_1, F_2, F_{3m} and F_{4m} be equipped with the topology given by the relative topology on

$$\{(a,b,c,d) \in \mathbf{R}^4;\ ad - bc = 1\}$$

induced by the usual topology on $\mathbf{R}^4$.

Let G_1 and G_2 be two groups whose elements are (some) bijections of intervals I_1 and I_2 onto themselves, respectively. We say that the groups G_1 and G_2 are C^k-conjugate (with respect to ϕ) for some positive integer k if there is a C^k-diffeomorphism ϕ of I_1 onto I_2, i.e. $\phi(I_1) = I_2$, $\phi \in C^k(I_1)$, $d\phi(x)/dx \neq 0$ on I_1, such that

$$G_2 = \phi \circ G_1 \circ \phi^{-1} := \{\phi \circ f \circ \phi^{-1}, f \in G_1\}.$$

If G_1 is equipped by a topology, the topology on G_2 is induced by the conjugacy.

For an element α of a group and an integer n define

$\alpha^{[0]}$ to be the unit element of the group,

$\alpha^{[n]} := \alpha^{[n-1]} \circ \alpha$ for positive n, and

$\alpha^{[n]} := (\alpha^{-1})^{[-n]}$ for negative n,

α^{-1} being the inverse to α; call $\alpha^{[n]}$ the nth iterate of α.

A group is called cyclic if it admits an element α all iterates of which from the whole group. The elements of this property are called generators of the group. If, in addition, $m \neq n$ implies $\alpha^{[m]} \neq \alpha^{[n]}$, then the group is an infinite cyclic group.

A group is (partially of linearly) ordered if the set of its elements is (partially or linearly) ordered and for any triple of its elements α, β and γ the relation $\alpha \leq \beta$ implies both $\alpha \circ \gamma \leq \beta \circ \gamma$ and $\gamma \circ \alpha \leq \gamma \circ \beta$. An ordered group is called archimedean if the following implication holds:

"whenever $\alpha^{[n]} \leq \beta$ for some elements α and β and for all integers n, then α is the unit element of the group".

Proposition 1. (O. Hölder [2]): *For each linearly ordered archimedean group there exists an order preserving isomorphism into a subgroup of the additive group of the reals.*

Corollary. *Each linearly ordered archimedean group is commutative.*

Let $\mathbf{SL}_2$ denote the set of all 2 by 2 real matrices with the determinants equal to 1.

Proposition 2. *The Jordan canonical form of* $A \in \mathbf{SL}_2$ *is just one from the following four mutually exclusive cases:*

I. $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} = \pm I;$

II. $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$, $\lambda \neq 0$, $\lambda \neq \pm 1$, $\lambda \in \mathbf{R}$;

III. $\begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}$;

IV. $\begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix}$, $\lambda \in (0,\pi) \cup (\pi, 2\pi)$.

A matrix from $\mathbf{SL}_2$ *that commutes with a matrix in case:*

I. *is any matrix from* S ;

II. *is just any matrix of the form*

$$\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix} \text{ for } \mu \neq 0,\ \mu \in \mathbf{R};$$

III. *is exactly of the form*

$$\begin{pmatrix} \pm 1 & 0 \\ \mu & \pm 1 \end{pmatrix},\ \mu \in \mathbf{R};$$

IV. *is just any matrix of the form*

$$\begin{pmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{pmatrix},\ \mu \in \mathbf{R}.$$

Proof. For $A \in \mathbf{SL}_2$ having the real elements, I and III are characterized by a double (real) characteristic root ± 1 whereas case II occurs exactly when the roots are real and different. Cases I or III correspond to the rank 0 or 1 of the matrix $A \mp I$. In case IV the characteristic roots are not real, $\cos \lambda \pm i \sin \lambda$, the Jordan form of A in the complex domain is

$$\begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix} .$$

The form of matrices that commute with those ones introduced under cases I, II, III and IV follows immediately from [4] or can be obtained by a straightforward computation.

From now let G denote a group of some C^n-diffeomorphisms of an open interval $I \subset \mathbf{R}$ onto itself, n being a positive integer. Moreover, we always suppose that the graphs of different elements of G do not intersect each other (on I).

3. THEOREM

If G *is* C^n*-conjugate to a closed subgroup of the group* F_1*, or* F_2*, or* F_{3m}*, or* F_{4m}*, then either* G *is trivial,*

or G *is an infinite cyclic group with a generator* $h_e \in C^n(I)$, $dh_e(x)/dx > 0$ *and* $h_e(x) \neq x$ *on* I,

or G *is* C^n*-conjugate to the group of all translations* $\{h_c\colon \mathbf{R} \to \mathbf{R},\ h_c(x) = x + c;\ c \in \mathbf{R}\}$.

Proof of this theorem was given in [7]. However, it is rather lengthy an involves many technical details and analytic investigations. Here we present a rather simple proof, basicaly relying upon Holder's result and the explicit form of commuting matrices in $\mathbf{SL}_2$.

Due to the supposition that different elements of G do not intersect each other on I, group G can be linearly ordered in the following way:

for h_1, $h_2 \in G$ we write $h_1 \leq h_2$ if either $h_1(x_0) < h_2(x_0)$ for some (then any) $x_0 \in I$ or $h_1(x_0) = h_2(x_0)$ (then $h_1 = h_2$).

Moreover, G is archimedean, because for $h \in G$, $h \neq id_I$ (the unit element in G) we have $h(x) \neq x$ on I and hence the limits

$$\lim_{i\to+\infty} h^{[i]}(x_0) \qquad \text{and} \qquad \lim_{i\to-\infty} h^{[i]}(x_0)$$

converge to different ends of the definition interval I for an arbitrary $x_0 \in I$.

Thus, due to Proposition 1 there exists an order preserving isomorphism of G onto a subgroup $\tilde{G}$ of the additive group $\mathbf{R}$.

If G is trivial then $G = \{id_I\}$ and $\tilde{G} = \{0\}$.

Let G be not trivial and $\tilde{G} = \{ie;\ i \in \mathbb{Z}\}$ for a fixed $e \in \mathbb{R}$, $e \neq 0$, i.e. $\tilde{G}$ is an infinite cyclic group generated by a nonzero number e. Mark as h_e this element of G corresponding in the above isomorphism to the number e. Evidently $h_e \in C^n(I)$, $dh_e(x)/dx > 0$ and $h_e(x) \neq x$ on I. Moreover

$$G = \{h_e^{[i]};\ i \in \mathbb{Z}\}\ ,$$

h_e being a generator of the infinite group G.

From now, let G be not trivial, neither it be an infinite cyclic group. Hence there exist two of its elements, h_1 and h_2 that do not belong to the same infinite cyclic subgroup of G. Both h_1 and h_2 are C^n-conjugate (with respect to ϕ) to the elements f_1 and f_2 from F_1, or F_2, or F_{3m}, or F_{4m}. Let

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

with $a_1d_1 - b_1c_1 = 1$ and $a_2d_2 - b_2c_2 = 1$ be the 2 by 2 matrices in the representation (1) of f_1 and f_2, respectively. Functions h_1 and h_2 are determined by these matrices up to a translation of f_1 and f_2 by $k_1\pi$ and $k_2\pi$; $k_1,k_2 \in \mathbb{Z}$. However

$$h_1 \circ h_2 = \phi \circ \mathrm{Arctan} \frac{a_1 \dfrac{a_2 \tan \phi^{-1} + b_2}{c_2 \tan \phi^{-1} + d_2} + b_1}{c_1 \dfrac{a_2 \tan \phi^{-1} + b_2}{c_2 \tan \phi^{-1} + d_2} + d_1} =$$

$$= \phi \circ \mathrm{Arctan} \frac{(a_1a_2 + b_1c_2)\tan \phi^{-1} + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)\tan \phi^{-1} + (c_1b_2 + d_1d_2)}$$

that shows that the group G is homomorphic to a subgroup

$\overline{G}$ in $\mathbf{SL}_2$. Now, instead of $\overline{G}$ in $\mathbf{SL}_2$ take such a subgroup G^* in $\mathbf{SL}_2$ conjugate to $\overline{G}$,

$$G^* = P\,\overline{G}\,P^{-1}, \quad \text{fixed} \quad P \in \mathbf{SL}_2$$

in which $P\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}P^{-1}$ corresponds to one of Jordan matrices

introduced in Proposition 2. It is easy to see that if G is conjugate to the subgroup $\phi^{-1} \circ G \circ \phi$ in F_1, or F_2, or F_{3m}, or F_{4m}, then there always exist a C^n- diffeomorphism ψ such that

$$\psi^{-1} \circ G \circ \psi$$

is a subgroup in F_1 or F_2, or F_{3m}, or F_{4m}, respectively, having G^* as its matrix representation.

Of course, groups $\overline{G}$ and G^* are isomorphic. Since bijections h_1 and h_2 commute, their corresponding matrices commute as well. According to Proposition 2 only the following cases can happen:

Case I.

$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ is the representation of h_1 (in G^*) and

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$ otherwise arbitrary,

corresponds to h_2 (in G^*). In this situation we may again change the matrix representation such that for a suitable $Q \in \mathbf{SL}_2$

we have $Q\begin{pmatrix} a & b \\ c & d \end{pmatrix}Q^{-1}$, the representative of h_2, of the Jordan form corresponding to the cases I - IV in Proposition 2; representation for h_1 remains the same because

$$Q\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}Q^{-1} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Let g_1 and g_2 with these matrix representations denote those functions that are conjugate to h_1 and h_2; A^0rctan f for f(0) = 0 denoting this branch of Arctan f for which A^0rctan f(0) = 0. Hence

$$g_1(t) = A^0\text{rctan}\,(\tan t) + k_1\pi = t + k_1\pi\ ,$$

where $k_1 \neq 0$ because $g_1 \neq \text{id}$. Since g_1 is a bijection of I onto I, we get $I = \mathbb{R}$, that means that in this situation the group G is conjugate to a subgroup of F_1 (and not of F_2, or F_{3m}, or F_{4m}). For g_2 we have the followng possibilities.

(Case I in Proposition 2):

$$g_2(t) = A^0\text{rctan}(\tan t) + k_2\pi = t + k_2\pi,\ k_2 \in \mathbb{Z},\ k_2 \neq 0,$$

that is imposssible because h_1 and h_2 as well as g_1 and g_2 cannot belong to the same cyclic group;

(case II in Proposition 2):

$$g_2(t) = A^0\text{rctan}(\lambda^2 \tan t) + k_2\pi,\ k_2 \in \mathbb{Z},$$

that again cannot happen since

$$g_1^{[n_1]} \circ g_2^{[n_2]}(t) = A^0\text{rctan}(\lambda^{2n_2}\tan t) + (n_1k_1 + n_2k_2)\pi$$

that intersects identity at 0 for $n_1 = k_2$ and $n_2 = -k_1 \neq 0$;

(case III in Proposition 2):

$$g_2(t) = A^0\text{rctan}\,\frac{\tan t}{\pm \tan t + 1} + k_2\pi$$

for which

$$g_1^{[n_1]} \circ g_2^{[n_2]} = A^0\text{rctan}\,\frac{\tan t}{\pm n_2 \tan t + 1} + (n_1k_1 + n_2k_2)\pi,$$

that again intersects identity at 0 for $n_1 = k_2$ and $n_2 =$

$= -k_1 \neq 0;$

(case IV in Proposition 2):

$$g_2(t) = \text{Arctan}\left(\frac{\cos\lambda\tan t + \sin\lambda}{-\sin\lambda\tan t + \cos\lambda}\right) = t + \lambda + k_2\pi, \quad \lambda \neq k\pi,$$

and

$$g_1^{[n_1]} \circ g_2^{[n_2]} = t + (n_1k_1 + n_2(k_2 + \lambda/\pi))\pi .$$

Since g_1 and g_2 do not belong to the same cyclic group, $(k_2 + \lambda/\pi)/k_1$ is irrational that shows that

$$\{n_1k_1 + n_2(k_2 + \lambda/\pi) \; ; \; n_1, n_2 \in \mathbf{Z}\}$$

is dense in $\mathbf{R}$. Because the group G is closed, that is preserved by any C^n-conjugacy, it is C^n-conjugate to the group of all translations of the reals:

$$\{t \mapsto t + c; \; c \in \mathbf{R}\}.$$

Case II.

$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$, $\lambda \neq 0$, $\lambda \neq \pm 1$, is the representative of h_1 (in G^*), and, according to Proposition 2,

$\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, $\mu \neq 0$, $\mu \neq \pm 1$ are the only representatives of h_2, since for $\mu = \pm 1$ the situation was already considered in Case I. Hence g_1 and g_2 conjugate to h_1 and h_2 can be expressed as

$$\begin{aligned} g_1(t) &= A^0\text{rctan}(\lambda^2 \tan t) + k_1\pi, \\ g_2(t) &= A^0\text{rctan}(\mu^2 \tan t) + k_2\pi, \text{ and} \qquad (2) \\ g_1^{[n_1]} \circ g_2^{[n_2]} &= \text{Arctan}(\lambda^{2n_1}\mu^{2n_2}\tan t) + (n_1k_1 + n_2k_2)\pi, \end{aligned}$$

the last function intersecting the identity at zero for suitable integers n_1, n_2, $n_1^2 > 0$ (if $k_1^2 + k_2^2 > 0$ we take

$n_1 = -k_2$, $n_2 = k_1$, otherwise $k_1 = k_2 = 0$ and we may take $n_1 \neq 0$ and n_2 arbitrary). Hence the group G cannot be conjugated to be a subgroup of F_1 that means that 0 is the left end of the definition interval of g_1 and g_2. This gives $k_1 = 0 = k_2$ in (2). Then also $\pi/2$ cannot be in the definition interval of $g_1(g_2)$ because $g_1(\pi/2) = \pi/2$. Thus functions g_1 and g_2 are defined on $(0,\pi/2)$. Now

$$g_1^{[n_1]} \circ g_2^{[n_2]}(t) = A^0\text{rctan}(e^{n_1 \ln \lambda^2 + n_2 \ln \mu^2} \tan t), \quad t \in (0,\pi/2),$$

and $\ln \lambda^2 / \ln \mu^2$ is irrational, otherwise g_1 and g_2 as well as h_1 and h_2 belong to the same cyclic group that was excluded from our considerations. Hence the set

$$\{n_1 \ln \lambda^2 + n_2 \ln \mu^2;\ n_1, n_2 \in \mathbf{Z}\}$$

is dense in $\mathbf{R}$ However the group

$$G_1 = \{A^0\text{rctan}(e^c \tan t),\ t \in (0,\pi/2);\ c \in \mathbf{R}\}$$

is conjugate to the translations

$$G_2 = \{x \mapsto x + c,\ x \in \mathbf{R};\ c \in \mathbf{R}\}$$

with respect to $\psi = \ln \circ \tan : (0,\pi/2) \to \mathbf{R}$ since $G_1 = A^0\text{rctan} \circ \exp(\ln \circ \tan t + c) = \psi^{-1} \circ G_2 \circ \psi$.

Case III.

$\begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}$ corresponds to h_1 (in G^*) and

$\begin{pmatrix} \pm 1 & 0 \\ \mu & \pm 1 \end{pmatrix}$ is the representation of h_2. Hence

$$g_1(t) = A^0\text{rctan}\left(\frac{\tan t}{\pm \tan t + 1}\right) + k_1\pi,$$

$$g_2(t) = A^0\text{rctan}\left(\frac{\tan t}{\pm\mu \tan t + 1}\right) + k_2\pi \quad \text{and}$$

$$g_1^{[n_1]} \circ g_2^{[n_2]} = A^0\text{rctan}\left(\frac{\tan t}{(\pm n_1 \pm \mu n_2)\tan t + 1}\right) +$$

$+ (n_1k_1 + n_2k_2)\pi$.

The same reasoning as in Case II in this proof gives $k_1 = 0 = k_2$ and zero is the left end of the interval of definition of g_1 and g_2. Then $g_1(\pi) = \pi$, $g_2(\pi) = \pi$, and G is conjugated to a subgroup of F_{31}. If μ is rational then g_1 and g_2 belong to the same cyclic group, the case already excluded. Hence μ is an irrational number and the set

$$\{n_1 + n_2\mu;\ n_1, n_2 \in \mathbf{Z}\}$$

is dense in $\mathbf{R}$. The group

$$G_1 = \{A^0\mathrm{rctan}\, \frac{\tan t}{-c \tan t + 1},\ t \in (0,\pi);\ c \in \mathbf{R}\}$$

is C^n-conjugate to

$$G_2 = \{x \mapsto x + c,\ x \in \mathbf{R};\ c \in \mathbf{R}\}$$

for any n. This can be seen from the fact that G_1 is conjugate to

$$G_3 = \{A^0\mathrm{rctan}\, (\tan s + c),\ s \in (-\pi/2, \pi/2);\ c \in \mathbf{R}\}$$

with respect to $s = t - \pi/2$ because

$$A^0\mathrm{rctan}\, \left(\frac{\tan(s + \pi/2)}{-c \tan(s + \pi/2) + 1}\right) - \frac{\pi}{2} = A^0\mathrm{rctan}(c + \tan s).$$

Moreover G_3 is conjugate to G_2 with respect to arctan: $\mathbf{R} \to (-\pi/2, \pi/2)$.

Finally Case IV

where $g_1(t) = t + \lambda + k_1\pi$, $\lambda \neq k\pi$, $t \in \mathbf{R}$,

and according to Proposition 2,

$$g_2(t) = t + \mu + k_2\pi.$$

Then

$g_1, g_2 \in F_1$ and

$$g_1^{[n_1]} \circ g_2^{[n_2]}(t) = t + (n_1(\lambda/\pi + k_1) + + n_2(\mu/\pi + k_2))\pi. \tag{3}$$

Since g_1 and g_2 do not belong to the same cyclic group, the quotient $(\lambda/\pi + k_1)/(\mu/\pi + k_2)$ is irrational and the set

$$\{n_1(\lambda/\pi + k_1) + n_2(\mu/\pi + k_2);\ n_1,n_2 \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. With respect to the fact that the group G is closed, relation (3) shows that G is C^n-conjugate to the group of all translations of $\mathbb{R}$ onto $\mathbb{R}$, Q.E.D.

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František Neuman
Mathematical Institute of the
Czechoslovak Academy of Sciences,
branch Brno, Mendelovo nám. 1
662 82 Brno, Czechoslovakia

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FUNCTIONS DECOMPOSABLE INTO FINITE SUMS OF PRODUCTS (OLD AND NEW RESULTS, PROBLEMS AND TRENDS)

František Neuman and Themistocles M. Rassias

In this paper we give an account of some of the most important developments concerning the problem of finding necessary and sufficient conditions for functions of n real variables to be decomposable into finite sums of products of one variable functions with minimal requirements on the regularity of the function. This problem goes back to J. d'Alembert.

Functions of certain special forms were investigated by several authors for centuries. One of such forms is a product of two functions of a single variable each, i.e.,

$$h(x,y) = f(x)g(y). \tag{1}$$

It is known at least from the time of J. d'Alembert [2], [3] in the year 1747, that each sufficiently smooth function h of the form (1) has to satisfy the following partial differential equation

$$\frac{\partial^2 \ln h}{\partial x \partial y} = 0. \tag{2}$$

A generalization of the form, namely to a finite sum of products of one-place functions

$$h(x,y) = \sum_{i=1}^{n} f_i(x)g_i(y) \tag{3}$$

was considered since the beginning of this century. In the year 1904 in the section *Arithmetics and Algebra* at the 3rd International Congress of Mathematicians in Heidelberg, Cyparissos Stéphanos from Athens presented as a necessary and sufficient condition for a function h to be of the form (3) the nonvanishing of the determinant

$$D^N(h) := \det \begin{pmatrix} h & \frac{\partial h}{\partial x} & \cdots & \frac{\partial^N h}{\partial x^N} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} & \cdots & \frac{\partial^{N+1} h}{\partial x^N \partial y} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^N h}{\partial y^N} & \frac{\partial^{N+1} h}{\partial x \partial y^N} & \cdots & \frac{\partial^{2N} h}{\partial x^N \partial y^N} \end{pmatrix} = 0 . \qquad (4)$$

His presentation with some further applications and consequences was published in [14] (see also [15]) in the same year. This three-page paper contains no proofs, and we have not succeeded in finding any paper of his, giving at least a hint of such a proof, the sufficiency part of which we consider as not completely trivial.

Indeed, in 1984 Themistocles M. Rassias sent a paper [12] for publication (that was published in 1986) containing a counter example: the function

$$h(x,y) = xy^2 + y|y| \text{ on } R^2$$

satisfies

$$\det \begin{pmatrix} h & \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} \end{pmatrix} = \det \begin{pmatrix} xy^2 + y|y| & y^2 \\ 2xy + 2|y| & 2y \end{pmatrix} = 0$$

for $N = 1$, but it is not of the form (1).

However, in the meantime, in 1980, František Neuman [8] and [9], found a correct version of a sufficient and necessary condition for smooth functions h to be of the form (3):

Theorem 1. Let I and J be unions of open intervals of the reals. If a function $h : I \times J \to \mathbb{R}$, having continuous derivatives $\frac{\partial^{k+j} h}{\partial x^k \partial y^j}$ for $k, j \leq N$, can be written in the form (3) on $I \times J$, then (4) is valid on $I \times J$. If, moreover, $f_i \in C^N(I), g_i \in C^N(J)$ and

$$\det \begin{pmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & \cdots & f_N' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{pmatrix} \neq 0 \qquad (5a)$$

for all $x \in I$, and

$$\det \begin{pmatrix} g_1 & g_2 & \cdots & g_N \\ g_1' & g_2' & \cdots & g_N' \\ \cdots & \cdots & \cdots & \cdots \\ g_1^{(N-1)} & g_2^{(N-1)} & \cdots & g_N^{(N-1)} \end{pmatrix} \neq 0 \qquad (5b)$$

for all $y \in J$, then

$$D^{N-1}(h) \neq 0 \text{ for all } (x,y) \in I \times J \tag{5}$$

holds.

If h satisfies $D^N(h) \equiv 0$ on $I \times J$ and (5), then h is of the form (3) with functions $f_i \in C^N(I)$ and $g_i \in C^N(J), i = 1, \dots, N$, complying with (5a) and (5b) (and thus $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ are linearly independent). All decompositions of h of the form

$$h(x,y) = \sum_{i=1}^{N} \bar{f}_i(x)\bar{g}_i(y)$$

are then exactly those for which

$$(\bar{f}_1, \dots, \bar{f}_N) = (f_1, \dots, f_N) \cdot C^T ,$$

and

$$(\bar{g}_1, \dots, \bar{g}_N) = (g_1, \dots, g_N) \cdot C^{-1} ,$$

where C is an arbitrary n by n nonsingular constant matrix, C^T and C^{-1} being its transpose and inverse respectively.

Remarks 1.

a. The above N-tuples $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ are in fact solutions of certain ordinary linear homogeneous differential equations in the corresponding variables. The observation has occured to be useful in further considerations for functions of more variables.

b. The above Theorem 1 is a version of Theorem 2 in Th. M. Rassias [12].

c. In the papers [8] and [9] F. Neuman has derived a necessary and sufficient condition for decomposition (3) of functions h even without any regularity condition.

Theorem 2. Let I and J be arbitrary nonempty sets. A function $h : I \times J \to \mathbb{R}$ can be written in the form (3) with linearly independent f_i and g_i if and only if the maximum of the rank of the matrices

$$(h(x_k, y_j))\,; k = 1, \dots, r\,; j = 1, \dots, s\,;$$

is N when $x_k \in I, y_j \in J$, and r, s are arbitrary integers. If, in addition, I and J are intervals, $h \in C^d(I \times J), d \geq 0$, then $f_i \in C^d(I)$ and $g_i \in C^d(J)$ for all $i = 1, \ldots, N$.

A simple algorithm verifying the criterion is also derived in [9], and topological properties of functions of the form (3) in L_2 are studied in [10].

J. Falmagne, a mathematical psychologist at New York University, asked (cf. [6], J. Aczél [1,p. 256]) about characterizations, by functional equations, of the functions of the form

$$h(x,y) = G\left(\sum_{i=1}^{N} f_i(x)g_i(y)\right), \tag{6}$$

$h : X \times Y \to \mathbb{R}, f_i : X \to \mathbb{R}, g_i : Y \to \mathbb{R}, X, Y$ arbitrary sets, $\{f_i\}$ independent, $\{g_i\}$ independent, $G : \mathbb{R} \to \mathbb{R}$ strictly monotonic (even continuity may be supposed).

To our best knowledge, only very little has been done in this subject. Let us mention just a few comments:

Remark 2. It can immediately be observed that the function h in (6) has exactly the same system of isohypsis, i.e., curves

$$\{(x,y) \in \mathbb{R}^2 : h(x,y) = c = \text{const.}, c \in \mathbb{R}\},$$

as that of the argument of G in (6) that is, a function of the form (3) for which we have characterizations in Theorems 1 and 2.

Remark 3. For $N = 1$ in (6), G, f and g of the class C^1 with $f(x) \cdot g'(y) \neq 0$, we have

$$h(x,y) = G(f(x)g(y)),$$

hence

$$\frac{\partial h}{\partial x} \Big/ \frac{\partial h}{\partial y} = \frac{f'(x)}{f(x)} \cdot \frac{g(y)}{g'(y)} = \varphi(x)\psi(y) \tag{7}$$

is of the form (1). Thus the left-hand side of (7) has to satisfy the condition (4) with $N = 1$.

In 1984, Th. M. Rassias [12] and in 1988, H. Gauchman and L. A. Rubel [7] considered functions h of the form (3) from several points of view. They derived some very interesting properties of such functions supposing their analyticity, n-times differentiability or merely continuity. Also convergence of sequences of these functions was studied. All three authors proposed a study of functions H of three variables of the form

$$H(x,y,z) = \sum_{i=1}^{N} A_i(x)B_i(y)C_i(z).$$

Th. M. Rassias even asked in P 286 [13] for a sufficient and necessary condition for functions of an arbitrary number of variables to be representable in finite many sums of products of one-place functions

$$H(x_1,\dots,x_m) = \sum_{i=1}^{N} A_{i1}(x_1)\dots A_{im}(x_m) \tag{8}$$

with minimal requirements on the regularity of H.

At the beginning of 1989 there were obtained definite results concerning also this problem. First F. Neuman [11] observed that for three or more variables it is convenient first to study decompositions of the form

$$H(x_1,x_2,x_3) = \sum_{i,j,k=1}^{N} c_{ijk}A_i(x_1)B_j(x_2)C_k(x_3) \tag{9}$$

(and its analogue for more variables) from which we get (8) by setting all $c_{ijk} = 0$ except of $c_{iii} = 1$. He obtained the following sufficient and necessary condition for smooth functions H to be of the form (9).

Theorem 3. The function $H : I \times J \times K \to \mathbb{R}$ with continuous $\frac{\partial^{3N} H}{\partial x_1^N \partial x_2^N \partial x_3^N}$ is of the form (9) with linearly independent N-tuples $\{A_i\}_{i=1}^N$, $\{B_j\}_{j=1}^N$, $\{C_k\}_{k=1}^N$ having the nonvanishing Wronskians if and only if H as a function of each single variable, i.e.,

$$x_i \mapsto H(x_1,x_2,x_3)\,, i \in \{1,2,3\}\,,$$

is a solution of just one ordinary linear homogeneous differential equation for any choice of other variables as parameters.

If a function is of the form (9) with *some* constants c_{ijk} then the question whether it is of the form with *given, prescribed* constants, e.g. like that in (8), can be answered by solving certain system of algebraic relations. Analogous results were derived in [11] for an arbitrary number of variables.

M. Čadek and J. Šimša have continued in these investigations in [4] and [5]. They derived:

Theorem 4. If

$$\begin{aligned} H(x,y,z) &= \sum_{i=1}^{m} A_i(x)\varphi_i(y,z) \\ &= \sum_{j=1}^{n} B_j(y)\psi_j(x,z) \\ &= \sum_{k=1}^{p} C_k(z)\chi_k(x,y) \end{aligned}$$

with linearly independent $\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n, \{C_k\}_{k=1}^p$ then

$$H(x,y,z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}A_i(x)B_j(y)C_k(z)\,.$$

By using this result (that can also be extended to any number of variables) they obtained a characterization of functions of the form (9) by means of certain functional determinants without explicitly referring to differential equations. By a similar manner they extended Theorem 2 from two variables to more variables in the case when no regularity condition on H is required at all. Their results in [5] concern again decompositions of functions of several variables this time in the form

$$H(x_1,\dots,x_l,x_{l+1},\dots,x_m) = \sum_{i=1}^{N} f_i(x_1,\dots,x_l)g_i(x_{l+1},\dots,x_m)$$

They introduced an original method of characterization of finite-dimensional spaces of functions of several variables that generalizes the notion of Wronskian for functions of one variable. By using this approach they obtained a characterization of finite dimensional linear spaces formed by functions of several variables by means of certain special systems of partial differential equations.

This is the present stage of the story that started at least 250 years ago, has had a very interesting development in this century and we hope still several important new results will be added to it in the future.

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František Neuman
Mathematical Institute
Czechoslovak Academy of Sciences
66282 Brno
Mendelovo Nám. 1
Czechoslovakia

Themistocles M. Rassias
Department of Mathematics
University of La Verne
P. O. Box 51105, Kifissia
Athens, Greece 14510

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ON RATIONAL MAPS BETWEEN K3 SURFACES

V. V. Nikulin

1. Introduction

Here, a K3 surface is a non-singular projective algebraic surface X over complex numbers field $\mathbb{C}$ with the trivial space of the regular 1-dimensional differential forms: $\Omega^1[X] = 0$, and the trivial sheaf of the regular 2-dimensional differential forms: $\Omega^2_X \cong \mathcal{O}_X$, where the $\mathcal{O}_X$ is the sheaf of regular functions on X. The last condition is equivalent to the existence of a regular non-zero 2-dimensional differential form ω_X which has no zeros on X.

Thanks to global Torelli theorem due to I. I. Piateckii-Shapiro and I. R. Shafarevich [PSh-Sh], we know very much about isomorphisms between K3 surfaces over the complex numbers field $\mathbb{C}$. Two K3 surfaces are isomorphic iff their periods are isomorphic.

Recently, I. R. Shafarevich posed an analogous question about rational maps between K3 surfaces: How can one know, using periods, when does a rational map between two K3 surfaces exist? A description of rational maps between K3 surfaces is interesting maybe from the view-point of the Arithmetic of K3 surfaces.

Let X be an algebraic K3 surface (over $\mathbb{C}$), let $H_X = H^2(X, \mathbb{Z})$, and let S_X and T_X be respectively the lattices of cohomology classes of algebraic and transcendental cycles on the surface X. By definition, T_X is the orthogonal complement to S_X in H_X with respect to the intersection pairing. Here and in what follows "lattice" means a "non-degenerate symmetric bilinear form over $\mathbb{Z}$". Hodge decomposition of $H_X \otimes \mathbb{C}$ induces a Hodge decomposition of $T_X \otimes \mathbb{C}$. It is defined by one-dimensional linear subspace

$H^{2,0}(X) \subset T_X \otimes \mathbb{C}$. I. R. Shafarevich posed the following

Question 1.1. Is it true that a rational map between K3 surfaces X and Y (i.e., an inclusion over $\mathbb{C}$ of the fields $\mathbb{C}(Y) \subset \mathbb{C}(X)$ of rational functions) exists iff there exist a positive $\lambda \in \mathbb{Q}$ and an isomorphism $\varphi : T_Y \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q}$ such that $\varphi(x \cdot y) = \lambda(x \cdot y)$ for any $x, y \in T_Y \otimes \mathbb{Q}$ (or φ is a similarity of quadratic forms over $\mathbb{Q}$), and $\varphi(H^{2,0}(Y)) = H^{2,0}(X)$?

Let $\gamma : X - - \to Y$ be a rational map between K3 surfaces. Then a resolution of indefinite points of γ gives a commutative diagram

$$\begin{array}{ccc} & Z & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ X & \overset{\gamma}{- - \to} & Y \end{array}$$

where Z is a non-minimal non-singular projective K3 surface, α is a birational morphism and β is a morphism. It gives the inclusion $\gamma^* = (\alpha^*)^{-1}\beta^* : T_Y(d) \to T_X$ of the lattices of a finite index for which $\gamma^*(H^{2,0}(Y)) = H^{2,0}(X)$ (γ^* preservers periods). Here d is the degree of γ and $M(d)$ is the lattice obtained, multiplying by d of the form of the lattice M. The inclusion γ^* does not depend on a choice of Z, α and β, and is the invariant of the rational map γ. Let $d = d'm^2$, where d' and m are the positive integers and d' is square-free. Then γ^* gives a canonical chain of inclusions

$$T_Y(d') \longleftarrow mT_Y(d') = T_Y(d'm^2) \xrightarrow{\gamma^*} T_X$$

of lattices of finite index. Here, we use the following notations: for $m \in \mathbb{Q}$, mM denotes the sublattice (or the overlattice) of the lattice M where $mM = \{mv | v \in M\}$ with the restriction on mM of the form of the lattice M. (We use the notation M^m to denote the orthogonal sum of m exemplary of the lattice M.) We canonically (by the obvious way) identify sublattice $mT_Y(d')$ of the lattice $T_Y(d')$ and the lattice $T_Y(d'm^2)$. This chain gives the isomorphism $\overline{\gamma^*} : T_Y(d') \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q}$ of forms over $\mathbb{Q}$, which we call the *modification corresponding to the rational map* γ. At first, the lattice T_X is replaced on some sublattice $T'_X \subset T_X$ (e.g., $T'_X = \gamma^*(T_Y(d'm^2))$ or $\overline{\gamma^*}(T_Y(d')) \cap T_X$), then T'_X is replaced on some overlattice $T_Y(d')$, and then $T_Y(d')$ is replaced on the lattice T_Y by dividing the form on d'.

We want to discuss here the following question which is similar to question 1.1.

Question 1.2. Let X and Y be K3 surfaces, d' be a square-free positive integer and $\varphi : T_Y(d') \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q}$ be an isomorphism of quadratic forms over $\mathbb{Q}$ (e.g., φ is an abstract modification of the lattices T_X and T_Y) and $\varphi(H^{2,0}(Y)) = H^{2,0}(X)$. Is it true, that then there exists a rational map $f : X - - \to Y$ such that $\varphi = \overline{f^*}$?

We say that an abstract modification φ above is trivial for a prime p iff $p \mid d'$ and φ induces an isomorphism $\varphi_p : T_Y(d') \otimes \mathbb{Z}_p \to T_X \otimes \mathbb{Z}_p$ of p-adic lattices. It is sufficient to answer the question 1.1 for every prime p only, i.e., for modifications φ, which are nontrivial for one prime p only. (One can deduce this from the epimorphicity of the Torelli map for K3 surfaces [Ku] and the following arithmetical fact: a primitive embedding of a lattice S into an unimodular indefinite lattice L exists iff for every prime p, a primitive embedding of the lattice $S \otimes \mathbb{Z}_p$ into $L \otimes \mathbb{Z}_p$ exists.)

The basic result of the paper is to show that the answer to the Question 1.2 is positive if $p = 2$ and $\operatorname{rk} T_X = \operatorname{rk} T_Y \leq 5$.

Theorem 1.3. Let X and Y be algebraic K3 surfaces with $\operatorname{rk} T_X = \operatorname{rk} T_Y \leq 5$, and $\varphi : T_Y(d) \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q}$ be an isomorphism of quadratic forms over $\mathbb{Q}$ (i.e., φ is an abstract modification of the lattices T_X and T_Y) for which $\varphi(H^{2,0}(Y)) = H^{2,0}(X), d \mid 2$, and φ induces an isomorphism $\varphi_p : T_Y(d) \otimes \mathbb{Z}_p \to T_X \otimes \mathbb{Z}_p$ of p-adic lattices for any $p \neq 2$.

Then there exists a sequence $X = X_1, X_2, \dots, X_{n+1} = Y$ of K3 surfaces and rational maps $f_i : X_i - - \to X_{i+1}$ of degree 2 such that the rational map $f = f_n \cdot \ldots \cdot f_2 \cdot f_1$ induces the modification φ, i.e., $\varphi = \overline{f^*}$.

See the proof of the theorem 3.1 below.

The proof of the theorem is based on two of our old papers [N2] and [N3]. If $h : X - - \to Y$ is a rational map of degree 2 between K3 surfaces, then the Galois involution ι of this map is a symplectic involution of the surface X, i.e., ι acts trivially in the space $H^{2,0}(X) = \Omega^2[X]$ of regular 2-forms of X. The map h is the composition of the quotient map $X \to X/\{\mathrm{id}, \iota\}$ and the minimal resolution of singularities $Y \to X/\{\mathrm{id}, \iota\}$. So, to set up the rational map of degree 2 of K3 surface X in other K3 surface, one should find a symplectic involution on X. In [N2] symplectic involutions (and, more generally, finite abelian symplectic groups) of K3 surfaces were described very completely, see Sec. 2. To investigate modifications under sequence of involutions of K3 surfaces, we use discriminant form technique developed in [N3]. Of course, constantly, we use global Torelli theorem for K3 surfaces [PSh-Sh]. We should say that results of [N2] and [N3] that we have mentioned above were used already by D. R. Morrison in [Mo] to prove that for K3 surface X with rk $T_X \geq 3$ a rational map of degree 2 in Kummer K3 surface exists (to prove this fact, he used also results of [N1] about the characterization of Kummer surfaces). But, to prove theorem 1.3, a more careful analysis than in [Mo] is required.

We want to remark that, we also prove the Theorem 2.2.7 below which gives the effective criterion for a preserving periods modification over 2 of transcendental lattices of K3 surfaces is defined by a composition of degree two rational maps between the K3 surfaces. We deduce Theorem 1.3 from this Theorem 2.2.7.

From the Theorem 1.3 and the characterization of Kummer surfaces in [N1], see also [Mo], we obtain the following theorem which was proved by I. R. Shafarevich and the author together.

Theorem 1.4 (V. V. Nikulin and I. R. Shafarevich). Let X and Y be algebraic K3 surfaces. Suppose that for all odd prime p there are primitive embeddings of p-adic lattices:

$$T_X \otimes \mathbb{Z}_p \subset U^3 \otimes \mathbb{Z}_p \quad \text{and} \quad T_Y \otimes \mathbb{Z}_p \subset U^3 \otimes \mathbb{Z}_p\,;$$

and for $p = 2$ there are embeddings of the quadratic forms over the field $\mathbb{Q}_2$:

$$T_X \otimes \mathbb{Q}_2 \subset U^3 \otimes \mathbb{Q}_2 \quad \text{and} \quad T_Y \otimes \mathbb{Q}_2 \subset U^3 \otimes \mathbb{Q}_2\,.$$

Here U is an even unimodular lattice of the signature (1,1). (Roughly speaking, X and Y have transcendental lattices of abelian surfaces over $\mathbb{Z}_p$ for any $p \neq 2$ and over $\mathbb{Q}_2$.)

Then the answer to Question 1.2 is positive for the K3 surfaces X and Y. (See the proof of Theorem 3.2 below.)

The proof of Theorem 1.3 shows that some success in the investigation of rational maps between K3 surfaces is connected with a construction of some concrete rational maps between K3 surfaces (similar to maps of degree 2, which we use here). They should play the same role as the factorization of abelian surfaces by the points of order p. Every rational map between abelian surfaces is a composition of such rational maps and of an automorphism.

See some further remarks to the Theorems 1.3 and 1.4 in Sec. 4.

At last, we would like to mention some results related with rational maps between K3 surfaces. In the situation of Question 1.2 (or 1.1), the cycle $Z_\varphi \in (T_X \otimes T_Y) \otimes \mathbb{Q}$ corresponding to φ belongs to $H^{2,2}(X \times Y, \mathbb{Q})$. Suppose that $d' = 1$. I. R. Shafarevich posed the following conjecture [Sh], which is a particular case of the Hodge conjecture: the cycle Z_φ is algebraic. This conjecture is proved now if rk $T_X \le 17$, and more generally, if the lattice S_X represents zero (or X has a pencil of elliptic curves). See [Shi-I] for rk $T_X = 2$, [Mo] for rk $T_X \le 3$, [Mu] for rk $T_X \le 11$, and [N4] for the case when the lattice S_X represents zero. Thus, this weaker conjecture is proved in much more generality now.

The Theorem 1.3 was inspired by our discussions with I. R. Shafarevich (by his initiative) on the rational maps problem for K3 surfaces. The Theorem 1.4 was deduced by I. R. Shafarevich and the author together. These theorems would not have appeared without Shafarevich's interest to this subject. We are very grateful to I. R. Shafarevich for his interest and support to this paper.

Notations for lattices and quadratic forms. Following [N3], we will use the following definitions and notations connected with lattices and quadratic forms.

We denote as $x \cdot y$ the value of the form of the lattice M for a pair $x, y \in M$, and $x^2 = x \cdot x$.

The lattice M is called *even* iff x^2 is even for any $x \in M$.

The *discriminant group* A_M of a lattice M is the $A_M = M^*/M$, where $M^* = \mathrm{Hom}(M, \mathbb{Z})$.

The *discriminant bilinear form* b_M of a lattice M is the symmetric bilinear pairing $b_M : A_M \times A_M \to \mathbb{Q}/\mathbb{Z}$, where $b_M(x^* + M, y^* + M) = x^* \cdot y^* + \mathbb{Z}, x^*, y^* \in M^*$. Here we extend linearly the bilinear form of M on the M^*. The form b_M is degenerate.

For an even lattice M the *discriminant quadratic form* $q_M : A_M \to \mathbb{Q}/2\mathbb{Z}$ is defined as $q_M(x^* + M) = (x^*)^2 + 2\mathbb{Z}$ for $x^* \in M^*$. The quadratic form q_M has the bilinear form b_M.

The *symbol* $\oplus$ denotes the orthogonal sum of lattices and bilinear and quadratic forms.

The *symbol* $(A)^{\perp}_{B}$ denotes the orthogonal complement to A in B.

The discriminant form of a lattice M is the orthogonal sum of its *p-components* (the restrictions of the form on the p-components of the group $\mathcal{A}_M$), which are defined by the discriminant forms of the p-adic lattices $M_p = M \otimes \mathbb{Z}_p$.

Every p-adic lattice is an orthogonal sum of the following *elementary p-adic lattices*: the lattice $K_{\theta}^{(p)}(p^k)$ of rank 1 has the matrix $\langle \theta p^k \rangle, \theta \in \mathbb{Z}_p^*$; the 2-adic lattice $U^{(2)}(2^k)$ of rank 2 has the matrix

$$\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix};$$

the 2-adic lattice $V^{(2)}(2^k)$ of rank 2 has the matrix

$$\begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}.$$

The discriminant quadratic forms of the p-adic lattices $K_{\theta}^{(p)}(p^k), U^{(2)}(2^k)$ and $V^{(2)}(2^k), k \geq 1$, are denoted as $q_{\theta}^{(p)}(p^k), u_{+}^{(2)}(2^k), v_{+}^{(2)}(2^k)$ respectively. Their bilinear forms are denoted as $b_{\theta}^{(p)}(p^k), u_{-}^{(2)}(2^k), v_{-}^{(2)}(2^k)$ respectively.

In this article we consider only even lattices and even 2-adic lattices. Thus, here, the term "discriminant form" denotes every time discriminant quadratic form.

For a finite abelian group $\mathcal{A}$ the symbol $l(\mathcal{A})$ denotes the minimal number of generators of $\mathcal{A}$. For a form q on a finite abelian group $\mathcal{A}$ we denote $\mathcal{A}_q = \mathcal{A}$ and $l(q) = l(\mathcal{A})$.

The discriminant $\operatorname{discr}(S)$ of a lattice S is the determinant of the matrix of S in some basis. A lattice S is called *unimodular* iff discr S is invertible. *The lattice* U is an even unimodular lattice of the signature (1,1). It is unique up to isomorphism. *The lattice* E_8 is an even unimodular lattice of the signature (0,8). It is unique up to isomorphisms too. *The signature* $(t_{(+)}, t_{(-)}, t_{(0)})$ of a quadratic form over $\mathbb{R}$ is the number of its positive, negative and zero squares. We do not show the number $t_{(0)}$ if the form is non-degenerate.

The *invariants* of a lattice S is a triplet $(t_{(+)}, t_{(-)}, q)$, where the $(t_{(+)}, t_{(-)})$ is a signature of the S and $q \cong q_S$, where q_S is the discriminant form of S. These invariants are equivalent to the genus of S.

An embedding $N \subset M$ of lattice S is called *primitive* iff the quotient-module M/N is a free module.

2. Compositions of Degree 2 Rational Maps between K3 Surfaces

Following [N2] (see [Mo] also), we will give here basic constructions connected with symplectic involutions of K3 surfaces.

2.1. Let X be a K3 surface and let ι be a symplectic involution of X. The following results are contained in [N2].

Let

$$S_\iota = \{x \in H_X \mid \iota(x) = -x\},$$

and

$$T^\iota = \{x \in H_X \mid \iota(x) = x\}.$$

The lattice S_ι is a negative-definite lattice of the rk $S_\iota = 8$, the discriminant group $A_{S_\iota} \cong (\mathbb{Z}/2\mathbb{Z})^8$, and S_ι has no elements δ with square $\delta^2 = -2$. By the classification of the definite unimodular lattices of rank ≤ 8 (see [Se], for example), $S_\iota = E_8(2)$. The lattice S_ι is the primitive sublattice of the lattice S_X. The lattice S_X is a primitive sublattice of the lattice $H_X = H^2(X, \mathbb{Z})$ also. Thus, we have a sequence of primitive embeddings of lattices:

$$S_\iota \subset S_X \subset H_X. \tag{2.1}$$

The lattice H_X is an even unimodular lattice of the signature (3,19). It follows (see [Se], for example) that $H_X \cong U^3 \oplus E_8^2$. The lattice S_ι has the unique (up to isomorphism) primitive embedding into the lattice H_X. It follows that $T^\iota = (S_\iota)^\perp_{H_X} \cong U^3 \oplus E_8(2)$. By (2.1), $T_X = (S_X)^\perp_{H_X}$ is a primitive sublattice of T^ι, and we have a sequence of primitive embeddings of lattices:

$$T_X \subset T^\iota \subset H_X. \tag{2.2}$$

Vice versa, suppose we have a primitive embedding $S \subset S_X$ of lattices, where $S \cong E_8(2)$. Then there exists $w \in W^{(2)}(S_X)$, such that $w(S) = S_\iota$ for some symplectic involution ι of X. Here $W^{(2)}(S_X)$ is the group generated by all reflections with respect to elements $\delta \in S_X$ with the square $\delta^2 = -2$.

The symplectic involution ι has precisely 8 fixed points, and the local action of ι in these points is the multiplication on -1. It follows, that

$X/\{\mathrm{id}, \iota\}$ has precisely 8 singular points of the type $\mathbb{A}_1$, which are the images under the quotient morphism $\pi : X \to X/\{\mathrm{id}, \iota\}$ of the fixed points. Let $\sigma : Y \to X/\{\mathrm{id}, \iota\}$ be the minimal resolution. The pre-images σ^{-1} of the singular points of $X/\{\mathrm{id}, \iota\}$ are non-singular rational curves $\Gamma_1, \dots, \Gamma_8$ of Y with divisor classes $e_1, \dots, e_8$, which generate the primitive negative-definite sublattices

$$Q_\iota = [e_1, \dots, e_8, (e_1 + \dots, +e_8)/2] \tag{2.3}$$

with the form $e_i \cdot e_j = -2\delta_{ij}$, of the lattice S_Y. So, we have the sequence of primitive embeddings of lattices:

$$Q_\iota \subset S_Y \subset H_Y. \tag{2.4}$$

It follows that the discriminant group $A_{Q_\iota} \cong (\mathbb{Z}/2\mathbb{Z})^6$, and the discriminant form $q_{Q_\iota} \cong u_+^{(2)}(2)^3$. Let $R^\iota = (Q_\iota)^\perp_{H_Y}$. By (2.4), we have the sequence

$$T_Y \subset R^\iota \subset H_X \tag{2.5}$$

of the primitive embeddings of the lattices. The lattices Q_ι and R^ι are the orthogonal complements one to another in the even unimodular lattice H_X. It follows [N3] that $q_{R^\iota} \cong -q_{Q_\iota} \cong -u_+^{(2)}(2)^3 \cong u_+^{(2)}(2)^3$, the lattice Q_ι has unique up to isomorphism primitive embedding in H_Y, and $R^\iota \cong U^3 \oplus Q_\iota$.

Let $\tau = \sigma^{-1}\pi : X - - \to Y$ be the corresponding rational map of degree 2. This map gives the embedding of the lattices

$$\tau^* : R^\iota(2) \to T^\iota, \tag{2.6}$$

which has the obvious property:

$$\tau^*(H^{2,0}(Y)) = H^{2,0}(X).$$

A lattice (or an 2-adic lattice) F is called *2-elementary* iff the discriminant group $A_F \cong (\mathbb{Z}/2\mathbb{Z})^a$. For 2-elementary lattices the following duality takes place: To a 2-elementary lattice F, the 2-elementary lattice $F^\times = F^*(2)$ is corresponding, and the canonical embedding $F \subset F^*$ gives the canonical embedding

$$F(2) \subset F^*(2) = F^\times, \tag{2.7}$$

and we have the following duality property:

$$(F^\times)^\times = (F^*(2))^*(2) = F. \tag{2.8}$$

The fundamental fact is that the embedding (2.6) is extended to the isomorphism (this extension is obviously unique) of the lattices:

$$\tau^* : R^{\iota}(2) \subset (R^{\iota})^{\times} \cong T^{\iota}, \tag{2.9}$$

where the embedding $R^{\iota}(2) \subset (R^{\iota})^{\times}$ is the canonical embedding (2.7). Thus, by (2.7) and (2.9) we have the following canonical isomorphisms of the lattices:

$$\tau^* : R^{\iota}(2) \cong (T^{\iota})^{\times}(2) = (T^{\iota})^*(4) = 2(T^{\iota})^* \subset T^{\iota}. \tag{2.10}$$

By (2.2), (2.5), (2.6), and (2.10), we have the following isomorphism, which describes the modification corresponding to the rational map $\tau : X - - \to Y$:

$$\tau^* : T_Y(2) \cong (T_X \otimes \mathbb{Q}) \cap (T^{\iota})^{\times}(2) = 2((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*) \subset T^{\iota}. \tag{2.11}$$

2.2. Here, we want to deduce from the properties 2.1 some general statements connected with K3 surfaces with symplectic involutions. It will be useful in what follows.

2.2.1. Let us consider the following general situation, connected with lattices. Suppose we have an even unimodular lattice L and two primitive sublattices $T \subset L, Q \subset L$ which are orthogonal one to another: $T \perp Q$. Let $[T \oplus Q]$ be the primitive sublattice in L generated by $T \oplus Q$. Then the subgroup

$$\Gamma_{[T\oplus Q]} = [T \oplus Q]/(T \oplus Q) \subset A_T \oplus A_Q$$

is an isotropic subgroup with respect to the quadratic form $q_T \oplus q_Q$ and $\Gamma_{[T\oplus Q]} \cap (A_T \oplus 0) = \Gamma_{[T\oplus Q]} \cap (0 \oplus A_Q) = 0 \oplus 0$. Let π_T and π_Q be the projections in A_T and A_Q respectively. Let

$$\mathfrak{H} = \pi_T(\Gamma_{[T\oplus Q]}) \subset A_T$$

be the subgroup of A_T. Then we have the inclusion

$$\xi : \mathfrak{H} \to A_Q$$

of the groups, where $\xi = \pi_Q(\pi_T)^{-1}$, and ξ gives the inclusion of the quadratic forms:

$$\xi : q_T \mid \mathfrak{H} \to -q_Q.$$

We would like to express the overlattice $T \subset ((Q_L^{\perp})^* \cap (T \otimes \mathbb{Q})$ of a finite index of the T using the subgroup $\mathfrak{H}$.

Lemma 2.2.1. $(((Q)_L^{\perp})^* \cap (T \otimes \mathbb{Q}))/T = \mathfrak{H} \subset \mathcal{A}_T$.

Proof. Let $P = (T \oplus Q)_L^{\perp}$. Then $T \oplus P \oplus Q \subset L$ is a sublattice of a finite index. For a sublattice $F \subset L$, we denote by $[F]$ a primitive sublattice $[F] = L \cap (F \otimes \mathbb{Q})$ of L generated by F. We have the subgroups

$$\Gamma_L = L/(T \oplus P \oplus Q) \subset \mathcal{A}_T \oplus \mathcal{A}_P \oplus \mathcal{A}_Q ,$$
$$\Gamma_{[T\oplus P]} = [T \oplus P]/(T \oplus P) \subset \mathcal{A}_T \oplus \mathcal{A}_P \subset \mathcal{A}_T \oplus \mathcal{A}_P \oplus \mathcal{A}_Q ,$$
$$\Gamma_{[T\oplus Q]} = [T \oplus Q]/(T \oplus Q) \subset \mathcal{A}_T \oplus \mathcal{A}_Q \subset \mathcal{A}_T \oplus \mathcal{A}_P \oplus \mathcal{A}_Q .$$

Here we identify $\mathcal{A}_T = \mathcal{A}_T \oplus 0 \oplus 0, \mathcal{A}_P = 0 \oplus \mathcal{A}_P \oplus 0, \mathcal{A}_Q = 0 \oplus 0 \oplus \mathcal{A}_Q$. Let π_T, π_P, π_Q be the corresponding projections in $\mathcal{A}_T, \mathcal{A}_P, \mathcal{A}_Q$ respectively. The subgroups $\Gamma_L, \Gamma_{[T\oplus P]}$, and $\Gamma_{[T\oplus Q]}$ are obviously isotropic with respect to the form $q_T \oplus q_P \oplus q_Q$.

It follows that we have to prove that

$$([T \oplus P]^*/(T \oplus P)) \cap \mathcal{A}_T = \pi_T(\Gamma_{[T\oplus Q]}) .$$

The lattice L is unimodular. It follows that $([T \oplus P]^*/(T \oplus P)) = (\pi_T \oplus \pi_P)(\Gamma_L)$. Thus, we have to prove that

$$(\pi_T \oplus \pi_P)(\Gamma_L) \cap \mathcal{A}_T = \pi_T(\Gamma_{[T\oplus Q]}).$$

This is equivalent to $\Gamma_L \cap (\mathcal{A}_T \oplus 0 \oplus \mathcal{A}_Q) = \Gamma_{[T\oplus Q]}$. This evidently follows from the fact that $[T \oplus Q]$ is a primitive sublattice of the L. □

2.2.2. Now, let us consider the case of Sec. 2.1 above when K3 surface X has a symplectic involution ι, and specify the situation of Sec. 2.2.1 to the case $L = H_X, T = T_X, Q = S^{\iota}$.

The primitive sublattice $M = [T_X \oplus S_\iota]$ in H_X, which is generated by the sublattice $T_X \oplus S_\iota$ of the lattice H_X, is defined by the inclusion of the forms

$$\xi : q_{T_X} \mid \mathfrak{H} \to -q_{S_\iota} = u_+(2)^4 , \tag{2.12}$$

where $\mathfrak{H}$ is a subgroup of the discriminant group $\mathcal{A}_{T_X}$. It is defined by the graphic $T_\xi = [T_X \oplus S_\iota]/(T_X \oplus S_\iota) \subset \mathcal{A}_{T_X} \oplus \mathcal{A}_{S_\iota}$ of the ξ, which is an isotropic subgroup of the form $q_{T_X} \oplus q_{S_\iota}$ in $\mathcal{A}_{T_X} \oplus \mathcal{A}_{S_\iota}$. The discriminant form

$$q_M = q_{T_X} \oplus q_{S_\iota} \mid ((\Gamma_\xi)^{\perp}_{q_{T_X} \oplus q_{S_\iota}} / \Gamma_\xi). \tag{2.13}$$

By (2.12), the $\mathfrak{H} \cong (\mathbb{Z}/2\mathbb{Z})^\alpha$ is a 2-elementary group, $\alpha \leq 8$, and also $\Gamma_\xi \cong (\mathbb{Z}/2\mathbb{Z})^\alpha$. Let $x_1, \dots, x_\alpha$ be a basis of Γ_ξ. By the inclusion (2.12), there exist a basis $x_1, \dots, x_\alpha$ of the isotropic group Γ_ξ and elements $y_1, \dots, y_\alpha$ of the $S + q_{S_\iota}$ such that we have with respect to the form $q_{T_X} \oplus q_{S_\iota}$: $[x_i, y_i] \perp [x_j, y_j]$ if $i \neq j$, and $[x_i, y_i] \cong u_+^{(2)}(2)$. It follows that

$$q_M \cong q_{T_X} \oplus u_+^{(2)}(2)^{4-\alpha} \quad \text{if} \quad \alpha \leq 4; \tag{2.14}$$

and

$$q_{T_X} \cong q'_{T_X} \oplus u_+^{(2)}(2)^{\alpha-4} \quad \text{and} \quad q_M \cong q'_{T_X} \quad \text{if} \quad \alpha > 4. \tag{2.15}$$

We used here the fact that the orthogonal term, $u_+^{(2)}(2)$ is splitting off uniquely up to isomorphism from a finite quadratic form. It follows that

$$l(q_{M_p}) = l(q_{(T_X)_p}) \text{ if } p \neq 2; \tag{2.16}$$

and

$$l(q_{M_2}) = l(q_{(T_X)_2}) + 8 - 2\alpha, \text{ if } p = 2. \tag{2.17}$$

Obviously,

$$\operatorname{rk} M = \operatorname{rk} T_X + 8. \tag{2.18}$$

The following conditions are sufficient and necessary for the existence of a primitive embedding of an even lattice with invariants $(t_{(+)}, t_{(-)}, q)$ into an indefinite even unimodular lattice with signature $(\ell_{(+)}, \ell_{(-)})$:

$$t_{(+)} \leq \ell(+), t_{(-)} \leq \ell_{(-)}; \tag{2.19}$$

$$t_{(+)} + t_{(-)} + l(q) \leq \ell_{(+)} + \ell_{(-)}; \tag{2.20}$$

$$(-1)^{\ell_{(+)} - t_{(+)}} |\mathcal{A}_q| \equiv \operatorname{discr} K(q_p) \bmod (\mathbb{Z}_p^*)^2 \tag{2.21}$$

for all odd prime p for which $t_{(+)} + t_{(-)} + l(q_p) = \ell_{(+)} + \ell_{(-)}$;

$$|\mathcal{A}_q| \equiv \pm \operatorname{discr} K(q_2) \bmod (\mathbb{Z}_2^*)^2 \tag{2.22}$$

if $t_{(+)} + t_{(-)} + l(q_2) = \ell_{(+)} + \ell_{(-)}$ and $q_2 \cong q_\theta^{(2)}(2) \oplus q'_2$. Here $K(q_p)$ is a p-adic lattice with the discriminant form q_p and $\operatorname{rk} K(q_p) = l(\mathcal{A}_{q_p})$ (the form $K(q_p)$ is unique up to isomorphism). See [N3, theorem 1.12.2].

By (2.14)–(2.22), the following conditions are sufficient and necessary for the existence of a primitive embedding of the lattice M corresponding to the isomorphim ξ into the lattice H_X:

$$\operatorname{rk} T_X + l(q_{(T_X)_p}) \leq 14 \tag{2.23}$$

for all odd prime p, and

$$|A_{T_X}| \equiv -\operatorname{discr} K(q_{(T_X)_p}) \operatorname{mod}(\mathbb{Z}_p^*)^2 \tag{2.24}$$

for all odd prime p for which $\operatorname{rk} T_X + l(q_{(T_X)_p}) = 14$;

$$\alpha \geq (\operatorname{rk} T_X + l(q_{(T_X)_2}))/2 - 3\,, \tag{2.25}$$

and

$$|A_{T_X}| \equiv \pm\operatorname{discr} K(q_{(T_X)_2}) \operatorname{mod}(\mathbb{Z}_2^*)^2 \tag{2.26}$$

if $\alpha = (\operatorname{rk} T_X + l(q_{(T_X)_2}))/2 - 3$ and $q_{(T_X)_2} \not\cong q_\vartheta^{(2)}(2) \oplus q'$.

The conditions (2.25), (2.26) and the strong inequalities

$$\operatorname{rk} T_X + l(q_{(T_X)_p}) < 14 \tag{2.27}$$

for all odd prime p are sufficient for the existence of a primitive embedding of the lattice M into the lattice H_X.

By the Lemma 2.2.1,

$$((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*)/T_X = \mathfrak{H}\,, \tag{2.28}$$

that defines the lattice $(T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*$. By (2.28) and (2.11) we get

Lemma 2.2.2. The $\tau^*(T_Y(2)) \subset T_X$ is defined by the following:

$$\tau^*(T_Y(2)) = 2((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*) \subset T_X \subset (T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*\,,$$

and

$$((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*)/T_X = \mathfrak{H} \subset A_{T_X}\,.$$

2.2.3. We can repeat results of 2.2.2 to obtain similar results for the K3 surface Y which has a rational map of degree two $\tau : X - - \to Y$ of a K3 surface X, defined by a symplectic involution ι of X. Here we apply results of the 2.2.1 to $L = H_Y, T = T_Y$, and $Q = Q_\iota$.

The primitive sublattice $M = [T_Y \oplus Q_\iota]$ in H_Y, which is generated by the sublattice $T_Y \oplus Q_\iota$ of the lattice H_Y, is defined by the inclusion of the forms

$$\xi : q_{T_Y} \mid \mathfrak{H} \to -q_{Q_\iota} = u_+(2)^3 , \tag{2.29}$$

where $\mathfrak{H}$ is a subgroup of the discriminant group $\mathcal{A}_{T_Y}$. It is defined by the graphic $\Gamma_\xi = [T_Y \oplus Q_\iota]/(T_Y \oplus Q_\iota) \subset \mathcal{A}_{T_Y} \oplus \mathcal{A}_{Q_\iota}$ of the ξ, which is an isotropic subgroup of the form $q_{T_Y} \oplus q_{Q_\iota}$ in $\mathcal{A}_{T_Y} \oplus \mathcal{A}_{Q_\iota}$. The discriminant form is

$$q_M = (q_{T_Y} \oplus q_{Q_\iota}) | ((\Gamma_\xi)^\perp_{q_{T_Y} \oplus q_{Q_\iota}} / \Gamma_\xi) . \tag{2.30}$$

By (2.29), $\mathfrak{H} \cong (\mathbb{Z}/2\mathbb{Z})^\beta$ is a 2-elementary group, $\beta \le 6$, and also $\Gamma_\xi \cong (\mathbb{Z}/2\mathbb{Z})^\beta$. Similarly to the case 2.2.2, we get:

$$q_M \cong q_{T_Y} \oplus u_+^{(2)}(2)^{3-\beta} , \text{ if } \beta \le 3 ; \tag{2.31}$$

and

$$q_{T_Y} \cong q'_{T_Y} \oplus u_+^{(2)}(2)^{\beta-3} \text{ and } q_M \cong q'_{T_Y} \text{ if } \beta > 3 . \tag{2.32}$$

It follows that

$$l(q_{M_p}) = l(q_{(T_Y)_p}) , \text{ if } p \ne 2 ; \tag{2.33}$$

and

$$l(q_{M_2}) = l(q_{(T_Y)_2}) + 6 - 2\beta , \text{ if } p = 2 . \tag{2.34}$$

Obviously,

$$\operatorname{rk} M = \operatorname{rk} T_Y + 8 . \tag{2.35}$$

By (2.19)–(2.22) and (2.31)–(2.35), the following conditions are sufficient and necessary for the existence of a primitive embedding of the lattice M corresponding to the inclusion ξ into the lattice H_Y:

$$\operatorname{rk} T_Y + l(q_{(T_Y)_p}) \le 14 \tag{2.36}$$

for all odd prime p, and

$$|\mathcal{A}_{T_Y}| \equiv -\operatorname{discr} K(q_{(T_Y)_p}) \bmod (\mathbb{Z}_p^*)^2 \tag{2.37}$$

for all odd prime p for which $\operatorname{rk} T_Y + l(q_{(T_Y)_p}) = 14$;

$$\beta \ge (\operatorname{rk} T_Y + l(q_{(T_Y)_2}))/2 - 4 , \tag{2.38}$$

and

$$|A_{T_Y}| \equiv \pm \operatorname{discr} K(q_{(T_Y)_2}) \bmod (\mathbb{Z}_2^*)^2, \tag{2.39}$$

if $\beta = (\operatorname{rk} T_Y + l(q_{(T_Y)_2}))/2 - 4$ and $q_{(T_Y)_2} \not\cong q_\theta^{(2)}(2) \oplus q'$.

The conditions (2.38), (2.39) and the strong inequalities

$$\operatorname{rk} T_Y + l(q_{(T_Y)_p}) < 14 \tag{2.40}$$

for all odd prime p are sufficient for the existence of a primitive embedding of the lattice M into the lattice H_Y.

2.2.4. Let X be a K3 surface. The pair $(T_X, H^{2,0}(X) \subset T_X \otimes \mathbb{C})$ is called *the transcendental periods* of the X. For two K3 surfaces X and Y, *an isomorphism of their transcendental periods* is an isomorphism $\varphi : T_X \cong T_Y$ of the lattices such that $(\varphi \otimes \mathbb{C})(H^{2,0}(X)) = H^{2,0}(Y)$. We say that a K3 surface X *is defined by its transcendental periods* iff every K3 surface X' with the transcendental periods isomorphic to that of X is isomorphic to X.

Lemma 2.2.3. Let Z be an algebraic K3 surface (over $\mathbb{C}$) which either has a symplectic involution or has a rational map of degree 2, $\tau : X - - \to Z$ of a K3 surfaces X.

Then Z is defined by its transcendental periods, and for any K3 surface Z' and an isomorphism $\varphi : T_{Z'} \cong T_Z$ of the transcendental periods, $\varphi = f^*$ for some isomorphism $f : Z \cong Z'$ of the surfaces.

Proof. Suppose that K3 surface X has a symplectic involution ι and let $\varphi : T_X \to T_{X'}$ be an isomorphism of the periods for K3 surface X'.

From the analog of Witt's theorem [N2], [N3], it follows that a primitive embedding of an even lattice K into an even unimodular lattice L is unique up to isomorphisms (for every two embeddings $i : K \subset L, i' : K \subset L$ we have $i' = gi$ for an automorphism g of L) if the conditions a), b), c) below take place:

a) the lattice $(K)^\perp_L$ is indefinite;

b) $\operatorname{rk} K + l(A_{K_p}) \le \operatorname{rk} L - 2$ for all prime $p \ne 2$;

c) either $\operatorname{rk} K + l(A_{K_2}) \le \operatorname{rk} L - 2$ or $q_{K_2} \cong q'_{K_2} \oplus u_+^{(2)}(2)$.

By (2.15), (2.23), and (2.25), the conditions a), b) and c) above hold for the primitive embedding $T_X \subset H_X$. It follows that the primitive embedding

$T_X \subset H_X$ is unique up to isomorphism. Thus, the isomorphism $\varphi : T_X \to T_{X'}$ of the lattices has an extension $\Phi : H_X \to H_{X'}$.

Let, for a K3 surface Z,

$$V(Z) = \{x \in S_Z \otimes \mathbb{R} \mid x^2 > 0\}$$

and let $V^+(Z)$ be a half cone of the $V(Z)$ which contains a polarization of the Z.

Suppose that $\Phi(V^+(X)) = V^+(X')$. Then, there exists an element $w \in W^{(2)}(X)$ such that $\Phi w(h_X) = h_{X'}$ for polarizations h_X and $h_{X'}$ of X and X'. w is trivial in T_X. From the global Torelli theorem [PSh-Sh], it follows that an isomorphism $f : X' \to X$ exists such that $f^* = \Phi w$. It follows that $f^* \mid T_X = \varphi$.

Suppose that $\phi(V^+(X)) = -V^+(X')$. In this case, let us find an automorphism Ψ of the lattice H_X such that $\Psi \mid T_X = \mathrm{id}_{T_X}$ and $\Psi(V^+(X)) = -V^+(X)$. Then we can replace Φ by $\Phi\Psi$ to reduce the case to the previous one.

The discriminant form $q_{S_X} \cong -q_{T_X}$ because $S_X = (T_X)^{\perp}_{H_X}$ and S_X is primitive in H_X. From this fact and (2.15), (2.23), (2.25), it follows that

$$\operatorname{rk} S_X \geq l(\mathcal{A}_{(S_X)_p}) + 8 \tag{2.41}$$

for all odd $p \neq 2$, and

$$\operatorname{rk} S_X \geq l(\mathcal{A}_{(S_X)_2}) + 16 - 2\alpha\,, \tag{2.42}$$

where $\alpha \leq 8$. By (2.15),

$$q_{(S_X)_2} = u_+^{(2)}(2) \oplus q'\,, \text{ if } \alpha \geq 5\,. \tag{2.43}$$

It follows (see [Kn] and [N3, theorem 1.13.2]) that a lattice with the same invariants $(t_{(+)}, t_{(-)}, q)$ as the lattice S_X is unique up to isomorphisms. From this fact and the criterion of the existence of an even lattice with given invariants $(t_{(+)}, t_{(-)}, q)$ (see [N3, theorem 1.10.1]), it follows that

$$S_X = S_1 \oplus S_2\,, \text{ where } S_1 \cong U \text{ or } S_1 \cong U(2)\,.$$

For the lattice S_1 the discriminant group $\mathcal{A}_{S_1} \cong (\mathbb{Z}/2\,\mathbb{Z})^a, a = 0$ or 2, is a 2-elementary group. It follows that there exists the automorphism Ψ of

H_X which is the (-id) in S_1 and which is identical in $(S_1)^{\perp}_{H_X}$. The Ψ gives an automorphism which we look for.

In the case when $Z = Y$ has a rational map of degree two

$$X - - \rightarrow Y$$

of the K3 surface X, the proof is the same if one uses 2.2.3. □

The (2.11) and the Lemma 2.2.2 show that the modification defined by a rational map of degree two $\tau : X - - \rightarrow Y$ of K3 surfaces is defined by a primitive embedding $T_X \subset T^{\iota}$ of the lattices where $T^{\iota} \cong U^3 \oplus E_8(2)$. The Lemma below shows that every such embedding is possible and reduces the problem of the description of modifications to a purely arithmetic one.

Let us denote $T \cong T^{\iota} \cong U^3 \oplus E_8(2)$.

Lemma 2.2.4. Let X be a K3 surface and $T_X \subset T \cong U^3 \oplus E_8(2)$ a primitive embedding of lattices.

Then there exists a symplectic involution ι of X such that for the corresponding rational map of degree two $\tau : X - - \rightarrow Y$ of K3 surfaces

$$\tau^* T_Y(2) = 2(T^* \cap (T_X \otimes \mathbb{Q})) \subset T_X .$$

Proof. In fact, in the proof of the Lemma 2.2.3, we have shown that a primitive embedding $T_X \rightarrow H_X$ of the lattices is unique up to isomorphisms, if a primitive embedding $T_X \subset T$ exists. It follows that an extension $T \subset H_X$ of the natural primitive embedding $T_X \subset H_X$ exists, where an embedding $T \subset H_X$ is also primitive. The lattice T is 2-elementary. It follows that the involution ϑ of the lattice H_X exists, which is identical in the lattice T and is the multiplication by (-1) in the lattice $S = (T_X)^{\perp}$. $q_S \cong -q_T \cong u_+^{(2)}(2)^4, \operatorname{rk} S = 8$. Then the lattice $S \cong S_1(2)$ where lattice S_1 is an even lattice. Particularly, the lattice S has no elements with the square (-2). It follows [N2] that there exists $w \in W^{(2)}(S_X)$ such that $w \vartheta w^{-1} = \iota^*$ for a symplectic involution ι of the X. The automorphism w gives the isomorphism $w : T \rightarrow T^{\iota}$ of the lattices which is identical in the lattice T_X. It follows that for the rational map corresponding to ι of degree two $\tau : X - - \rightarrow Y$ of K3 surfaces we have (see (2.11)) that

$$\tau^* T_Y(2) = 2((T^{\iota})^* \cap (T_X \otimes \mathbb{Q})) = 2(T^* \cap (T_X \otimes \mathbb{Q})) . \qquad \square$$

By the results above, we get

Theorem 2.2.5. Let X be an algebraic K3 surface.

If X has a rational map of degree two $\tau : X - - \to Y$ in a K3 surface Y then the following condition (i) holds:

(i) rk $T_X + l(q_{(T_X)_p}) \leq 14$ for all odd prime p, and $|A_{T_X}| \equiv -\mathrm{discr}\, K\, (q_{(T_X)_p}))\mathrm{mod}\, (\mathbb{Z}_p^*)^2$ for all odd prime p for which rk $T_X + l(q_{(T_X)_p}) = 14$;

If the condition (i) holds, then there is the bijection between modifications $\tau^* : T_Y(2) \to T_X$ corresponding to rational maps of degree two $\tau : X - - \to Y$ between K3 surfaces X and Y, and pairs $(\mathfrak{H}, \vartheta)$ defined below.

Here $\mathfrak{H} \cong (\mathbb{Z}/2\mathbb{Z})^\alpha$ is a 2-elemenatry subgroup $\mathfrak{H} \subset A_{(T_X)_2}$ such that the condition (ii) below holds.

(ii) There exists an embedding $\xi : q_{T_X} \mid \mathfrak{H} \to u_+^{(2)}(2)^4$ of the finite quadratic forms, and

$$\alpha \geq (\mathrm{rk}\, T_X + l(q_{(T_X)_2}))/2 - 3,$$

and

$$|A_{T_X}| \equiv \pm \mathrm{discr}\, K(q_{(T_X)_2}) \bmod (\mathbb{Z}_2^*)^2$$

if $\alpha = (\mathrm{rk}\, T_X + l(q_{(T_X)_2}))/2 - 3$ and $q_{(T_X)_2} \cong q_\vartheta^{(2)}(2) \oplus q'$.

For the lattice $T_X \subset \widetilde{\mathfrak{H}} \subset T_X^*$ defined by the equality $\widetilde{\mathfrak{H}}/T_X = \mathfrak{H}$, the ϑ is an isomorphism of the lattices

$$\vartheta : T_Y(2) \xrightarrow{\sim} 2\,\widetilde{\mathfrak{H}} \subset T_X,$$

such that $\vartheta(H^{2,0}(Y)) = H^{2,0}(X)$. For any $\mathfrak{H}$ satisfying the condition (ii) there exists a K3 surface Y and an isomorphism ϑ with these properties. $\vartheta = \tau^*$ for a rational map $\tau : X - - \to Y$ of degree two.

Proof. We leave the reader to deduce it from the Lemmas above. □

2.2.5. Let us define the composition of modifications which will correspond to the composition of rational maps.

Let T_1, T_2, T_3 be lattices and $\varphi_1 : T_1(d_1) \otimes \mathbb{Q} \to T_2 \otimes \mathbb{Q}, T_2(d_2) \otimes \mathbb{Q} \to T_3 \otimes \mathbb{Q}$ be isomrophisms of symmetric bilinear forms over $\mathbb{Q}$, where d_1, d_2 are square-free positive integers. In other words, we have two abstract modifications of the lattices T_1, T_2, T_3. Let $d_1d_2 = m^2(d_1d_2)'$ where m and $(d_1d_2)'$ are integers and $(d_1d_2)'$ is square free. Then the sequence of inclusions of lattices

$$T_1((d_1d_2)') = (1/m)T_1(d_1d_2) \supset T_1(d_1d_2)$$

is defined. It gives the identification of the forms over $\mathbb{Q}$

$$T_1((d_1d_2)') \otimes \mathbb{Q} = (1/m)T_1(d_1d_2) \otimes \mathbb{Q} = T_1(d_1d_2) \otimes \mathbb{Q},$$

and the isomorphism $\overline{\varphi_2\varphi_1}$ of the forms

$$\begin{aligned}\overline{\varphi_2\varphi_1} : T_1((d_1d_2)') \otimes \mathbb{Q} =& (1/m)T_1(d_1d_2) \otimes \mathbb{Q} = T_1(d_1d_2) \\ & \otimes \mathbb{Q} \xrightarrow{\varphi_1} T_2(d_2) \otimes \mathbb{Q} \xrightarrow{\varphi_2} T_3 \otimes \mathbb{Q}\end{aligned}$$

is called the composition of the modifications φ_1, φ_2.

Suppose that $f_1 : X_1 - - \to X_2, f_2 : X_2 - - \to X_3$ are two rational maps between algebraic surfaces. Then the modification $\overrightarrow{f_2f_1}^{\bullet}$ corresponding to the composition f_1f_2 of the rational maps is obviously the composition of the modifications $\overrightarrow{f_1}^{\bullet}, \overrightarrow{f_2}^{\bullet}$.

2.2.6. Using the results above, we want to describe modifications corresponding to rational maps $f : X - - \to Y$ between K3 surfaces X and Y which are compositions $f = f_n \dots f_1$ of rational maps $f_1, f_2, \dots, f_n$ of the degree two. A composition of any two rational maps of this type is a rational map of this type also. Thus, these rational maps define the *category* $\mathcal{K}$ of the rational maps.

Lemma 2.2.6. Let $f : X - - \to Y$ be a rational map between K3 surfaces X and Y, which is a composition $f = f_n \cdot \dots \cdot f_1$ of the rational maps of degree two, $f_1 : X_1 = X - - \to X_2, \dots, f_n : X_n - - \to X_{n+1} = Y$ between the non-singular algebraic surfaces $X_1, \dots, X_{n+1}$ (i.e., $f \in \mathcal{K}$).

Then the minimal models of the surfaces $X_1, \dots, X_{n+1}$ are K3 surfaces. So, we can choose birationally the surfaces $X_1, \dots, X_{n+1}$ being K3 surfaces.

Proof. Rational maps $f_1, \dots, f_n$ give the isomorphisms

$$H^{2,0}(X) = H^{2,0}(X_1) \cong H^{2,0}(X_2) \cong \dots \cong H^{2,0}(X_{n+1}) \cong H^{2,0}(Y),$$

because $H^{2,0}(X) \cong^{2,0} (Y) \cong \mathbb{C}$. It follows that Galois involutions $\iota_1, \dots, \iota_n$ of the maps $f_1, \dots, f_n$ are trivial in the spaces

$$H^{2,0}(X) = H^{2,0}(X_1) \cong H^{2,0}(X_2) \cong \dots \cong H^{2,0}(X_{n+1}) \cong H^{2,0}(Y).$$

Then the involution ι_1 is a symplectic involution of the K3 surface $X_1 = X$. Let Y be the minimal resolution of the singularities of $X/\{\mathrm{id}, \iota\}$. We know (see [N2] and also 2.1) that the surface Y is a K3 surface. The surface X_2 is birationally isomorphic to the surface Y, and its minimal model is a K3 surface. Thus, we can suppose that $X_2 = Y$ is a K3 surface. In such a way, we obtain the proof using the induction. □

Using the Theorem 2.2.5 and the Lemma 2.2.6, we obtain the following description of the modifications corresponding to rational maps from the category $\mathcal{K}$ between K3 surfaces.

Theorem 2.2.7. Let X be an algebraic K3 surface.

If X has a rational map $f : X - - \to Y$ in a K3 surface Y which is a composition of rational maps of degree two, and $\deg f > 1$, then the condition (i) of Theorem 2.2.5 holds for T_X.

Let for T_X the condition (i) of the theorem 2.2.5 holds, a positive integer $d|2$, and Y is a K3 surface.

Then modifications $\overline{f^*} : T_Y(d) \otimes \mathbb{Q} \xrightarrow{\sim} T_X \otimes \mathbb{Q}$ corresponding to rational maps $f : X - - \to Y$ which are compositions $f = f_n \dots f_1$ of rational maps $f_1, \dots, f_n$ of degree two ($d = 1$ if n is even, and $d = 2$ if n is odd) are defined by sequences $(T_1, \mathfrak{H}_1), (T_2, \mathfrak{H}_2), \dots, (T_n, \mathfrak{H}_n)$ of pairs and by the isomorphisms ϑ defined below. Every such sequence and every ϑ are possible.

Here, $T_i, i = 1, \dots, n$, are sublattices of the maximal rank in the form $T_X \otimes \mathbb{Q}$ for i odd, and in the form $T_X(1/2) \otimes \mathbb{Q}$ for i even. Here $\mathfrak{H}_i \cong (\mathbb{Z}/2\mathbb{Z})^{a_i}$ is a 2-elementary subgroup $\mathfrak{H}_i \subset A_{T_i}$. The lattices T_i are defined by induction. The sublattice $T_1 = T_X \subset T_X \otimes \mathbb{Q}$. For $1 \le i \le n$ the sublattice $T_{i+1}(2) = 2\widetilde{\mathfrak{H}}_i \subset T_i$, where $\widetilde{\mathfrak{H}}_i/T_i = \mathfrak{H}_i$. It gives the inclusion $T_{i+1} \subset T_X(1/2) \otimes \mathbb{Q}$ if i is odd, and the inclusion $T_{i+1} \subset T_X(1/4) \otimes \mathbb{Q} = (1/2)T_X \otimes \mathbb{Q} = T_X \otimes \mathbb{Q}$, if i is even. For every pair $(T_i, \mathfrak{H}_i), 1 \le i \le n$, condition (ii) of the Theorem 2.2.5 should be true (one should replace in the condition the T_X by T_i, and $\mathfrak{H}$ by $\mathfrak{H}_i$).

The $\vartheta : T_Y \to T_{n+1}$ is an isomorphism of the lattice which induces the isomorphism of the periods, i.e., $\vartheta(H^{2,0}(Y)) = H^{2,0}(X) \subset T_X \otimes \mathbb{C}$. For the sequence $(T_i, \mathfrak{H}_1), (T_2, \mathfrak{H}_2), \dots, (T_n, \mathfrak{H}_n)$ satisfying the condition above there exists such K3 surface Y and an isomorphism ϑ.

The modification $\overline{f^*}$ defined by the sequence and the ϑ is the composition of the ϑ and of the inclusion of the sublattice $T_{n+1} \subset T_X \otimes \mathbb{Q}$ for n even and $T_{n+1} \subset T_X(1/2) \otimes \mathbb{Q}$ for n odd under multiplication of the forms by $d = 2$ for n odd.

Proof. The Theorem follows from Theorem 2.2.5 using compositions of rational maps and modifications above (it is more difficult to formulate this theorem than to deduce it from the Theorem 2.2.5). □

Remark 2.2.8. From Theorem 2.2.7, we obtain the following sequence of sublattices of the form $T_X \otimes \mathbb{Q}$:

$$T_1 \supset T_2(2) \subset T_3 \supset T_4(2) \subset \dots \quad \text{in} \quad T_X \otimes \mathbb{Q},$$

where $(1/2)T_{i+1}(2)/T_i = \mathfrak{H}_i$ for all odd i, and $T_{i+1}(1/2)/T_i = \mathfrak{H}_i$ for i even.

Theorem 2.2.7 reduces the description of modifications corresponding to rational maps between K3 surfaces from the category $\mathcal{K}$ to the purely algebraic problem. We will use the Theorem 2.2.7 for the proof of the basic Theorem 3.1 of the paper (Theorem 1.3. of the Introduction) in the following paragraph.

3. Rational Maps between K3 Surfaces with the Transcendental Lattice of the Rank ≤ 5.

Here we prove the basic theorems (the Theorem 1.3 and 1.4 of the Introduction) of the paper.

Theorem 3.1. Let X and Y be algebraic K3 surfaces with $\operatorname{rk} T_X = \operatorname{rk} T_Y \leq 5$, and $\varphi : T_Y(d) \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q}$ be an isomorphism of quadratic forms over $\mathbb{Q}$ (i.e., φ is an abstract modification of the lattices T_X and T_Y) for which $\varphi(H^{2,0}(Y)) = H^{2,0}(X), d|2$, and φ induces an isomorphism $\varphi_p : T_Y(d) \otimes \mathbb{Z}_p \to T_X \otimes \mathbb{Z}_p$ of p-adic lattices for any prime $p \neq 2$.

Then there exists a sequence $X = X_1, X_2, \ldots, X_{n+1} = Y$ of K3 surfaces and rational maps $f_i : X_i - - \rightarrow X_{i+1}$ of degree 2 such that the rational map $f = f_n \cdot \ldots \cdot f_2 \cdot f_1$ induces the modification φ, i.e., $\varphi = \overline{f^*}$.

Proof. We divide it on several steps.

3.1. We denote $T = T_X$ and $\tilde{T} = \varphi(T_Y) \subset T \otimes \mathbb{Q}(1/d)$. Using Theorem 2.2.7 and Remark 2.2.8, one should find a sequence of the $\mathbb{Z}$-sublattices of the form $T \otimes \mathbb{Q}$:

$$T = T_1 \supset T_2(2) \subset T_3 \supset \ldots T_{n+1}(d) = \tilde{T}(d), \tag{3.1}$$

where n is odd if $d = 2$, and n is even if $d = 1$, such that the conditions of Theorem 2.2.7 hold. A sequence which satisfy the conditions of Theorem 2.2.7 is called further an acceptable.

By the condition of Theorem 3.1, $T \otimes \mathbb{Z}_p = \tilde{T} \otimes \mathbb{Z}_p$ for any odd prime p. According to Theorem 2.2.7, quotient modules of the modules of the sequence (3.1) should be 2-groups. Thus, one should find the sequence (3.1) over ring $\mathbb{Z}_2$ only. One has the obvious inequality $l(A_{(T_X)_p}) \leq \operatorname{rk} T_X \leq 5$ for every p. Then $l(A_{(T_X)_p}) + \operatorname{rk} T_X < 14$. Thus, the condition (i) of Theorem 2.2.7 is true, and for a construction of the sequence (3.1) we should satisfy condition (ii) of Theorem 2.2.7 only.

3.2. At first, for $\operatorname{rk} T \leq 5$, we will construct an acceptable sequence $T = T_1, \ldots, T_{m+1} = T'$ of lattices such that m is odd and $T' = 2T(1/2) \subset T \otimes \mathbb{Q}(1/2)$. Thus, the lattice, $T' \cong T(2)$. We consider the most difficult cases $\operatorname{rk} T = 4$ and 5.

Let $\operatorname{rk} T = 4$.

Let (over $\mathbb{Z}_2$)$T = S_1 \oplus S_2 \oplus R(2)$ where S_1, S_2 are lattices of rank 1, and R is an even lattice of rank 2. Let $\{\zeta_1\}$ be a basis of the S_1, $\{\zeta_2\}$ a basis of the S_2, and $\{\zeta_3, \zeta_4\}$ a basis of the lattice $R(2)$. Let us prove that the following sequence of lattices is acceptable:

$$\begin{aligned} T_1 =& \ [\zeta_1, \zeta_2, \zeta_3, \zeta_4], T_2 = [2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4](1/2), \\ T_3 =& [\zeta_1, 2\zeta_2, \zeta_3, \zeta_4], T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2). \end{aligned}$$

In this case the subgroup $\mathfrak{H}_1 = [\zeta_1, \zeta_2, \zeta_3/2, \zeta_4/2]/[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$, and, evidently, there exists an embedding of the forms $q_{T_1} | \mathfrak{H}_1 \rightarrow u_+^{(2)}(2)^4$. We have: $\alpha_1 = 2 > 1 \geq (\mathrm{rk}\, T_1 + l(q_{T_1}))/2 - 3$ since $4 = \mathrm{rk}\, T_1 \geq l(q_{T_1})$. It proves the condition (ii) of Theorem 2.2.7 for the pair $(T_1,\ \mathfrak{H}_1)$. The lattice $T_2 = S_1(2) \oplus S_2(2) \oplus R$, and $\alpha_2 = 1$. In this case $\mathfrak{H}_2 = [\zeta_1, 2\zeta_2, \zeta_3, \zeta_4]/[2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4]$, and evidently an embedding $q_{T_2} | \mathfrak{H}_2 \to u_+^{(2)}(2)^4$ of the forms exists. Since the lattice R is even then either R is unimodular or $l(\mathcal{A}_R) = 2$. If the lattice R is unimodular, then $\alpha_2 = 1 > (\mathrm{rk}\, T_2 + l(q_{T_2}))/2 - 3$. If R is not unimodular, then we have the equality $\alpha_2 = 1 = (\mathrm{rk}\, T_2 + l(q_{T_2}))/2 - 3$. And we should prove the congruence (where we consider the lattice T_2 as a lattice over $\mathbb{Z}$):

$$|\mathcal{A}_{T_2}| \equiv \pm \mathrm{discr}\, K(q_{(T_2)_2}) \bmod (\mathbb{Z}_2^*)^2 .$$

In this case, $K(q_{(T_2)_2}) \cong (T_2)_2 = T_2 \otimes \mathbb{Z}_2$, and this congruence holds because

$$\mathrm{discr}\, T_2 = \pm |\mathcal{A}_{T_2}| .$$

for the lattice T_2 over $\mathbb{Z}$. $\alpha_3 = 1$, and the proof of the condition (ii) for $(T_3,\ \mathfrak{H}_3)$ is the same.

The same proof of the condition (ii) should be produced in all cases which we consider below. We will leave this procedures to the reader.

Now, suppose that the lattice T does not have a representation of the type above. From the decomposition of 2-adic lattices in an orthogonal sum of lattices of the rank 1 and 2, one obtains that it is possible only in the following two cases which we consider at once.

The case $T = R_1(2^m) \oplus R_2(2^n)$, where R_1, R_2 are an even unimodular lattices of rank two, $m \geq 0, n \geq 0$. Let $\{\zeta_1, \zeta_2\}$ be a basis of the lattice $R_1(2^m)$ and $\{\zeta_3, \zeta_4\}$ a basis of the $R_2(2^n)$. If $m = n = 0$ then the sequence of lattices

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4],\ T_2 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2)$$

is acceptable. Suppose that $n \geq 1$. Then the following sequence of the lattices is acceptable:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]\ ,\ T_2 = [2\zeta_1, 2\zeta_2, \zeta_3, 2\zeta_4](1/2),$$
$$T_3 = [2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4],\ T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2).$$

The case $T = S_1 \oplus S_2 \oplus R$, where S_1, S_2 are even lattices of rank one, and R is an unimodular lattice of the rank two. If one of the lattices

$S_1(1/2), S_2(1/2)$ is not even, then the following sequence of the lattices is acceptable:

$$T_1 = T\,,\ T_2 = 2T(1/2)\,.$$

Now suppose that the lattice $S_2(1/2)$ is even. Let $\{\zeta_1\}$ be a basis of S_1, $\{\zeta_2\}$ be a basis of S_2, and $\{\zeta_3, \zeta_4\}$ be a basis of the lattice R. Then the following sequence is acceptable:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]\,,\ T_2 = [2\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4](1/2)\,,$$
$$T_3 = [\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4]\,,\ T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2)\,.$$

Let rk $T = 5$.

Suppose that $T = S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus S_5$, where rk $S_i = 1$, and the lattices $S_4(1/2)$ and $S_5(1/2)$ are even. Let $\{\zeta_i\}$ be a basis of S_i. Then the following sequence of lattices is acceptable:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_2 = [2\zeta_1, 2\zeta_2, 2\zeta_3, \zeta_4, \zeta_5](1/2)\,,$$
$$T_3 = [2\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, \zeta_5](1/2)\,,$$
$$T_5 = [\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4, \zeta_5]\,,\ T_6 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,.$$

Let $S = S_1 \oplus S_2 \oplus S_3 \oplus R$, where S_1, S_2, S_3 are lattices of rank 1, rk $R = 2$, and the lattices $S_3(1/2)$ and $R(1/2)$ are even. Let $\{\zeta_1\}$ be a basis of S_1, $\{\zeta_2\}$ be a basis of S_2, $\{\zeta_3\}$ be a basis of S_3, and $\{\zeta_4, \zeta_5\}$ be a basis of R. In this case the following sequence of lattices is acceptable:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_2 = [2\zeta_1, 2\zeta_2, 2\zeta_3, \zeta_4, \zeta_5](1/2)\,,$$
$$T_3 = [2\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_4 = [2\zeta_1, 2\zeta_2, \zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,,$$
$$T_5 = [2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_6 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,.$$

Now suppose that the lattice T has no representations of the types above. Then only the following cases are possible. We consider them at once.

The case $T = S \oplus R_1(2^m) \oplus R_2(2^n), m \geq 0, n \geq 0$, where rk $S = 1$ and R_1, R_2 are even unimodular lattices of the rank 2. Let $\{\zeta_1\}$ be a basis of S, $\{\zeta_2, \zeta_3\}$ of $R_1(2^m)$, $\{\zeta_4, \zeta_5\}$ of $R_2(2^n)$. Suppose that $m \geq 1$. Then we obtain the following acceptable sequence:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_2 = [2\zeta_1, \zeta_2, \zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,,$$
$$T_3 = [2\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_4 = [2\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)$$
$$T_5 = [2\zeta_1, \zeta_2, \zeta_3, \zeta_4, 2\zeta_5]\,,\ T_6 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,.$$

Suppose that $m = n = 0$. If the lattice $S(1/2)$ is not even, then we obtain the following acceptable sequence:

$$T_1 = T\,,\, T_2 = 2T(1/2)\,.$$

If the lattice $S(1/2)$ is even, then the following sequence is acceptable:

$$T_1 = [\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]\,,\; T_2 = [\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,,$$
$$T_3 = [\zeta_1,\zeta_2,\zeta_3,2\zeta_4,2\zeta_5]\,,\; T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,.$$

The case $T = R \oplus S_1 \oplus S_2 \oplus S_3$, where R is an even unimodular lattice of the rank 2, and S_1, S_2, S_3 are lattices of the rank one. The case, when all lattices $S_1(1/2), S_2(1/2), S_3(1/2)$ are not even is reduced to the previous case, because then $S_1 \oplus S_2 \oplus S_3 = R'(2) \oplus S'$, where R' is an even unimodular lattice of rank 2 and S' is a lattice of rank 1. Thus, we can suppose that the lattice $S_3(1/2)$ is even. Let $\{\zeta_1, \zeta_2\}$ be a basis of R, $\{\zeta_3\}$ of S_1, $\{\zeta_4\}$ of S_2, and $\{\zeta_5\}$ of S_3. Suppose that one of the lattices $S_1(1/2)$ or $S_2(1/2)$ is not even. In this case, we have the following acceptable sequence:

$$T_1 = [\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]\,,\; T_2 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, \zeta_5](1/2)\,,$$
$$T_3 = [\zeta_1,\zeta_2,2\zeta_3,2\zeta_4,\zeta_5]\,,\; T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,.$$

Suppose now that the lattice $S_2(1/2)$ is even (together with the lattice $S_3(1/2)$). Then the following sequence is acceptable:

$$T_1 = [\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]\,,\; T_2 = [2\zeta_1, 2\zeta_2, 2\zeta_3, \zeta_4, \zeta_5](1/2)\,,$$
$$T_3 = [\zeta_1,\zeta_2,2\zeta_3,\zeta_4,\zeta_5]\,,\; T_4 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, \zeta_5](1/2)\,,$$
$$T_5 = [\zeta_1,\zeta_2,2\zeta_3,2\zeta_4,\zeta_5]\,,\; T_6 = [2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)\,.$$

It finishes the proof of the statement.

3.3. Here, for a lattice T of rk $T \le 5$ and with an even lattice $T(1/2)$, we will construct an acceptable sequence $T = T_1, \dots, T_m = T''$ of lattices such that m is odd and $T'' = T(1/2) \subset T \otimes \mathbb{Q}(1/2)$.

Suppose that rk $T \le 4$. Then the following sequence is acceptable:

$$T_1 = T\,, T_2 = T(1/2)\,.$$

Suppose that rk $T = 5$.

Let $T = R_1(2) \oplus R_2(2) \oplus S(4)$, where the lattices R_1, R_2, S are even and rk $R_1 = \mathrm{rk}\, R_2 = 2$, rk $S = 1$. Let (ζ_1, ζ_2) be a basis of $R_1(2)$, $\{\zeta_3, \zeta_4\}$ be a basis of $R_2(2)$, and $\{\zeta_5\}$ of $S(4)$. Then the following sequence is acceptable:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_2 = [2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2)\,,$$
$$T_3 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5/2]\,,\ T_4 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2)\,.$$

Let $T = R_1(2) \oplus R_2(4) \oplus S(2)$ where the lattices R_1, R_2, S are even. Let $\{\zeta_1, \zeta_2\}$ be a basis of $R_1(2)$, $\{\zeta_3, \zeta_4\}$ be a basis of $R_2(4)$, and $\{\zeta_5\}$ be a basis of $S(2)$. Then the following sequence is acceptable:

$$T_1 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]\,,\ T_2 = [2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2)\,,$$
$$T_3 = [\zeta_1, \zeta_2, \zeta_3/2, \zeta_4/2, \zeta_5]\,,\ T_4 = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2)\,.$$

Now suppose that lattice T has no representations of the type above. Then $T = R_1(2) \oplus R_2(2) \oplus S(4)$, where R_1, R_2 are even unimodular lattices and rk $R_1 = \mathrm{rk}\, R_2 = 2$, S is an odd unimodular lattice and rk $S = 1$. Then the following sequence is acceptable:

$$T_1 = T\,,\ T_2 = T(1/2)\,.$$

It finishes the proof of the statement.

3.4. Here we will finish the proof of the Theorem. We consider the most difficult case rk $T_X = \mathrm{rk}\, T = 5$.

Let us reduce the case $d = 2$ to the case $d = 1$. Using Sec. 3.2, we can find an acceptable sequence $T = T_1, \dots, T_m$, such that $T_m = 2T(1/2)$. In the case $d = 2$ both lattices T_m and $\tilde{T}$ are contained in the one form $T(1/2) \otimes \mathbb{Q}$. It is sufficient to find an acceptable sequence for $T = T_m$ and $\tilde{T}$ where both lattices are contained in the one form $T(1/2) \otimes \mathbb{Q}$. Thus, we have to deal with the case $d = 1$ now:

Now suppose that $d = 1$. Then both lattices T and $\tilde{T}$ are lattices of the one quadratic form $T \otimes \mathbb{Q}$. Let $S = T \cap \tilde{T}$. Thus, we have the following sequence of inclusions of the lattices of the form $T \otimes \mathbb{Q}$:

$$T \supset S \subset \tilde{T}\,.$$

Using results 3.2, we can find an acceptable sequence

$$T = T_1, \dots, T_{2m} = 2T \subset T \otimes \mathbb{Q}\,.$$

Using results 3.3, we can find an acceptable sequence

$$2\tilde{T} = S_1, \dots, S_{2n} = \tilde{T} \subset T \otimes \mathbb{Q}\,.$$

Thus, it is sufficient to find an acceptable sequence with the first term $2T$ and with the final term $2\tilde{T}$. The lattices $2T$ and $2\tilde{T}$ are more convenient because the lattice $2T \cong T(4)$ and the lattice $2\tilde{T} \cong \tilde{T}(4)$ where T and $\tilde{T}$ are even lattices.

Thus, it is sufficient to find an acceptable sequence for the lattices $T \cong T'(4)$ and $\tilde{T} \cong \tilde{T}'(4)$ where T' and $\tilde{T}'$ are even lattices. Further, we suppose that it is true.

The quotient group T/S is a finite abelian 2-group. It follows that there exists a sequence of sublattices of the form $T \otimes \mathbb{Q}$:

$$T = S_1 \supset S_2 \supset \dots \supset S_a = S\,,$$

for which $S_i/S_{i+1} \cong \mathbb{Z}/2\mathbb{Z}\,, i = 1, \dots a-1$. Let S_i' be a sublattice of $T \otimes \mathbb{Q}$ which satisfies the condition:

$$S_i \supset S_{i+1} \supset S_i' \supset 2S_i\,, \text{ and } S_i'/2S_i \cong (\mathbb{Z}/2\mathbb{Z})^2\,.$$

Then, evidently

$$S_{i+1}/S_i' \cong (\mathbb{Z}/2\mathbb{Z})^2\,.$$

Let us show that the sequence of the lattices

$$S_i\,, S_i'(1/2)\,, S_{i+1}$$

is acceptable.

The lattice $S_i = M(4)$ where M is an even lattice (since it is true for the lattice T and $S_i \subset T$). Then, the sublattice S_i' is constructed from the subgroup $\mathfrak{H} = (1/2)S_i' = (1/2)S_i'/S_i \subset A_{S_i}$, $\mathfrak{H} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $q_{S_i}|\,\mathfrak{H} = 0$. It follows that there exists an embedding of the forms:

$$q_{S_i}|\,\mathfrak{H} \to u_+(2)^4\,.$$

We have: $l(\mathcal{A}_{(S_i)_2}) = 5$ because $S_i = M(4)$ where M is a lattice. So, we have the equality: $2 = (\mathrm{rk}\, S_i + l(\mathcal{A}_{(S_i)_2})/2 - 3$. Thus, we should prove the congruence for the lattice S_i over $\mathbb{Z}$:

$$|\mathcal{A}_{S_i}| \equiv \pm \mathrm{discr}\, K(q_{(S_i)_2}) \bmod (\mathbb{Z}_2^*)^2 .$$

Since $S_i = M(4)$, in this case $K(q_{(S_i)_2}) \cong S_i \otimes \mathbb{Z}_2$. It follows that discr $S_i = \pm|\mathcal{A}_{S_i}|$, and the condition (ii) of the Theorem 2.2.7 is true.

The lattice $S_i'(1/2) \subset S_i(1/2) \subset T(1/2) = T'(2)$, where T' is an even lattice. Using this fact, in the same way as above, one proves that the sequence of the lattices $S_i'(1/2), S_{i+1}$ is acceptable. Corresponding to this sequence the subgroup $\mathfrak{H}$ of the discriminant group of the lattice $S_i'(1/2)$ is $\mathfrak{H} = S_{i+1}(1/2)/S_i'(1/2) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

In such a way, we obtain an acceptable sequence of sublattices of $T \otimes \mathbb{Q}$:

$$T = S_1 \supset S_1'(2) \subset S_1 \supset \ldots \subset S_{a-1} \supset S_{a-1}'(2) \subset S_a = S .$$

The quotient group $\tilde{T}/S$ is a finite abelian 2-group also. Then we can find a sequence of sublattices of the form $T \otimes \mathbb{Q}$:

$$S = P_1 \subset P_2 \subset \ldots \subset P_{b-1} \subset P_b = \tilde{T}$$

with $P_{i+1}/P_i \cong \mathbb{Z}/2\mathbb{Z}, 1 \leq i \leq b-1$. Let P_i' be a sublattice of the form $T \otimes \mathbb{Q}$ which satisfy the condition:

$$2P_{i+1} \subset P_i' \subset P_i \text{ and } P_i/P_i' \cong (\mathbb{Z}/2\mathbb{Z})^2 .$$

Let us show that the sequence of lattices

$$P_i, P_i'(1/2), P_{i+1}$$

is acceptable.

The lattice $P_i = M(4)$ where M is an even lattice, since it holds for $\tilde{T}$, and P_i is a sublattice of the $\tilde{T}$. Then the lattice $P_i'(1/2)$ is constructed from the subgroup $\mathfrak{H} = (1/2)P_i'/P_i \subset \mathcal{A}_{P_i'}\, \mathfrak{H} \cong (\mathbb{Z}/2\mathbb{Z})^3$ and $q_{P_i} | \mathfrak{H} = 0$. It follows that there exists an embedding of the forms:

$$q_{P_i} | \mathfrak{H} \to u_+^{(2)}(2)^4 .$$

Since rk $P_i = 5$, then we have the strong inequality:

$$3 > (\mathrm{rk}\, P_i + l(\mathcal{A}_{(P_i)_2})/2 - 3 = 2 .$$

It proves the condition (ii) of Theorem 2.2.7, and the sequence of lattices $P_i, P_i'(1/2)$ is acceptable.

The lattice $P_i'(1/2) \subset P_i(1/2) \cong M(2)$, where the lattice M is even. Using this fact, in the same way as above, one proves that the sequence of the lattices $P_i'(1/2), P_{i+1}$ is acceptable. Corresponding to this sequence, the subgroup $\mathfrak{H}$ of the discriminant group of the lattice $P_i'(1/2)$ is $\mathfrak{H} = P_{i+1}(1/2)/P_i'(1/2) \cong (\mathbb{Z}/2)^3$.

In such a way, we obtain an acceptable sequence of the lattices of the form $T \otimes \mathbb{Q}$:

$$S = P_1 \supset P_1'(2) \subset P_2 \supset \ldots \subset P_{b-1} \supset P_{b-1}'(2) \subset P_b = \tilde{T}.$$

This finishes the proof of the Theorem. □

From Theorem 3.1 and the theory of Kummer surfaces, we obtain the following theorem (Theorem 1.3 of the Introduction). This theorem was proved by I. R. Shafarevich and the author together.

Theorem 3.2. (V. V. Nikulin and I. R. Shafarevich). Let X and Y be algebraic K3 surfaces. Suppose that for all odd prime p there are primitive embeddings of p-adic lattices:

$$T_X \otimes \mathbb{Z}_p \subset U^3 \otimes \mathbb{Z}_p \text{ and } T_Y \otimes \mathbb{Z}_p \subset U^3 \otimes \mathbb{Z}_p;$$

and for $p = 2$ there are embeddings of the quadratic forms over the field $\mathbb{Q}_2$:

$$T_X \otimes \mathbb{Q}_2 \subset U^3 \otimes \mathbb{Q}_2 \text{ and } T_Y \otimes \mathbb{Q}_2 \subset U^3 \otimes \mathbb{Q}_2.$$

Let for the positive square-free integer d we have an isomorphism $\varphi : T_Y(d) \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q}$ of quadratic forms over $\mathbb{Q}$ (an abstract modification) and $\varphi(H^{2,0}(Y)) = H^{2,0}(X)$.

Then there exists a rational map $f : X \to Y$ such that $\varphi = \overline{f^*}$.

Proof. One can see very easily that for any odd prime p we have an isomorphism: $U \otimes \mathbb{Z}_p \cong U(2) \otimes \mathbb{Z}_p$, and that $U \otimes \mathbb{Q}_2 \cong U(2) \otimes \mathbb{Q}_2$. It follows that for any odd prime p there are primitive embeddings

$$T_X \otimes \mathbb{Z}_p \subset U(2)^3 \otimes \mathbb{Z}_p \text{ and } T_Y \otimes \mathbb{Z}_p \subset U(2)^3 \otimes \mathbb{Z}_p$$

and

$$T_X \otimes \mathbb{Q}_2 \subset U(2)^3 \otimes \mathbb{Q}_2 \text{ and } T_Y \otimes \mathbb{Q}_2 \subset U(2)^3 \otimes \mathbb{Q}_2 .$$

The lattice $U(2)^3$ is unique in its genus (it follows from the classification of the unimodular lattices). Then, there exist embeddings of the lattices $T_X \subset U(2)^3$ and $T_Y \subset U(2)^3$ such that these embeddings are primitive over all odd prime p. Let T_1 be the primitive sublattice of $U(2)^3$, generated by T_X, and T_2 be the primitive sublattice of $U(2)^3$ generated by T_Y. We have the natural identifications $T_X \otimes \mathbb{Q} = T_1 \otimes \mathbb{Q}$ and $T_Y \otimes \mathbb{Q} = T_2 \otimes \mathbb{Q}$ of the quadratic forms over $\mathbb{Q}$ such that for all odd prime p we have $T_X \otimes \mathbb{Z}_p = T_1 \otimes \mathbb{Z}_p$ and $T_Y \otimes \mathbb{Z}_p = T_2 \otimes \mathbb{Z}_p$ under the identifications. Surfaces X and Y are algebraic. It follows that $\operatorname{rk} T_X = \operatorname{rk} T_Y \leq 5$ since there are embeddings $T_X \subset U(2)^3$ and $T_Y \subset U(2)^3$. From the prove of Theorem 3.1, it follows that there are K3 surfaces X_1 and Y_1, and rational maps $g_1 : X - - \to X_1$ and $g_2 : Y_1 - - \to Y$, which are compositions of the rational maps of degree two, and isomorphisms of the lattices $\vartheta_1 : T_{X_1} \cong T_1$ and $\vartheta_2 : T_2 \cong T_{Y_1}$ such that $\overline{g_1^*} = \vartheta_1 \otimes \mathbb{Q}$ and $\overline{g_2^*} = \vartheta_2 \otimes \mathbb{Q}$ under the identifications above of the quadratic forms over $\mathbb{Q}$:$T_X \otimes \mathbb{Q} = T_1 \otimes \mathbb{Q}$ and $T_Y \otimes \mathbb{Q} = T_2 \otimes \mathbb{Q}$. Under the identifications, the preserving periods modification $\varphi : T_Y(d_1) \otimes \mathbb{Q} \cong T_X \otimes \mathbb{Q}$ defines the preserving periods modification

$$\varphi_1 = (\vartheta_1 \otimes \mathbb{Q})^{-1} \cdot \varphi \cdot (\vartheta_2 \otimes \mathbb{Q})^{-1} : T_{Y_1}(d_1) \otimes \mathbb{Q} \cong T_{X_1} \otimes \mathbb{Q} .$$

The lattices $T_{X_1} \cong T_1$ and $T_{Y_1} \cong T_2$ have primitive embeddings into the lattice $U(2)^3$. It follows from the criterion of [N1] for K3 surface to be Kummer surface and [N3] (see [Mo]) that both K3 surfaces X_1 and Y_1 are Kummer surfaces. We recall that if A is an abelian surface and ι is a multiplication by -1 on A, then the minimal resolution Z of singularities of the surface $A/\{1, -1\}$ is called Kummer surface. This surface is an algebraic K3 surface. It is not difficult to prove that the statement of the theorem is true for the abelian surfaces and homomorphisms of abelian surfaces. The transcendental lattices of Z and A are naturally identified: $T_Z = T_A(2)$, and under this identification $H^{2,0}(Z) = H^{2,0}(A)$. It follows that the theorem is true for Kummer surfaces (every homomorphism between abelian surfaces gives the rational map of the corresponding Kummer surfaces and the corresponding modification of their transcendental periods). Thus, there exists a rational map $h : X_1 - - \to Y_1$, and $\overline{h^*} = \varphi_1$. Then the rational map $g_2 \cdot h \cdot g_1 : X - - \to Y$ gives the modification φ. □

Remark 3.3. It is very easy to reformulate the conditions of Theorem

3.2 using discriminant forms:

$$\operatorname{rk} T_X + l(q_{(T_X)_p}) \le 6$$

for all odd prime p, and

$$|\mathcal{A}_{T_X}| \equiv -\operatorname{discr} K(q_{(T_X)_p}) \bmod (\mathbb{Z}_p^*)^2$$

for all odd prime p for which $\operatorname{rk} T_X + l(q_{(T_X)_p}) = 6$;

$$\operatorname{rk} T_X + l(\tilde{q}_{(T_X)_2}) \le 6\,,$$

and

$$|\mathcal{A}_{T_X}| \equiv \pm\operatorname{discr} K(\tilde{q}_{(T_X)_2})$$

if $\operatorname{rk} T_X + l(\tilde{q}_{(T_X)_2}) = 6$ and $\tilde{q}_{(T_X)_2} \not\cong q_\theta^{(2)}(2) \oplus q'$. (Here $\tilde{q}_{(T_X)_2}$ is the discriminant form of a maximal even overlattice of the lattice $T_X \otimes \mathbb{Z}_2$).

Remark 3.4. The condition of the Theorem 3.2 holds if $\operatorname{rk} T_X = \operatorname{rk} T_Y \le 3$. Thus, in this case Theorem 3.2 is true.

4. Several Remarks

We want to give here several remarks about the results obtained above.

4.1. Theorem 3.1 (or the Theorem 1.3 of the Introduction) is not true for $\operatorname{rk} T_X = 6$. If $(T_X)_2 = T_X \otimes \mathbb{Z}_2 \cong V^{(2)}(1)^3$, then the condition (ii) of the Theorem 2.2.5 does not hold. Thus, the surface X has no rational maps of degree two into other K3 surfaces, and Theorem 3.1 is not true for the surface X and any other K3 surface Y (for example for $Y = X$).

4.2. Let us remark that every abstract modification $\varphi : T_1(d) \otimes \mathbb{Q} \to T_2 \otimes \mathbb{Q}$ of the lattices defines the inverse modification $\varphi^{-1} : T_2(d) \otimes \mathbb{Q} \to T_1 \otimes \mathbb{Q}$. Thier composition (in the sense of 2.2.5) $\varphi^{-1} \cdot \varphi : T_1 \otimes \mathbb{Q} \to T_1 \otimes \mathbb{Q}$ should be the identical map. Thus, a rational map $f : X - - \to Y$ of surfaces gives also an inverse modification $\overline{f^*}^{-1} : T_X(d_1) \otimes \mathbb{Q} \to T_Y \otimes \mathbb{Q}$.

For $\operatorname{rk} T_X = \operatorname{rk} T_Y = 6$ we obtain the following variant of Theorem 3.1: An abstract modification $\varphi : T_X(d) \otimes \mathbb{Q} \to T_Y \otimes \mathbb{Q}$ satisfying conditions of Theorem 3.1 is a composition of the modifications corresponding to rational maps of degree two between K3 surfaces and of their inverse. The proof of the statement is similar to the proof of Theorem 3.1.

4.3. For rk $T_X = 7$ the statement above is not true. There are K3 surfaces with rk $T_X = 7$ such that for the lattice T_X, condition (i) of Theorem 2.2.5 does not hold. This K3 surface has no symplectic involutions and has no rational maps of degree two $Z \dashrightarrow X$ of a K3 surface Z.

4.4. Results of the paper show that it is very important in questions 1.1 and 1.2 to construct some examples of rational maps between K3 surfaces. Here we used rational maps of degree two between K3 surfaces and rational maps between Kummer surfaces which are induced by the homomorphisms between abelian surfaces. All other rational maps between K3 surfaces in this paper were compositions of these rational maps.

It would be very interesting to describe rational maps $f : X \dashrightarrow Y$ of degree 3 between K3 surfaces. If f is a Galois map then f is defined by the action of the abelian symplectic group of order 3 on the surface X, and all these actions and the corresponding quotient maps f are described in [N2]. In this case, rk $T_X = \operatorname{rk} T_Y \leq 10$, and these maps are very rare. But a description of the non-normal rational maps f of degree 3 is unknown now.

We do not know examples of rational maps $f : X \dashrightarrow Y$ of degree > 1 between general (with rk $S_X = \operatorname{rk} S_Y = 1$) K3 surfaces X and Y.

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V. V. Nikulin
Steklov Mathematical Institute
ul. Vavilova 42
Moscow 117966, GSP-1 USSR

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SOME CLASSES OF VARIATIONAL INEQUALITIES

Muhammad Aslam Noor

ABSTRACT

Variational inequality theory is an effective technique to study a wide class of problems arising in various branches of pure and applied sciences. In recent years, this theory has been extended and generalized in various directions. The main aim of this paper is to introduce and study a new class of variational inequalities, which includes and generalizes the previous known results. Projection technique is used to suggest and propose a new unified and general algorithm for these classes of variational inequalities. Convergence and sensitivity analysis is also considered.

1. INTRODUCTION

It is well known that variational principles enable us to study many unrelated problems arising in different branches of pure and applied sciences in a unified and general framework. In recent years, these principles have been enriched by the discovery of variational inequality theory. Variational inequalities were introduced by Stampacchia [1] and Fichera [2] in the early 1960's to study the problems in potential theory and mechanics respectively. Since then, this subject has been developed in several directions using new and powerful methods. The variety of problems to which variational inequality techniques may be applied is impressive and amply representative for the richness of the field. Some of these developments have made mutually enriching contacts with other areas of mathematical and engineering sciences including elasticity, transportation and economics equilibrium theory, nonlinear programming and operations research, see Kikuchi and Oden [3], Baiocchi and Capelo [4], Crank [5], and Rodrigues [6] and the reference therein for mathematical and physical modelling and applications. This theory was developed simultaneously not only to study the fundamental facts about the qualitative behaviour of solutions of nonlinear problems, but also to solve them more efficiently numerically. In fact, this theory provides us a sound basis for computing the approximate solution of many moving and free boundary values problems in a unified framework.

In 1971, Baiocchi reformulated the flow problems through porous media in terms of variational inequalities by using a transformation, see Oden and Kikuchi [7] for formulation and numerical results. Since then, variational inequalities have made a tremendous impact in this field and related areas. Recently Kikuchi and Oden [3] have shown that the general problem of equilibrium of elastic bodies in contact with rigid foundation on which frictional forces are developed, can be characterized by a class of variational inequalities. It is worthmentioning that the formulation of contact problems as variational inequalities was originally studied and considered by Duvaut and Lions [8]. One of the main advantages of the variational inequality formulation is that the location of the free boundary (contact area) becomes an instrinsic part of the solution and no special devices are needed to locate it. In most cases, the existence of solutions to such problems is an open problem. Some special cases have considered by Noor [9,10], Demkowicz and Oden [11], and Duvaut and Lions [8].

Equally important is the area of mathematical programming known as the complementarity theory, which was introduced and studied by Lemke [12] in 1964. Cottle and Dantzig [13] defined the complementarity problem and called it the fundamental problem. A survey paper by Lemke [14] outlines the early theoretical results, most of which were motivated and inspired by applications to equilibrium type problems in operations research and game theory. For most recent results and applications, see [3,4,5,6,15]. The relationship between a variational inequality problem and a complementarity problem has been noted by Lions [16], Lions and Stampacchia [17] and Mancino and Stampacchia [18]. However, it was Karamardian [19,20], who showed that if the set involved in a variational inequality problem and complementarity problem is a convex cone, then both problems are equivalent. This interrelation between these problems is very useful and has been successfully applied to use the variational inequality technique to suggest and analyze constructive algorithms for complementarity problems by Ahn [21] and Noor [22, 23]. For related work, see Rassias [24], where one can find global variational methods for variational problems of more than one variables.

It is clear that the theory so far developed in recent years is applicable for considering free and moving boundary problems of even order. Nothing is known for the case of odd order boundary problems. Tonti [25] has developed a very general theory to derive the variational principles for both odd and even order boundary problems. Inspired and motivated by the applications of variational principles in the theory of differential equations, the author has developed iterative algorithms for certain classes of variational inequalities related with odd order differential boundary problems. It is well known that all these classes are generalization of the variational inequality introduced by Lions and Stampacchia in 1967. It is natural to consider the unification of these different generalizations. In this paper, we introduce a new class of variational inequalities, which unifies many of the previously known classes. Projection technique is used to suggest an iterative algorithm for this class. Various special cases have been discussed. We have given only a brief introduction of this fast growing interesting field of pure and applied sciences. The interested reader is advised to explore this field further. It is our hope that this brief introduction may inspire and motivate the readers to discover new and innovative applications of variational inequalities in other areas of sciences. Despite of all the activities going on in this subject, still many open problems remain to be considered, especially the sensitivity analysis for variational inequalities. Furthermore, the development and refinement of algorithms for finding the approximate solutions of variational inequalities need further research work.

In Section 2, we review the relevant literature and formulate a new general class of variational inequalities. Projection technique is used to suggest to an iterative algorithm, which is the subject of Section 3 convergence analysis is discussed in Section 4. Sensivity analysis is studied in Section 5.

2. BASIC RESULTS AND FORMULATIONS

Let H be a real Hilbert space with its dual space H', whose inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Let K be a closed convex nonempty set in H. We also denote by $\langle\cdot,\cdot\rangle$, the pairing between H' and H.

Given a continuous operator $T : H \longrightarrow H'$, we consider the problem of finding $u\varepsilon K$ such that

$$\langle Tu, v-u\rangle \geq 0, \qquad \text{for all} \quad v\varepsilon K. \tag{2.1}$$

The inequality of type (2.1) is known as variational inequality introduced and studied by Stampacchia and Fichera in 1964. Lions and Stampacchia [17] proved the existence of unique solution of (2.1) using essentially the projection technique. We note that if T is a linear symmetric operator, then the solution $u\varepsilon K$ satisfying (2.1) is equivalent to find the minimum of the functional $I[v]$, defined by

$$I[v] = \frac{1}{2}\langle Tv,v\rangle , \tag{2.2}$$

on the convex set K in H.

For the case, when $K = H$, then problem (2.1) is equivalent to find $u\varepsilon H$ such that

$$\langle Tu,v\rangle = 0, \qquad \text{for all } v\varepsilon H \tag{2.3}$$

The problem (2.3) is known as the weak formulation of boundary value problems, where T is any differential or integral operator associated with the given problem, see Lions [16].

In the formulation of the variational inequality, the underlying convex set K does not depend upon the solution. In many applications, the convex set K also depends implicitly on the solution u itself. In this case, the variational inequality (2.1) is known as the quasi-variational inequality, which is a generalization of the variational inequality (2.1). This useful generalization was considered and studied by Bensoussan and Lions [26]. To be more specific, a quasi-variational inequality problem is indeed a problem of the type:

Given a point-to set mapping $K:u \longrightarrow K(u)$, which associates a closed convex subset $K(u)$ of H with any element u of H, find $u\varepsilon K(u)$ such that

$$\langle Tu, v-u\rangle \geq 0, \qquad \text{for all } v\varepsilon K(u) \tag{2.4}$$

In many important applications, see Mosco [27], Bensoussan and Lions [26] and Baiocchi and Capelo [4], the set $K(u)$ is of the following form

$$K(u) = m(u) + K, \tag{2.5}$$

where m is a point-to-point mapping and K is a closed nonempty convex set of H. Note that if the point-to-point mapping m is zero, then quasi-variational inequality problem (2.4) is exactly the variational inequality problem (2.1). It has been shown by Noor [28], Noor and Noor [29], Glowinski, Lions and Tremolieres [30] that the solution of the problems (2.4) and (2.1) can be obtained from the iterative methods using the project techniques.

In 1975, Noor [28] extended the variational inequality problem (2.1) to study a class of mildly nonlinear elliptic boundary value problems having constraints. Given nonlinear operators $T,A:H \longrightarrow H'$, we consider the problem of finding $u \varepsilon K$ such that

$$\langle Tu,v-u \rangle \geq \langle A(u), v-u \rangle, \quad \text{for all } v \varepsilon K \qquad (2.6)$$

The inequalities of the type (2.6) are known as the (strongly) mildly nonlinear variational inequalities. It is worth mentioning that unilateral contact problems involing contact laws of monotone nature do not lead to the formulation of variational inequalities directly. However, it has been shown by Papagiotopoulos [31], using the notions of Clarke's generalized gradient and Rockafellor's upper subderivative, that the nonconvex unilateral contact problems can only be characterized by a class of variational inequalities of type (2.6). For the existence, iterative methods and finite element approximate solutions of inequalities (2.6), see Noor [28,32,33].

The quasi mildly nonlinear variational inequality problem is to find $u \varepsilon K(u)$ such that

$$\langle Tu, v-u \rangle \geq \langle A(u), v-u \rangle, \quad \text{for all } v \varepsilon K(u) \qquad (2.7)$$

This generalization is again due to Noor [34]. For the related work, also see Mosco [27]. It is obvious that the problems (2.4), and (2.6) are two different generalizations of the variational inequalities (2.1) introduced by Stampacchia [1]. Clearly the problem (2.7) is most general and includes (2.1), (2.4) and (2.7) as special cases.

We would like to point out that all these classes of variational inequalities are applicable to study the boundary value problems of even order. The present form of variational inequalities cannot be used to study the odd order constraint boundary value problems. This fact alone motivated us to extend and generalize the present variational inequality theory. In this case, we consider problem of the following form: Given, $T,g:H \longrightarrow H'$, consider the problem of finding $u\varepsilon H$ such that $g(u)\varepsilon K$ and

$$\langle Tu, g(v)-g(u)\rangle \geq 0, \qquad \text{for all } g(v)\varepsilon K. \qquad (2.8)$$

The inequality (2.8) is known as general nonlinear variational inequality. This problem is due to Noor [35]. The variational inequality problem (2.8) has been extended by Noor [36] to the include the case, when the convex set also depends upon the solution implicitly. The general quasi variational inequality problem is to find $u\varepsilon H$ such that $g(u)\varepsilon K(u)$ and

$$\langle Tu, g(v)-g(u)\rangle \geq 0, \qquad \text{for all } g(v)\varepsilon K(u) \qquad (2.9)$$

Motivated and inspired by the research work going on in this area, Noor [37] considered and studied the more general case, which enable us to include the odd order (strongly) mildly nonlinear boundary value problems subject to some constraints. Given $T,A,g:H \longrightarrow H'$ nonlinear operators, we consider the problem of finding $u\varepsilon H$ such that $g(u)\varepsilon K$ and

$$\langle Tu, g(v) - g(u)\rangle \geq \langle A(u), g(v) - g(u)\rangle \quad \text{for all } g(v)\varepsilon K \qquad (2.10)$$

which are known as general mildly nonlinear variational inequalities, see Noor [37] for itertive method and applications.

We note that for $g = I$, the identity operator, the variational inequalities problems (2.8), (2.9) and (2.10) are exactly the same as the problems (2.1), (2.4) and (2.6). These problems enables us to study both the even and odd order boundary value problems in a unified and general framework.

It is clear that the variational inequalities problems (2.4), (2.6) - (2.10) are different generalizations of the original variational inequality problem (2.1). It is natural to consider the unification of these problems and study them in a general framework. This is the main motivation to consider the problem of the type:

Find $u \in H$ such that $g(u) \in K(u)$ and

$$\langle Tu, g(v) - g(u)\rangle \geq \langle A(u), g(v) - g(u)\rangle, \quad \text{for all } g(v) \in K(u) \tag{2.11}$$

Special Cases

We know consider a special case, which is itself a very important and active field of research. We consider the case, when the convex set K is a convex cone. Let

$$K^* = \{v \in H', (v,u) \geq 0, \quad \text{for all } u \in K\}$$

be the polar (dual) cone of K in H. The corresponding problems are as:

Find $u \in K$ such that

$$Tu \in K^* \quad \text{and} \quad (u,Tu) = 0 . \tag{2.12}$$

Such types of problems are known as linear and nonlinear complementarity problems depending upon whether the operator T is linear or nonlinear. These problems are originally due to Lemke [12] and Cottle and Dantzig [13].

The quasi complementarity problem is to find such that

$$Tu \in K^* \quad \text{and} \quad (u,Tu) = 0 . \tag{2.13}$$

Such types of problems have been studied by Dolcetta [38], Pang [39], and Noor [22,23] using different techniques.

The mildly nonlinear complementarity problem is to find $u \in K$ such that

$$(Tu - A(u)) \in K^* \quad \text{and} \quad (u, Tu - A(u)) = 0. \tag{2.14}$$

The problem (2.14) has been studied by Noor [40,23] using the technique of variational inequalities. Iterative algorithms for problem (2.14) are considered in [41] along with convergence analysis.

Noor [42] has also studied the general quasi complementarity problem of the type:

Find $u \in K(u)$ such that

$$Tu \in K^*(u) \quad \text{and} \quad (u, Tu) = 0 . \tag{2.15}$$

Here $K^*(u)$ is the polar cone of $K(u)$ in H.

The general complementarity problem is to find $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^* \quad \text{and} \quad (Tu, g(u)) = 0. \tag{2.16}$$

This problems is due to Oettli and Noor [43]. These problems have been further generalized and extended as follows:

Find $u \in H$ such that

$$g(u) \in K, \ (Tu - A(u)) \in K^* \quad \text{and} \quad (g(u), Tu - A(u)) = 0 \tag{2.17}$$

This problem appears to be new one. Note that if the operator $A(u) \equiv 0$, then problem (2.17) is exactly the one studied by Oettli and Noor [43]. The related general mildly nonlinear quasi complementarity problem is to find $u \in H$ such that

$$g(u) \in K(u), \quad (Tu - A(u)) \in K^* \quad \text{and} \ (g(u), Tu - A(u)) = 0 \tag{2.18}$$

Note that if $A(u) \equiv 0$ and $K(u)$ is independent of u, that is $K(u) \equiv K$, then problem (2.18) reduces to the problem (2.16).

From the above discussions, we conclude that the general strongly nonlinear quasi variational inequality problem (2.11) is more general and includes all the previous ones as special cases.

3. ITERATIVE ALGORITHMS

We, in this section, show that the problem (2.11) is equivalent to a fixed point problem. The fixed point formulation is then used to suggest a general iterative type algorithm for computing the solution of the quasi variational inequalities and its various special cases.

<u>Lemma 3.1</u>: If K(u) is defined by the relation (2.5), then $u \in K(u)$ is a solution of (2.11) if and only if $u \in K(u)$ satisfies the following relation

$$g(u) = P_K[g(u) - \rho\Lambda(Tu - A(u)) - m(u)] + m(u), \qquad (3.1)$$

for some $\rho > 0$. Here P_K is the projection of H into K and m is any arbitrary point-to-point mapping. Λ is the canonical isomorphiom from H' onto H such that for all $v \in H$ and $f \in H'$,

$$\langle f, u\rangle = (\Lambda f, v). \qquad (3.2)$$

<u>Proof</u>: Its proof is similar to that of Lemma 3.1 in Noor [43]. See also Chan and Pang [44].

Lemma 3.1 implies that the problem (2.11) is equivalent to finding a fixed point of

$$u = F(u) ,$$

where

$$F(u) = u-g(u) + m(u) + P_K[g(u) - \rho\Lambda(Tu - A(u)) - m(u)], \qquad (3.3)$$

with a positive constant ρ. The fixed point formulation enables us to propose the following general and unified iterative algorithm for the quasi variational inequalities (2.11).

<u>Algorithm 3.1</u>

For given $u_o \in H$, compute u_{n+1} by the iterative scheme:

$$u_{n+1} = u_n - g(u_n) + m(u_n) + P_K[g(u_n) - \rho A(Tu_n - A(u_n)) - m(u_n)]$$

$$n = 0,1,2,\ldots \qquad (3.4)$$

for $\rho > 0$.

Special Cases

I. If the point-to-point mapping m is zero, then Algorithm 3.1 is exactly the same as discussed in Noor [37].

Algorithm 3.2

For given $u_o \varepsilon H$, compute u_{n+1} by the iterative scheme.

$$u_{n+1} = u_n - g(u_n) + P_K[g(u_n) - \rho A(Tu_n - A(u_n))], \quad n=0,1,2\ldots$$

II: If the nonlinear operator $A(u) \equiv 0$, then Algorithm 2.1 is equivalent to:

Algorithm 3.3

For given $u_o \varepsilon H$, compute u_{n+1} by the scheme.

$$u_{n+1} = u_n - g(u_n) + m(u_n) + P_K[g(u_n) - \rho AT_n - m(u_n)], \quad n=0,1,2,$$

For the convergence analysis of Algorithm 3.3, see Noor [36].

III: If the nonliner operator $A(u) \equiv 0$ and $m(u) \equiv 0$, then Algorithm 3.1 reduces to the following.

Algorithm 3.4

For given $u_o \varepsilon H$, find u_{n+1} from the iterative scheme

$$u_{n+1} = u_n - g(u_n) + P_K[g(u_n) - \rho ATu_n], \quad n=0,1,2,\ldots.$$

This result is again due to Noor [45].

IV: If g=I, the identity operator, and m(u) = 0, then Algorithm 3.1 is exactly the one discussed in Noor [33] and Noor and Noor [29].

Algorithm 3.5

For given $u_o \varepsilon H$, find u_{n+1} from the scheme.

$$u_{n+1} = P_K[u_n - \rho(Tu_n - A(u_n)], \quad n = 0,1,2,\ldots.$$

V: If g=I, the identity operator, m(u) = 0, and A(u) ≡ 0, then Algorithm 3.1 becomes:

Algorithm 3.6

For given $u_o \varepsilon H$, find u_{n+1} from the scheme.

$$u_{n+1} = P_K[u_n - \rho\ Tu_n], \quad n = 0,1,2,\ldots..$$

This algorithm is mainly due to Glowiniski, Lions and Tremolieres [30] and Noor and Noor [29].

VI. If g=I, the identity operator, and A(u) ≡ 0, then Algorithm 3.1 reduces to the one that is proposed by Noor [45], see also Chan and Pang [44].

Algorithm 3.7

For given $u_o \varepsilon H$, find u_{n+1} from the iterative scheme

$$u_{n+1} = m(u_n) + P_K[u_n - \rho\, Tu_n - m(u_n)], \quad n = 0,1,2,\ldots.$$

For the corresponding complementarity problems, these algorithms can be suggested with the same convergence criteria. From the above discussions and observations, it is clear that Algorithm 3.1 proposed in this paper is more general and includes many previously known algorithms for various classes of variational inequalities and complementarity problems as special cases, which are mainly due to Cryer [46], Mangasarian [47], Ahn [21], Chan and Pang [44], Pang [39], Noor [22,23,25] and Fang [48].

4. CONVERGENCE ANALYSIS

In this section, we study those conditions under which the approximate solution obtained from Algorithm 3.1 converges to the exact solution of the general quasi variational inequality (2.11). For this purpose, we need the following concepts.

Definition 4.1: An operator $T:H \longrightarrow H'$ is said to be

(i) Strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u-v\rangle \geq \alpha\|u-v\|^2, \quad \text{for all } u,v \in H \tag{4.1}$$

(ii) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u-v\|, \quad \text{for all } u,v \in H. \tag{4.2}$$

From (4.1) and (4.2), it follows that $\alpha \leq \beta$.

We now state and prove the main result of this paper.

<u>Theorem 4.1</u>: Let the operators $T,g:H \longrightarrow H'$ be both strongly monotone and Lipschitz continuous respectively. If the operator A and the point-to-point mapping m are also both Lipschitz continuous, then

$$u_n \longrightarrow u \quad \text{strongly in } H,$$

$$\text{for} \quad \left|\rho - \frac{\alpha+\gamma(k-1)}{\beta^2 - \gamma^2}\right| < \frac{\sqrt{(\alpha+\gamma(k-1))^2 - (\beta^2 - \gamma^2)k(2-k)}}{\beta^2 - \gamma^2}, \; k < 1,$$

$$\alpha > \gamma(1-k) + \sqrt{(\beta^2 - \gamma^2)k(2-k)}, \quad \text{and} \quad \gamma(1-k) < \alpha,$$

where u_{n+1} and u are solutions satisfying (3.4) and (2.11) respectively.

<u>Proof</u>: From Lemma 3.1, we conclude that the solution u of (2.11) can be characterized by the relation (3.1). Hence from (3.1) and (3.4), we have.

$$\begin{aligned} ||u_{n+1} - u|| &= ||u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u) + P_K[g(u_n) - m(u_n) \\ &\quad - \rho A(Tu_n - A(u_n))] - P_K[g(u) - m(u) - \rho A(Tu - A(u))]|| \\ &\le 2||u_n - u - (g(u_n) - g(u))|| + 2||m(u_n) - m(u)|| \\ &\qquad + ||u_n - u - \rho A(Tu_n - A(u_n))|| + \rho||A(u_n) - A(u)||, \end{aligned} \tag{4.3}$$

using the fact that P_K is a non-expansive operator [4].

Since T, g are both strongly monotone and Lipschitz continuous, so by using the technique of Noor [45], we have

$$||u_n - u - (g(u_n) - g(u))||^2 \leq (1 - 2\delta + \sigma^2)\ ||u_n - u||^2 \quad (4.4)$$

and

$$||u_n - u - \rho A(Tu_n - Tu))||^2 \leq (1-2\rho\alpha + \beta^2\rho^2)\ ||u_n - u||^2. \quad (4.5)$$

From (4.3), (4.4), (4.5) and by using the Lipschitz continuity of the operator A and mapping m, we obtain

$$\begin{aligned} ||u_{n+1} - u_n|| &\leq \{(2\sqrt{1-2\delta+\sigma^2})\ 2\xi + \rho\gamma + (\sqrt{(1-2\alpha\rho+\beta^2\rho^2)})\}\ ||u_n - u\ || \\ &= \{k + \rho\gamma + t(\rho)\}\ ||u_n - u||\ . \\ &= \theta||u_n - u||, \end{aligned}$$

where

$$\theta = k + \rho\gamma + t(\rho),$$

with

$$k = 2\xi + 2\sqrt{1 - 2\delta + \sigma^2}\ , \quad t(\rho) = \sqrt{1-2\alpha\rho+\rho^2\beta^2}\ .$$

Now $t(\rho)$ assumes its minimum value for $\bar{\rho} = \frac{\alpha}{\beta^2}$ with $t(\rho) = \sqrt{1 - \frac{\alpha^2}{\beta^2}}$. We have to show that $\theta < 1$. For $\rho = \bar{\rho}$, $k+\rho\gamma+t(\rho)<1$ implies that $k < 1$ and $\alpha > \gamma(1-k) + \sqrt{(\beta^2-\gamma^2)k(2-k)}$. Thus it follows that $\theta = k + \rho\gamma + t(\rho) < 1$ for all ρ with

$$\left|\rho - \frac{\alpha+\gamma(k-1)}{\beta^2 - \gamma^2}\right| < \frac{\sqrt{(\alpha+\gamma(k-1))^2 - (\beta^2-\gamma^2)k(2-k)}}{\beta^2 - \gamma^2}, \quad k < 1$$

$$\alpha > \gamma(1-k) + \sqrt{(\beta^2 - \gamma^2)\ k(2-k)} \quad \text{and} \quad \gamma(1-k) < \alpha\ .$$

Since $\theta < 1$, so the fixed point problem (3.1) has a unique solution u and consequently, the iterative solution u_{n+1} obtained from (3.4) converges to u, the exact solution of the problem (2.11).

5. SENSITIVITY ANALYSIS

We now study the sensitivity analysis for the general quasi variational inequality problem (2.11). Sensitivity analysis for variational inequalities has been studied by Dafermos [49], Kyparisis [50,51], Qiu and Magnanti [52] and Tobin [53] using different methods. We mainly follow the projection technique used by Dafarmos [49] and Noor [54] for the study of the sensitivity analysis. This approach has strong geometric flavour. To formulate the problems, let M be an open subset of H in which the parameter λ takes values and assume that $\{K_\lambda(u) : \lambda \in M\}$ is a family of closed convex subsets of H. The parametric general quasi variational inequality problem is to find $u \in H$ such that $g(u) \in K_\lambda(u)$ and

$$\langle T(u,\lambda),\ g(v) - g(u)\rangle \geq \langle A(u,\lambda),\ g(v) - g(u)\rangle, \qquad (5.1)$$

for all $g(v) \in K_\lambda(u)$, where $T(u,\lambda)$ and $A(u,\lambda)$ are given operators defined on the set of (u,λ) with $\lambda \in M$. We also assume that for some $\bar{\lambda} \in M$, the problem (5.1) admits a solution $\bar{u}$.

We want to investigate those conditions under which, for each λ in a neighbourhood of $\bar{\lambda}$, the problem (5.1) has a unique solution $u(\lambda)$ near $\bar{u}$ and the function $u(\lambda)$ is continuous and differentiable. We assume that X is the closure of a ball in H centered at u.

We also need the following concepts.

Definition 5.1: The operator $T(u,\lambda)$ defined on $X \times M$ is said to be locally, for all $\lambda \in M$, $u,v \in X$;

(a) Strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T(u,\lambda) - T(v,\lambda),\ u-v\rangle \geq \alpha \|u-v\|^2, \qquad (5.2)$$

(b) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|T(u,\lambda) - T(v,\lambda)\| \leq \beta\|u-v\|, \tag{5.3}$$

From Lemma 3.1, we conclude that problem (5.1) can be transformed to the fixed point problem of the map:

$$F(u,\lambda) = u - g(u) + m(u) + P_{K_\lambda}[g(u) - \rho(T(u,\lambda) - A(u,\lambda)) - m(u)], \tag{5.4}$$

for all $\lambda \varepsilon M$, some $\rho>0$ and m is a point-to-point mapping.

Since we are interested in the case, when the solution of the problem (5.1) lie in the interior of X, so we consider the map $F^*(u,\lambda)$ defined by

$$F^*(u,\lambda) = u - g(u) + m(u) + P_{K_\lambda \cap X}[g(u) - \rho(T(u,\lambda) - A(u,\lambda)) - m(u)], \tag{5.5}$$

for all $(u,\lambda) \varepsilon X\times M$.

First of all, we show that the map $F^*(u,\lambda)$ has a fixed point, which is the motivation of our next result.

Lemma 5.1: For all $u,v\varepsilon X$, and $\lambda\varepsilon M$, we have

$$\|F^*(u,\lambda) - F^*(v,\lambda)\| \leq \theta\|u-v\|,$$

where $\theta = k+t(\rho)<1$ for $\gamma(1-k) < \alpha$, $k<1$, $\alpha > \gamma(1-k) + \sqrt{(\beta^2-\gamma^2)k(2-k)}$ and

$$\left|\rho - \frac{\alpha+\gamma(k-1)}{\beta^2-\gamma^2}\right| < \frac{\sqrt{(\alpha+\gamma(k-1))^2 - (\beta^2-\gamma^2)k(2-k)}}{\beta^2-\gamma^2}$$

with

$$k = 2\xi + 2\sqrt{1-2\delta+\sigma^2} \text{ and } t(\rho) = \sqrt{1-2\alpha\rho+\beta^2\rho^2}.$$

Proof: Its proof is similar to that of Lemma 3.1

Remark 5.1: From Lemma 5.1, it is clear that the map $F^*(u,\lambda)$ defined by (5.5) has a unique fixed point $u(\lambda)$, that is $u(\lambda) = F^*(u,\lambda)$. We also know that by assumption, the function $\bar{u}$, for $\lambda = \bar{\lambda}$ is a solution of problem (5.1). Again using Lemma 5.1, we see that $\bar{u}$ is a fixed point of $F^*(u,\lambda)$ and it is also a fixed point of $F^*(u,\bar{\lambda})$. Consequently, we have $u(\bar{\lambda}) = \bar{u} = F^*(u(\bar{\lambda}),\bar{\lambda})$.

We now show that the solution $u(\lambda)$ of the parametric variational inequality (5.1) is continuous (Lipschitz continuous).

Lemma 5.2: If the operators $T(\bar{u},\lambda)$, $A(\bar{u},\lambda)$, $g(\bar{u})$, $m(\bar{u})$ and the map $\lambda \longrightarrow P_{K_\lambda \cap X}[g(\bar{u}) - \rho(T(\bar{u},\bar{\lambda}) - A(\bar{u},\bar{\lambda})) - m(\bar{u})]$ are continuous (Lipschitz continuous), then the solution $u(\lambda)$ satisfying (5.1) is continuous (Lipschitz continuous) at $\lambda = \bar{\lambda}$.

Proof: For $\lambda \in M$ and using Lemma 5.1, we have

$$\|u(\lambda) - u(\bar{\lambda})\| \leq \|F^*(u(\lambda),\lambda) - F^*(u(\bar{\lambda}),\lambda\| + \|F^*(u(\bar{\lambda}),\bar{\lambda}) - F^*(u,\bar{\lambda}),\bar{\lambda})\|$$
$$\leq \theta\|u(\lambda) - u(\bar{\lambda})\| + \|F^*(u,(\bar{\lambda}),\lambda) - F^*(u(\bar{\lambda}),\bar{\lambda})\| \quad (5.6)$$

From (5.5) and the fact that the projection map is non-expansive, we have

$$\|F^*(u(\bar{\lambda}),\lambda) - F^*(u(\bar{\lambda}),\bar{\lambda})\| \leq \rho\|T(u(\bar{\lambda}),\lambda) - T(u(\bar{\lambda}),\bar{\lambda})\|$$
$$+ \rho\|A(u(\bar{\lambda}),\lambda) - A(u(\bar{\lambda}),\bar{\lambda}\|$$
$$+ \|P_{K_\lambda \cap X}[g(u(\bar{\lambda})) - \rho(T(u(\bar{\lambda}),\bar{\lambda})$$
$$- A(u(\bar{\lambda}),\bar{\lambda})) - m(\bar{u})] - P_{K_{\bar{\lambda}} \cap X}[g(u(\bar{\lambda}))$$
$$- \rho(T(u(\bar{\lambda}),\bar{\lambda}) - A(u(\bar{\lambda}),\bar{\lambda})) - m(\bar{u})]\|. \quad (5.7)$$

From (5.6), (5.7) and remark 5.1, we have

$$\|u(\lambda) - u(\bar{\lambda})\| \le \frac{\rho}{1-\theta}\left(\|T(\bar{u},\lambda) - T(\bar{u},\bar{\lambda})\| + \|A(\bar{u},\lambda) - A(\bar{u},\bar{\lambda})\|\right)$$

$$+ \frac{1}{1-\theta}\|P_{K_\lambda \cap X}[g(\bar{u}) - m(\bar{u}) - \rho(T(\bar{u},\bar{\lambda}) - A(\bar{u},\bar{\lambda}))]$$

$$- P_{K_{\bar{\lambda}} \cap X}[g(\bar{u}) - \rho(T(\bar{u},\bar{\lambda}) - A(\bar{u},\bar{\lambda})) - m(\bar{u})\|$$

the required result.

Similarly using the technique of Dafermos [49], we can show that there exists a neighbourhood $N \subset M$ of λ such that for $\lambda \in N$, $u(\lambda)$ is the unique solution of the parametric general quasi variational inequality (5.1) in the interior of X.

On the basis of the above results and observations, we obtain the main result of this section.

Theorem 5.1:

Let u be the solution of the parametric general quasi variational inequality (5.1) at $\lambda = \bar{\lambda}$ amd $T(u,\lambda)$ be the locally strongly monotone Lipschitz continuous operator for all $u,v \in X$. If the operators $T(\bar{u},\lambda)$, $A(\bar{u},\lambda)$, $g(\bar{u})$ and the ma $\lambda \to P_{K_\lambda \cap X}[g(\bar{u}) - \rho(T(\bar{u},\bar{\lambda}) - A(\bar{u},\bar{\lambda})) - m(\bar{u})]$ are continuous (Lipschitz continuous) at $\lambda = \bar{\lambda}$, then there exists a neighbourhood $N \subset M$ of $\bar{\lambda}$ such that for $\lambda \in N$, the problem (5.1) has a unique solution $u(\lambda)$ in the interior of X, $u(\bar{\lambda}) = \bar{u}$ and $u(\lambda)$ is continuous (Lipschitz continuous) at $\lambda = \bar{\lambda}$.

Remark 5.2: We would like to point out that the function $u(\lambda)$ as defined in Theorem 5.1 is continuously differentiable on some neighbourhood N of $\bar{\lambda}$. Its proof follows from the technique of Dafermos [49].

We have already shown that if the convex set K is a convex cone in H, then variational inequality problem and the generalized complementarity problem are equivalent. One can study the sensitivity analysis for the parametric generalized quasi complementarity problem of the type find u∈H such that $g(u) \in K_\lambda(u)$ and

$$(T(u,\lambda) - A(u,\lambda)) \in K^*_\lambda(u) \text{ and } (g(u), T(u,\lambda) - A(u,\lambda) = 0.$$

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Muhammad Aslam Noor
Mathematics Department
College of Science
King Saud University
P.O. Box 2455, Riyadh 11451
Saudi Arabia

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THE AHLFORS LAPLACIAN ON A RIEMANNIAN MANIFOLD

Bent Ørsted *Antoni Pierzchalski*

ABSTRACT

Motivated by the theory of quasi–conformal mappings, we define a second order elliptic operator L on the vector fields on a Riemannian manifold M. The kernel of L is the space of conformal Killing vector fields, and we investigate the spectral properties of L under conformal deformations of the metric. In particular we find the conformal variation of the constant term in the asymptotic expansion of the heat kernel of L. This variation is proportional to the log term in the expansion for a related non–elliptic operator. One of our main applications of L is to construct families of smooth quasi–conformal deformations of transformations of M.

0. INTRODUCTION

The classical notion of quasi–conformal transformations, both infinitesimal and global, can in a natural way be extended to the category of Riemannian manifolds [8]. A key role is played by the first–order differential operator $SX = L_X g - \frac{2}{n}(\operatorname{div} X)g$, where X is a vector field on a Riemannian manifold (M,g) and L_X denotes the Lie derivative. We introduce the second–order diffe-

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rentiatial operator $L = S^*S$, where S^* is the formal adjoint of S, and investigate some of its geometrical properties. L will be called the Ahlfors Laplacian of the Riemannian manifold.

L is shown to be elliptic, and the kernel of L consists exactly of the conformal Killing vector fields on M; for that reason (among others) the Lie algebra of conformal vector fields on M forms a finite–dimensional space. The spectrum of L turns out to have some interesting geometrical properties related to conformal deformations of the metric. We find the transformation rule for L under such deformations and apply this to the conformal deformations of the asymptotic expansion of the trace of the operator $\exp(-tL)$. It follows that under a certain technical assumption the coefficient to t^0 in this expansion is a conformal invariant of M for M even–dimensional. This technical assumption is somewhat mysterious; we hope to relate it to geometric properties of M, see the remark at the end of Chapter 3. Specifically we find a relation between the variation of $\operatorname{tr}\exp(-tL)$ and a trace of the heat kernel for SS^*; as a by–product we get asymptotic expansions for $\operatorname{tr}\omega\exp(-tSS^*)$, ω a function. Finally we apply the same semigroup $\exp(-tL)$ to an arbitrary vector field X to obtain a family X_t converging uniformly to a conformal vector field as $t \to \infty$; this provides a partial solution to the problem of finding quasi–conformal deformations of transformations on a Riemannian manifold.

Let us finally mention that from our formula for L it follows directly (as observed in [12]) that a manifold of negative Ricci curvature does not admit any conformal Killing vector fields.

Apart from the examples we give in this paper, we hope that the Ahlfors Laplacian will have further applications in the study of conformal and quasi–conformal geometry on manifolds. L and its spectrum certainly encode much global information, in a natural way generalizing the case of Riemann surfaces.

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CONTENTS

1. BASIC PROPERTIES OF S AND S^*

Let M be a Riemannian manifold of dimension n with a Riemannian metric g. For simplicity, we assume that all manifolds and mappings are smooth, i.e. of class C^∞. ∇ denotes the Levi-Civita connection of the metric g. We extend it in a natural way to the whole tensor algebra of M, denoted by the same letter ∇. $\mathscr{X}$ and $\mathscr{M}$ denote the spaces of all vector fields on M and of all symmetric trace free tensor fields of type (0.2) on M, respectively. $\mathscr{D}^1$ denotes the space of all differential 1-forms on M. $\mathscr{X}_0$, $\mathscr{M}_0$, $\mathscr{D}^1_0$ denote the corresponding spaces of tensor fields with compact support. At each point $p \in M$, T_p and T^*_p denote the tangent and the cotangent space at p, respectively.

If $e_1,\dots,e_n$ is a base for T_p and $\omega^1,\dots,\omega^n$ its dual base in T^*_p we define the scalar product $g(v,w)$ of covectors $v = v_i\omega^i$, $w = w_j\omega^j$ (summation convention) as follows:

$$g(v,w) = g^{ij}v_iv_j$$

where (g^{ij}) is the inverse matrix of $(g_{ij}) = (g(e_i,e_j))$. Similarly, if v,w are two symmetric tensors $v = v_{ij}\omega^i \otimes \omega^j$, $w = w_{kl}\omega^k \otimes \omega^l$ we define $g(v,w)$ as follows

$$g(v,w) = g^{ik}g^{jl}v_{ij}w_{kl}\,.$$

The fact that we use the same letter g for the extended metric should not be confusing.

Now we define the global scalar product (.,.):

$$(V,W) = \int_M g(V,W) \quad V,W \in \mathscr{X} \text{ (or } \mathscr{D}^1\text{, or } \mathscr{M}\text{)}$$

if V or W is of a compact support. The integral is taken with respect to the Lebesgue measure on M generated by g.

An investigation of quasi–conformal deformations of a Riemannian manifold leads in a natural way to the Ahlfors operator S defined on the space $\mathscr{X}$ of all vector fields (= deformations) Z on M as follows:

$$SZ = L_Z g - \frac{2}{n} \operatorname{div} Z \cdot g \tag{1.1}$$

where L_Z is the Lie derivative in direction Z and $\operatorname{div} Z = \operatorname{tr}(X \to \nabla_X Z)$ is the divergence of Z (cf. [1], [9]). SZ is then a symmetric trace free tensor field of type (0.2).

The norm of SZ is a good measure of the rank of quasi–conformality of the deformation Z in the sense that the rank of quasi–conformality of the one–parameter group of transformations generated by Z may be estimated by the norm of SZ (cf. [11], [8]).

In the case $M = \mathbb{R}^2$ (= $\mathbb{C}$), S reduces (if we use a complex notation), to the Cauchy–Riemann operator. It might therefore be regarded as a multidimensional generalization, making sense for all dimensions.

Its formal adjoint S^* is the operator of divergence type acting on the space $\mathscr{M}$, see (1.5).

Consider now the following two operators of second order

$$S^*S$$

and

$$SS^*.$$

$L = S^*S$ is strongly elliptic in the sense that its leading symbol is positive (see the end of this chapter). In the case $n = 2$ it reduces to the Laplace–Beltrami operator: $2(\delta d + d\delta)$. We decided therefore to call L the **Ahlfors Lapla-**

cian of M. Being natural generalizations of classical operators, S and S^*S have many nice properties and, we think, are interesting in their own right.

The operator SS^* is semi–elliptic in the sense that its symbol is nonnegative, see the end of this chapter.

Studying transformation formulas it is more convenient to have the operators S and S^*S act on forms rather than on vector fields:

Let α be a 1–form and Z a vector field on M dual to each other in the sense that

$$\alpha(X) = g(Z,X), \quad X \in \mathscr{X}.$$

Since

$$\operatorname{div} Z = -\,\delta\alpha$$

where, in coordinates, $\delta\alpha = -\nabla^i\alpha_i$, and since

$$(L_Z g)(X,Y) = (\nabla\alpha)(X,Y) + (\nabla\alpha)(Y,X)$$

we get that

$$L_Z g - \tfrac{2}{n}\operatorname{div} Z\, g = 2\nabla^s\alpha + \tfrac{2}{n}\,\delta\alpha\cdot g \tag{1.2}$$

where $\nabla^s\alpha$ is a symmetrized version of $\nabla\alpha$, i.e.

$$(\nabla^s\alpha)(X,Y) = \tfrac{1}{2}[(\nabla\alpha)(X,Y) + (\nabla\alpha)(Y,X)].$$

Consider the differential operator $S : \mathscr{D}^1 \to \mathscr{M}$ defined by

$$S\alpha = 2\nabla^s\alpha + \tfrac{2}{n}\,\delta\alpha\, g\,. \tag{1.3}$$

Then, by (1.2), $S\alpha = SZ$ where SZ is defined by (1.1). It is easy to see that $S\alpha \in \mathscr{M}$, i.e. $S\alpha$ is a symmetric trace free tensor field of type (0.2). Indeed, symmetry is obvious, and if $X_1,\dots,X_n$ is a local orthonormal frame then

$$\mathrm{tr}S\alpha = \sum_i (S\alpha)(X_i,X_i) = \sum_i [2(\nabla^s\alpha)(X^i,X^i) + \tfrac{2}{n}\,\delta\alpha\, g(X^i,X^i)] = -2\delta\alpha + \tfrac{2}{n}\,\delta\alpha\cdot n = 0.$$

The kernel N_S of S, i.e. the space of all 1–forms α such that $S\alpha = 0$ consists exactly of the conformal Killing 1–forms.

Example 1.4 [1]. In the case $M = \mathbb{R}^n$, $n \geq 3$, $\alpha \in N_S$ if and only if in the Cartesian system $x = (x^1,\dots,x^n)$

$$\alpha(x) = a_k dx^k + b_{jk}x^j dx^k + 2c_j x^j x^s \delta_{sk} dx^k - x^j \delta_{js} x^s c_k dx^k$$

where $a = (a_1,\dots,a_n)$, $c = (c_1,\dots,c_n)$ are constant vectors and $b = (b_{ij})$ is a constant matrix such that $b_{ij} + b_{ji} - \frac{2}{n}\,\delta_{ij}\mathrm{tr}b = 0$, $i,j = 1,\dots,n$.

Now for an arbitrary two–tensor field $\varphi \in \mathcal{M}$ define

$$S^*\varphi = 2\delta\varphi \tag{1.5}$$

where $\delta\varphi$ is the 1–form defined locally by $\delta\varphi_j = -\nabla^i\varphi_{ij}$. Then

$$S^* : \mathcal{M} \to \mathcal{D}^1$$

is a first order linear differential operator formally adjoint to S in the sense that

$$(S\alpha,\varphi) = (\alpha,S^*\varphi) \qquad \alpha \in \mathcal{D}^1,\ \varphi \in \mathcal{M}$$

if only α or φ are of compact support [9].

The two above operators S and S^* define two second order differential operators

$$S^*S : \mathcal{D}^1 \to \mathcal{D}^1$$

and

$$SS^* : \mathscr{M} \to \mathscr{M}.$$

By (1.3), (1.5) and the local expression for δ, we get the following local expression for S^*S and SS^*:

$$S^*S\alpha_j = -2\nabla^i\nabla_i\alpha_j - 2\nabla^i\nabla_j\alpha_j - \frac{4}{n} d\alpha_j, \quad \alpha \in \mathscr{D}^1 \tag{1.6}$$

and

$$SS^*\varphi_{ij} = -2\nabla_i\nabla^k\varphi_{kj} - 2\nabla_j\nabla^k\varphi_{ki} + \frac{4}{n}\nabla^l\nabla^k\varphi_{kl}g_{ij}, \quad \varphi \in \mathscr{M}. \tag{1.7}$$

By (1.6) we can get two other useful formulas for S^*S expressing its relationship to the Laplace–Beltrami operator and the Ricci tensor of M. We have namely

$$S^*S\alpha = -4R\alpha + 2\Delta\alpha + \frac{2n-4}{n} d\delta\alpha \tag{1.8}$$

or, equivalently

$$S^*S\alpha = -4R\alpha + 2\delta d\alpha + \frac{4n-4}{n} d\delta\alpha \tag{1.9}$$

where $\Delta = \delta d + d\delta$ is the Laplace–Beltrami operator, R is the Ricci tensor and $R\alpha$ denotes the 1–form defined by

$$R\alpha(X) = R(Z,X) \qquad X \in \mathscr{X} \tag{1.10}$$

where Z is the vector field dual to α ($\alpha(X) = g(Z,X)$, $X \in \mathscr{X}$).

Now we would like to observe how S, S^*, S^*S and SS^* transform under a conformal change of the Riemannian metric g. To this aim assume that $\bar{g}$ is another Riemannian metric on M conformally related to g in the sense that there exists a positive function Ω on M such that

$$\bar{g} = \Omega^2 g. \tag{1.11}$$

All objects related to the metric $\bar{g}$ will be denoted by letters with "–" over them.

Observe first that if a is a function on M then (cf. [9])

$$S a\alpha = aS\alpha + 2da \otimes^s \alpha - \frac{2}{n}\, \alpha(\nabla a) \cdot g \tag{1.12}$$

and

$$\overset{*}{S} a\varphi = a\overset{*}{S}\varphi - 2\varphi(\nabla a, \cdot) \tag{1.13}$$

where $\otimes^s$ denotes the symmetrized tensor product: $(da \otimes^s \alpha)(X,Y) = \frac{1}{2}(da(X)\alpha(Y) + da(Y)\alpha(X))$ and ∇a is the gradient of a.

As we need some auxiliary formulae we will prove the following

Lemma 1.14 [10]. For arbitrary $\alpha \in \mathscr{D}^1$ and $\varphi \in \mathscr{M}$ we have:

$$\bar{S}\alpha = S\alpha - \frac{4}{\Omega}\, d\Omega \otimes^s \alpha + \frac{4}{n\Omega}\, \alpha(\nabla\Omega)g \tag{1.15}$$

and

$$\bar{\overset{*}{S}}\varphi = \frac{1}{\Omega^2}\, \overset{*}{S}\varphi - \frac{2n-4}{\Omega^3}\, \varphi(\nabla\Omega, \cdot) \tag{1.16}$$

where $\nabla\Omega$ denotes the gradient of Ω (with respect to g).

Proof. First of all we are going to derive that if $\alpha \in \mathscr{D}^1$ then

$$\bar{\delta}\alpha = \frac{1}{\Omega^2}\, \delta\alpha - \frac{n-2}{\Omega^3}\, \alpha\,(\nabla\alpha). \tag{1.17}$$

Using the transformation formula for the Levi–Civita connection (cf. [7]):

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{\Omega}(d\Omega(X)Y + d\Omega(Y)X) - \frac{1}{\Omega}\, \nabla\Omega g(X,Y) \tag{1.18}$$

we get that in local coordinates $(x^1,\dots,x^n)$:

$$\bar{\delta}\alpha = -\bar{\nabla}^i\alpha_i = -\bar{g}^{ij}\bar{\nabla}_j\alpha_i = -\bar{g}^{ij}(\bar{\nabla}_i\alpha)(X_j)$$

$$= -\bar{g}^{ij}[X_i(\alpha(X_j)) - \alpha(\bar{\nabla}_iX_j)]$$

$$= -\bar{g}^{ij}[X_i(\alpha(X_j)) - \alpha(\nabla_iX_j + \tfrac{1}{\Omega}(d\Omega(X_i)X_j + d\Omega(X_j)X_i) - \tfrac{1}{\Omega}\nabla\Omega g_{ij}]$$

$$= \tfrac{1}{\Omega^2}g^{ij}[X_i(\alpha(X_j)) - \alpha(\nabla_{i\,j})] + \tfrac{1}{\Omega^3}g^{ij}[d\Omega(X_i)\alpha(X_j) + \alpha(X_i)d\Omega(X_j) - \alpha(\nabla\Omega)g_{ij}]$$

$$= \tfrac{1}{\Omega^2}\delta\alpha - \tfrac{1}{\Omega^3}[\alpha(\nabla\Omega) + \alpha(\nabla\Omega) - n\alpha(\nabla\Omega)]$$

$$= \tfrac{1}{\Omega^2}\delta\alpha - \tfrac{n-2}{\Omega^3}\alpha(\nabla\Omega),$$

where $X_i = \frac{\partial}{\partial x^i}$ $i = 1,\dots,n$, i.e. (1.17).

Similarly we get

$$\bar{\nabla}\alpha = \nabla\alpha - \tfrac{2}{\Omega}d\Omega \otimes^s \alpha + \tfrac{1}{\Omega}\alpha(\nabla\Omega)g. \tag{1.19}$$

Indeed, $(\bar{\nabla}\alpha)(X_i,X_j) = (\bar{\nabla}_{X_j}\alpha)X_i = X_j(\alpha(X_i)) - \alpha(\bar{\nabla}_{X_j}X_i)$

$$= X_j(\alpha(X_i)) - \alpha(\nabla_{Y_j}X_i + \tfrac{1}{\Omega}(d\Omega(X_i)X_j + d\Omega(X_j)X_i) - \tfrac{1}{\Omega}\nabla\Omega g(X_i,X_j))$$

$$= X_j(\alpha(X_i)) - \alpha(\nabla_{X_j}X_i) - \tfrac{1}{\Omega}(d\Omega(X_i)\alpha(X_j)$$
$$+ \alpha(X_i)d\Omega(X_j)) + \tfrac{1}{\Omega}\alpha(\nabla\Omega)g(X_i,X_j)$$

$$= (\nabla\Omega)(X_i,X_j) - \tfrac{2}{\Omega}(d\Omega \otimes^s \alpha)(X_i,X_j) + \tfrac{1}{\Omega}\alpha(\nabla\Omega)g(X_i,X_j)$$

i.e. (1.19).

Consequently

$$\bar{\nabla}^s\alpha = \nabla^s\alpha - \frac{4}{\Omega}\, d\Omega \otimes^s \alpha + \frac{2}{\Omega}\, \alpha(\nabla\Omega) g. \tag{1.20}$$

Using now (1.3), (1.17) and (1.20) we obtain

$$\begin{aligned}\bar{S}\alpha \quad &= \bar{\nabla}^s\alpha + \frac{2}{n}\, \bar{\delta}\alpha\cdot g = \nabla^s\alpha - \frac{4}{\Omega}\, d\Omega \otimes^s \alpha + \frac{2}{n}\, \alpha(\nabla\Omega)\cdot g \\ &\quad + \frac{2}{n}\left(\frac{1}{\Omega^2}\, \delta\alpha - \frac{n-2}{\Omega^3}\, \alpha(\nabla\Omega)\right)\Omega^2 g \\ &= \nabla^s\alpha + \frac{2}{n}\, \delta\alpha - \frac{4}{\Omega}\, d\Omega \otimes^s \alpha + \frac{4}{n\Omega^3}\, \alpha(\nabla\Omega) g\end{aligned}$$

and the (1.15) is proved.

In the same way we get (1.16). Indeed,

$$\begin{aligned}\bar{S}^*\varphi_j &= 2\bar{\delta}\varphi_j = -2\bar{\nabla}^i\varphi_{ij} = -2g^{ik}\bar{\nabla}_k\varphi_{ij} \\ &= 2\bar{g}^{ik}[X_k\varphi(X_i,X_j) - \varphi(\bar{\nabla}_k X_i,X_j) - \varphi(X_i,\nabla_k X_j)] \\ &= -2\bar{g}^{ik}[X_k\varphi(X_i,X_j) \\ &\quad - \varphi(\nabla_k X_i + \frac{1}{\Omega}(d\Omega(X_k)X_i + d\Omega(X_i)X_k) - \frac{1}{\Omega}\nabla\Omega g_{ki}, X_j) \\ &\quad - \varphi(X_i, \nabla_k X_j + \frac{1}{\Omega}(d\Omega(X_k)X_j + d\Omega(X_j)X_k) - \frac{1}{\Omega}\nabla\Omega g_{kj})] \\ &= -\frac{2}{\Omega^2}\, g^{ik}[X_k\varphi(X_i,X_j) - \varphi(\nabla_k X_i,X_j) - \varphi(X_i,\nabla_k X_j)] \\ &\quad + \frac{2}{\Omega^3}\, g^{ik}[d\Omega(X_k)\varphi(X_i,X_j) + d\Omega(X_i)\varphi(X_k,X_j) - g_{ki}\varphi(\nabla\Omega,X_j)\end{aligned}$$

$$+ d\Omega(X_k)\varphi(X_i,X_j) + d\Omega(X_j)\varphi(X_i,X_k) - g_{kj}\varphi(X_i,\nabla\Omega)]$$

$$= \frac{2}{\Omega^2}\,\delta\varphi_j + \frac{2}{\Omega^3}\,[\varphi(\nabla\Omega,X_j) + \varphi(\nabla\Omega,X_j) - n\varphi(\nabla\Omega,X_j)$$
$$+ \varphi(\nabla\Omega,X_j) + 0 - \varphi(\nabla\Omega,X_j)]$$

$$= \frac{1}{\Omega^2}\,S^*\varphi_j - \frac{2n-4}{\Omega^3}\,\varphi(\nabla\Omega,X_j)$$

which completes the proof. q.e.d.

Now we can prove the following

Theorem 1.21. If $\bar{g}$ is a Riemannian metric conformally related to g in the sense of (1.11), then for arbitrary $\alpha \in \mathscr{D}^1$ and $\varphi \in \mathscr{M}$ the following transformation formulas hold:

$$\Omega^{-2}\bar{S}\Omega^2\alpha = S\alpha \tag{1.22}$$

$$\Omega^n\bar{S}^*\Omega^{-n}\bar{S}\Omega^2\alpha = S^*S\alpha \tag{1.23}$$

$$\Omega^n\bar{S}^*\Omega^{-n+2}\varphi = S^*\varphi \tag{1.24}$$

$$\Omega^{-2}\bar{S}\Omega^{n+2}\bar{S}^*\Omega^{-n+2}\varphi = SS^*\varphi \tag{1.25}$$

hold, where $n = \dim M$.

Proof. Replace S by $\bar{S}$ and a by Ω^2 in (1.12):

$$\bar{S}\Omega^2\alpha = \Omega^2\bar{S}\alpha + 2d\Omega^2 \otimes^s \alpha - \frac{2}{n}\,\alpha(\bar{\nabla}\Omega^2)\bar{g}\,.$$

Since $d\Omega^2 = 2\Omega d\Omega$, $\bar{\nabla}\Omega^2 = 2\Omega\bar{\nabla}\Omega = 2\Omega\,\frac{1}{\Omega^2}\,\nabla\Omega = \frac{2}{\Omega}\,\nabla\Omega$ and $\bar{g} = \Omega^2 g$, we obtain

$$\bar{S}\Omega^2\alpha = \Omega^2\bar{S}\alpha + 4\Omega d\Omega \otimes^s \alpha - \frac{4}{n}\,\Omega\alpha(\nabla\Omega)g\,.$$

Using now Lemma 1.14 we get

$$\bar{S}\Omega^2\alpha = \Omega^2(S\alpha - \frac{4}{\Omega}\,d\Omega \otimes^s \alpha + \frac{4}{n\Omega}\,\alpha(\nabla\Omega)g) + 4\Omega d\Omega \otimes^s \alpha - \frac{4\Omega}{n}\,\alpha(\nabla\Omega)\cdot g$$

which is equivalent to (1.12).

Replace now S^* by $\bar{S}^*$, ∇ by $\bar{\nabla}$ and a by Ω^{-n+2} in (1.13):

$$\bar{S}^*\Omega^{-n+2} = \bar{S}^*\Omega^{-n+2}\bar{S}^*\varphi - 2\varphi(\bar{\nabla}\Omega^{-n+2},\cdot)\ .$$

Since

$$\bar{\nabla}\Omega^{-n+2} = (-n+2)\Omega^{-n+1}\bar{\nabla}\Omega = (-n+2)\Omega^{-n-1}\nabla\Omega\ ,$$

we get

$$\bar{S}^*\Omega^{-n+2}\varphi = \Omega^{-n+2}\bar{S}^*\varphi + (2n-4)\Omega^{-n-1}\varphi(\nabla\Omega,\cdot)\ .$$

Using now Lemma 1.14, we get

$$\bar{S}^*\Omega^{-n+2}\varphi = \Omega^{-n+2}(\frac{1}{\Omega^2}\,S^*\varphi - \frac{2n-4}{\Omega^3}\,\varphi(\nabla\Omega,\cdot))$$

$$+ (2n-4)\Omega^{-n+1}\varphi(\nabla\Omega,\cdot)$$

which is equivalent to (1.24).

Finally, combining formulas (1.22) and (1.24) we get (1.23) and (1.25).

q.e.d.

Assume now that $\bar{g}$ and g are conformally related in the sense of (1.11), where

$$\Omega = e^{u\omega}\ ,\quad u \in \mathbb{R}\ .$$

Converting formulas (1.22) – (1.25) we get

$$\bar{S} = e^{2u\omega}Se^{-2u\omega}$$

$$\overline{S^*S} = e^{-nu\omega}S^*e^{n\omega}Se^{-2u\omega}$$

$$\bar{S}^* = e^{-nu\omega}S^*e^{(n-2)u\omega}$$

$$\overline{SS^*} = e^{2u\omega}Se^{-(n+2)u\omega}.$$

By differentiation with respect to u we get the following

Corollary 1.26

$$(\bar{S})^{\cdot} = 2\omega S - 2S\omega \tag{1.27}$$

$$(\overline{S^*S})^{\cdot} = -n\omega S^*S - 2S^*S\omega + nS^*\omega S \tag{1.28}$$

$$= -2S^*S\omega - 2ni(\omega)S$$

where $i(\omega)$ denotes interior derivative

$$(\bar{S}^*)^{\cdot} = -n\omega S^* + (n-2)S^*\omega \tag{1.29}$$

$$(\overline{SS^*})^{\cdot} = 2\omega SS^* - (n+2)S\omega S^* + (n-2)SS^*\omega \tag{1.30}$$

where $\cdot = \left.\frac{d}{du}\right|_{u=0}$.

We will conclude this chapter by deriving inequalities for the leading symbols of S^*S and SS^*.

Let $p \in M$ and $\omega \in T_p^*$. The symbol (at p) of a differential operator on a vector bundle ξ

$$L : C^\infty(\xi) \to C^\infty(\xi)$$

is the mapping $\sigma_L(\omega) = \sigma_L(p,\omega)$ defined by

$$\sigma_L(\omega)s = -L(a^2 s)_p\,, \quad s \in C^\infty(\xi)$$

where a is a function in a neighbourhood of p with

$$a(p) = 0, \quad da(p) = \omega \tag{1.31}$$

Of course, this definition does not depend on the choice of a.

Take $\alpha \in \mathscr{D}^1$ and $\varphi \in \mathscr{M}$. If a satisfies (1.31) then, by (1.12) and (1.13)

$$\sigma_{S^*S}(\omega)\alpha = -S^*S(a^2\alpha)_p = S^*(a^2 S\alpha + 4a\,da \otimes^s \alpha - \tfrac{4}{n}\,a\alpha(\nabla a)g)_p$$

$$= (8(da \otimes^s \alpha)(\nabla a,\cdot))_p - \tfrac{8}{n}\,\alpha(\nabla a)g(\nabla a,\cdot))_p$$

$$= 8\omega \otimes^s \alpha(\omega^\#,\cdot) - \tfrac{8}{n}\,\alpha(\omega^\#)g(\omega^\#,\cdot)$$

$$= 4(\omega(\omega^\#)\alpha + \tfrac{n-2}{n}\,\alpha(\omega^\#)\omega)$$

where $\omega^\#$ is the vector dual to ω and, similarly,

$$\sigma_{SS^*}(\omega)\varphi = -SS^*(a^2\varphi)_p = -S(a^2 S^*\varphi - 4a\varphi(\nabla a,\cdot))_p$$

$$= 8da \otimes^s \varphi(\nabla a,\cdot) - \tfrac{8}{n}\,\varphi(\nabla a,\nabla a)g$$

$$= 8\omega \otimes^s \varphi(\omega^\#,\cdot) - \tfrac{8}{n}\,\varphi(\omega^\#,\omega^\#)g\,.$$

Consequently,

$$g(\sigma_{S^*S}(\omega)\alpha,\alpha)_p = 4(\| \omega \|_p^2 \| \alpha \|_p^2 + \tfrac{n-2}{n} (\alpha(\omega^\#))^2)$$

and

$$g(\sigma_{SS^*}(\omega)\varphi,\varphi) = 8 \| \varphi(\omega^\#,\cdot) \|_p^2 .$$

Since $(\alpha(\omega^\#))^2 \leq \| \omega \|_p^2 \| \alpha \|_p^2$, we get the following inequalities

$$4 \| \omega \|_p^2 \leq g(\sigma_{S^*S}(\omega)\alpha,\alpha)_p \leq \tfrac{8(n-1)}{n} \| \omega \|_p^2 \| \alpha \|_p^2$$

which says that $L = S^*S$ is strongly elliptic, and

$$g(\sigma_{SS^*}(\omega)\varphi,\varphi)_p \geq 0$$

which says that SS^* is semi–elliptic. It would be very interesting for our purposes in Chapter 3 to know, which parts of the theory of elliptic operators extend to semi–elliptic operators. For example, whether SS^* is hypo–elliptic, and whether its heat kernel admits an asymptotic expansion for small time parameter. At any rate, SS^* has presumably a large kernel, but the same non–zero spectrum as S^*S.

2. THE HEAT KERNEL OF THE AHLFORS LAPLACIAN

Recall that $L = S^*S$ is an elliptic second order differential operator on the vector fields (or by canonical duality on the one–forms) on our compact Riemannian n–dimensional manifold M with metric tensor g. The spectrum of L is non–negative

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \tag{2.1}$$

with $\lambda_k \to \infty$ as $k \to \infty$. As in standard elliptic theory [5] we get for $t > 0$ an asymptotic relation

$$\sum_{k=1}^{\infty} e^{-t\lambda_k} \sim \sum_{i=0}^{\infty} a_i t^{(2i-n)/2}, \quad t \downarrow 0 \tag{2.2}$$

where the left side in (2.2) is the sum over all the eigenvalues in (2.1) counted with multiplicity, so it is just the trace in L^2 of the operator

$$\exp(-tL) = \sum_{k-1}^{\infty} e^{-t\lambda_k} \varphi_k \otimes \varphi_k^*, \tag{2.3}$$

where $\{\varphi_k | k = 1,2,...\}$ is an orthonormal basis of the Hilbert space of square–integrable vector fields. The coefficients a_i in the right hand side of (2.2) are integrals of local expressions in the jets of L, in our case local invariants in the metric, its inverse and its derivatives:

$$a_i = \int_M U_i \, d\,\text{vol}. \tag{2.4}$$

U_i has level $2i$ in the sense that if we make a uniform dilatation of the metric: $\bar{g} = A^2 g$ for $0 < A \in \mathbb{R}$, then $\bar{U}_i = A^{-2i} U_i$.

It is to be expected that the information encoded via L in the spectrum and the coefficients (2.4) is of particular relevance for conformal and quasi–conformal geometry. Let us pause to make two very elementary observations of this nature; they are well–known, see [12].

Proposition 2.5.
(1) The kernel of L consists exactly of the conformal Killing vector fields; these therefore span a finite dimensional Lie algebra.
(2) If M is of negative Ricci curvature, then the kernel of L is zero; hence M does not admit any conformal Killing vector fields in this case.

Proof. (1) A vector field X is conformal if and only if $SX = 0$; this means $0 = (SX,SX) = (S^*SX,X) = (LX,X)$ which again is equivalent to $LX = 0$. Note that by elliptic regularity theory there also are no weak solutions to $SX = 0$ other than smooth ones. For (2) we apply the formula (1.8) acting on one–forms; if $R < 0$ there is only a trivial kernel for L, since the last two terms are positive semi–defiinite operators.

q.e.d.

Example 2.6. Let $M = S^n$ be the standard n–sphere; it admits the maximal conformal group $0(n+1,1)$ and isometry group $0(n+1)$. A conformal vector field X is an isometry if and only if $\operatorname{div} X = -\delta\alpha = 0$, where α is the corresponding one–form. On such an α

$$L\alpha = 2\delta d\alpha - 4R\alpha$$

with $R = (n-1)$ and the possible eigenvalues of δd equal to $(k+1)(k+n-2)$, $k = 1,2,3,\ldots$. Hence $L\alpha = 0$ corresponds to $k = 1$, since $2\cdot 2(n-1) - 4(n-1) = 0$. Similarly, on $d\alpha = 0$ we have the spectrum of $d\delta$ equal to $(k+n-1)k$ and so $L\alpha = 0$ happens exactly when $k = 1$; these are the purely conformal vector fields. This method (see [4]) gives us the whole spectrum of L on S^n : The eigenvalues are with $k = 1,2,3,\ldots$

$$\begin{aligned} &\lambda = 2(k+1)(k+n-2) - 4(n-1) \quad \text{on co–closed 1–forms} \\ &\lambda = \frac{4n-4}{n}(k+n-1)k - 4(n-1) \quad \text{on closed 1–forms.} \end{aligned} \tag{2.7}$$

The multiplicities are as in [4].

Remark 2.8. Using Hodge theory as in the previous example, one could similarly compute the spectrum of L for the compact symmetric spaces. Another interesting class of examples is that of hypersurfaces in $\mathbb{C}^n$, see [9].

The heat semigroup is an infinitesimaly smoothing operator converging to the projection onto the conformal Killing vector fields; we shall return to this in Chapter 4 as another example of the interplay between the functional analysis of L and the quasi–conformal geometry of M. In the following Chapter 3 we consider the dependence $\lambda_k = \lambda_k(g)$ of the eigenvalues on the metric and the corresponding dependence $a_i = a_i(g)$, especially under conformal deformations. Note that the kernel of L is clearly conformally invariant.

3. THE ASSOCIATED CONFORMAL VARIATIONS

In this chapter we shall introduce the pseudo–differential calculus for constructing parametrices and the approximate heat kernel of L; for references see [6] and [3] – we shall adapt the notation and the concepts primarily from these. In particular we shall follow the latter in the consideration of a one–parameter family of metrics $\bar{g} = g(u) = \exp(u\omega)g$ $(u \in \mathbb{R})$, where ω is a fixed smooth function on M, and g the Riemannian metric. Corresponding to this deformation we get the Ahlfors Laplacian $L = L(u)$ depending on u, the eigenvalues $\lambda_k = \lambda_k(u)$ and the coefficients (2.4) $a_i = a_i(u)$, etc. By a dot we denote the u–differentiation of quantities at $u = 0$. Our aim is via (2.2), (2.3) and consideration of their u–derivatives to see under which conditions $(a_{n/2})^{\cdot} = 0$, i.e. when $a_{n/2}$ is a conformal invariant (n even). For this we need the variational formulas in Chapter 1 for $\dot{L}$ and the operator calculus in [3].

First we establish the variational version of (2.2):

Proposition 3.1. Let the metric g depend on $u \in \mathbb{R}$ as above; then the L^2–trace of the heat semi–group for L is differentiable in u, as are the coefficients a_i in (2.4), and we have the asymptotic expansion

$$(\mathrm{tr}\exp(-tL))^{\cdot} \sim \sum_{i=o}^{\infty} \dot{a}_i\, t^{(2i-n)/2}, \quad t \downarrow 0. \tag{3.2}$$

Proof. This is just a reformulation of Theorem 3.3 of [3]; the symbol class is as in 3.1 of [3] and the vector bundle just the tangent bundle of M. Note that the term–by–term differentiation of an asymptotic series is a delicate matter, as simple examples will demonstrate.

q.e.d.

Our next result is also taken from [3]; it is the formula of Ray and Singer generalized to the operator L. Since $\exp(-tL)$ is infinitely smoothing, the various formulas and formal manipulations are valid.

Proposition 3.3. With notation as above, we have (at $u = 0$)

$$(\mathrm{tr}\,\exp(-tL))^{\cdot} = -t\cdot \mathrm{tr}\dot{L}\,\exp(-tL). \tag{3.4}$$

Now we can combine (3.2) and (3.4) using one formula for L from Chapter 1; recall that (acting on 1–forms)

$$\dot{L} = -2L\omega - nS^{*}\omega S - n\omega L.$$

This may be substituted into (3.4), and using cyclic permutations under the trace ($\exp(-tL)$ is infinitely smoothing, and S and S^{*} are first–order differential operators) we get as $t \downarrow 0$

$$\sum_{i=o}^{\infty} \dot{a}_i\, t^{(2i-n)/2} \sim -t\cdot \mathrm{tr}(-2L\omega + nS^{*}\omega S - n\omega L)\exp(-tL) \tag{3.5}$$

$$= -nt\cdot \mathrm{tr}S^{*}\omega S\,\exp(-tL) + (n+2)t\cdot \mathrm{tr}\,\omega L\exp(-tL).$$

We shall treat the two terms in (3.5) separately: The second term in (3.5) is $n + 2$ times

$$t\cdot \mathrm{tr}\,\omega\exp(-tL) \tag{3.6}$$

$$= \quad -t\,\frac{d}{dt}\,\mathrm{tr}\,\omega\exp(-tL)$$

$$\sim \quad -t\,\frac{d}{dt}\sum_{i=0}^{\infty} t^{(2i-n)/2}\int_M \omega\, U_i\, d\mathrm{vol}$$

$$\sim \quad \frac{1}{2}\sum_{i=0}^{\infty} (n-2i)t^{(2i-n)/2}\int_M \omega\, U_i\, d\mathrm{vol}$$

as we may differentiate the asymptotic expansion in t term–by–term (see [3] (3.20) and the preceeding argument). For the first term in (3.5), it is $-n$ times

$$
\begin{aligned}
& t\cdot \mathrm{tr} S^* \omega S \exp(-tL) && (3.7)\\
= \quad & t\cdot \mathrm{tr}\, \omega S \exp(-tL) S^* \\
= \quad & t\cdot \mathrm{tr}\, \omega \exp(-tSS^*) SS^* \\
= \quad & t\cdot \mathrm{tr}\, \omega SS^* \exp(-tSS^*).
\end{aligned}
$$

We have thus arrived at the corresponding heat semi–group for SS^*, which, however, is not an elliptic operator. It has a possibly infinite–dimensional kernel, does not have the elliptic regularity properties of S^*S. But it does have the same non–zero spectrum as S^*S with the same multiplicities ($\alpha \to S\alpha$ provides the isomorphism between the eigenspaces). Combining the three previous equations (3.5), (3.6) and (3.7) we arrive at

$$
\begin{aligned}
& \sum_{i=0}^{\infty} \dot{a}_i\, t^{(2i-n)/2} && (3.8)\\
\sim \quad & \frac{n+2}{2} \sum_{i=0}^{\infty} (n-2i) t^{(2i-n)/2} \int_M \omega\, U_i \,\mathrm{dvol} \\
- \quad & nt\cdot \mathrm{tr}\, \omega\, SS^* \exp(-tSS^*), \quad t \downarrow 0.
\end{aligned}
$$

Proposition 3.9. For every smooth function ω on M there is an asymptotic expansion of

Proposition 3.3. With notation as above, we have (at $u = 0$)

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$$\dot{L} = -2L\omega - nS^{*}\omega S - n\omega L.$$

This may be substituted into (3.4), and using cyclic permutations under the trace ($\exp(-tL)$ is infinitely smoothing, and S and S^{*} are first–order differential operators) we get as $t \downarrow 0$

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$$t\cdot \mathrm{tr}\omega\exp(-tL) \tag{3.6}$$

$$= \quad -t\,\frac{d}{dt}\,\mathrm{tr}\omega\exp(-tL)$$

$$\sim \quad -t\,\frac{d}{dt}\sum_{i=0}^{\infty} t^{(2i-n)/2}\int_M \omega\, U_i\, d\mathrm{vol}$$

$$\sim \quad \frac{1}{2}\sum_{i=0}^{\infty}(n-2i)t^{(2i-n)/2}\int_M \omega\, U_i\, d\mathrm{vol}$$

as we may differentiate the asymptotic expansion in t term–by–term (see [3] (3.20) and the preceeding argument). For the first term in (3.5), it is $-n$ times

$$t\cdot \mathrm{tr} S^{*}\omega S \exp(-tL) \tag{3.7}$$

$$= \quad t\cdot \mathrm{tr}\,\omega S \exp(-tL)S^{*}$$

$$= \quad t\cdot \mathrm{tr}\,\omega \exp(-tSS^{*})SS^{*}$$

$$= \quad t\cdot \mathrm{tr}\,\omega SS^{*} \exp(-tSS^{*}).$$

We have thus arrived at the corresponding heat semi–group for SS^{*}, which, however, is not an elliptic operator. It has a possibly infinite–dimensional kernel, does not have the elliptic regularity properties of $S^{*}S$. But it does have the same non–zero spectrum as $S^{*}S$ with the same multiplicities $(\alpha \to S\alpha$ provides the isomorphism between the eigenspaces). Combining the three previous equations (3.5), (3.6) and (3.7) we arrive at

$$\sum_{i=0}^{\infty} a_i\, t^{(2i-n)/2} \tag{3.8}$$

$$\sim \quad \frac{n+2}{2} \sum_{i=0}^{\infty} (n-2i)t^{(2i-n)/2} \int_M \omega\, U_i\, \mathrm{dvol}$$

$$- \quad nt\cdot \mathrm{tr}\, \omega\, SS^{*} \exp(-tSS^{*}), \quad t \downarrow 0.$$

Proposition 3.9. For every smooth function ω on M there is an asymptotic expansion of

$$\operatorname{tr} \omega SS^* \exp(-tSS^*) \tag{3.10}$$

$$= -\frac{d}{dt} \operatorname{tr}_0 \omega \exp(-tSS^*)$$

$$\sim \sum_{i=0}^{\infty} b_i t^{(2i-n-2)/2}, \quad t \downarrow 0.$$

Furthermore, (3.10) may be integrated (in t) to obtain an asymptotic expansion of

$$\operatorname{tr}_0 \omega \exp(-tSS^*) \tag{3.11}$$

$$\sim \sum_{i=0}^{\infty} c_i t^{(2i-n)/2} + b \log t, \quad t \downarrow 0.$$

Here tr_0 denotes the L^2–trace on the orthogonal complement to the kernel of S^*, i.e.

$$\operatorname{tr}_0 A = \operatorname{tr} PAP$$

with P the orthogonal projection onto $(\ker S^*)^\perp = (\text{range } S)^-$.

Proof. The first equality in (3.10) is obtained by working with the orthonormal basis $\{\lambda_k^{-1} S\varphi_k\}$ of $(\ker S^*)^\perp = (\text{range } S)^-$, with $\{\varphi_k\}$ an orthonormal basis consisting of eigenfunctions of S^*S with non–zero eigenvalue λ_k. If ω_{kl} is the matrix of ω we get for (3.10)

$$\sum_k \omega_{kk} \lambda_k e^{-t\lambda_k} = -\frac{d}{dt} \sum_k \omega_{kk} e^{-t\lambda_k}$$

summing as indicated only over non–zero eigenvalues. The asymptotic expansion in (3.10) follows from (3.8), and the integrated form (3.11) by the (allowed) term–by–term integration. The coefficients c_i and b are again integrated local invariants. Note that they depend on ω.

q.e.d.

At this point we conjecture that the log t–term in (3.11) is absent in general; to prove it, one would need a theory of heat kernels etc. for semi–elliptic operators like SS^*. We suspect optimistically that SS^* is in fact sub–elliptic and that methods from that theory will establish this conjecture. What we can immediately assert is the following

Theorem 3.12. Let M be an even–dimensional compact Riemannian manifold with metric tensor g and Ahlfors Laplacian L. Then the coefficient $a_{n/2}$ in the asymptotic expansion

$$\operatorname{tr}\exp(-tL) \sim \sum_{i=0}^{\infty} a_i\, t^{(2i-n)/2}$$

is invariant under conformal deformations $\bar{g} = \Omega^2 g$ of the metric, if and only if the asymptotic expansion (3.11) never has a log t–term. More generally, we have (3.12) below.

Proof. The absence of the log t–term means exactly that the right–hand side of (3.8) contains no constant term in t. Thus in this case $\dot{a}_{n/2} = 0$. In general

$$\dot{a}_{n/2} = n \cdot b \tag{3.12}$$

so the result is clear.

q.e.d.

Remark. As a weaker conjecture we offer that $b = 0$ under either some geometric conditions on M, or conditions on ω.

$$\operatorname{tr} \omega SS^* \exp(-tSS^*) \tag{3.10}$$

$$= -\frac{d}{dt} \operatorname{tr}_0 \omega \exp(-tSS^*)$$

$$\sim \sum_{i=0}^{\infty} b_i t^{(2i-n-2)/2}, \quad t \downarrow 0.$$

Furthermore, (3.10) may be integrated (in t) to obtain an asymptotic expansion of

$$\operatorname{tr}_0 \omega \exp(-tSS^*) \tag{3.11}$$

$$\sim \sum_{i=0}^{\infty} c_i t^{(2i-n)/2} + b \log t, \quad t \downarrow 0.$$

Here tr_0 denotes the L^2–trace on the orthogonal complement to the kernel of S^*, i.e.

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is invariant under conformal deformations $\bar{g} = \Omega^2 g$ of the metric, if and only if the asymptotic expansion (3.11) never has a log t–term. More generally, we have (3.12) below.

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Remark. As a weaker conjecture we offer that $b = 0$ under either some geometric conditions on M, or conditions on ω.

4. APPLICATIONS TO QUASI–CONFORMAL DEFORMATIONS

In this chapter we will treat an application of L and the corresponding semi–group close to the original motivation for introducing L. Namely, we shall apply L and exp(–tL) to the problem of finding quasi–conformal deformations of a given transformation.

Suppose X is a smooth vector field on our Riemnnian manifold M, compact as usual. Then X can be expanded in terms of eigenvectors X_k for S^*S:

$$X = \sum_{k=0}^{\infty} X_k$$

where $S^*SX_k = \lambda_k X_k$. If S^*S has a kernel, then the first eigenvalue will be zero, and the corresponding projection of X onto the zero eigenspace we shall call X_0. If S^*S does not have a kernel, we set $X_0 = 0$.

Now we can apply the semi–group for L to X and obtain

$$\exp(-tL)X = \sum_{k=0}^{\infty} e^{-t\lambda_k} X_k \tag{4.1}$$

$$= \quad X_0 + \sum_{k=1}^{\infty} e^{-t\lambda_k} X_k$$

where $0 < \lambda_1 \leq \lambda_2 \leq \dots$. (4.1) is clearly a smoothing out of X, so that for t large, the result is very close to being a conformal Killing vectorfield. Indeed, (4.1) is convergent in L^2 to X_0. By using the Sobolev inequalities this convergence may be made uniform and even with control over the quasi–conformal modulus of the family of deformations

$$X(t) = \exp(-tL)X. \tag{4.2}$$

Theorem 4.3. Let X be a smooth vector field on M and consider the family of deformations as in (4.1) and (4.2). Then we have the following estimates:

(i) $$\|X(t) - X_0\|_2^2 \leq e^{-2t\lambda_1} \|X\|_2^2$$

(ii) $$\|X(t) - X_0\|_{2,s}^2 \leq (e^{-2t\lambda_1} + (\tfrac{s}{2t})^s e^{-s}) \|X\|_2^2 \quad (s > 0)$$

where $\|\cdot\|_{2,s}$ denotes the Sobolev norm.

(iii) For any $s > n/2$ there is a $C > 0$ such that

$$\|X(t) - X_0\|_\infty \leq C(e^{-2t\lambda_1} + (\tfrac{s}{2t})^s e^{-s})^{1/2} \|X\|_2.$$

(iv) For the quasi–conformal modulus we have

$$\|SX(t)\|_\infty \leq C(e^{-2t\lambda_1} + (\tfrac{s+1}{2t})^{s+1} e^{-s-1})^{1/2} \|X\|_2$$

where $0 < C$ is a constant depending only on $s > n/2$.

Proof. (i) is clear from (4.1). For (ii) we use the elliptic operator L to get the Sobolev norm

$$\|X\|_{2,s}^2 = \|L^{s/2}X\|_2^2 + \|X\|_2^2$$

$$= ((I + L^s)X,X)$$

(inner product in the space of L^2 vector fields). Then

$$\|X(t) - X_0\|^2_{2,s} = ((I + L^s)(X(t) - X_0), X(t) - X_0)$$

$$= \sum_{k=1}^{\infty} (1 + \lambda_k^s) e^{-2t\lambda_k} \|X_k\|_2^2$$

$$\leq (e^{-2t\lambda_1} + (\tfrac{s}{2t})^s e^{-s}) \|X\|_2^2 .$$

This last estimate follows by considering the function $\lambda^s e^{-2t\lambda}$.

In (iii) we invoke the Sobolev inequalities [6] using (ii), and finally in (iv) the fact that S is a differential operator of order one, and therefore continuous between the Sobolev spaces in question:

$$\|SX(t)\|_\infty \leq C'' \|SX(t)\|_{2,s}$$

$$= C'' \|S(X(t) - X_0)\|_{2,s}$$

$$= C' \|X(t) - X_0\|_{2,s+1}$$

$$= C(e^{-2t\lambda_1} + (\tfrac{s+1}{2t})^{s+1} e^{-s-1}) \|X\|_2)$$

with C'', C', C positive constants and $s > n/2$. Here Sobolev's inequality was used in the space (image of S) of symmetric, trace-free 2-tensors.

q.e.d.

Remark 4.4. In the argument above, X may just be a square-integrable vector field (not necessarily smooth); the family $X(t)$ $(t > 0)$ will by the smoothing property of $\exp(-tL)$ consist of smooth vector fields, and

$$\|X(t) - X\|_2 \to 0 \text{ as } t \to 0.$$

On the other hand, the limit as $t \to \infty$ still behaves as in Theorem 4.3. Thus by (iv) in particular, $X(t)$ provides a very natural family of quasi-conformal deformations of X.

From the formulas in Chapter 1, writing L as a sum of positive-definite operators plus -4 times the Ricci curvature, we get the following estimate for λ_1:

$$\lambda_1 \geq -4R. \qquad (4.5)$$

This is to be understood in the sense of pointwise inequalities for the eigenvalues of R.

Corollary 4.6. Suppose M is of Ricci curvature $\leq -R_0$, where R_0 is a positive constant (so in particular there are in this case no conformal Killing vector fields). Then the first eigenvalue of L is $\lambda_1 \geq 4R_0$, and the quasi-conformal family $X(t)$ in Theorem 4.3 converges to zero. Furthermore,

$$\|SX(t)\|_\infty \leq C\cdot(e^{-8tR_0} + (\tfrac{s+1}{2t})^{s+1}e^{-s-1})^{1/2}\,\|X\|_2$$

for $s > n/2$ and C the Sobolev constant from (iv) Theorem 4.3.

In general, an estimate of the first non-zero eigenvalue would give the exact rate of decay in Theorem 4.3.

We shall finish our discussion with the analogue of Theorem 4.3 for the case of global transformations of M. We shall not carry out the details in maximal generality (the consideration of homeomorphisms instead of smooth transformations, measurable vector fields instead of smooth ones etc.).

Recall from [9] the fact that if the vector field X is k-quasi-conformal, i.e. $\|SX\|_\infty \leq k$, then the corresponding one-parameter family F_s of transformations $F_s = \exp(sX)$ are K-quasi-conformal with $K = \exp(k^2|s|/2)$. We can therefore get global quasi–conformal deformations by deforming the generator X.

Theorem 4.7. Consider the one–parameter group $F(t)$ of transformations of M generated by a smooth vector field X. Then the family of deformations

$$F(t) = \exp(X(t)) \tag{4.8}$$

with $X(t)$ as in Theorem 4.3 is a family of K_t–quasi–conformal transformations with (notation as in [8])

$$K_t \leq \exp \tfrac{1}{2}(e^{-2t\lambda_1} + (\tfrac{s+1}{2t})^{s+1} e^{-s-1})\|X\|_2^2) . \tag{4.9}$$

Proof. This is just the global estimate corresponding to (iv), Theorem 4.3. q.e.d.

The estimate (4.9) gives some control over the behaviour of the family (4.8). Note that we have thus arrived at a very natural family of deformations of global transformations F connected to the identity in the diffeomorphism group of M. The limit $F(\infty)$ is conformal, so we may record the following

Corollary 4.10. Suppose the transformation F is in the group generated by one–parameter groups as in Theorem 4.7; suppose furthermore that the orientation–preserving conformal diffeomorphisms of M form a connected group. Then there is a quasi–conformal family $F(t)$ with $F(0) = F$, converging pointwise to the identity, and with quasi–conformal modulus satisfying an estimate

$$K_t \leq \exp(Ae^{-Bt} + C(s)t^{-s})$$

for $A,B, > 0$, $s > \frac{n}{2}$, and $C(s) > 0$.

Proof. This follows by repeated use of (4.9), since F is now a product of transformations of the form in Theorem 4.7.

q.e.d.

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Bent Ørsted
Sonderforschungsbereich 170
"Geometrie und Analysis"
D–3400 Göttingen
West Germany
and
Department of Mathematics
Odense University
DK–5230 Odense M
Denmark

Antoni Pierzchalski
Inst. of Mathematics
Łódz University
Banacha 22
90–238 Łódz
Poland

Constantin Carathéodory: An International Tribute (pp. 1049–1074)
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UNIFORM STABILIZATION OF THE EULER-BERNOULLI EQUATION WITH FEEDBACK OPERATOR ONLY IN THE NEUMANN BOUNDARY CONDITION

N. Ourada and R. Triggiani

__Abstract__. We study the uniform stabilization problem for the Euler-Bernoulli equation defined in a smooth, bounded domain Ω of R^n, with just one suitable dissipative boundary feedback operator acting on the Neumann B.C., while the Dirichlet B.C. is kept homogeneous. The uniform stabilization results which we present are fully consistent with recently established exact controllability and optimal regularity theories, which in fact motivate the choice of the function spaces in the first place. In particular, if the dissipative feedback operator acts on the entire boundary Γ, no geometrical conditions on Ω are needed.

1. __Introduction, Preliminaries, and Statement of Main Results__

1.1. __Introduction and Literature__

Let Ω be an open, bounded domain in R^n, where typically $n \geq 2$, with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_1 relatively open with Γ_0 possibly empty, while Γ_1 non-empty.

In Ω we consider the Euler-Bernoulli mixed problem in $w(t,x)$ on an arbitrary time interval $(0,T]$ with homogeneous Dirichlet boundary condition and non-homogeneous forcing term (control function) acting only in (possibly a part of) the Neumann boundary condition:

$$
\begin{cases}
w_{tt}+\Delta^2 w = 0 & \text{in } (0,T]\times\Omega = Q; & (1.1a)\\
w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \text{in } \Omega; & (1.1b)\\
w|_\Sigma \equiv 0 & \text{in } (0,T]\times\Gamma = \Sigma; & (1.1c)\\
\frac{\partial w}{\partial \nu}|_{\Sigma_0} \equiv 0 & \text{in } (0,T]\times\Gamma_0 = \Sigma_0; & (1.1d)\\
\frac{\partial w}{\partial \nu}|_{\Sigma_1} = g & \text{in } (0,T]\times\Gamma_1 = \Sigma_1; & (1.1e)
\end{cases}
$$

Data $\{w_0,w_1,g\}$ in $L_2(\Omega)\times H^{-2}(\Omega)\times L_2(\Sigma)$ produce a unique solution $\{w,w_t\} \in C([0,T];L_2(\Omega)\times H^{-2}(\Omega))$ [Lio.1], [Lio.2], an optimal regularity result. Indeed, exact controllability on [0,T], T > 0 arbitrary, on the state space $L_2(\Omega)\times H^{-2}(\Omega)$ within the class of $L_2(\Sigma)$-controls g holds true as well [Lio.1], [Lio.2]. It should be noted at the outset that the case where a control function acts in the Dirichlet B.C. (1.1c), while the control g in (1.1e) may or may not be zero has also been studied and is, in fact, quite different from the case (1.1) of the present paper (the function spaces are different, the multipliers are different): here results of exact controllability on an arbitrary T > 0 on (appropriate) spaces of optimal regularity are obtained in [L-T.1], [L-T.2], [L-T.3], while results of uniform stabilization on such spaces are obtained in [B-T.1], by means of an explicitly, dissipative feedback acting on the velocity w_t. Finally, we remark that the abstract theory of the linear quadratic regulator problem and corresponding algebraic Riccati equation as in [FLT.1] (which extends the case of the wave equation with Dirichlet control as in [L-T.4]) covers both the above problem (1.1), as well as the case of a control function acting in the Dirichlet B.C. (1.1c). As a result, it produces in both cases a feedback operator, based on the algebraic Riccati operator acting on the full pair $\{w,w_t\}$, which yields uniform stabilization in the spaces of optimal regularity and exact controllability (see [FLT.1; Appendix D] and [L-T.5]). Indeed, it is precisely the foregoing theory of exact controllability on spaces of optimal regularity which guarantees the Finite Cost Condition of the quadratic cost problem over an infinite horizon, and thus allows for the application of the Riccati theory in [FLT.1].

One problem that still needs investigation to complete the overall theory is the problem of uniform stabilization of the dynamics (1.1) by means of an explicit, dissipative feedback operator based on w_t. The present paper is devoted precisely to this problem. The results which we shall obtain in our Theorem 1.2 are fully consistent with recent results of exact controllability [Lio.1], [Lio.2], obtained directly via H.U.M. rather than through stabilization, both in regard of the spaces of optimal regularity and in regard of the properties of the triple $\{\Omega,\Gamma_0,\Gamma_2\}$, in particular, in regard of the lack of geometrical conditions on Ω (except for smoothness of Γ), if g in (1.1e) is applied to all of Γ (i.e., $\Gamma_0 = \emptyset$). Indeed, according to a well-known result for time reversible systems [R.1], the uniform stabilization results given here imply corresponding exact controllability results: these precisely coincide with those in [Lio.1], [Lio.2] as far as spaces and properties of the triple $\{\Omega,\Gamma_0,\Gamma_1\}$ in terms of a radial field are concerned. However, in Remarks 1.2 and 1.3 we point out possible generalizations of our uniform stabilization result (hence of exact controllability) which involve suitably small perturbations of a radial field.

1.2. Preliminaries and Choice of Dissipative Feedback

Throughout this paper, we let A: $L_2(\Omega) \supset \mathcal{D}(A) \to L_2(\Omega)$ be the positive self-adjoint operator defined by

$$Af = \Delta^2 f, \quad \mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{1.2}$$

We have [G.1]

$$\mathcal{D}(A^{1/4}) = H_0^1(\Omega); \ \mathcal{D}(A^{1/2}) = H_0^2(\Omega) = \left\{f \in H^2(\Omega): f|_\Gamma = \frac{\partial f}{\partial\nu}|_\Gamma = 0\right\} \tag{1.3}$$

with equivalent norms. Thus, for $f \in \mathcal{D}(A^{1/4}) = H_0^1(\Omega)$,

$$\|f\|_{\mathcal{D}(A^{1/4})} = \|A^{1/4}f\|_{L^2(\Omega)}, \text{ equivalent to } \|f\|_{H^1(\Omega)},$$

$$\text{in turn equivalent to } \{\int_\Omega |\nabla f|^2 d\Omega\}^{1/2} \tag{1.4}$$

by Poincaré inequality. Also, for $f \in \mathcal{D}(A^{1/2}) = H_0^2(\Omega)$,

$$\|f\|_{\mathcal{D}(A^{1/2})} = \|A^{1/2}f\|_{L_2(\Omega)} = \{\int_\Omega |\Delta f|^2 d\Omega\}^{1/2}. \tag{1.5}$$

As mentioned in the introduction, our optimal space will be

$$Z = L_2(\Omega)\times H^{-2}(\Omega) = L_2(\Omega)\times[\mathcal{D}(A^{1/2})]' . \tag{1.6}$$

<u>Choice of feedback operator</u>. With $g = 0$ in (1.1e), the resulting homogeneous problem generates a unitary group on $L_2(\Omega)\times[\mathcal{D}(A^{1/2})]'$ described by the map $\{w_0,w_1\} \to \{w,w_t\}$. It is justified in Section 1.4, see below (1.25), that the following choice of a feedback operator $\mathcal{F}(w_t)$ on $\Sigma_1 = (0,\infty)\times\Gamma_1$:

$$\frac{\partial w}{\partial \nu}\Big|_{\Sigma_1} = g = \mathcal{F}(w_t) = -\tilde{G}_2^* w_t = -\tilde{G}_2^* AA^{-1} w_t = [\Delta A^{-1} w_t]_{\Sigma_1} \tag{1.7}$$

provides a reasonable <u>candidate</u> for the uniform stabilization problem of (1.1), in the sense that the closed loop feedback dynamics with (I.7) used as (1.1e) is well-posed in the semigroup sense in Z and all of its solutions originating in Z decrease as $t \to \infty$ in the Z-norm. To show that they decrease to zero, and, in fact, in the uniform norm $\mathcal{L}(Z)$ is the major task of this paper. In (1.7) we have set $\tilde{G}_2$ to be the operator (Green map)

$$\tilde{G}_2 g_2 = y \Leftrightarrow \begin{cases} \Delta^2 y = 0 & \text{in } \Omega; & (1.8a) \\ y|_\Gamma = 0 & \text{in } \Gamma; & (1.8b) \\ \frac{\partial y}{\partial \nu}\Big|_{\Gamma_0} = 0,\ \frac{\partial y}{\partial \nu}\Big|_{\Gamma_1} = g_2; & & (1.8c) \end{cases}$$

$$\tilde{G}_2 : L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \cap H_0^1(\Omega), \tag{1.9}$$

while $\tilde{G}_2^*$ is the adjoint of $\tilde{G}_2$ in the sense that

$$(\tilde{G}_2 g_2, v)_{L_2(\Omega)} = (g_2, \tilde{G}_2^* v)_{L_2(\Gamma_1)}, \ \forall \ g_2 \in L_2(\Gamma_1), \ v \in L_2(\Omega). \tag{1.10}$$

Moreover, it is proved by Green's theorem that [L-T.2]

$$\tilde{G}_2^* Af = \begin{cases} 0 & \text{in } \Gamma_0 \\ -[\Delta f]_{\Gamma_1} & \text{in } \Gamma_1 \end{cases} \qquad f \in \mathcal{D}(A). \tag{1.11}$$

Thus, (1.11) justifies the last step in (1.7).

1.3. <u>The Feedback System: Statement of Main Results</u>

By virtue of (1.7), the resulting candidate feedback system, whose stability properties in Z we shall investigate, is

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } (0,T)\times\Omega = Q; & (1.12a) \\ w(0,\cdot) = w_0, \ w_t(0,\cdot) = w_1 & \text{in } \Omega; & (1.12b) \\ w|_\Sigma \equiv 0 & \text{in } (0,T)\times\Gamma = \Sigma; & (1.12c) \\ \frac{\partial w}{\partial \nu}\Big|_{\Sigma_0} \equiv 0 & \text{in } (0,T)\times\Gamma_0 = \Sigma_0; & (1.12d) \\ \frac{\partial w}{\partial \nu}\Big|_{\Sigma_1} = [\Delta(A^{-1} w_t)]_{\Sigma_1} & \text{in } (0,T)\times\Gamma_1 = \Sigma_1; & (1.12e) \end{cases}$$

Using the techniques of [T.1], [L-T.7], [L-T.2], [B-T.1], etc., problem (1.12) can be re-written more conveniently in abstract form as

$$w_{tt} = -Aw - A\tilde{G}_2 \tilde{G}_2^* w_t; \tag{1.13a}$$

$$\frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix} = \mathcal{A}\begin{vmatrix} w \\ w_t \end{vmatrix} \quad \text{on } Z. \tag{1.13b}$$

$$\mathcal{A} = \begin{vmatrix} 0 & I \\ -A & -A\tilde{G}_2\tilde{G}_2^* \end{vmatrix} : \quad \mathcal{D}(\mathcal{A}) = \{y \in Z: \mathcal{A}y \in Z\}. \tag{1.14}$$

A more explicit description of $\mathcal{D}(\mathcal{A})$ will be given in the subsequent analysis, see (2.3)-(2.5) below.

<u>Theorem 1.1</u>. (i) (Well-posedness on Z) The operator $\mathcal{A}$ in (1.14) is dissipative on $\mathcal{D}(\mathcal{A}) \subset Z = L_2(\Omega)\times[\mathcal{D}(A^{1/2})]'$ (see (1.6)) and satisfies here the range condition: range $(\lambda I-\mathcal{A}) = Z$ for all $\lambda > 0$. Thus, by Lumer-Phillips theorem, $\mathcal{A}$ generates a strongly continuous contraction semigroup $e^{\mathcal{A}t}$ on Z. The resolvent operator $R(\lambda,\mathcal{A})$ is given by

$$R(\lambda,\mathcal{A}) = \begin{bmatrix} \dfrac{I-V^{-1}(\lambda)}{\lambda} & V^{-1}(\lambda)A^{-1} \\ & \\ -V^{-1}(\lambda) & \lambda V^{-1}(\lambda)A^{-1} \end{bmatrix} \tag{1.15}$$

$$V(\lambda) = I+\lambda\tilde{G}_2\tilde{G}_2^* + \lambda^2A^{-1}, \tag{1.16}$$

at least for Re $\lambda > 0$, and moreover, $R(\lambda,\mathcal{A})$ is compact on Z. The resolvent set of $\mathcal{A}$ satisfies $\rho(\mathcal{A}) \supset \{\lambda: \text{Re } \lambda \geq 0\}$ if Γ_1 satisfies the uniqueness property (1.21) in Remark 1.1 below, which is certainly the case if $\Gamma_1 = \Gamma$.

(ii) (L_2-boundedness in time of the feedback operator) For $\{w_0,w_1\} \in Z$ the solution w of (1.12), or (1.13), satisfies

$$\frac{dE(t)}{dt} = -2\int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma = -2\|\tilde{G}_2^* w_t\|^2_{L_2(\Gamma_1)}$$
$$= -2\|\Delta(A^{-1}w_t)\|^2_{L_2(\Gamma_1)} \leq 0; \tag{1.17}$$

$$E(t)-E(0) = -2\int_0^t \int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma\, dt = -2\int_0^t \|\tilde{G}_2^* w_t\|^2_{L_2(\Gamma_1)} dt; \tag{1.18}$$

$$\int_0^\infty \int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma\, dt = \int_0^\infty \|\tilde{G}_2^* w_t\|^2_{L_2(\Gamma_1)} dt \le \tfrac{1}{2} E(0), \tag{1.19}$$

where throughout the paper we set for convenience

$$E(t) \equiv \left\| e^{\mathcal{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|^2_{L_2(\Omega)\times[\mathcal{D}(A^{1/2})]'} = \|w(t)\|^2_{L_2(\Omega)} + \|A^{-1/2} w_t(t)\|^2_{L_2(\Omega)}. \quad \blacksquare \tag{1.20}$$

__Remark 1.1__. The point $i\mu$, μ real and $\mu \neq 0$, of the imaginary axis belongs to the resolvent set $\rho(\mathcal{A})$ of the operator $\mathcal{A}$ in (1.14), provided the following uniqueness property holds true: If $\phi(x)$ is a (smooth) function which satisfies

$$\Delta^2 \phi = \mu^2 \phi \quad \text{in } \Omega; \tag{1.21a}$$

$$\phi|_\Gamma = \frac{\partial \phi}{\partial \nu}\Big|_\Gamma = 0 \quad \text{in } \Gamma; \tag{1.21b}$$

$$\Delta\phi|_{\Gamma_1} = 0 \quad \text{in } \Gamma_1; \tag{1.21c}$$

then, in fact,

$$\phi \equiv 0 \quad \text{in } \Omega. \tag{1.21d}$$

Plainly, the above uniqueness holds true if $\Gamma_1 = \Gamma$: in this case one readily obtains also the fourth boundary condition $\frac{\partial(\Delta\phi)}{\partial \nu}\Big|_\Gamma = 0$ from the second condition in (1.21b), see e.g., [L-T.8, Remark 2.1], and a standard uniqueness result yields the conclusion (1.21d).

More generally, a sufficient condition on the inactive portion of the boundary Γ_0 for the uniqueness (1.21) to hold true is that Γ_0 be as in (1.23) below. ■

The next Theorem 1.2 gives a uniform stabilization result, in particular when the feedback (1.12e) is active on the entire boundary Γ, i.e., when $\Gamma_0 = \emptyset$. If, instead $\Gamma_0 \neq \emptyset$, then Γ_0 is assumed of the special form as in (1.23) below. Theorem 1.2 then recovers the exact controllability results obtained directly by H.U.M. in [Lio.1], [Lio.2] (same spaces and same assumption on Γ_0), by a direct application of [R.1].

<u>Theorem 1.2</u>. (Uniform stabilization: the radial field case)

(a) Let $\Gamma_0 = \emptyset$ so that the feedback (1.12e) acts on all of Γ. Then, the feedback system (1.12), equivalently the abstract system (1.14), is uniformly (exponentially) stable on the space Z given by (1.6): there exist constants $\delta > 0$ and $M = M_\delta \geq 1$ such that

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_Z = \left\| e^{At} \begin{vmatrix} w_0 \\ w_t \end{vmatrix} \right\|_Z \leq Me^{-\delta t} \left\| \begin{vmatrix} w_0 \\ w_t \end{vmatrix} \right\|_Z, \quad t \geq 0. \tag{1.22}$$

(b) More generally, the uniform decay (1.22) holds true if we take

$$\Gamma_0 = \Gamma_-(x_0) = \{x \in \Gamma: (x-x_0)\cdot\nu \leq 0\} \tag{1.23}$$

for some point $x_0 \in R^n$, where ν = unit normal vector pointed outward. ∎

<u>Remark 1.2</u>. The uniform stabilization result (1.22) in Theorem 1.2 may be (slightly) generalized to linear vector fields

$$h_i(x) = a_i(x_i - x_{0,i}) \text{ for some } x_0 = [x_{0,1}, \ldots, x_{0,n}] \in R^n, \tag{1.24}$$

where the coefficients $\{a_i\}$ are constant, and there is a constant $m > 0$ such that the corresponding differences satisfy the condition that

$$\sup_i |a_i - m| \tag{1.25}$$

is sufficiently small. See Section 2.5. A further generalization is pointed out in the subsequent Remark 1.3. ∎

<u>Remark 1.3</u>. The uniform stabilization result (1.22) in Theorem 1.2 may be generalized to the case of a triplet $\{\Omega, \Gamma_0, \Gamma_1\}$ satisfying:

there exists a vector field $h(x) = [h_1(x), \ldots, h_n(x)] \in [C^3(\overline{\Omega})]^n$ such that

(i)
$$\int_\Omega \Delta q \left(\sum_{i=1}^n \nabla h_i \cdot \nabla q_{x_i} \right) d\Omega \geq \rho \int_\Omega (\Delta q)^2 d\Omega, \tag{1.26}$$

where $q(x)$ is an $H_0^2(\Omega)$-function on Ω satisfying therefore

$$q|_{\Gamma} = \frac{\partial q}{\partial \nu}|_{\Gamma} = 0, \tag{1.27}$$

and $\rho > 0$ is a suitable constant, possibly depending on h(x), Ω, and q(x);

(ii) either the (elliptic) uniqueness property (1.21) holds true; or else the corresponding dynamical uniqueness property: if $\psi(t,x)$ solves

$$\psi_{tt}+\Delta^2\psi = 0 \qquad \text{in } (0,T]\times\Omega = Q; \tag{1.28a}$$

$$\psi|_{\Sigma} = \frac{\partial\psi}{\partial\nu}|_{\Sigma} = 0 \qquad \text{in } (0,T]\times\Gamma = \Sigma; \tag{1.28b}$$

$$\Delta\psi|_{\Sigma_1} = 0 \qquad \text{in } (0,T]\times\Gamma_1 = \Sigma_1, \tag{1.28c}$$

then, in fact,

$$\psi \equiv 0 \qquad \text{in } Q. \tag{1.28d}$$

Either uniqueness property (1.21) or (1.28) holds true in case Γ_0 is given by (1.23).

The linear vector field in (1.24), (1.25) satisfies the inequality (1.26), and, moreover, the quantity

$$M_h = \max\{|\Delta h_i|, |\Delta(\text{div } h)|, |\nabla(\text{div } h)|\} \tag{1.29}$$

is zero in this case. Thus, see Section 2.5, no lower order terms are involved in inequality (2.8) below: it is precisely to absorb the lower order term $\| |\nabla p| \|_{C([0,T];L_2(\Omega))}$ of inequality (2.8) that condition (ii) is invoked. More general perturbations of the radial field than the linear field (1.24), (1.25) can be given which satisfy inequality (1.26), but then the uniqueness property (1.21) or (1.28) comes into play; see Section 2.5. ∎

1.4. Sketch of Proof of Theorem 1.1

The proof of much of Theorem 1.1 is very similar to the proof of analogous results for wave equations with Dirichlet control [L-T.6], or Neumann control [T.2], and for Euler-Bernoulli equations with control in the Dirichlet B.C. [B-T.1]. Thus, details are omitted and only some distinctive features of problem (1.12) will be mentioned. According to techniques as in [T.1], [L-T.7], etc., for waves, or [B-T.1], [L-T.2], etc., for Euler-Bernoulli problem, the abstract differential equation which models problem (1.1) is

$$w_{tt} = -Aw + A\tilde{G}_2 g, \tag{1.30}$$

(recall (1.8)) whose corresponding first-order system is

$$\frac{d}{dt}\begin{vmatrix} w \\ w_t \end{vmatrix} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \begin{vmatrix} w \\ w_t \end{vmatrix} + \begin{vmatrix} 0 \\ A\tilde{G}_2 g \end{vmatrix}. \tag{1.31}$$

Since $\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}$ is skew-adjoint on Z, (1.6), Eq. (1.31) plainly suggests to take $g = -\tilde{G}_2^* w_t$ (modulo a positive function which we shall take identically 1) for feedback stabilization (as anticipated in (1.7)), for this choice then makes the corresponding feedback operator $\mathcal{A}$ in (1.14) dissipative on Z; indeed, for $y = [y_1, y_2] \in \mathcal{D}(\mathcal{A})$:

$$\text{Re}(\mathcal{A}y, y)_Z = -(A\tilde{G}_2\tilde{G}_2^* y_2, y_2)_{[\mathcal{D}(A^{1/2})]'} = -(\tilde{G}_2\tilde{G}_2^* y_2, y_2)_{L_2(\Omega)}$$

$$= -\|\tilde{G}_2^* y_2\|^2_{L_2(\Gamma_1)} \le 0. \tag{1.32}$$

The proof of Theorem 1.1 follows closely the above references; in particular, for the compactness of $R(\lambda,\mathcal{A})$ one may follow the argument of [B-T.1; Theorem 1.1, Step 2]. The only point which needs further explanation is the claim that the imaginary axis is in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. That $0 \in \rho(\mathcal{A})$ is immediate, as the resolvent is compact.

Next, if $\mu \neq 0$ real, an argument as in the above references yields that if $i\mu$ is an eigenvalue of $\mathcal{A}$, then Eqs. (1.21a-b-c) hold true, and if the uniqueness property (1.21d) applies, then we have a contradiction. To show the final statement of Remark 1.1--that the uniqueness property (1.21) holds true for Γ_0 as in (1.23)--we proceed as follows. We apply the multipliers $(s-s_0)\cdot\nabla\phi$ and ϕ to problem (1.21a). This produces (see subsequent Section 2.2)

$$\int_{\Gamma}(\Delta\phi)^2(x-x_0)\cdot\nu d\Gamma = 4\int_{\Omega}(\Delta\phi)^2 d\Omega. \tag{1.33}$$

Then, using (1.21c) and (1.23), we obtain

$$0 = \int_{\Gamma_1}(\Delta\phi)^2(x-x_0)\nu d\Gamma \geq \int_{\Gamma}(\Delta\phi)^2(x-x_0)\cdot n d\Gamma, \tag{1.34}$$

and then $\Delta\phi \equiv 0$ in Ω follows from (1.33): this, along with the first condition in (1.21b) yields $\phi \equiv 0$ in Ω as desired. ■

2. Proof of Theorem 1.2

2.1. Preliminaries and a Change of Variable $w \to p$

With reference to the 'energy' $E(t)$ of the w-problem (1.12) defined in (1.20) our task is, as usual, to show that: There exists a time $0 < T < \infty$ such that

$$E(T) \leq r\, E(0),\ r < 1; \quad \text{or } \|e^{\mathcal{A}T}\|_{\mathcal{L}(Z)} < 1, \tag{2.1}$$

$Z = L_2(\Omega)\times[\mathcal{D}(A^{1/2})]'$, norm-equivalent to $L_2(\Omega)\times H^{-2}(\Omega)$, after which the uniform decay (1.22) is then established. To prove (2.1), it will suffice, as usual, to show that: There exists a time $0 < T < \infty$ and a corresponding constant $c_T > 0$ such that

$$E(T) \leq c_T \int_0^T\int_{\Gamma_1}\left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma_1\, dt \tag{2.2}$$

for (2.2), combined with the non-increasing property (1.17) of $E(t)$, will then yield (1.22). Our subsequent effort is aimed at establishing (2.2). To this end, we use the idea introduced in [L-T.6] which consists in lifting the low (though optimal) topology $L_2(\Omega)\times H^{-2}(\Omega)$ for the solution $\{w(t),w_t(t)\}$ of (1.12) to the level $H_0^2(\Omega)\times L_2(\Omega)$ suitable for multipliers techniques for the pair $\{p(t),p_t(t)\}$, where p is the dependent variable of a new problem.

This idea was also successfully used in [B-T.1] in the study of uniform stabilization of problem (1.1) with feedback also in the Dirichlet B.C. (1.1c). (But the transformation $w \to p$ in [B-T.1] is different from the one in (2.8) below for the present problem (1.12), due to the different topologies involved; moreover, for the very same reasons, the multipliers used in the p-problem in [B-T.1] are different from the multipliers used in the present paper.)

Unless otherwise noted, we take henceforth $\{w_0,w_1\} \in \mathcal{D}(\mathcal{A})$ and show the estimate (2.2) with constant c_T independent of $\{w_0,w_1\}$. It follows readily from (1.14) that $\mathcal{A}z = \{z_2, -A[z_1+\tilde{G}_2\tilde{G}_2^*z_2]\} \in Z = L_2(\Omega)\times[\mathcal{D}(A^{1/2})]'$ implies

$$\{z_1,z_2\} \in \mathcal{D}(\mathcal{A}) \Rightarrow \begin{cases} z_2 \in L_2(\Omega); & (2.3) \\ z_1+\tilde{G}_2\tilde{G}_2^*z_2 \in \mathcal{D}(A^{1/2}) = H_0^2(\Omega), & (2.4) \end{cases}$$

and thus, by (1.9),

$$\tilde{G}_2\tilde{G}_2^*z_2 \in H^3(\Omega) \cap H_0^1(\Omega), \text{ hence } z_1 \in H^2(\Omega) \cap H_0^1(\Omega), \quad (2.5)$$

upon using (2.4). Since $\{w_0,w_1\} \in \mathcal{D}(\mathcal{A})$ implies $\{w(t),w_t(t)\} \in C([0,T];\mathcal{D}(\mathcal{A}))$ by Theorem 1.1(i), we then have by (2.3) and (2.5),

$$\begin{vmatrix} w_0 \\ w_t \end{vmatrix} \in \mathcal{D}(\mathcal{A}) \Rightarrow \begin{cases} w(t) \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) & (2.6) \\ w_t(t) \in C([0,T];L_2(\Omega)) & (2.7) \end{cases}$$

Following the idea in [L-T.6], [B-T.1], we then introduce a new variable p in the present case by setting

$$A^{1/2}p = A^{-1/2}w_t; \text{ i.e.,}$$

$$p = A^{-1}w_t \in \begin{cases} C([0,T];\mathcal{D}(A^{1/2})) \text{ if } \{w_0,w_1\} \in Z; & (2.8a) \\ C([0,T];\mathcal{D}(A)) \text{ if } \{w_0,w_1\} \in \mathcal{D}(\mathcal{A}). & (2.8b) \end{cases}$$

From (2.8) and (1.13a), we obtain

$$p_t = A^{-1}w_{tt} = -w-\tilde{G}_2\tilde{G}_2^*w_t \in \begin{cases} L_2(0,T;L_2(\Omega)) & \text{if } \{w_0,w_1\} \in Z; & (2.9a) \\ C([0,T];\mathcal{D}(A^{1/2})) & \text{if } \{w_0,w_1\} \in \mathcal{D}(\mathcal{A}), & (2.9b) \end{cases}$$

where the regularity in (2.9) follows from (1.18), (2.6), and (2.4). Hence, from (2.9) we obtain via (2.8) (left),

$$p_{tt} = -w_t-\tilde{G}_2\tilde{G}_2^*w_{tt} = -Ap-\tilde{G}_2\tilde{G}_2^*w_{tt}. \tag{2.10}$$

In terms of the scalar function $p(t,x)$, $x \in \Omega$, corresponding to the vector-valued $p(t) = p(t,\cdot)$, the abstract equation (2.10) can be rewritten as the following Euler-Bernoulli problem:

$$\begin{cases} p_{tt}+\Delta^2 p = F; & (2.11a) \\ p(0,\cdot) = p_0 = A^{-1}w_1;\ p_t(0,\cdot) = p_1 = A^{-1}w_{tt}(0); & (2.11b) \\ p|_\Sigma \equiv 0; & (2.11c) \\ \frac{\partial p}{\partial \nu}|_\Sigma \equiv 0; & (2.11d) \end{cases}$$

where $p_0 \in \mathcal{D}(A)$ and $p_1 = -(w_0+\tilde{G}_2\tilde{G}_2^*w_1) \in \mathcal{D}(A^{1/2})$, by (2.8b) and (2.9b), respectively, and where the homogeneous boundary conditions in (2.11c-d) are a consequence of $p \in \mathcal{D}(A^{1/2}) = H_0^2(\Omega)$ from (2.8) and (1.3). In (2.11a) we have set

$$F = -\tilde{G}_2\tilde{G}_2^*w_{tt} = -\tilde{G}_2\tilde{G}_2^*AA^{-1}w_{tt} = \tilde{G}_2[\Delta p_t]_{\Sigma_1}. \tag{2.12}$$

In the sequel, we shall have to consider pointwise values of $p_t(t)$: these make sense by (2.9b) for initial data $\{w_0,w_1\} \in \mathcal{D}(\mathcal{A})$ as assumed, while from (2.9a) the pointwise meaning of $p_t(t)$ in $L_2(\Omega)$ is lost for general initial data in Z. In the analysis below of the p-system (2.11), we shall crucially use from (2.8) (left) and (2.9) (left), respectively, and (1.5),

$$\|w_t\|_{[\mathcal{D}(A^{1/2})]'} = \|A^{-1/2}w_t\|_{L_2(\Omega)} = \|A^{1/2}p\|_{L_2(\Omega)} = \left\{\int_\Omega (\Delta p)^2 d\Omega\right\}^{1/2}; \tag{2.13}$$

$$p_t = -w + \mathcal{O}(\|\tilde{G}_2^* w_t\|_{L_2(\Gamma_1)}), \tag{2.14}$$

where $\frac{\partial w}{\partial \nu} = -\tilde{G}_2^* w_t \in L_2(0,\infty;L_2(\Gamma_1))$ from (1.19), since G_2 is bounded on $L_2(\Gamma)$ (see (1.9)). Recalling the 'energy' E(t) of the w-problem from (1.20), we have via (2.13), (2.14),

$$E(t) = \int_\Omega p_t^2(t)+(\Delta p(t))^2 d\Omega + \mathcal{O}(\|\tilde{G}_2^* w_t\|^2_{L_2(\Gamma_1)}). \tag{2.15}$$

In (2.14), (2.15), the symbol $\mathcal{O}$ means, as usual, bounded above by a constant, in fact, independent of T. Dependence of constants on T will always be noted explicitly.

2.2. Integral Identities for the p-Problem (2.11)

Throughout, we let $Q = (0,T]\times\Omega$, $\Sigma = (0,T]\times\Gamma$, etc.

Proposition 2.1. Let $h(x) = [h_1(x),\dots,h_n(x)] \in [C^3(\bar{\Omega})]^n$ be a given vector field, and let $\{w_0,w_1\} \in \mathcal{D}(\mathcal{A})$ so that $\{p_0,p_1\} \in \mathcal{D}(A)\times\mathcal{D}(A^{½})$. Then the solution p of problem (2.11) satisfies the following identity:

$$\begin{aligned} ½\int_\Sigma (\Delta p)^2 h\cdot\nu \, d\Sigma &= 2\int_Q \Delta p\left[\sum_{i=1}^n \nabla h_i\cdot\nabla p_{x_i}\right]dQ \\ &+ \int_Q \Delta p[\Delta h_1,\dots,\Delta h_n]\cdot\nabla p \, dQ \\ &+ ½\int_Q [p_t^2-(\Delta p)^2]\text{div}\, h \, dQ \\ &- \int_Q F\, h\cdot\nabla p\, dQ + [(p_t(t),h\cdot\nabla p(t))_\Omega]_0^T. \quad \blacksquare \end{aligned} \tag{2.16}$$

Remark 2.1. We note explicitly that the following identities hold true:

$$\mathrm{div}(H\nabla p) = \sum_{i=1}^{n} \nabla h_i \cdot \nabla p_{x_i} + \nabla p \cdot \nabla(\mathrm{div}\ h); \tag{2.17}$$

$$\mathrm{div}(H^T\nabla p) = \sum_{i=1}^{n} \nabla h_i \cdot \nabla p_{x_i} + [\Delta h_1, \ldots, \Delta h_n] \cdot \nabla p, \tag{2.18}$$

where $H = H(x)$ is the $n\times n$ matrix with (i,j)-entry $\frac{\partial h_i}{\partial x_j}$ and H^T its transpose, so that (2.17) and (2.18) imply

$$\mathrm{div}[(H+H^T)\nabla p] = 2\sum_{i=1}^{n} \nabla h_i \cdot \nabla p_{x_i} + \nabla p \cdot \nabla(\mathrm{div}\ h) + [\Delta h_1, \ldots, \Delta h_n] \cdot \nabla p, \tag{2.19}$$

and hence (2.16) can be rewritten as

$$\begin{aligned} \tfrac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu \, d\Sigma &= \tfrac{1}{2}\int_Q [p_t^2 - (\Delta p)^2]\mathrm{div}\ h\, dQ + \int_Q \Delta p\ \mathrm{div}[(H+H^T)\nabla p]dQ \\ &- \int_Q \Delta p\ \nabla p\cdot\nabla(\mathrm{div}\ h)dQ - \int_Q F\ h\cdot\nabla p\ dQ \\ &+ [(p_t(t), h\cdot\nabla p(t))_\Omega]_0^T. \end{aligned} \tag{2.20}$$

<u>Proof of Proposition 2.1</u>. One uses the multiplier $h\cdot\nabla p$ as in [Lio.2], [Lag.1], [L-T.8], [T.3]. (We use a general field h, even though we shall specialize later to radial fields in our principal result, Theorem 1.2, mostly for the benefit of including in our arguments the generalizations pointed out in Remarks 1.2 and 1.3.) ∎

We now handle the first integral on the right of (2.20).

<u>Proposition 2.2</u>. Under the assumptions of Proposition 2.1, the solution p of problem (2.11) satisfies the following identity:

$$\begin{aligned} \int_Q [p_t^2 - (\Delta p)^2]\mathrm{div}\ h\, dQ &= -\int_Q F\ p\ \mathrm{div}\ h\ dQ + \int_Q p\Delta p\ \Delta(\mathrm{div}\ h)dQ \\ &+ 2\int_Q \Delta p\ \nabla p\cdot\nabla(\mathrm{div}\ h)dQ + [(p_t(t), p(t)\mathrm{div}\ h)]_\Omega]_0^T. \end{aligned} \tag{2.21}$$

<u>Remark 2.2</u>. We note explicitly for future use that identity (2.21) continues to hold true if we set div h ≡ 1 in it; i.e., if in the proof we multiply Eq. (2.11a) simply by p rather than p div h. ∎

<u>Proof of Proposition 2.2</u>. One uses the multiplier p div h [Lio.2], [Lag.1], [L-T.8]. ∎

We next insert (2.21) into (2.16), or respectively, (2.20), and obtain the final identity of the p-system.

<u>Proposition 2.3</u>. Under the assumptions of Proposition 2.1, the solution p of (2.12) satisfies the following identity (from (2.16)):

$$\frac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu \, d\Sigma = 2\int_Q \Delta p\left[\sum_{i=1}^n \nabla h_i\cdot\nabla p_{x_i}\right]dQ + \int_Q \Delta p[\Delta h_1,\dots,\Delta h_n]\cdot\nabla p \, dQ$$

$$+ \frac{1}{2}\int_Q p\Delta p \, \Delta(\text{div } h)dQ + \int_Q \Delta p \, \nabla p\cdot\nabla(\text{div } h)dQ$$

$$- \int_Q F \, h\cdot\nabla p \, dQ - \frac{1}{2}\int_Q F \, p \text{ div } h \, dQ + b_{0,T}; \qquad (2.22)$$

$$b_{0,T} = [(p_t, h\cdot\nabla p)_\Omega]_0^T + \frac{1}{2}[(p_t, p \text{ div } h)_\Omega]_0^T, \qquad (2.23)$$

where (2.22) can be rewritten (from (2.18)) more concisely as

$$\frac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu \, d\Sigma = \int_Q \Delta p \text{ div}[(H+H^T)\nabla p]dQ + \frac{1}{2}\int_Q p\Delta p \, \Delta(\text{div } h)dQ$$

$$- \int_Q F \, h\cdot\nabla p \, dQ - \frac{1}{2}\int_Q F \, p \text{ div } h \, dQ + b_{0,T}. \quad \blacksquare \qquad (2.24)$$

The analysis below will show *a-fortiori* that the terms in identity (2.24) are well defined by establishing appropriate estimates thereof.

2.3. <u>Analysis of Terms Involving F and the Boundary Data</u> $b_{0,T}$

The crucial term in (2.24) is the one involving F $h\cdot\nabla p$.

<u>Proposition 2.4</u>. Let the assumptions of Proposition 2.1 hold true.

(a) Then, the following identity is satisfied:

$$-\int_Q F\, h\cdot\nabla p\, dQ - \tfrac{1}{2}\int_Q F\, p\, \mathrm{div}\, h\, dQ + b_{0,T}$$

$$= -[(w, h\cdot\nabla p)_\Omega]_0^T - \tfrac{1}{2}[(w,\ p\,\mathrm{div}\, h)]_\Omega]_0^T$$

$$- \int_0^T (\tilde{G}_2\tilde{G}_2^* w_t, h\cdot\nabla p_t)_\Omega dt - \tfrac{1}{2}\int_0^T (\tilde{G}_2\tilde{G}_2^* w_t, p_t\ \mathrm{div}\, h)_\Omega dt \quad (2.25)$$

where

$$(\tilde{G}_2\tilde{G}_2^* w_t, h\cdot\nabla p_t)_\Omega + \tfrac{1}{2}(\tilde{G}_2\tilde{G}_2^* w_t, p_t\ \mathrm{div}\, h)_\Omega$$

$$= -\tfrac{1}{2}(\tilde{G}_2\tilde{G}_2^* w_t, p_t\ \mathrm{div}\, h)_\Omega - (p_t, h\cdot\nabla(\tilde{G}_2\tilde{G}_2^* w_t))_\Omega \quad (2.26a)$$

$$= \mathcal{O}\left[\|\tilde{G}_2^* w_t\|_{L_2(\Gamma_1)}\|p_t\|_{L_2(\Omega)}\right], \quad (2.26b)$$

with constant in $\mathcal{O}$ depending on h and $\|A^{1/4}\tilde{G}_2\|$.

(b) The following estimate holds true for the right hand side of (2.25), with E(t) as in (1.20) where $\varepsilon > 0$ is arbitrary:

$$-\int_Q F h\cdot\nabla p\, dQ - \tfrac{1}{2}\int_Q F p\ \mathrm{div}\, h\, dQ + b_{0,T} \ge -C_h[E(T)+E(0)]$$

$$- \frac{1}{\varepsilon} C_h \int_0^T \|\tilde{G}_2^* w_t\|^2_{L_2(\Gamma_1)} dt - \varepsilon\int_0^T \|p_t\|^2_{L_2(\Omega)} dt. \quad (2.27)$$

__Proof__. (a) Recalling $F = -\tilde{G}_2\tilde{G}_2^* w_{tt}$ from (2.12) and integrating by parts in t, we obtain

$$-\int_Q F\, h\cdot\nabla p\, dQ = \int_0^T (\tilde{G}_2\tilde{G}_2^* w_{tt}, h\cdot\nabla p)_\Omega dt$$

$$= [(\tilde{G}_2\tilde{G}_2^* w_t, h\cdot\nabla p)_\Omega]_0^T - \int_0^T (\tilde{G}_2\tilde{G}_2^* w_t, h\cdot\nabla p_t)_\Omega dt, \quad (2.28)$$

and using now $\tilde{G}_2\tilde{G}_2^*w_t = -w-p_t$ from (2.9), we obtain from (2.28),

$$-\int_Q F\ h\cdot\nabla p\ dQ + [(p_t,h\cdot\nabla p)_\Omega]_0^T = -[(w,h\cdot\nabla p)_\Omega]_0^T - \int_0^T(\tilde{G}_2\tilde{G}_2^*w_t,h\cdot\nabla p_t)_\Omega dt. \qquad (2.29)$$

Similarly,

$$-\int_Q F\ p\ \text{div}\ h\ dQ + [(p_t,p\ \text{div}\ h)_\Omega]_0^T = -[(w,p\ \text{div}\ h)_\Omega]_0^T - \int_0^T(\tilde{G}_2\tilde{G}_2^*w_t,p_t\ \text{div}\ h)_\Omega dt. \qquad (2.30)$$

Then (2.29) and (2.30) lead to (2.25). Finally, using the identity

$$\int_\Omega vh\cdot\nabla\psi\ d\Omega = \int_\Gamma v\psi h\cdot\nu\ d\Gamma - \int_\Omega \psi h\cdot\nabla v\ d\Omega - \int_\Omega v\psi\ \text{div}\ h\ d\Omega, \qquad (2.31)$$

a consequence of the divergence theorem, with $v = \tilde{G}_2\tilde{G}_2^*w_t$ and $\psi = p_t$ along with the B.C. $p_t|_\Gamma = 0$ from (2.11c), we readily verify identity (2.26a), from which estimate (2.26b) follows at once.

(b) Estimate (2.27) follows readily from (2.25), (2.26b), recalling (2.13) and the definition (1.20) of E(t). ∎

2.4. Completion of the Proof of Theorem 1.2: The Radial Field Case and Absence of Geometrical Conditions if $\Gamma_0 = \phi$

Step 1. We specialize to the radial field $h(x) = x-x_0$ as in the assumption

$$H(x) = \text{Identity};\quad \text{div}\ h \equiv \dim\Omega = n;\quad \nabla(\text{div}\ h) \equiv 0, \qquad (2.32)$$

so that the basic identity (2.22), or its more concise form (2.24), becomes

$$\tfrac{1}{2}\int_\Sigma(\Delta p)^2 h\cdot\nu\ d\Sigma = 2\int_Q(\Delta p)^2 dQ - \int_Q Fh\cdot\nabla p\ dQ - \tfrac{1}{2}\int_Q Fp\ \text{div}\ h\ dQ + b_{0,T}. \qquad (2.33)$$

Next, identity (2.21) and Remark 2.2 give

$$\int_Q (\Delta p)^2 dQ = \int_Q p_t^2 dQ + \int_Q Fp\, dQ - [(p_t,p)_\Omega]_0^T, \tag{2.34}$$

which inserted in (2.33) produces the identity

$$\tfrac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu\, d\Sigma = 2\int_Q p_t^2 dQ - \int_Q Fh\cdot\nabla p\, dQ - \tfrac{1}{2}\int_Q Fp \operatorname{div} h\, dQ$$

$$+ 2\int_Q Fp\, dQ - 2[(p_t,p)_\Omega]_0^T + b_{0,T}. \tag{2.35}$$

Summing up (2.33) and (2.35) results in

$$\tfrac{1}{2}\int_\Sigma (\Delta p)^2 h\cdot\nu\, d\Sigma = 2\int_Q [(\Delta p)^2 + p_t^2] dQ - 2\Big\{\int_Q Fh\cdot\nabla p\, dQ + \tfrac{1}{2}\int_Q Fp \operatorname{div} h\, dQ - b_{0,T}\Big\}$$

$$+ 2\int_Q Fp\, dQ - 2[(p_t,p)_\Omega]_0^T. \tag{2.36}$$

<u>Step 2</u>. We now recall (2.15) for the first term on the right of (2.36); estimate (2.27) for the last term in $\{\cdot\}$ of (2.36); and a similar estimate for the last two terms in (2.36) (which are, in fact, contained in (2.27)). We obtain for the right hand side (R.H.S.) of (2.36):

$$\text{R.H.S. of (2.36)} \geq (2-\varepsilon)\int_0^T E(t)dt - C_h[E(T)+E(0)]$$

$$- \frac{C_h}{\varepsilon}\int_0^T \|\tilde{G}_2^* w_t\|^2_{L_2(\Gamma_1)} dt. \tag{2.37}$$

Recalling (1.18), we rewrite (2.37) as

$$\text{R.H.S. of (2.36)} \geq (2-\varepsilon)\int_0^T E(t)dt - 2C_h E(T) - C_{h,\varepsilon}\int_0^T\int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma\, dt. \tag{2.38}$$

<u>Step 3</u>. From (1.7) or (1.12e) and (2.8), we have

$$\frac{\partial w}{\partial \nu}\Big|_{\Sigma_1} = [\Delta A^{-1} w_t]_{\Sigma_1} = [\Delta p]_{\Sigma_1}, \tag{2.39}$$

so that with the definition of $\Gamma_0 = \Gamma_-(x_0)$ given by (1.22), we have for the left hand side (L.H.S.) of (2.36)

$$\text{L.H.S. of (2.36)} = \int_{\Sigma} (\Delta p)^2 (x-x_0)\cdot\nu d\Sigma$$

$$\leq \int_{\Sigma_1} (\Delta p)^2 (x-x_0)\cdot\nu d\Sigma_1 \leq C_h \int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Sigma_1. \tag{2.40}$$

Combining (2.38) with (2.40), we obtain

$$C_{h,\varepsilon} \int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma\, dt \geq (2-\varepsilon)\int_0^T E(t)dt - 2C_h\, E(T)$$

$$\geq [(2-\varepsilon)T - 2C_h]E(T), \tag{2.41}$$

where in the last step of (2.41) we have used the dissipativity property (1.17) of E(t). Taking T sufficiently large in (2.41) yields the desired estimate (2.2), Theorem 1.2 is proved. ∎

2.5. <u>Final Remarks on Possible Generalizations</u>

In this section we elaborate on the possible generalizations pointed out in Remarks 1.2 and 1.3.

<u>Concerning Inequality (1.26)</u>. Let $h(x) \in [C^3(\overline{\Omega})]^n$ be a vector field, and let H(x) be the n×n matrix with (i,j)-entry $\frac{\partial h_i}{\partial x_j}$ as in Section 2.2. If h(x) is radial, then $H(x) \equiv$ identity. We then consider the following perturbation H(x) of the identity matrix:

(i) Let the off-main diagonal terms $\frac{\partial h_i}{\partial x_j}$, $i \neq j$, be sufficiently small in the sup-norm;

(ii) let the main diagonal terms satisfy the conditions that

$$\sup \left[\frac{\partial h_i}{\partial x_j}(x)-m\right] \equiv d$$

be sufficiently small, for a constant $m > 0$.

Then, inequality (1.26) holds true, if $q \in H_0^2(\Omega)$ as assumed.

In fact, we may write

$$\Delta q\left(\sum_{i=1}^{n} \nabla h_i \cdot \nabla q_{x_i}\right) = m(\Delta q)^2 + (\Delta q)Q(x); \tag{2.43}$$

$$Q(x) = \sum c_{ij}(x) q_{x_i x_j}; \tag{2.44}$$

$$\int_\Omega \Delta q\left(\sum_{i=1}^{n} \nabla h_i \cdot \nabla q_{x_i}\right) d\Omega \geq m\int_\Omega (\Delta q)^2 dQ - \int_\Omega |(\Delta q)Q| d\Omega. \tag{2.45}$$

But for each i, j,

$$\|q_{x_i x_j}\|_{L_2(\Omega)} \leq \|q\|_{H^2(\Omega)} \leq C\|q\|_{\mathcal{D}(A^{1/2})} = c\|A^{1/2} q\|_{L_2(\Omega)} = c\|\Delta q\|_{L_2(\Omega)}, \tag{2.46}$$

since $q \in H_0^2(\Omega) = \mathcal{D}(A^{1/2})$, with equivalent norms (see (1.3) and (1.5)), where c is a constant of equivalence. Hence, from (2.44), (2.46), we obtain

$$\int_\Omega |(\Delta q)Q| d\Omega \leq k\left\{\int_\Omega (\Delta q)^2 d\Omega\right\}^{1/2}, \tag{2.47}$$

so that (2.45), (2.47) imply

$$\int_\Omega \Delta q\left(\sum_{i=1}^{n} \nabla h_i \cdot \nabla q_{x_i}\right) d\Omega \geq (m-k)\int_\Omega (\Delta q)^2 d\Omega. \tag{2.48}$$

The constant k depends on $\operatorname{supp}\left|\frac{\partial h_i}{\partial x_j}\right|$, $i \neq j$; on d in (2.42); on c in (2.46); and if these quantities are sufficiently small with respect to m, we may obtain $m-k = \rho > 0$ as desired. This situation occurs, in particular, for linear fields as in (1.24), (1.25) of Remark 1.2.

Modifications of the Proof of Section 2 for a Vector Field Satisfying Inequality (1.26). We return to the basic identity (2.22). If inequality (1.26) holds true, we readily find for the right hand side (R.H.S) of (2.22):

$$\text{R.H.S. of (2.22)} \geq (2\rho-\varepsilon)\int_Q (\Delta p)^2 dQ - \frac{M_h}{\varepsilon}\int_Q |\nabla p|^2 dQ$$

$$- \int_Q Fh\cdot\nabla p \, dQ - \tfrac{1}{2}\int_Q F\, p \operatorname{div} h \, dQ + b_{0,T}, \tag{2.49}$$

where the constant M_h is defined in (1.29) (and is zero if h is linear as in (1.24), (1.25)). The proof now proceeds as in Section 2 and yields the inequality

$$C_{h,\varepsilon}\int_0^T\int_{\Gamma_1} (\Delta p)^2 d\Sigma + \frac{M_h T}{\varepsilon}\, \| |\nabla p| \|^2_{C([0,T];L_2(\Omega))}$$

$$\geq [(2\rho-\varepsilon)T - 2C_h]\, \|\{p_0,p_1\}\|^2_{\mathcal{D}(A^{1/2})\times L_2(\Omega)} \tag{2.50}$$

counterpart of (2.41) (recall that $\frac{\partial w}{\partial \nu} = \Delta p$ on Σ_1, by (2.39)), where we have also used (1.18) for $t = T$ and (2.13), (2.14); where we now have an additional (lower order) term $|\nabla p|$. We absorb this as follows.

Lemma 2.1. Inequality (2.50) implies that: there is a constant C_T such that

$$\| |\nabla p| \|_{C([0,T];L_2(\Omega))} \leq C_T \int_0^T\int_{\Gamma_1} (\Delta p)^2 d\Sigma. \tag{2.51}$$

Proof. The proof follows similar arguments e.g., [Lio.1-2], [L-T.2-3], etc., with the following novelty: The contradiction argument

$$\int_0^T \int_{\Gamma_1} (\Delta p_n)^2 d\Sigma \to 0; \tag{2.51}$$

$$\| |\nabla p_n| \|_{C([0,T];L_2(\Omega))} \equiv 1, \tag{2.52}$$

for solutions p_n of the non-homogeneous p-problem (2.11), leads in the usual way to the result that $\{p_n\}$ are uniformly bounded in $L_\infty(0,T;\mathcal{D}(A^{\frac{1}{4}}))$, and hence, by compactness [S.1],

$$p_n \to \text{limit function } \tilde{p}, \text{ strongly in } L_\infty(0,T;\mathcal{D}(A^{\frac{1}{4}})), \tag{2.53}$$

so that from (2.12),

$$\| |\nabla \tilde{p}| \|_{C([0,T];L_2(\Omega))} = 1. \tag{2.54}$$

Now, the limit $\tilde{p}$ satisfies $\Delta\tilde{p}|_{\Sigma_1} = 0$ by (2.11), and hence the corresponding right hand side function $\tilde{F}$ in (2.11), (2.12) becomes $\tilde{F} = \tilde{G}_2[\Delta\tilde{p}_t]_{\Sigma_1} \equiv 0$. Thus, the limit problem for $\tilde{p}$ becomes *homogeneous* on the right hand side (as in the corresponding exact controllability question):

$$\begin{aligned} &\tilde{p}_{tt} + \Delta^2\tilde{p} = 0 && \text{in } Q; \\ &\tilde{p}|_\Sigma = \frac{\partial\tilde{p}}{\partial\nu}|_\Sigma = 0 && \text{in } \Sigma; \\ &\Delta\tilde{p}|_{\Sigma_1} = 0 && \text{in } \Sigma_1. \end{aligned} \tag{2.55}$$

It is here that the uniqueness property (1.28) is invoked to obtain $\tilde{p} \equiv 0$ in Q, a contradiction with (2.14). ∎

The rest of the proof proceeds as in Section 2, following (2.41).

Finally, we remark that if, instead of criterion (2.1), one uses the equivalent criterion (Datko's theorem):

$$\int_0^\infty E(t)dt \leq C\, E(0),$$

then absorption of the lower order terms can be done as in [B-T.1, Theorem 1.3b]: but this requires that the imaginary axis belongs to the resolvent set $\rho(A)$ of A. Hence, in this case, the elliptic uniqueness property (1.21) is needed. ■

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N. Ourada and R. Triggiani
Department of Applied Mathematics
Thornton Hall
University of Virginia
Charlottesville, VA 22903 USA

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STRONG G-CONVERGENCE OF NONLINEAR ELLIPTIC OPERATORS AND HOMOGENIZATION

Alexander Pankov

General questions on G-convergence are investigated for arbitrary order nonlinear elliptic operators of divergence form. The theorems on strong G-compactness, on locallity of strong G-convergence, and on convergence of arbitrary solutions are obtained. As an application, some results on homogenization are stated for rapidly oscillated nonlinear elliptic operators.

Introduction

Various problems on highly inhomogeneous media give rise to homogenization theory for partial differential equations and to the study of more general questions on G-convergence of operators (and on Γ-convergence of functionals). Now there is an extensive theory dealing with linear problems (see [2 -4, 21, 25, 26] and the references therein). As for nonlinear problems the Γ-convergence of integral functionals is well-studied (see, for example, [6, 28] contained important references). For nonvariational problems there is a method of construction of formal asymptotics [2, 3]. However, the convergence problem (i.e., justification of homogenization procedure) is studied only in the two following cases: 1) for periodic second order equations of divergence form which are linear with respect to leading derivatives, and 2) for second order nonlinear problem in fine-grained domains [8, 22–24].

The aim of the paper is to study general properties of G-convergence for arbitrary order nonlinear elliptic operators of divergence form and, as consequence, to obtain convergence results for corresponding homogenization problems. The main results were announced in [13, 14] (a detailed presentation was given in [17]), but the proofs presented here are simplified with respect to the original ones. We note here the paper [18] dealing with the same problems, but only for monotone second order operators and under much more restrictive assumptions. Later on, we deal with operators whose power rate of growth is equal to $p-1$, where $p \geq 2$. As for the case $1 < p < 2$ we need to modify conditions (1.2) and (2.3) below (see, for example, [13]).

Our starting point is [25]. In this paper the important concept of strong G-convergence was introduced and investigated for linear arbitrary order elliptic operators. As a consequence, the justification of homogenization procedure was done for that case. Namely, this concept, extended to our situation in a suitable way, is the central of our approach. As for the techniques, it is based on the monotonicity method (the standard reference is [11]) in spite of the fact that we consider very general elliptic operators which are monotonic in a leading part only.

The contents of the paper is as follows. Section 1 is preliminary and deals with G-convergence of abstract monotone operators. Some versions of the main results are known, but we need in good estimates for a G-limit operator. In Sec. 2 we introduce the concept of strong G-convergence and state the main results. For invertible operators this concept may be introduced in the same way as in the linear case. But, as we work with non-invertible (in general) operators, we need to modify the definition. In more restrictive situation similar treating of G-convergence may be found in [19]. The proofs of the main results are contained in Secs. 3 and 4. In the first of them we study the simplest case, where the operators under consideration contain only the leading terms and the leading derivatives. It is the central technical point of the paper. Then, in Sec. 4 we pass to the general case. We note here that our approach differ from that of [25] (in linear case). It is based on a direct construction of a strong G-limit operator and does not contain any version of so-called condition (N). It seems that this approach is more transparent and technically simple (a similar scheme was used in [27, 29] for some linear cases). Finally, Sec. 5 deals with some homogenization problems.

Now, we note the paper [16], (the case when the leading terms and the lowest one have essentially different growth rates) and the papers [9, 10] (nonlinear parabolic operators).

1. G-convergence of Monotone Operators

Let V be a separable reflexive Banach space over $\mathbb{R}$ and V' be its dual. We denote by $(.\ ,\ .)$ the canonical bilinear form (pairing) on $V' \times V$. Assuming $p \geq 2$ and fixing $\lambda_0 > 0, \lambda_1 \geq 0, \varkappa > 0, h_0 \geq 0, \Theta > 0$ we consider a class $M = M(\lambda_0, \lambda_1, h_0, \varkappa, \Theta)$ of operators $A : V \dashrightarrow V'$ such that

$$\|Av\|_*^{p'} \leq \lambda_0.\|v\|^p + \lambda_1\,, \tag{1.1}$$

$$(Av - Aw\,, v - w) \geq \kappa\|v - w\|^p\,, \tag{1.2}$$

$$\|Av - Aw\|_*^{p'} \leq \Theta.H(v,w)^{1-s/p}.\|v - w\|^s\,, \tag{1.3}$$

where $0 < s \leq p'$ and $H(v,w,\dots) = h_0 + \|v\|^p + \|w\|^p + \dots$. Here and later on $\|\ \ \|_*$ stands for the norm in V' and $p^{-1} + (p')^{-1} = 1$. Inequalities (1.2) (with $w = 0$) and (1.1) imply the following coercivity inequality

$$(Av, Av) \geq d_0.\|v\|^{p'} - K.\lambda_1\,, \tag{1.4}$$

where $d_0 > 0$ and $K > 0$ depend on $\varkappa$ only. In particular all such operators are invertible [11].

One say that a sequence $A^k : V \dashrightarrow V'$ of invertible operators G-converges to an invertible operator $A : V \dashrightarrow V'$, if $(A^k)^{-1}f \dashrightarrow A^{-1}f$ weakly in V for any $f \in V'$. We write $A^k \overset{G}{\dashrightarrow} A$ for this situation.

Theorem 1.1. For any sequence $A^k \in M$ there is a subsequence $A^{k'}$ such that $A^{k'} \overset{G}{\dashrightarrow} A$, and inequalities (1.2) and

$$\|Av\|_*^{p'} \leq \bar{\lambda}_0.\|v\|^p + K.\lambda_1\,, \tag{1.5}$$

$$\|Av - Aw\|_*^{p'} \leq \bar{\Theta}.\|H_1(v,w)^{1-\bar{s}/p}.\|v - w\|^{\bar{s}}\,, \tag{1.6}$$

hold. Here $\bar{\lambda}_0, \bar{\Theta}$ and K depend on $\lambda_0, \varkappa$ and $\Theta, \mathcal{H}_1(\cdot) = \mathcal{H}(\cdot) + \lambda_1$, and

$$\bar{s} = sp/(p^2 - sp + s)\,.$$

Proof. It is easy to see that the operators $R_k = (A^k)^{-1}$ are uniformly bounded and equicontinuous on any ball in V'. Hence, using the diagonal procedure we may assume that there is a limit operator $Rf = \lim R_k f$

(weakly in V). It is not hard to see that this operator is bounded and continuous.

If we pass to the limit in inequality (1.4) (with A replaced by A^k) and use the weak lower semicontinuity of the norm, we obtain the inequality

$$(f, Rf) \geq d_0 . \|Rf\|^p - K.\lambda_1 . \tag{1.7}$$

Moreover, if in the previous argument we take into account (1.1) and then pass to the limit, we obtain the inequality

$$(f, Rf) \geq \lambda_0^{-1} .d_0 .\|f\|_*^{p'} - (\lambda_0 + K) .\lambda_1 . \tag{1.8}$$

Hence, R is coercive. In the similar way (1.2) and (1.3) imply strict monotonicity of R and, as a consequence, its invertibility.

Now we set $A = R^{-1}$. Inequality (1.8) together with the Yung inequality implies (1.5). Similarly, (1.7) implies

$$\|Rf\|^p \leq K .(\|f\|_*^{p'} + \lambda_1) . \tag{1.9}$$

For $v, w \in V$ we set $v_k = R_k A v$ and $w_k = R_k A w$. Since A^k satisfied inequality (1.2), $Av = A^k v_k$ and $Aw = A^k w_k$., we have

$$(Av - Aw\, , v_k - w_k) \geq x .\|v_k - w_k\|^p . \tag{1.10}$$

Passing to the limit, we see that our limit operator A satisfies inequality (1.2). Now, by (1.9).

$$H(v_k, w_k) \leq K.H_1(v, w) .$$

By definition of v_k and w_k and by (1.3) we obtain the inequality

$$\begin{aligned} \|Av - Aw\|_*^{p'} &\leq \Theta .H(v_k, w_k)^{1-s/p} .\|v_k - w_k\|^s \\ &\leq \bar{\Theta} .H_1(v, w)^{1-s/p} .\|v_k - w_k\|^s . \end{aligned}$$

Using (1.10) to estimate $\|v_k - w_k\|$ in the last inequality and passing to the limit, we obtain

$$\begin{aligned} \|Av - Aw\|_*^{p'} &\leq \bar{\Theta} .H_1(v, w)^{1-s/p} .(Av - Aw\, , v - w)^{s/p} \\ &\leq \bar{\Theta} .H_1(v, w)^{1-s/p} .\|Av - Aw\|_*^{s/p} .\|v - w\|^{s/p} . \end{aligned}$$

This implies (1.6) and the theorem is proved. □

2. Strong G-convergence. Main Results

In a bounded domain $Q \subset \mathbb{R}^n$ we shall consider differential operators of the form

$$Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha A_\alpha(x, \delta^m u), \tag{2.1}$$

where $\partial_j = \partial/\partial x_j, \partial = (\partial_1, \dots, \partial_n), \delta^m u$ is the collection of all partial derivatives of u of order not greater than m (the number of them will be denoted by M), and the usual conventions on multi-indices are used. Now we set $\partial^m = \{\partial^\alpha\}_{|\alpha|=m}$. The numbers of members in $\partial^m u$ and $\partial^{m-1}u$ will be denoted by M_1 and M_2 respectively. The following notations will be used also:

$$A_\alpha(x, \delta^{m-1}u, \partial^m) = A_\alpha(x, \delta^m u),$$
$$A_\alpha(x, \eta, \xi) = A_\alpha(x, \xi), \xi = (\eta, \xi) \in R^M = R^{M_2} \times R^{M_1}.$$

If A_α does not depend on lower derivatives, we write simply $A_\alpha(x, \partial^m u)$ and $A_\alpha(x, \xi)$. Later on we will use the following convention. If we consider an operator of the form (2.1) labeled by some mark, then the same mark will be preassigned to the "coefficients" of the operator. For example, $A^k_\alpha(x, \xi)$ are the coefficients of A^k, etc.

Let $V = W_0^{m,p}(Q)$ and $\overline{V} = W^{m,p}(Q)$ be usual Sobolev spaces. The space V is endowed with the norm

$$\|u\| = \|u\|_Q = \left[\sum_{|\alpha|=m} \|\partial^m\|_p^p \right]^{1/p}$$

where $\| \quad \|_p = \| \quad \|_{p,Q}$ is the usual $L^P(Q)$-norm.

Now we give a precise description of the operator classes we shall consider later on. We assume that $A_\alpha(|\alpha| \leq m)$ satisfies the Carathéodory condition, i.e., $A_\alpha(x, \xi)$ is measurable in $x \in Q$ for all $\zeta \in \mathbb{R}^M$ and is continuous in ζ for almost all $x \in Q$. It is assumed also that for almost all $x \in Q$ the following inequalities are valid:

$$|A_\alpha(x, \zeta)|^{p'} \leq c_0 . |\zeta|^p + c(x) \tag{2.2}$$

where $p \geq 2, c_0 > 0$, and $c \in L^1(Q)$ is nonnegative;

$$\sum_{|\alpha|=m} [A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi')] . (\xi_\alpha - \xi'_\alpha) \geq \kappa |\xi - \xi'|^p, \tag{2.3}$$

where $x > 0$;

$$|A_\alpha(x,\zeta) - A_a(x,\zeta')|^{p'} \le \Theta.[(h(x) + |\zeta|^p + |\zeta'|^p).\nu(|\eta - \eta'|) + (h(x) + |\zeta|^p + |\zeta'|^p)^{1-s/p}.|\xi - \xi'|^s], \tag{2.4}$$

where $\Theta > 0, 0 < s \le p', \zeta = (\eta,\xi), \zeta' = (\eta',\xi'), h \in L^1(Q)$ is nonnegative and $\nu(r)$ is a continuity modulus, i.e., a nondecreasing continuous function on $[0,+\infty)$ such that $\nu(0) = 0, v(r) > 0$ if $r > 0$, and $\nu(r) = 1$ if $r \ge 1$.

Operators satisfying the present conditions act continuously in the following way [11]: $A : V \dashrightarrow V' = W^{-m,p'}(Q)$ and $A : \overline{V} \dashrightarrow V'$. Now we fix the constant $p \ge 2$. By specification of another parameters, which appear in (2.2)–(2.4), we obtain the operator class $E = E(c_0, c, \kappa, h, \Theta, \nu, s)$. If we replace (2.3) by the inequality

$$\sum_{|\alpha|\le m} [A_\alpha(x,\zeta) - A_\alpha(x,\zeta')].(\zeta_\alpha - \zeta'_\alpha) \ge \kappa.|\xi_\alpha - \xi'_\alpha|^p. \tag{2.5}$$

where $\zeta = (\eta,\xi), \zeta' = (\eta',\xi')$, we obtain the subclass $DM = DM(c, c, \kappa, h, \Theta, v, s)$. Moreover we define $DM_0(c_0, c, \kappa, h, \Theta, \nu, s) \subset DM$ by the following conditions: $A_\alpha \equiv 0$ if $|\alpha| < m, A_\alpha(x,\zeta) = A_\alpha(x,\xi), \zeta = (\eta,\xi)$ if $|\alpha| = m$. Evidently, DM_0 does not depend on ν.

Note that a union of classes $E(\cdot)$ (or $DM(\smile)$, or $DM_0(..)$) when their parameters belong to compact subsets of corresponding spaces $L^1(Q)$ or $\mathbb{R}_+$), is contained in some class of the same type.

Now we introduce the concept of strong G-convergence. First let us consider the case when $A^k, k \in \mathbb{N}$, and A are invertible operators (from V into V') of the form (2.1) (for example, $A^k, A \in DM$ [11]). For $u \in V$ we set $u_k = (A^k)^{-1}Au$ and then we define "generalized gradients" as

$$\Gamma_\alpha(u) = A_\alpha(x, \delta^m u), \Gamma^k_\alpha(u) = A^k_\alpha(x, \delta^m u_k), |\alpha| \le m.$$

(Here $A^k_\alpha(x,\zeta)$ are "coefficients" of A^k). It is easy to see that Γ_α and Γ^k_α act continuously from V into $L^{p'}(Q)$. One says that the sequence A^k strongly G-converges to $A(A^k \overset{G}{\Longrightarrow} A)$, if $A^k \overset{G}{\dashrightarrow} A$ and $\Gamma^k_\alpha(u) \dashrightarrow \Gamma_\alpha(u)(|\alpha| \le m)$ weakly in $L^{p'}(Q)$ for any $u \in V$.

Operators from the classes E are noninvertible in general. Hence we need to modify the previous definition to cover the case of such operators. We do this as follows. Denote by A_0 the leading part of A,

$$A_0 u = \sum_{|\alpha|=m} (-1)^m \partial^\alpha A_\alpha(x, \delta^m u).$$

Associated with A there is the operator $A : V \times V \dashrightarrow V'$, acting by the formula

$$A(u,v) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \partial^\alpha A_\alpha(x, \delta^{m-1} v, \delta^m u).$$

Extractly as above, its leading part $A_0(u,v)$ may be defined. We denote by $A_1(u,v)$ the sum of lower order terms of $A(u,v)$ and set $A_1(u) = A_1(u,u)$. We note that for $A \in E$ all the operators of the form $u \mapsto A_0(u,v), v \in V$, belong to some DM_0 (depending on v) and, as a consequence, are invertible. This simple fact is the key to more general definition of strong G-convergence which will be given now.

Consider operators $A^k, k \in \mathbb{N}$, and A, belonging to some classes E (their parameters may depend on an operator). For $u, v \in V$ we define $u_k \in V$ as a unique solution of the equation $A_0^k(u_k, v) = A_0(u,v)$. (Such u_k is well defined). Then we set

$$\Gamma_\alpha(u,v) = A_\alpha(x, \delta^{m-1} v, \partial^m u),$$
$$\Gamma_\alpha^k(u,v) = A_\alpha^k(x, \delta^{m-1} v, \partial^m u).$$

where $|\alpha| \le m$. We say that the sequence A^k strongly G-converges to A, if for any $v \in V$ the operators $A_0^k(\cdot, v)$ are G-convergent to $A_0(\cdot, v)$ and for any $u, v \in V$ we have $\Gamma_\alpha^k(u,v) \dashrightarrow \Gamma_\alpha(u,v) (|\alpha| \le m)$ weakly in $L^{p'}(Q)$.

For strong G-convergence on DM we shall use the first definition only. Equivalence of our two definitions on DM will be stated later on (it is obvious on DM_0).

The principal result on strong G-convergence is the following compactness theorem.

Theorem 2.1. For any sequence $A^k \in E(c_0, c, \kappa, h, \Theta, \nu, s)$ there exists a subsequence $A^{k'}$ such that $A^{k'} \stackrel{G}{\Longrightarrow} A$ and $A \in E(\bar{c}_0, \bar{c}, \kappa, \bar{h}, \bar{\Theta}, \nu, \bar{s})$, where $\bar{s} = sp/(p^2 - sp + s)$.

For any subdomain $Q' \subset Q$, expression (2.1) defines an operator $A|Q'$ which maps $W_0^{m,p}(Q')$ into $W^{-m,p'}(Q')$. Strong G-convergence is local in the sense of the following result.

Theorem 2.2. Assume that $A^k \in \varepsilon(c_0, c, \kappa, h, \Theta, \nu s)$ and $A^k \stackrel{G}{\Longrightarrow} A$. Then $A^k|Q' \stackrel{G}{\Longrightarrow} A|Q'$ for any $Q' \subset Q$.

It is the following property (convergence of arbitrary solutions) which is especially important in homogenization problems.

Theorem 2.3. Under the conditions of Theorem 2.2 let $v_k \in \overline{V}$ be such that $v_k \dashrightarrow u$ weakly in $\overline{V}$ and $A^k v_k \dashrightarrow f$ in V'. Then $Au = f$ and $A^k_\alpha(x, \delta^m v_k) \dashrightarrow A_\alpha(x, \delta^m u)$ weakly in $L^{p'}(Q)(|\alpha| \leq m)$.

In particular, this result implies the equivalence of our two definitions of strong G-convergence on DM.

For operator (2.1) the energy density is defined by the formula

$$E(u)(x) = \sum_{|\alpha|\leq m} A_\alpha(x, \delta^m u(x)) \,.\partial^\alpha u(x)\,.$$

Theorem 2.4. Under the conditions of Theorem 2.3 $E^k(v_k) \dashrightarrow E(u)$ weakly in the distribution space $D'(Q)$.

The proofs of all these statements are contained in the next two sections.

Remark 2.5. In the case when A^k is a Euler operator of some integral functional Φ^k and $A^k \overset{G}{\dashrightarrow} A$, A is also a Euler operator of an integral functional Φ. In the case, when A^k (and A) belong to E, the functionals Φ^k and Φ are not convex in general. However, they are convex if $A^k \in DM$ (as a consequence $A \in DM$). Then it is not hard to see that Φ is the Γ-limit of Φ^k (for Γ-convergence see, for example, [6, 28]). □

3. Proofs of the Main Results: Operators of the Class DM_0.

First of all we note that operators from DM (and, as a consequence, from DM_0) belong to $M(\lambda_0, \lambda_1, h_0, x, \Theta)$, where $\lambda = M_1 c_0$,

$$\lambda_1 = \lambda_1(Q) = \int_Q \lambda(x)dx\,,\ \lambda(x) = M_1\,.c(x)\,,$$

$$h_0 = h_0(Q) = \int_Q h(x)dx$$

remember that $V = W_0^{m,p}(Q)$). Later in this section we consider operators from DM_0 only.

We need the following technical result.

Lemma 3.1. Let $A^k \in DM_0, A^k u_k \dashrightarrow f, A^k v_k \dashrightarrow g$, strongly in V' with $\{u_k\}$ and $\{v_k\}$ being bounded in V, and $z_k = u_k - v_k \dashrightarrow 0$ weakly in V. Then $f = g, z_k \dashrightarrow 0$ in $W^{m,p}_{\text{loc}}(Q)$. Moreover, for any (independent in η) functions $C^k(x,\xi)$ satisfying the Carathédory condition and inequalities of the type (2.2), and (2.3) (in particularly for A^k_α) we have $C^k(x,\partial^m u_k) - C^k(x,\partial^m u_k) \dashrightarrow 0$ weakly in $L^{p'}(Q)$ and with respect to the measure.

Proof. For any $\phi \in C_0^\infty(Q), 0 \le \phi \le 1$, we have, by (2.3),

$$\sum_{|\alpha|=m} \int_Q Z^k_\alpha \cdot \phi \cdot \partial^\alpha z_k dx \ge C. \left(\sum_{|\alpha|=m} \|\phi \partial^\alpha z_k\|_p \right)^p ,$$

where $Z^k_\alpha = A^k_\alpha(x, \partial^m u_k) - A^k_\alpha(x, \partial^m v_k)$. Moreover,

$$\sum_{|\alpha|=m} \int_Q Z^k_\alpha . \partial^\alpha(\phi z_k) .dx = \left(A^k u_k - A^k u_k \, , \phi z_k\right) \to 0 .$$

Since the sequence Z^k_α is bounded in $L^{p'}(Q)$, then the Leibnitz formula and Sobolev inbedding theorem imply $z_k \dashrightarrow 0$ in $W^{m,p}_{\text{loc}}(Q)$. Hence $\delta^m z_k \dashrightarrow 0$ by measure. From these and from estimates (2.2) and (2.4) for C^k it is not hard to deduce that $C^k(x,\partial^m u_k) - C^k(x, \partial^m v_k) \dashrightarrow 0$ by measure. As this difference is bounded in $L^{p'}(Q)$, it converges to zero weakly. When $C^k = A^k_\alpha$ this implies that $A^k u_k - A^k v_k \dashrightarrow 0$ weakly in V'. Hence $f = g$ and the proof is complete. □.

Now we need some preliminary constructions.

Let $X_1 = L^p(Q)^{M_1}$; its members will be written as $\psi = (\psi_\alpha)_{|\alpha|=m}$. We have $A = \overline{A} \circ \partial^m$, where $\overline{A} : X_1 \dashrightarrow V'$ is given by

$$\overline{A}\psi = \sum_{|\alpha|=m} (-1)^m \partial^\alpha A_\alpha(x, \psi) .$$

For, $\psi, x \in X_1$ we set $\overline{A}_\psi x = A(\psi + x)$ and $A_\psi = \overline{A}_\psi \circ \partial^m$, the operator A_ψ is well-defined on V (and on $\overline{V}$). It is easy to see that

$$A_{\psi+\partial^m w}(u) = A_\psi(u+w)\,, \tag{3.1}$$
$$\|A_\psi u\|_*^{p'} \leq \bar{\lambda}_0 .(\|u\|^p + \|\psi\|^p) + \lambda_1 \tag{3.2}$$
$$(A_\psi u - A_\psi w\,, u - w) \geq \kappa .\|u-w\|^p\,, \tag{3.3}$$
$$\|A_\psi u - A_\psi w\|_*^{p'} \leq \bar{\theta} .H(u,w,\psi)^{1-s/p} .\|u-w\|^s\,. \tag{3.4}$$

where $u, v \in V$ and $\psi \in X_1$. For any $\psi \in X_1$ the operator $A_\psi : V \dashrightarrow V'$ is invertible. Hence there is an operator $R : V' \times X_1 \dashrightarrow V$ defined by $R(f,\psi) = A_\psi^{-1} f$. We have

$$R(f, \psi + \partial^m w) = R(f,\psi) + w\,, \tag{3.5}$$
$$\|R(f,\psi)\|^p \leq K .(\|f\|^{p'} + \|\psi\|^{p'} + \lambda_1) \tag{3.6}$$

(comp. with (1.9)).

The following statement is a straightforward consequence of inequalities (3.2), (3.6) and (2.3).

Lemma 3.2. The operator R is Holderian on any ball of the space $V' \times X_1$ uniformly with respect to $A \in DM_0(c_0, c, \kappa, h, \Theta, s)$.

It is the following result that is a central point of the section. (and, in some sense, of all our study).

Lemma 3.3. Any sequence $A^k \in DM_0(c_0, c, \kappa, h, \Theta, s)$ contains subsequence which is strongly G-convergent. The limit operator belongs to some DM_0 (with, possibly different parameters; the parameter $\bar{s}$ is the same as in Theorem 1.1).

Proof. The proof is divided into several steps.

Step 1. By theorem 1.1 and Lemma 3.2 we may assume (passing to a subsequence, if it is needed) that $A_\psi^k \overset{G}{\dashrightarrow} A_\psi (\psi \in X_1)$, where $A_\psi : V \dashrightarrow V'$ is an (abstract) operator. We set $A = A_0$ and $R(f,\psi) = A_\psi^{-1} f$. Thus we have $R^k(f,\psi) \dashrightarrow R(f,\psi)$ weakly in V

(here $R^k(f,\psi) = (A^k_\psi)^{-1}f$) according to our previous conventions. Theorem 1.1 implies also, that A_ψ satisfies inequality (3.3) and

$$\|A_\psi u\|_*^{p'} \le \bar{\lambda}_0(\|u\|^p + \|\psi\|^p) + K\,.\lambda_1\,, \tag{3.7}$$
$$\|A_\psi u - A_\psi v\|_*^{p'} \le \bar{\Theta}\,.H_1(u,v,\psi)^{1-\bar{s}/p}\,.\|u-v\|^{\bar{s}}\,. \tag{3.8}$$

It is easy to see that the just constructed operators R and A satisfy equations (3.5) and (3.1) respectively.

Now we define $\overline{A} : X_1 \to V'$ by the formula $\overline{A}\psi = A_\psi(0)$. By (3.1), $A = \overline{A}\circ\partial^m$, and , by (3.7)

$$\|A\psi\|_*^{p'} \le \bar{\lambda}_0\,.\|\psi\|^p + K\,.\lambda_1\,. \tag{3.9}$$

Finally, A may be extended to the space $\overline{V}$ as $\overline{A}\circ\partial^m$.

Step 2. For $\psi \in X_1$ we set

$$\psi_k = \psi + \partial^m R^k(A\psi,\psi) = \psi + \partial^m u_k^1\,. \tag{3.10}$$

when $\psi = \partial^m u$, we have $\psi_k = \partial^m u_k$ with $u_k = u + u_k^1$. Evidently, $\psi_k ---\to \psi$ weakly in $X_1(u_k ---\to u$ weakly in $V)$. Now we define the operator $\overline{\Gamma}^k_\alpha : X_1 ---\to L^{p'}(Q), |\alpha| = m$, by the formula

$$\overline{\Gamma}^k_\alpha(\psi) = A^k_\alpha(x,\psi_k)\,.$$

Using (3.9), inequality (3.6) for R^k and inequality (2.2) for A^k_α we obtain

$$\|\overline{\Gamma}^k_\alpha(\psi)\|_{p}\,,p' \le \bar{\lambda}_0\,.\|\psi\|^p + K\,.\lambda_1\,. \tag{3.11}$$

Set also $\Gamma^k_\alpha(u) = \overline{\Gamma}^k_\alpha(\partial^m u)$ for $u \in \overline{V}$.

Step 3. Let $Q_1 \subset Q, \psi, \chi \in X_1$ and $\psi|Q_1 = \chi|Q_1$. Then $(A\psi)|Q_1 = (A\chi)|Q_1$ (i.e., A is a local operator). Indeed, we set $\chi_k = \chi + \partial^m v_k^1$, where $v_k = R^k(A\chi,\chi)$ (comp. with (3.10)). Since $\psi_k - \chi_k ---\to \psi - \chi$ weakly in X_1, then $(u_k^1 - v_k^1)|Q_1 --\to 0$ weakly in $W^{m,p}(Q_1)$. Now we note that

$$A^k\psi(u_k^1)|Q_1 = A(\psi)|_{Q_1}$$

and

$$A^k_\chi(v^1_k)|Q_1 = A(\chi)|_{Q_1}\,.$$

In Q_1 the operators A^k_ψ and A^k_χ coincide. Hence, applying Lemma 3.1, we obtain the required results.

For $Q_1 \subset Q$, passing once more to a subsequence, we may construct a G-limit operator $A_{(1)}$ and corresponding operators $\overline{\Gamma}^k_{(1),\alpha}$. As above, it may be stated that $A_{(1)}(\psi|Q_1) = (A\psi)|Q_1$. In particular the passage to a subsequence at this point is really superfluous. Additionally, if $\overline{\Gamma}^k_\alpha(\psi)$ converges weakly in $L^{p'}(Q)$ to some operators $\overline{\Gamma}_\alpha(\psi)$ (for any $\psi \in X_1$), then the operators $\overline{\Gamma}_\alpha$ are local (in the same sense as A) and generalized gradients $\overline{\Gamma}^k_{(1),\alpha}$ associated with Q_1 (generally, they disagree with the restriction of $\overline{\Gamma}^k_\alpha$ to Q_1) converge weakly to $\overline{\Gamma}_\alpha|Q_1$.

Step 4. By (3.11) we may assume (using the diagonal procedure) that there is a dense countable set in X_1 of ψ's such that the sequence $\{\overline{\Gamma}^k_\alpha(\psi)\}$ converges weakly in $L^{p'}(Q)$. Then, in fact, this remains valid for all $\psi \in X_1$ and, consequently, the operators $\overline{\Gamma}_\alpha : X_1 \dashrightarrow L^{p'}(Q), |\alpha| = m$, are well defined such that $\overline{\Gamma}^k_\alpha \dashrightarrow \overline{\Gamma}_\alpha$ weakly in $L^{p'}(Q)$. To prove this it is sufficient to see that, by Lemma 3.2 and condition (2.4), the operators $\overline{\Gamma}^k_\alpha$ are continuous uniformly with respect to k.

By (3.11) we have

$$\|\overline{\Gamma}_\alpha(\psi)\|^{p'}_{p'} \leq \overline{\lambda}_0 \,.\|\psi\|^p + K\,.\lambda_1\,. \tag{3.12}$$

Moreover,

$$\|\overline{\Gamma}_\alpha(\psi) - \overline{\Gamma}_a(\chi)\|^{p'}_{p'} \leq \overline{\theta}\,.H_1(\psi, x)^{1-s/p} \cdot \|\psi - \chi\|^s\,. \tag{3.13}$$

Indeed, let ψ_k be defined by (3.1) and $\chi_k = \chi + \partial^m v^1_k$ be defined in the similar way with ψ replaced by χ. We set $Z^k_\alpha = \overline{\Gamma}^k_\alpha(\psi) - \overline{\Gamma}^k_\alpha(\chi), Z^1_k = u^1_k - v^1_k, \sigma = \psi - \chi$ and $\sigma_k = \psi_k - \chi_k$. By (3.6) and (2.4) we have

$$\|Z^k_\alpha\|^{p'}_{p'} \leq \overline{\theta} \cdot H_1^{1-s/p} \cdot \|\sigma_k\|^s \tag{3.14}$$

where $H_1 = H_1(\psi, \chi)$. For $y = \overline{A}\psi - \overline{A}\chi$ there is a representation

$$y = \sum_{|\alpha|=m} (-1)^m \partial^\alpha Z^k_\alpha\,.$$

Therefore, using (2.3), (3.6) and (3.14), we obtain

$$
\begin{aligned}
(y, z_k^1) &= \int_Q \sum_{|\alpha|=m} Z_\alpha^k \sigma_{k\alpha} dx - \int_Q \sum_{|\alpha|=m} Z_\alpha^k \sigma_\alpha dx \\
&\geq \varkappa \,.\|\sigma_k\|^p - \overline{\theta}\,.[H_1^{1-s/p}\,.\|\sigma_k\|^s]^{1/p'}\,.\|\sigma\|\,.
\end{aligned}
$$

By using the Yung inequality this perimits us to obtain an upper bound for $\|\sigma_k\|$ in terms of $\|\sigma\|$ and (y, Z_k^1). Now put together this bound and (3.14) and then pass to the limit using the weak convergence $Z_k --- \to 0$. Thus we obtain (3.13). As a consequence, the operators $\overline{\Gamma}_\alpha, |\alpha| = m$, are continuous. Also we note that these operators are local (see the end of step 3).

Now we show that

$$
\sum_{|\alpha|=m} \int_Q [\overline{\Gamma}_\alpha(\psi) - \overline{\Gamma}_\alpha(\chi)]\,.(\psi_\alpha - \chi_\alpha) dx \geq \varkappa \,.\|\psi - \chi\|^p\,. \tag{3.15}
$$

We use the notations introduced after (3.14) and set $Z_\alpha = \overline{\Gamma}_\alpha(\psi) - \overline{\Gamma}_\alpha(\chi)$. As in the proof of Lemma 3.1,

$$
\sum_{|\alpha|=m} \int_Q Z_\alpha^k \phi \sigma_{k\alpha} dx \to \sum_{|\alpha|=m} \int_Q Z_\alpha^k \phi \sigma_\alpha dx
$$

for any $\phi \in C_0^\infty(Q)$ such that $0 \leq \phi \leq 1$. Since A^k and R^k satisfy inequalities (1.3) and (3.12) respectively, the left hand side here may be estimated below by $\kappa\|\phi\sigma_k\|^p$. As $\liminf \|\phi\sigma_k\| \geq \|\phi\sigma\|$, we obtain, passing to the limit, that

$$
\sum_{|\alpha|=m} \int_Q Z_\alpha \phi \sigma_\alpha dx \geq \varkappa \,.\|\phi\sigma\|^p\,.
$$

Since ϕ is arbitrary, this implies (3.15).

Finally, passing to the limit in the identity

$$
\sum_{|\alpha|=m} \int_Q \overline{\Gamma}_\alpha^k(\psi) \partial^\alpha v dx = (\overline{A}^k \psi_k, v) = (\overline{A}\psi, v)\,, v \in V\,, \psi \in X_1\,.
$$

gives rise to the representation

$$
\overline{A}\psi = \sum_{|\alpha|=m} (-1)^m \partial^\alpha \overline{\Gamma}_\alpha(\psi)\,.
$$

In particularly,

$$Au = \sum_{|\alpha|=m} (-1)^m \partial^\alpha \Gamma_\alpha(u), \tag{3.16}$$

where $\Gamma_\alpha = \overline{\Gamma}_\alpha \circ \partial^m$.

We note that estimates (3.12), (3.13) and (3.15) are valid for any subdomain $Q_1 \subset Q$ instead of Q with $\lambda_1 = \lambda_1(Q_1)$ and $H_1 = H_{1,Q_1}$.

Step 5. Set $A_\alpha(x,\xi) = \overline{\Gamma}_\alpha(\xi)(x), |\alpha| = m$, where $\xi \in \mathbb{R}^{M_1}$ in the right hand side is viewed as an element of X_1. These functions are measurable for all ξ and

$$|A_\alpha(x,\xi)|^{p'} \leq \overline{\lambda}_0|\xi|^p + \mathcal{K}.\lambda(x), \tag{3.17}$$

$$|A_\alpha(x,\xi) - A_\alpha(x,\xi')|^{p'} \leq \overline{\theta}.(h_1(x)|\xi|^p + |\xi'|^p)^{1-\bar{s}/p}.|\xi - \xi'|^{\bar{s}}, \tag{3.18}$$

where $h_1(x) = \lambda(x) + h(x)$. Indeed, let x_0 be a common Lebesque point of the functions $h_1(x), A_\alpha(x,\xi)$ and $A_\alpha(x,\xi)$, and Q_ε be a ball of the radius ε centered at x_0. Take (3.13) with $Q = Q_\varepsilon, \psi = \xi$ and $x = \xi'$ and divide the result by mesQ_ε. Now, passing to the limit as $\varepsilon - -- \longrightarrow 0$ we obtain (3.18). Similarly, (3.12) and (3.15) imply (3.17) and (2.3) respectively.

By virtue of (3.16), all we need now is to show that $\overline{\Gamma}_\alpha(\psi)(x) = A_\alpha(x,\psi(x))$ for $\psi \in X_1$ and almost all $x \in Q$. Since $A_\alpha(x,\xi)$ is continuous in ξ, then almost all points are common Lebesque points of $A_\alpha(x,\xi), \xi \in \mathbb{R}^{M_1}$ (see [8], Lemma 17.1). Therefore, the same is true for common Lebesque points of the functions $A_\alpha(x,\xi), \xi \in \mathbb{R}^{M_1}, h_1, \overline{\Gamma}_\alpha(\psi)$ and ψ. Let x_0 be such a point and Q_ε be an ε-ball around x_0. Now taking (3.13) with $Q = Q_\varepsilon$, given ψ and $\chi = \varepsilon$, where $\xi = \psi(x_0)$, and passing to the limit, as above, we obtain the required result. □

The arguments we used at the end of step 3 and at the beginning of step 4 give rise also to the following

Lemma 3.4. Let $A^k \in DM_0(c_0, c, \kappa, h, \Theta, s)$ and $A^k \overset{G}{\Longrightarrow} A$ in Q. Then $A^k|Q_1 \overset{G}{\Longrightarrow} A|Q_1$ for any subdomain $Q_1 \subset Q$. □

Lemma 3.5. Let $A^k \in DM_0(c_0, c, \kappa, h, \Theta, s)$ and $A^k \overset{G}{\Longrightarrow} A$. Assume that $A^k v_k - -- \longrightarrow f$ in V', where $v_k \in \overline{V}$ and $v_k - - - - \longrightarrow u$ weakly.

Then $Au = f$ and $A^k_\alpha(x, \partial^m v_k) \dashrightarrow A_\alpha(x, \partial^m u), |\alpha| = m$, weakly in $L^{p'}(Q)$.

Proof. Let $u_k = u + u^1_k$ be defined by (3.10). Then $u_k \dashrightarrow u$ and $u_k - v_k \dashrightarrow 0$ weakly in $\overline{V}$, and $A^k u_k = Au = g$. Applying Lemma 3.1, we complete. □

4. Proofs of the Main Results: General Case

To prove Theorems 2.1–2.3 in full generality we need the following comparison lemma for G-limit operators.

Lemma 4.1. Let $A^k, B^k \in DM_0(c_0, c, x, h, \Theta, s), A^k \overset{G}{\Longrightarrow} A$ and $B^k \overset{G}{\Longrightarrow} B$. Assume that for bounded sequences $\{\gamma_k\} \subset L^1(Q)$ and $\{\delta_k\} \subset L^\infty(Q)$ of nonnegative functions we have

$$|A^k_\alpha(x,\xi) - B^k_\alpha(x,\xi)|^{p'} \leq (\gamma_k(x) + |\xi|^p) \cdot \delta_k(x)\,, |\alpha| = m\,, \tag{4.1}$$

$\gamma_k \dashrightarrow \gamma$ strongly in $L^1(Q)$ and $\delta_k \dashrightarrow \delta$ almost everywhere. Then

$$|A_\alpha(x,\xi) - B_\alpha(x,\xi)|^{p'} \leq \overline{\theta}\,.(\gamma(x) + h_1(x) + |\xi|^p)\,.\delta(x)\,, |\alpha| = m\,, \tag{4.2}$$

where $h_1(x) = h(x) + \lambda(x)$.

Proof. Without loss of generality we may suppose that $\gamma_k = \gamma$ and $\delta_k = \delta$. Indeed, if our statement is valid in that case, we may apply it with γ_k and δ_k replaced by $\sup\{\gamma, \gamma_k, k \geq k_0\}$ and $\sup\{\delta, \delta_k, k \geq k_0\}$ respectively and pass to the infimum in the inequality of the type (4.2) which we obtain. Similarly, we may assume $\delta(x)$ being a step-function with open subsets as foots of steps. Therefore, by locality it is sufficient to examine the case $\delta(x) = \delta_k(x) = 1$ only.

Now let $\psi = \xi$. Consider $\psi_k = \psi + \partial^m u^1_k$ being constructed by formula (3.10). Also we construct $\chi_k = \psi + \partial^m v^1_k$ by the similar formula for the operators B^k. We set $y = \overline{A}\psi - \overline{B}\chi = \overline{A}^k\psi_k - \overline{B}^k\chi_k$ and we use the notations we introduce in the proof of (3.13). Evidently,

$$(y, z_k^1) = \int_Q \sum_{|\alpha|=m} [A_\alpha^k(x,\psi_k) - B_\alpha^k(x,\chi_k)] \cdot \partial^\alpha z_k^1 dx$$
$$= \int_Q \sum_{|\alpha|=m} Z_\alpha^k \cdot \sigma_{\alpha k} dx - \int_Q \sum_{|\alpha|=m} [A_\alpha^k(x,\chi_k) - B_\alpha^k(x,\chi)_k]\sigma_{\alpha k} dx .$$

To estimate the second integral here we use the Yung inequality. Then we have

$$(y, z_k^1) \geq \kappa\|\sigma_k\|^p - \varepsilon\|\sigma_k\|^p - C . L_k \geq (\kappa/2) . \|\sigma_k\|^p - C . L_k .$$

where

$$L_k = \int_Q (\gamma(x) + |\chi_k(x)|^p) dx .$$

Hence

$$\|\sigma_k\|^p \leq \overline{\theta} . [(y, z_k^1) + L_k] .$$

Now, (2.4) and (4.1) imply

$$\|A_\alpha^k(x,\psi_k) - B_\alpha^k(x,\chi_k)\|_{p'}^{p'p/s} \leq \overline{\theta} . [L_k^{p/s} + H_{(k)}^{p/s-1} . \|\sigma_k\|^p] .$$

where $H_{(k)} = H(\psi_k, \chi_k)$. By inequality (3.6) for R^k-type operators associated with A^k and B^k, we have

$$H_{(k)} \leq C . H_1(\psi) , \|\chi_k\|^p \leq C . H_1(\psi)$$

(for definition of H_1 see Theorem 1.1). Evidently,

$$H_1(\psi) \leq L = \int_Q (\gamma(x) + h_1(x) + |\xi|^p dx .$$

Therefore

$$\|A_\alpha^k(x,\psi_k) - B_\alpha^k(x,\chi_k)\|_p^{p'p/s} \leq \overline{\theta} . \{L_k^{p/s} + H_1(\psi)^{p/s-1} .$$
$$[y, z_k^1) + L_k]\} \leq \overline{\theta} . \{L^{p/s} + L^{p/s-1} . (y, z_k^1)\} .$$

Passing to the limit and taking into account that $z_k^1 - - - - \rightarrow 0$ weakly, we obtain

$$\|A_\alpha(x,\xi) - B_\alpha(x,\xi)\|_{p',Q}^{p'} \leq \overline{\theta} . \int_Q (\gamma(x) + h_1(x) + |\xi|^p) dx ,$$

By locality, this inequality still valid with Q replaced by any subdomain $Q_1 \subset Q$. This implies the required result. □

Proofs of the theorems 2.1 and 2.2. For $\psi \in X_2 = L^p(Q)^{M_2}$ we set

$$A^k_{0,\psi}(u) = \sum_{|\alpha|=m} (-1)^m \partial^\alpha A^k_\alpha(x, \psi, \partial^m u).$$

By Lemma 3.3 we assume that $A^k_{0,\psi} \overset{G}{\Longrightarrow} A_{0,\psi}$, where $A_{0,\psi}$ is an operator of the class DM_0 and ψ runs a dense countable subset $\Lambda \subset X_2$. Now we note that in fact the last is true for all $\psi \in X_2$. Indeed, for any $\psi \in X_2$ we have $A^{k'}_{0,\psi} \overset{G}{\Longrightarrow} A_{0,\psi}$ for a subsequence $\{k'\}$. Let $\psi_j \in \Lambda$ and $\psi_j \dashrightarrow \psi$ in X_2. Additionally we may assume that $\psi_j \dashrightarrow \psi$ almost everywhere. By Lemma 4.1 with $\gamma_k = h(x) + |\psi(x)|^p + |\psi_j(x)|^p$ and $\delta_k(x) = \nu(|\psi(x)| + |\psi_j(x)|)$ we have

$$\begin{aligned} &|A_{\alpha,\psi_j}(x,\xi) - A_{\alpha,\psi}(x,\xi)|^{p'} \\ &\le \theta\,.(h_1(x) + |\psi(x)|^p + |\psi_j(x)|^p)\,.\nu(|\psi(x) - \psi_j(x)|)\,. \end{aligned}$$

Hence $A_{\alpha,\psi_j}(x,\xi) \dashrightarrow A_{\alpha,\psi}(x,\xi)$, for almost all $x \in Q$. Thus, the passage to the subsequence is superfluous and we obtain the required result. Moreover

$$A_{0,\psi} \in DM_0(\bar{c}_0, \bar{c} + |\psi|^p, \bar{\kappa}, h_1 + |\psi|^p\,, \bar{\theta}, \bar{s})\,.$$

Now we set

$$\overline{\Gamma}^k_\alpha(u, \psi) = A^k_\alpha(x, \psi, \partial^m u_k)\,.|\alpha| \le m\,,$$

where $u_k \in V$ is the unique solution of the equation $A^k_{0,\psi}(u_k) = A_{0,\psi}(u)$. For $|\alpha| = m$ these are the generalized gradients for the set of operators $\{A^k_{0,\psi}, A_{0,\psi}\}$. Hence $\Gamma^k_\alpha(u,\psi) \dashrightarrow \Gamma_\alpha(u,\psi) = A_{\alpha,\psi}(x, \partial^m u)$, $|\alpha| = m$, weakly in $L^{p'}(Q)$. Moreover, we may consider $\overline{\Gamma}^k_\alpha(\psi', \psi), \overline{\Gamma}_\alpha(\psi', \psi)$ with $\psi' \in X_1$ in a similar way as in Sec. 3. In addition

$$\|\overline{\Gamma}^k_\alpha(\psi', \psi)\|^{p'}_p \le \overline{\lambda}_0\,.(\|\psi\|^p + \|\psi'\|^p) + \mathcal{K}\,.\lambda_1\,. \tag{4.3}$$

For a fixed ψ the operators $\overline{\Gamma}^k_\alpha$ are continuous in the first variable uniformly with respect to k. (This is stated in the proof of Lemma 3.3 (step

4) for the case $|\alpha| = m$. The case $|\alpha| < m$ is quite similar). Therefore, for any fixed ψ (and, then, for a countable dense set of such ψ's), passing to a subseuence, if necessary, we may assume that $\overline{\Gamma}^k_\alpha(\psi', \psi) \longrightarrow \overline{\Gamma}_\alpha(\psi', \psi)$ weakly in $L^{p'}(Q)(|\alpha| \leq m)$. By remark 4.2 this is true, really, for all $\psi \in X_2$. Additionally, the operators $\overline{\Gamma}_\alpha, |\alpha| \leq m$, are local with respect to ψ' (see the proof of Lemma 3.3, step 3); their locallity with respect to ψ is obvious.

Now we define the operator A by the formula

$$Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha \overline{\Gamma}_\alpha(\partial^m u, \delta^{m-1} u).$$

It is of the form (2.1) and belongs to some class E. This may be proved by a simple modification of steps 4 and 5 of the proof of Lemma 3.3. It is obvious that $A^k \overset{G}{\Longrightarrow} A$.

Lemma 3.4 and previous considerations give rise to the locallity of strong G-convergence (theorem 2.2). □

Proof of theorem 2.3. The Sobolev embedding theorem implies that $\chi_k = \delta^{m-1} v_k \longrightarrow \chi = \delta^{m-1} u$ strongly in X_2. Then we have $A^k_{0,\chi_k} \overset{G}{\Longrightarrow} A_{0,\chi}$. Indeed, by Lemma 3.3, passing to a subsequence, we may assume that $A^k_{0,\chi_k} \overset{G}{\Longrightarrow} \hat{A}$. We set $\gamma_k(x) = h(x) + |\chi(x)|^p + |\chi_k(x)|^p$ and $\delta(x) = \nu(|\chi_k(x) - \chi(x)|)$ and then apply Lemma 4.1 to the sequences $\{A^k_{0,\chi_k}\}$ and $\{A^k_{0,\chi}\}$. We have $\hat{A} = A_{0,\chi}$ and the passage to a subsequence is superfluous.

Now let $\psi = \partial^m u$ and $\psi_k = \partial^m u_k$ be constructed by formula (3.10) (with A and A^k replaced by $A_{0,\chi}$ and A^k_{0,χ_k} respectively). By definition of strong G-convergence for operators from DM_0

$$A^k_\alpha(x, \delta^{m-1} v_k, \partial^m u_k) \longrightarrow A_\alpha(x, \delta^{m-1} u, \partial^m u)\,, |\alpha| = m \tag{4.4}$$

weakly in $L^{p'}(Q)$. Passing to a subsequence we may assume that this is true for $|\alpha| \leq m$ (comp the proof of theorem 2.1).

Now we write

$$A_{0,\chi_k}(v_k) = f_k - A^k_1(v_k).$$

Obviously, $\{A^k_1(v_k)\}$ is bounded in $W^{-m+1,p'}(Q)$. Hence, we may assume that $A^k_1(v_k) \longrightarrow g$ weakly in that space and, as a consequence, strongly in V'. By Lemma 3.5

$$A_{0,\chi}(u) = A_0(u) = f - g. \tag{4.5}$$

Since $u_k \dashrightarrow u, v_k \dashrightarrow v$ weakly in $\overline{V}$ and $A^k_{0,\chi_k}(u_k) = A_{0,\chi}(u) = f - g$, we can apply Lemma 3.1. Then, taking $C^k(x,\xi) = A^k_\alpha(x, \delta^{m-1}v_k(x), \xi)$ we obtain that

$$A^k_\alpha(x, \delta^{m-1}v_k, \partial^m v_k) - A^k_\alpha(x, \delta^{m-1}v_k, \partial^m u_k) \rightarrow 0 . |\alpha| \leq m,$$

weakly in $L^{p'}(Q)$. This and (4.4) imply that

$$A^k_\alpha(x, \delta^m v_k) \dashrightarrow A_\alpha(x, \delta^m u), |\alpha| \leq m,$$

weakly in $L^{p'}(Q)$. In particular, we obtain that $g = \lim A^k_1(v_k) = A_1(u)$. Hence, by (4.5) $A(u) = A_0(u) + A_1(u) = f$ and the theorem is proved. □.

Proof of theorem 2.4. The proof is similar to that for corresponding linear results [25]. □.

Remark 4.3. Using the techniques we present here it is not hard to see that $A^k \overset{G}{\Longrightarrow} A$ iff $A^k_\psi \overset{G}{\Longrightarrow} A_\psi$ for any $\psi \in L^p(Q)^M$, where $A_\psi(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^m A_\alpha(x, \psi + \delta^m u)$. □ (4.6)

Remark 4.4. The proof of the following statement is similar to that of Theorem 2.3. Let $A^k \overset{G}{\Longrightarrow} A$ and $v, u_k \in W^{m,p}(Q)$. Assume that $u_k \dashrightarrow u$ weakly in $W^{m,p}(Q)$ and $A^k_0(u_k, v) \dashrightarrow f$ strongly in $W^{-m,p'}(Q)$. Then $A_0(u,v) = f$ and $A^k_\alpha(x, \delta^{m-1}v, \partial^m u_k) \dashrightarrow A_\alpha(x, \delta^{m-1}v, \partial^m u), |\alpha| \leq m$, weakly in $L^{p'}(Q)$. □.

On the set of operators (2.1) satisfying (2.2) we define the metric

$$\rho(A^1, A^2) = \sup_{\substack{x \in Q, \zeta \in R^m \\ |\alpha| \leq m}} (c + |\zeta|^p)^{-p'} . |A^1_\alpha(x,\zeta) - A^2_\alpha(x,\zeta)| \tag{4.7}$$

It is not hard to see, that $E(c_0, c, \kappa, h, \Theta, \nu, s)$ is complete with respect to that metric.

Proposition 4.5. Let $A^k_l \in E(c_0, c, \kappa, h, \Theta, \nu, s)$. Assume that $A^k_l \overset{G}{\Longrightarrow} A_l, A^k_1 \dashrightarrow A^k$ uniformly with respect to k, and $A_1 \rightarrow A$. Then $A^k \overset{G}{\Longrightarrow} A$.

The proof is simple, but quite tedious, and we omit it.

5. Homogenization

We consider a family of operators

$$A^\varepsilon u = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha A_\alpha(\varepsilon^{-1}x, \delta^m u)\,, \varepsilon > 0\,. \tag{5.1}$$

We assume that the functions $A_\alpha(x,\zeta)$, defined $\mathbb{R}^n \times \mathbb{R}^M, |\alpha| \le m$, are 1-periodic in $x \in \mathbb{R}^n$, and satisfy the Carathéodory condition and inequalities (2.2)–(2.4) with $c(x) = c, h(x) = h$. As a consequence, $A^\varepsilon \in E(c_0, c, \kappa, h, \Theta, \nu, s), \varepsilon > 0$ for any domain $Q \subset \mathbb{R}^n$.

To determine the homogenized operator of family (5.1) (i.e., the strong G-limit of the family) we consider the following auxiliary equation

$$\sum_{|\alpha|=m} (-1)^m \partial^\alpha A_\alpha(y, \eta, \xi + \partial_y^m N) = 0\,. \tag{5.2}$$

For any $\zeta = (\eta, \xi) \in \mathbb{R}^M = \mathbb{R}^{M_2} \times \mathbb{R}^{M_1}$ there is a 1-periodic in y generalized solution $N(y,\zeta)$ of (5.2) which is unique up to an additive constant. Indeed, let W be the space of 1-periodic functions from $W^{m,p}_{\text{loc}}(\mathbb{R}^n)$, factorized by constants. Then the left hand part of (5.2) defines an operator $U_\zeta : W --- \to W'$ which is continuous, strictly monotone and coercive. Thus, the required result follows.

Now we set

$$\hat{A}_\alpha(\zeta) = \langle A_\alpha(y, \eta, \xi + \partial_y^m N(y,\zeta))\rangle\,, \tag{5.3}$$

where $\langle f\rangle$ is the mean value of the 1-periodic function. Since $\partial^m N(\cdot,\zeta) \in L^p_{\text{loc}}(\mathbb{R}^n)^{M_1}$ and is 1-periodic, then the function $\hat{A}_\alpha$ is well-defined.

The following homogenization theorem is valid.

Theorem 5.1. For any bounded domain $Q \subset \mathbb{R}^n$ we have $A^\varepsilon \xrightarrow{G} \hat{A}$ as $\varepsilon ----- \to 0$, where

$$\hat{A}u = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha \hat{A}_\alpha(\delta u)^m\,. \tag{5.4}$$

Proof. By Theorem 2.1 for any sequence $\varepsilon' \longrightarrow 0$ there is a subsequence $\varepsilon' \longrightarrow 0$ such that $A^{\varepsilon'} \xrightarrow{G} \tilde{A}$. Now it is sufficient to prove that $\tilde{A}_\alpha(x,\zeta) = \hat{A}_\alpha(\zeta), |\alpha| \le m$.

To do this we set $N^\varepsilon(x,\zeta) = \varepsilon^m N(\varepsilon^{-1}x,\zeta)$. It is not hard to see that $N^\varepsilon(\cdot,\zeta) \longrightarrow 0$ weakly in $W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for any ζ. Moreover,

$$A^\varepsilon_{0,\zeta}(N^\varepsilon, 0) = 0\,,$$

where $A^\varepsilon_{0,\zeta}(u,v)$ is the leading part of the sheafted operator $A^\varepsilon_\zeta(u,v)$. By Remark 4.3, $A^\varepsilon_\zeta \xrightarrow{G} \tilde{A}_\zeta$. Using Remark 4.4, we obtain that $\tilde{A}_{0,\zeta}(0) = 0$ and

$$A_\alpha(\varepsilon^{-1}x,\eta,\xi + \partial^m N^\varepsilon(x,\zeta)) \to \tilde{A}_\alpha(x,\zeta)\,, |\alpha| \le m\,,$$

weakly in $L^{p'}(Q)$ for any bounded $Q \subset \mathbb{R}^n$. On the other hand

$$A_\alpha(\varepsilon^{-1}x,\eta,\xi + \partial^m N^\varepsilon(x,\zeta)) = A_\alpha(y,\eta,\xi + \partial^m_y N(y,\zeta))|_{y=\varepsilon^{-1}x}\,.$$

By (5.3) this converges weakly in $L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$ to $\hat{A}_\alpha(\zeta)$ and we complete the proof. □.

Now we discuss a statistical homogenization theorem for homogeneous random operators. Let (Ω,μ) be a probabilistic space. On Ω we consider an n-dimensional dynamical system $T(x), x \in \Omega$. This means, that for any $x \in \mathbb{R}^n$ it is given a measurable transformation $T(x)$ of Ω satisfying the following conditions:

(1) $T(x), x \in \mathbb{R}^n$, is measure preserving;
(2) $T(0) = \mathrm{id}$ and $T(x+y) = T(x) \circ T(y)$ for $x, y \in \mathbb{R}^n$;
(3) the map $T : \mathbb{R}^n \times \Omega \longrightarrow \Omega, T : (x,\omega) \longrightarrow T(x)\omega$, is measurable.

The formula $(U(x)f)(\omega) = f(T(x)\omega)$ defines a n-parameter group of isometries in the space $L^p(\Omega)$: Later on we assume that this group is strongly continuous. The latter is valid if the space $L^p(\Omega)$ is separable (see, for example, the proof of von Neuman theorem in [20], Theorem VIII. 9). For simplicity we assume also the dynamic system T being ergodic, i.e., any measurable T-invariant function on Ω is constant. By $\langle . \rangle$ we denote the mean value, i.e.,

$$\langle f \rangle = \int_\Omega f(\omega)d\mu(\omega)\,.$$

Now let us consider the functions $A_\alpha(\omega,\zeta)$, $|\alpha| \le m$, on $\Omega \times \mathbb{R}^M$ satisfying the Carathéodory condition. Also we assume that inequalities (2.2)–(2.4) are valid with x replaced by $\omega \in \Omega$ (here $c(\omega) = c$ and $h(\omega) = h$). Then for almost all $\omega \in \Omega$ the operator

$$A^\varepsilon(\omega)u = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha A_\alpha(T(\varepsilon^{-1}x)\omega, \delta^m u)\,, \varepsilon > 0\,, \tag{5.5}$$

is well-defined. Moreover, $A^\varepsilon \in E(c_0, c, \kappa, h, \Theta, \nu, s)$.

Theorem 5.2. For any bounded domain $Q \subset \mathbb{R}^n$ and for almost all $\omega \in \Omega$ we have $A^\varepsilon(\omega) \overset{G}{\Longrightarrow} \hat{A}$ as $\varepsilon \longrightarrow 0$. Moreover, the coefficients of $\hat{A}$ does not depend on $x \in \mathbb{R}^n$ and $\omega \in \Omega$.

The proof may be carried out along the same lines as Theorem 5.1 (with corresponding technical complications). We describe it briefly. First of all we define "derivatives" along dynamical system T (more precisely, along its trajectories). We denote by $\hat{\partial} = (\hat{\partial}_1, \dots, \hat{\partial}_n)$ the collection of generators of the group $U(x)$. There a dense subspace $\varphi \subset L^p(\Omega)$ which is contained in the domains of all the operators $\hat{\partial}^\alpha = \hat{\partial}_1^{\alpha_1} \dots \hat{\partial}_n^{\alpha_n}, \alpha \in \mathbb{Z}_+^n$ (comp [25]). Moreover, the operators $\hat{\partial}^\alpha, \alpha \in \mathbb{Z}_+^n$, are mutually commuting (in any reasonable sense).

Now we denote by $W^{m,p}$ the completion of φ with respect to the seminorm

$$\|f\| = \left(\sum_{|\alpha|=m} \|\hat{\partial}^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} . \tag{5.6}$$

This is a Banach space (factorization by the kernel of the seminorm takes place automatically). The operator $\hat{\partial}^m : W^{m,p} \longrightarrow L^p(\Omega)^M, \hat{\partial}^m f = \{\hat{\partial}^\alpha f\}_{|\alpha|=m}$, is an isometric embedding. In particular, the space $W^{m,p}$ is reflexive. Its dual will be denoted by $W^{m,p'}$. By duality the operators $\hat{\partial}^\alpha : L^{p'}(\Omega) \longrightarrow W^{-m,p'}, |\alpha| = m$, may be defined.

Instead of (5.2) we use now the following equation

$$\sum_{|\alpha|=m} (-1)^m \hat{\partial}^\alpha A_\alpha(\omega, \eta, \xi + \hat{\partial}^m N) = 0\,. \tag{5.7}$$

As above, equation (5.7) has a unique solution $N(\cdot,\zeta) \in W^{m,p}$ for any $\zeta \in \mathbb{R}^M$. Since $\hat{\partial}^m N(\cdot,\zeta) \in L^p(\Omega)^{M_1}$, the functions

$$\hat{A}_\alpha(\zeta) = \langle A_\alpha(\omega, \eta, \xi + \hat{\partial}^m N(\omega,\zeta))\rangle\,, |\alpha| = m\,, \tag{5.8}$$

are well-defined. The homogenized operator $\hat{A}$ is given by the formula

$$\hat{A}u = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial^\alpha \hat{A}_\alpha(\delta^m u). \tag{5.9}$$

Now we approximate $N(\cdot,\zeta)$ by $N^\delta(\cdot,\zeta) \in \varphi$ up to a small $\delta > 0$ (approximation with respect to $W^{m,p}$-norm). To prove, that $A^\varepsilon(\omega) \stackrel{G}{\Longrightarrow} \hat{A}$ we use $N^{\varepsilon,\delta}(x,\zeta) = \varepsilon^m N^\delta(T(\varepsilon^{-1}x)\omega,\zeta)$ instead of $N^\varepsilon(x,\zeta)$ and the statistical ergodic theorem instead of elementary arguments dealing with periodic functions.

In particular, we can take $\Omega = \mathbb{R}^n_B$, the so-called Bohr compactification of $\mathbb{R}^n$ [15]. This gives rise to a statistical homogenization theorem for periodic operators. But in this situation there is a more precise result. Assume that the functions $A_\alpha(x,\zeta), |\alpha| \leq m$, are continuous in $\zeta \in \mathbb{R}^M$ for any $x \in \mathbb{R}^n$, the functions $(1+|\zeta|^{p-1})^{-1}A_\alpha(x,\zeta)$ are almost periodic in $x \in \mathbb{R}^n$ uniformly with respect to $\zeta \in \mathbb{R}^M$, and inequalities defined by (5.1). Then the following individual homogenization theorem holds.

Theorem 5.3. For any bounded domain $Q \subset \mathbb{R}^n$ we have $A^\varepsilon \stackrel{G}{\Longrightarrow} \hat{A}$. Moreover, the operator $\hat{A}$ is translation-invariant.

Proof. Set $\Omega = \mathbb{R}^n_B$. The functions $A_\alpha, |\alpha| \leq m$, may be extended to continuous functions on $\mathbb{R}^n_B \times \mathbb{R}^M$. These extensions (we will denote they by A_α) satisfy all the conditions of Theorem 5.2. Here $T(x)\omega = \omega + x$ for $x \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^n_B$ (we remember that $\mathbb{R}^n_B$ is a compact abelian group and $\mathbb{R}^n \subset \mathbb{R}^n_B$). Therefore, for the family of operators

$$A^\varepsilon(\omega)u = \sum_{|\alpha|=m} (-1)^{|\alpha|}\partial^\alpha A_\alpha(\omega + \varepsilon^{-1}x, \delta^m u), \varepsilon > 0,$$

the statistical homogenization theorem is valid. In other words there is a measurable subset $\Omega_0 \subset \Omega$ such that $\mu(\Omega_0) = 1$ and $A^\varepsilon(\omega) \stackrel{G}{\Longrightarrow} \hat{A}$ for $\omega \in \Omega_0$. The operator $A^\varepsilon(\omega)$ depends continuously in $\omega \in \Omega$ uniformly with respect to $\varepsilon > 0$. Since Ω_0 is dense in Ω. Proposition 4.5 (more precisely, its version for nets, because $\mathbb{R}^n_B$ is nonmetrizable) applies and we complete the proof. □.

Remark 5.4. The operator $\hat{A}$ may be constructed by formulas (5.3), (5.4), where $N(y,\zeta)$ is almost periodic (in the sense of Besicovitch) solution for (5.2). □.

Remark 5.5. All the results of the section may be extended to the case when coefficients of operators under consideration are highly oscillated along a slow background, i.e., $A^{\varepsilon}_{\alpha}(x,\delta^m u) = A_{\alpha}(\varepsilon^{-1}x, x, \delta^m u)$. □.

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Alexander A. Pankov
Institute of Applied Problems in Mechanics and Mathematics
Ukrainian Academy of Sciences
Naucnaja 3-B
(290 047) LVOV
USSR

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"ON THE WELL–POSEDNESS AND RELAXABILITY OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS"

N.S. Papageorgiou

Abstract

In this work we examine the relation existing between well–posedness (sensitivity) and relaxability of nonlinear distributed parameter systems. We introduce the notion of "strong calmness" which describes the dependence of the value of the problem on perturbations of the state constraints and we show that it is equivalent to "relaxability". We also present an equivalent, control free description of the relaxed problem and we prove a density result.

1. Introduction.

It is well known that if we want to derive necessary conditions for optimality, things do simplify if we have some convexity hypothesis at our disposal and this partly motivates the introduction of the relaxed system, wherein the original dynamical equations are replaced by their convexified versions. Another important reason to consider the relaxed system, is that it always has a solution under very mild hypotheses. But then we need to know if and when the relaxation process introduces new better solutions or leaves the value of the problem unchanged and an original optimal control, optimal for the relaxed system too (relaxability). On the other hand, given the optimal control problem, it is important–especially when state constraints are present – to have a precise mathematical formulation to express the fact that the original problem is well posed in the sense that arbitrarily small perturbations of the data, do not drastically change the value of the problem. The first to formalize this stability concept was Clarke [1], who for this purpose introduced the notion of calmness. He then proved that calmness implies relaxability for finite dimensional systems with no state constraints (see theorem 2 in [1]). In this paper we consider nonlinear distributed parameter systems with state constraints. We introduce a stronger notion of calmness and we show that it is in fact equivalent to relaxability. We present the results without proof. A detailed exposition will appear elsewhere.

2. Strong calmness.

Let Y be any Banach space. We will be using the following notations:
$P_{f(c)}(Y) = \{A \subseteq Y$: nonempty, closed, (convex)$\}$ and $P_{(w)k(c)}(Y)$ $= \{A \subseteq Y$: nonempty, (w–) compact, (convex)$\}$.

Now let H be a separable Hilbert space and X a dense linear subspace carrying the structure of a separable, reflexive Banach space and with the embedding $X \hookrightarrow H$ compact. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, where all embeddings are continuous, dense and compact. So (X, H, X^*) is a Gelfand triple. By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of H, X^*), while by $<\cdot,\cdot>$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$ we will denote the inner product of H. Recall that $<\cdot,\cdot>\Big|_{XxH} = (\cdot,\cdot)$. We will model the control space using a separable Banach space Y.

Consider the following nonlinear, distributed parameter optimal control problem of Lagrange type:

$$\left[\begin{array}{l} J(x,u) = \int_0^b L(t,x(t),\ u(t))\,dt \to \inf = p(0) \\ \text{s.t. } \dot{x}(t) + A(t,x(t)) = f(t,x(t),\ u(t)) \text{ a.e.} \\ x(0) = x_0,\ \ x(t) \in C(t),\ \ u(t) \in U(t) \text{ a.e.} \end{array}\right] (*)$$

We will need the following hypotheses on the data of (*).

H(A): $A: T \times X \to X^*$ is a map s.t. (1) $t \to A(t,x)$ is measurable,
(2) $x \to A(t,x)$ is sequentially continuous from X_w into X^*_w (where X_w (resp. X^*_w) denotes the space X (resp. X^*) with the w–topology),
(3) $x \to A(t,x)$ is monotone, (4) $<A(t,x), x> \geq c_1 \|x\|^2$ a.e. $c_1 > 0$ and
(5) $\|A(t,x)\|_* \leq g(t) + c_2 \|x\|^2$ a.e. with $g(\cdot) \in L^\infty_+$, $c_2 > 0$,
H(f): $f: T \times H \times Y \to H$ is a map s.t. (1) $t \to f(t,x,u)$ is measurable,
(2) $(x,u) \to f(t,x,u)$ is sequentially continuous from $H \times Y_w$ into H_w and
(4) $|f(t,x,u)| \leq a(t) + b(|x| + \|u\|)$ a.e. with $a(\cdot) \in L^2_+$, $b > 0$,

H(U): $U: T \to P_{fc}(Y)$ is an L^2–integrably bounded multifunction,

H(C): $C: T \to P_f(H)$ is an L^1–integrably bounded multifunction with $x_0 \in C(0) \cap X$,

H(L): $L: T \times H \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper measurable integrand s.t. $\phi(t) \leq L(t,x,u)$ a.e. with $\phi(\cdot) \in L^1$,

H_a: There exist $(x,u) \in W(T) \times L^2(Y)$ satisfying the constraints of (*) s.t. $J(x,u) < \infty$.

Recall that $W(T) = \{x(\cdot) \in L^2(X): \dot{x}(\cdot) \in L^2(X^*)\}$. We know that $W(T) \subseteq C(T,H)$ (see for example Lions [4]). To problem (*) we associate the following perturbed problem

$$J(x,u) = \int_0^b L(t,x(t), u(t))\, dt \to \inf = P(\epsilon)$$

$$\left\{ \begin{array}{l} \text{s.t. } \dot{x}(t) + A(t,x(t)) = f(t,x(t), u(t)) \text{ a.e.} \\ \quad x(0) = x_0, \int_0^b d_H(x(t), C(t))\, dt \leq \epsilon,\ u(t) \in U(t) \text{ a.e.} \end{array} \right\} (*)_\epsilon$$

Let $V = \{m: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ s.t. $m(\cdot)$ is nondecreasing and $\lim_{\epsilon \downarrow 0} m(\epsilon) = m(0) = 0\}$. We will say that (*) is "strongly calm" if and only if there exists $m(\cdot) \in V$ s.t. $\lim_{\epsilon \downarrow 0} \frac{P(\epsilon) - P(0)}{m(\epsilon)} > -\infty$. Note that Clarke [1] defined "calmness" using $m(\epsilon) = \epsilon$.

Using our stronger notion of calmness we can easily check that:

Proposition 2.1: If hypotheses H(A), H(f), H(U), H(C), H(L) and H_a hold, then $P(\cdot)$ is right continuous at zero iff (*) is strongly calm.
So strong calmness is equivalent to well–posedness.

Let $m(\cdot) \in V$ and set $K(m) = \inf\{J(x,u) + m(\int_0^b d_H(x(t), C(t))\, dt)$ s.t. $\dot{x}(t) + A(t,x(t)) = f(t,x(t), u(t))$ a.e., $x(0) = x_0$, $u(t) \in U(t)$ a.e.$\}$. We have:

Proposition 2.2: If hypotheses H(A), H(f), H(U), H(C), H(L) and H_a hold, then $P(\cdot)$ is right continuous iff there exists $m \in V$ s.t. $P(0) = K(m)$ iff (*) is strongly calm.

Also to problem (*) we can associate the following penalized version, in which we perturb the cost criterion instead of the constraints.

$$\left[\begin{array}{l} J(x,u) + \frac{1}{\epsilon} \int_0^b d_H(x(t),\ C(t))\,dt \to \inf = Q(\epsilon) \\ \text{s.t. } \dot{x}(t) + A(t,x(t)) = f(t,x(t),u(t)) \text{ a.e.} \\ \qquad x(0) = x_0,\ u(t) \in U(t) \text{ a.e.} \end{array}\right] (*)'_\epsilon$$

Observe that for $\epsilon = 0$, $(*)'_\epsilon$ reduces to (*) and so $Q(0) = P(0)$. The next proposition tells us that the previous analysis is also valid for the above "penalized" problem.

<u>Proposition 2.3</u>: <u>If</u> hypotheses $H(A)$, $H(f)$, $H(U)$, $H(C)$, $H(L)$ and H_a hold, <u>then</u> $Q(\cdot)$ is right continuous at zero <u>iff</u> $P(\cdot)$ is.

3. <u>Relaxability and strong calmness</u>.

In this section we will show that well–posedness in the sense of right continuity of $P(\cdot)$ at zero and thus strong calmness of the original problem is equivalent to the "relaxability" of the system. To this end we introduce the following relaxed system:

$$\left[\begin{array}{l} J_r(x,\lambda) = \int_0^b \int_Y L(t,x(t),\ z)\ \lambda(t)\ (dz)\ dt \to \inf = P_r(0) \\ \text{s.t. } \dot{x}(t) + A(t,x(t)) = \int_Y f(t,x(t),\ z)\ \lambda(t)(dz) \text{ a.e.} \\ \qquad x(0) = x_0,\ \lambda(\cdot) \in S_\Sigma^Y,\ x(t) \in C(t) \end{array}\right] (*)_r$$

where $\Sigma(t) = \{\lambda \in M_+^1(Y): \lambda(U(t)) = 1\}$, with $M_+^1(Y)$ being the space of probability measures on $M_+^1(Y)$. We will say that (*) is "relaxable" <u>iff</u> $P(0) = P_r(0)$.

We will need the following stronger hypotheses on the data

<u>$H(f)_1$</u>: $f: T \times H \times Y \to H$ is a map s.t. (1) $t \to f(t,x,u)$ is measurable, (2) $(x,u) \to f(t,x,u)$ is sequentially continuous from $H \times Y_w$ into H_w, (3) $|f(t,x,u) - f(t,y,u)| \le k_M(t)\ |x-y|$ a.e. for all $\|u\| \le M$ with $k_M(\cdot) \in L_+^1$ and (4) $|f(t,x,u)| \le a(t) + b(t)\ (|x| + \|u\|)$ a.e. with $a(\cdot), b(\cdot) \in L_+^2$.

<u>$H(U)_1$</u>: $U: T \to P_{fc}(Y)$ is a measurable multifunction s.t. $U(t) \subseteq W$ a.e. with $W \in P_{wkc}(Y)$.

$\underline{H(L)_1}$: $L: T \times H \times Y \to \mathbb{R}$ is an integrand s.t. (1) $(t,x,u) \to L(t,x,u)$ is measurable, (2) $(x,u) \to L(t,x,u)$ is continuous from $X \times W_w$ into $\mathbb{R}$, where W_w denotes the set W with its relative w–topology, and (3) for every $B \subseteq H$ compact, $t \to \inf[L(t,x,u): x \in B, u \in W]$, belongs in L^1.

With a weaker hypothesis on L (namely joint measurability, lower semicontinuity on $H \times W_w$ and for each $B \subseteq H$ compact $-\infty < \int_0^b \inf [L(t,x,u): x \in B, u \in W]\, dt$, we can show that $(x,\lambda) \to J_r(x,\lambda)$ is l.s.c. on $C_B \times (L^\infty(T, M(W_w)), w^*)$, where $C_B = \{x(\cdot) \in C(T, X_w): x(t) \in B, t \in T\}$ and $M(W_w)$ is the space of Radon measures on W_w. Using this fact, we can prove the following theorem relating well–posedness and relaxability.

Theorem 3.1: If hypotheses $H(A)$, $H(f)_1$, $H(U)_1$, $H(C)$, $H(L)_1$ and H_a hold, then $P(\cdot)$ is right continuous iff (*) is relaxable.

So combining theorem 3.1 with the results of section 2, we get the following complete characterization of relaxability:

Theorem 3.2: If hypotheses $H(A)$, $H(f)_1$, $H(U)_1$, $H(C)$, $H(L)_1$ and H_a hold, then the following statements are equivalent:
(1) problem (*) is relaxable, (2) $P(\cdot)$ is right continuous at zero, (3) $Q(\cdot)$ is right continuous at zero, (4) problem (*) is strongly calm.

The proofs of these results are based on some density results concerning the trajectories of the controlled evolution equations proved by the author in [5] and [6].

4. An alternative form of the relaxed problem.

Let $p: T \times H \times X^* \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by $p(t,x,v) = \inf\{L(t,x,u): v + A(t,x) = f(t,x,u), u \in U(t)\}$, with $\inf \emptyset = +\infty$. So $p(t,x,v)$ represents the minimum cost of producing velocity v at time t using admissible controls and given that the state of the system is x. Using $p(\cdot,\cdot,\cdot)$ we can have the following control free formulation of the relaxed problem:

$$\hat{J}_r(x) = \int_0^b p^{**}(t,x(t),\dot{x}(t))\,dt \to \inf = \hat{P}_r(0)$$

$$\left.\begin{cases} \text{s.t. } \dot{x}(t) + A(t,x(t)) \in \overline{\text{conv}}\, F(t,x(t)) \text{ a.e.} \\ x(0) = x_0,\ u(t) \in U(t) \text{ a.e., } x(t) \in C(t) \end{cases}\right\} (\hat{*})_r$$

with $F(t,x) = \cup\{f(t,x,u): u \in U(t)\}$ and $p^{**}(\cdot,\cdot,\cdot)$ is the second convex conjugate of $p(t,\cdot,\cdot)$. The next theorem tells us that problems $(*)_r$ and $(\hat{*})_r$ have the same value.

<u>Theorem 4.1</u>: <u>If</u> hypotheses $H(A)$, $H(f)$, $H(U)_1$, $H(C)$, $H(L)_1$ and H_a hold, <u>then</u> $P_r(0) = \hat{P}_r(0)$.

The proof of this result is rather involved and is based on properties of Radon measures, of measurable multifunctions and of the convex conjugates. Also an interesting byproduct of the proof is that $(*)_r$ and $(\hat{*})_r$ have equivalent dynamics, hence the same trajectory sets. Furthermore from theorem 4.1 we deduce that system $(*)$ is relaxable <u>iff</u> $P(0) = \hat{P}_r(0)$.

5. <u>A density result</u>.

The problem of whether the original trajectories are dense in the relaxed ones, can not be answered using the results of [5] because of the presence of state constraints. In fact it is a nontrivial problem and in this section we present a solution to it.

We need the following lemmata, which are also interesting in their own as general results about multifunctions and convex sets.

<u>Lemma I</u>: <u>If</u> Z is a Banach space, $C: T \to P_{fc}(Z)$ is a Hausdorff continuous multifunction with bounded values and for all $t \in T$ int $C(t) \neq \emptyset$, <u>then</u> the set of continuous selectors of C, denoted by $CS(C)$, is nonempty and int $CS(C) = CS(\text{int } C)$.

The first conclusion of the lemma follows from Michael's selection theorem, while the proof of the second conclusion is based on the fact that $t \to \text{bd } C(t)$ is Hausdorff continuous too (see DeBlasi–Pianigiani [2]).

<u>Lemma II</u>: <u>If</u> Z is a Banach space, $A, B \subseteq Z$ are nonempty, $\bar{A}$ is convex, B is closed convex with $\operatorname{int} B \neq \emptyset$ and $A \cap \operatorname{int} B \neq \emptyset$, <u>then</u> $\overline{A \cap B} = \bar{A} \cap B$.

The proof of this lemma is based on some simple convex analytic arguments.

Now we are ready for the density result in the presence of state constraints. We were able to prove it for systems with linear dynamics $\dot{x}(t) + A(t)\,x(t) = B(t)\,u(t)$ a.e., $x(0) = x_0$, $u \in S^1_U = \{$integrable selectors of $U(\cdot)\}$. We will need the following hypotheses:

<u>$H(A)_1$</u>: $A: T \times X \to X^*$ is a map s.t. (1) $A(t)(\cdot)$ is linear, monotone, (2) $\|A(t)x\|_* \leq k(t)\,\|x\|$ a.e. with $k(\cdot) \in L^1_+$ (i.e. $A(t)(\cdot) \in \mathscr{L}(X, X^*)$), (3) $<A(t)\,x, x> \geq c_1\,\|x\|^2$ $c_1 > 0$, (4) $\|A(t')x - A(t)\,x\|_* \leq m|t'-t|\,\|x\|$ $m > 0$.

<u>$H(B)$</u>: $B \in L^2(T, \mathscr{L}(Y, X^*))$, <u>$H(U)_2$</u>: $U(\cdot)$ is L^2–integrably bounded,

<u>$H(C)_1$</u>: $C: T \to P_{fc}(H)$ is h–continuous.

From Tanabe [7] (section 5.4) we know that under $H(A)_1$ $\{A(t)\}_{t \in T}$ generates a strongly continuous evolution operator $\Phi(t,s) \in \mathscr{L}(H)$ $0 \leq s \leq t \leq b$ and a trajectory $x(\cdot) \in W(T)$ can be written as $x(t) = \Phi(t,0)\,x_0 + \int_0^t \Phi(t,s)\,B(s)\,u(s)\,ds$, $t \in T$. We will assume the following about $\Phi(\cdot,\cdot)$:

<u>H_c</u>: $\Phi(t,s)$ is compact for $t - s > 0$.

Let $\hat{S}(x_0)$ be the set of trajectories of the original problem and $\hat{S}_r(x_0)$ the set of the relaxed ones.

<u>Theorem 5.1</u>: <u>If</u> hypotheses $H(A)_1$, $H(f)_1$, $H(U)_1$, $H(C)_1$, H_c hold and there exists $x(\cdot) \in S(x_0)$ s.t. $x(t) \in \operatorname{int} C(t)$ $t \in T$, <u>then</u> $\hat{S}_r(x_0) = \hat{S}(x_0)$, the closure in $C(T,H)$.

The proof is based on the two lemmata above and the unconstrained density results proved in [5].

Our work extends to distributed parameter systems those of Dontchev–Morduhovic [3] and Zolezzi [8].

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N. S. Papageorgiou
University of California
1015 Department of Mathematics
Davis, California 95616
U.S.A

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GENERALIZED SPECTRUM FOR THE DIMENSION: THE APPROACH BASED ON CARATHEODORY'S CONSTRUCTION

Ya. B. Pesin*

ABSTRACT

We use a generalization of the classical Caratheodory's construction for introduction and study of the general spectrum for the dimensions. It is a one-parameter family of characteristics of a dimension type which is widely used at present in various physical investigations. We show in the two-dimensional case that the generalized spectrum calculated to a measure which is invariant under a smooth dynamical system and has non-zero Lyapunov exponents does not depend on the parameter and is equal to the Hausdorff dimension of the measure.

1. Introduction.

There is a deep connection between the complexity of topological structure of the invariant set and dynamical properties of a system acting on it. It generates a relation between a dimension of the invariant set and characteristics of dynamics such as entropy and Lyapunov exponents. In the investigation of such a kind not only the classical Hausdorff dimension but many other quantities are used. They have many other features in common with the general notion of dimension and therefore are called dimensionlike characteristics. On the other hand, it is much easier to calculate many of them by means of a computer. That is why they are widely used in physical investigations. In [4], a general approach for introduction of many different dimensionlike characteristics was given based on the classical Caratheodory construction; various general properties were studied and formulae for calculating some of them were obtained.

Recently a new type of dimension was introduced in [7] (cf. also [1], [2], [8]) and became very popular with physicists. It is a one-parameter family of quantities called the generalized spectrum for the dimensions. Leaving in the frameworks of a general Caratheodory approach we will give two different definitions of the generalized spectrum. According to them we will introduce two different families of dimensionlike characteristics which can be used with equal success in physical applications. In these two cases we will obtain formulae for the calculation of the of the generalized spectrum for two-dimensional diffeomorphisms preserving a measure with non-zero Lyapunov exponents: namely we will show that the generalized spectrum of the measure does not depend on the parameter and is equal to the Hausdorff dimension of the measure. It is in accordance with the conjecture formulated in [1],[2].

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2. A General Construction of Dimensionlike Characteristics.

We describe, with some modifications, the general approach for introducing dimensionlike characteristics given in [4].

Let X be a set, F be a collection of subsets in X. Assume that there are three functions $\eta, \phi, \psi : F \to \mathbf{R}^+$ satisfying the following conditions:

A1. $\emptyset \in F$, $\phi(\emptyset) = 0$;

A2. there exists $\delta > 0$ such that $\phi(U) < 1$ for any $U \in F$ with $\psi(U) \leq \delta$;

A3. for any $Z \subset X$ and $\varepsilon > 0$ there exists a finite or countable collection $G \subset F$ which covers Z (i.e., $\bigcup_{U \in G} U \supset Z$) with $\psi(G) \stackrel{\text{def}}{=} \sup\{\psi(U) : U \in G\} \leq \varepsilon$.

For a real α, we set

$$M(\alpha, Z, \varepsilon) = \inf_{G \subset F} \left\{ \sum_{U \in G} \eta(U)\phi(U)^\alpha \right\}$$

where the infimum is taken over all finite or countable collections $G \subset F$ with $\psi(G) \leq \varepsilon$ which cover Z. It is easy to see that $M(\alpha, Z, \varepsilon)$ does not decrease when ε tends to 0. Therefore the limit exists

$$m_c(\alpha, Z) = \lim_{\varepsilon \to 0} M(\alpha, Z, \varepsilon).$$

One can show (cf. [4]) that the function $m_c(\alpha, \cdot)$ is an upper measure on X which is called the α-upper Caratheodory measure on X. Further, the function $m_c(\cdot, Z)$ (for a fixed Z) has the following property: there exists a change-over value α_c such that $m_c(\alpha, Z) = \infty$ for $\alpha < \alpha_c$ and $m_c(\alpha, Z) = 0$ for $\alpha > \alpha_c$. The value α_c is called the Caratheodory dimension of Z and is denoted by $\dim_c Z$. It evidently depends on F, η, ϕ, ψ.

We set

$$R(\alpha, Z, \varepsilon) = \inf_{G \subset F} \left\{ \sum_{U \in G} \eta(U)\phi(U)^\alpha \right\}$$

where the infimum is taken over all finite or countable collections $G \subset F$ covering Z with $\psi(U) = \varepsilon$ for all $U \in G$. Let

$$r_c^l(\alpha, Z) = \liminf_{\varepsilon \to 0} R(\alpha, Z, \varepsilon), \quad r_c^u(\alpha, Z) = \limsup_{\varepsilon \to 0} R(\alpha, Z, \varepsilon).$$

These functions have the following property: there exist change-over values α_c^l, α_c^u such that $r_c^{l,u}(\alpha, Z) = \infty$ for $\alpha \leq \alpha_c^{l,u}$, $r_c^{l,u}(\alpha, Z) = 0$ for $\alpha > \alpha_c^{l,u}$. The values $\alpha_c^{l,u}$ are called the lower and the upper Caratheodory capacities of Z and are denoted by $\mathrm{Cap}_c^{l,u} Z$.

Let μ be a Borel measure on X. Following [4] we set

$$\dim_c \mu = \inf\{\dim_c Z : Z \subset X, \mu(Z) = 1\},$$

$$\mathrm{Cap}_c^{l,u} \mu = \liminf_{\delta \to 0}\{\mathrm{Cap}_c^{l,u} Z : Z \subset X, \mu(Z) \geq 1 - \delta\}.$$

These quantities are called respectively the measure Caratheodory dimension and the lower and upper measure Caratheodory capacities.

For a fixed $x \in X$ we set

$$d^l_{c,\mu,\alpha}(x) = \liminf_{\varepsilon\to 0} \frac{\alpha \ln \mu(U)}{\ln \eta(U) + \alpha \ln \phi(U)},$$
$$d^u_{c,\mu,\alpha}(x) = \limsup_{\varepsilon\to 0} \frac{\alpha \ln \mu(U)}{\ln \eta(U) + \alpha \ln \phi(U)}.$$

where the infinum and supremum are taken over all $U \ni x$ for which $\psi(U) \leq \varepsilon$. The quantities $d^{l,u}_{c,\mu,\alpha}(x)$ are called respectively the α-lower and α-upper pointwise Caratheodory dimension of measure μ at point x. It is worthwhile to emphasize that in general they depend on α.

We formulate some properties of the dimensionlike characteristics introduced above. The proofs can be found in [4].

PROPOSITION 1.
1) $\dim_c \emptyset = \mathrm{Cap}^l_c \emptyset = \mathrm{Cap}^u_c \emptyset = 0$;
2) $\dim_c Z_1 \leq \dim_c Z_2$, $\mathrm{Cap}^{l,u}_c Z_1 \leq \mathrm{Cap}^{l,u}_c Z_2$ *if* $Z_1 \subset Z_2 \subset X$;
3) $\dim_c \left(\bigcup_{i\geq 0} Z_i\right) = \sup_{i\geq 0} \dim_c Z_i$,

$$\mathrm{Cap}^{l,u}_c \left(\bigcup_{i\geq 0} Z_i\right) \geq \sup_{i\geq 0} \mathrm{Cap}^{l,u}_c Z_i, \quad Z_i \subset X;$$

4) $\dim_c Z \leq \mathrm{Cap}^l_c Z \leq \mathrm{Cap}^u_c Z$;
5) *if μ is a Borel measure on X then*

$$\dim_c \mu \leq \mathrm{Cap}^l_c \mu \leq \mathrm{Cap}^u_c \mu.$$

We consider the problem of coinciding the quantities $\dim_c \mu$, $\mathrm{Cap}^u_c \mu$. The first result in this direction was obtained for the so-called classical dimensionlike characteristics in [6]. The general case was studied if [4]. We formulate the additional conditions on μ which are close to ones given in [4].

PROPOSITION 2. *Let μ be a Borel measure on X with the following properties*
1) *for μ-almost every $x \in X$*

$$d^l_{c,\mu,\alpha}(x) = d^u_{c,\mu,\alpha}(x) \stackrel{\text{def}}{=} d_\alpha(x);$$

2) *there exists $\beta \neq 0$ such that $d_\beta(x) = \beta$ for μ-almost evry $x \in X$;*
3) *there exists $\beta_0 > 0$ such that for μ-almost every $x \in X$ the function $d_\alpha(x)$ is twice differentiable over $\alpha \in [\beta - \beta_0, \beta + \beta_0]$ and*

$$\frac{d}{d\alpha} d_\beta(x) \begin{cases} < 1 & \text{if } \beta > 0, \\ > 1 & \text{if } \beta < 0; \end{cases}$$

4) *for μ-almost every $x \in X$ there exists a number $\varepsilon(x) > 0$ such that*

$$\eta(U)\phi(U)^{\alpha} < 1$$

for any $U \in F$ for which $U \ni x$ and $\psi(U) \le \varepsilon(x)$;

5) *for any Z with $\mu(Z) > 0$, $\lambda > 1$, $t > 0$ there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \le \varepsilon_1$ there is $G \subset F$ satisfying the following properties: G covers Z, $\psi(U) \le \varepsilon$ for any $U \in G$ and*

$$\sum_{U \in G} \mu(U)^{\lambda} \le t. \tag{1}$$

Then $\dim_c \mu = \mathrm{Cap}_c^l \mu = \mathrm{Cap}_c^u \mu = \beta$.

PROOF: The proof follows closely the arguments given in [4]. First we will show that $\dim_c \mu \ge \beta$. Let Λ be the set of points $x \in X$ for which conditions 1,2, 3, 4 hold and let $Z \subset \Lambda$ be a set with $\mu(Z) = 1$. For given $\beta_0 > \gamma > 0$, $\rho > 0$ we set $Z_{\rho,\gamma}$ to be the set of $x \in Z$ such that: 1) $\rho \le \varepsilon(x)$; 2)$\ln \mu(U)/(\ln \eta(U) + (\beta - \gamma) \ln \phi(U)) \ge 1$ for any $U \in F$, $U \ni x$, $\psi(U) \le \rho$; 3) $\frac{d}{d\alpha} d_{\alpha}(x) < 1$ if $\beta > 0$ and $\frac{d}{d\alpha} d_{\alpha}(x) > 1$ if $\beta < 0$ for all $\alpha \in [\beta - \gamma, \beta + \gamma]$; 4) $\eta(U)\phi(U)^{\alpha} < 1$ for $\alpha \in [\beta - \gamma, \beta + \gamma]$. It is obvious that $Z_{\rho_1,\gamma_1} \subset Z_{\rho_2,\gamma_2}$ if $\rho_1 \ge \rho_2$, $\gamma_1 \ge \gamma_2$. It follows from condition 2 that

$$\bigcup_{\rho > 0, \gamma > 0} Z_{\rho,\gamma} = Z.$$

Therefore there exist $\rho_0 > 0$, $\gamma_0 > 0$ such that $\mu(Z_{\rho,\gamma}) \ge 1/2$ for any $0 < \rho < \rho_0$, $0 < \gamma < \gamma_0$. Fix $0 < \rho \le \rho_0$, $0 < \gamma \le \gamma_0$, $x \in Z_{\rho,\gamma}$ and take $U \in F$ such that $U \ni x$, $\psi(U) \le \varepsilon$. We have from the definition of the set $Z_{\rho,\gamma}$ and condition 3 that

$$\mu(U) \le \eta(U)\phi(U)^{\beta - \gamma}. \tag{2}$$

Let now $G \subset F$ cover $F_{\rho,\gamma}$ and $\psi(G) \le \varepsilon$. We have from (2) that

$$\sum_{U \in G} \eta(U)\phi(U)^{\beta - \gamma} \ge \sum_{U \in G} \mu(U) \ge \mu(Z_{\rho,\gamma}) \ge \frac{1}{2}.$$

It follows from this that $M(\beta - \gamma, Z_{\rho,\gamma}, \varepsilon) \ge \frac{1}{2}$. This implies that $m(\beta - \gamma, Z_{\rho,\gamma}) \ge \frac{1}{2}$, hence $\dim_c Z_{\rho,\gamma} \ge \beta - \gamma$. Therefore,

$$\dim_c Z \ge \dim_c Z_{\rho,\gamma} \ge \beta - \gamma.$$

As Z is an arbitrary set of full measure it follows that $\dim_c \mu \ge \beta - \gamma$. This implies the desired result because γ can be taken arbitrarily small.

Now we will show that $\mathrm{Cap}_c^u \mu \le \beta$. For given $\beta_0 > \gamma > 0$, $\rho > 0$, we set $Z_{\rho,\gamma}$ to be the set of $x \in Z$ such that: 1) $\rho \le \varepsilon(x)$; 2) $\ln \mu(U)/(\ln \eta(U) + \frac{\beta + \gamma}{\beta + 2\gamma} \ln \phi(U)) \ge 1$ for any $U \in F$, $U \ni x$,

$\psi(U) \le \rho$; 3) $\frac{d}{d\alpha}d_\alpha(x) < 1$ if $\beta > 0$ and $\frac{d}{d\alpha}d_\alpha(x) > 1$ if $\beta < 0$ for all $\alpha \in [\beta-\gamma, \beta+\gamma]$; 4) $\eta(U)\phi(U)^\alpha < 1$ for $\alpha \in [\beta-\gamma, \beta+\gamma]$. It is obvious that $Z_{\rho_1,\gamma_1} \subset Z_{\rho_2,\gamma_2}$ if $\rho_1 \ge \rho_2$, $\gamma_1 \ge \gamma_2$. It follows from condition 2 that $\bigcup_{\rho>0,\gamma>0} Z_{\rho,\gamma} = Z$. Therefore, for given $\delta > 0$ there exist $\rho_0 > 0$, $\gamma_0 > 0$ such that $\mu(Z_{\rho,\gamma}) \ge 1-\delta$ for any $0 < \rho \le \rho_0$, $0 < \gamma \le \gamma_0$. Fix $0 < \rho \le \rho_0$, $0 < \gamma \le \gamma_0$, $x \in Z_{\rho,\gamma}$ and let $U \in F$ be a set such that $U \ni x$, $\psi(U) = \epsilon$. We have from condition 3 that

$$\eta(U)\phi(U)^{\beta+2\gamma} \le \mu(U)^\lambda \tag{3}$$

where $\lambda = (\beta+2\gamma)(\beta+\gamma) < 1$. Choose an arbitrary $t > 0$ and take ϵ_0 in accordance with condition 4. Then for $0 < \epsilon \le \epsilon_0$ we take $G \subset F$ covering $Z_{\rho,\gamma}$ and such that $\psi(U) = \epsilon$ for any $U \in G$ and G satisfying (1). It follows from (1) and (3) that

$$\sum_{U\in G} \eta(U)\phi(U)^{\beta+2\gamma} \le \sum_{U\in G} \mu(U)^\lambda \le t.$$

We have from this that $R(\beta+2\gamma, Z_{\rho,\gamma}\epsilon) \le t$. This implies that $r_c^u(\beta+2\gamma, Z_{\rho,\gamma}) \le t$. As t is arbitrarily small, we have that $\mathrm{Cap}_c^u Z_{\rho,\gamma} \le \beta+2\gamma$. Taking into consideration that $\mu(Z_{\rho,\gamma}) \ge 1-\delta$ for arbitrarily small δ and γ is also arbitrarily small we have $\mathrm{Cap}_c^u \mu \le \beta$. The proposition is proved.

As in [4] and [6] we introduce the so-called classical dimensionlike characteristics setting: F is the collection of open balls in X, $\eta(U) = 1$, $\phi(U) = \psi(U) = \mathrm{diam}\, U$, $U \in F$. The Caratheodory dimension and the lower and upper Caratheodory capacities of a set Z coincide respectively with the Hausdorff dimension and the lower and upper capacities of Z. We denote them by $\dim_H Z, C^{l,u}(Z)$. If μ is a Borel measure on X then the Caratheodory dimensionlike characteristics of μ introduced above are the measure Hausdorff dimension, $\dim_H \mu$, the lower and upper measure capacities $C^{l,u}(\mu)$, the lower and upper pointwise dimensions $d_\mu^{l,u}(x)$. It is easy to see that

$$d_\mu^l(x) = \liminf_{\epsilon\to 0} \frac{\ln \mu(B(x,\epsilon))}{\ln \epsilon}, \quad d_\mu^u(x) = \limsup_{\epsilon\to 0} \frac{\ln \mu(B(x,\epsilon))}{\ln \epsilon}.$$

The following result is a direct consequence of Proposition 2 (cf. also [6]).

Proposition 3. *Assume that for μ-almost every $x \in X$*

$$d_\mu^l(x) = d_\mu^u(x) \stackrel{\text{def}}{=} d.$$

Then $\dim_H \mu = C^l(\mu) = C^u(\mu) = d$.

We also formulate the result belonging to L.-S. Young which allows us to calculate the classical dimensionlike characteristics of a measure in the two-dimensional case.

Let M be a two-dimensional smooth compact Riemann manifold, $f: M \to M$ a C^2-diffeomorphism preserving an ergodic Borel probability measure. Denote by χ_μ^1, χ_μ^2 the Lyapunov characteristic exponents of μ and assume that $\chi_\mu^1 > 0 > \chi_\mu^2$ (cf. [4], [6]).

PROPOSITION 4. *For μ-almost every $x \in M$*

$$d^l_\mu(x) = d^u_\mu(x) = h_\mu(f)\left(\frac{1}{\chi^1_\mu} - \frac{1}{\chi^2_\mu}\right) \stackrel{\text{def}}{=} d$$

where $h_\mu(f)$ is the metric entropy of f. In particular,

$$\dim_H \mu = C^l(\mu) = C^u(\mu) = d.$$

3. Definitions of the Generalized Spectrum for the Dimension.

The generalized spectrum for the dimensions was originally introduced in [7] (cf. also [2], [8]). Another approach was given in [1] for the case of expanding maps. We give two versions of the definition of the generalized spectrum using the procedure described in the previous section.

1. Let X be a compact metric space, μ a Borel measure on X, F a collection of open balls. We set for a fixed real q

$$\eta(U) = \nu(U)^q, \ \phi(U) = \psi(U) = \operatorname{diam} U.$$

It is easy to verify that they satisfy conditions A1–A3. Thus the dimensionlike characteristics constructed by them are defined (they depend on q and ν). For $q = 0$ they are the classical dimensionlike characteristics. One can show that they are equal to zero if $q = 1$. Therefore, we will assume that $q \neq 1$ and will use the following notations and names:

$$\dim_{q,\nu} Z = \frac{1}{1-q}\dim_c Z \text{— the generalized dimension of order } q \text{ of } Z;$$

$$C^{l,u}_{q,\nu}(Z) = \frac{1}{1-q}\operatorname{Cap}^{l,u}_c Z \text{—the generalized lower and upper capacities of order } q \text{ of } z;$$

$$\dim_{q,\nu} \mu = \frac{1}{1-q}\dim_c \mu \text{—the generalized dimension of order } q \text{ of } \mu;$$

$$C^{l,u}_{q,\nu}(\mu) = \frac{1}{1-q}\operatorname{Cap}^{l,u}_c \mu \text{—the generalized lower and upper capacities of order } q \text{ of } \mu;$$

$$d^{l,u}_{q,\nu,\mu,\alpha}(x) = \frac{1}{1-q}d^{l,u}_{c,\mu,\alpha}(x) \text{—the generalized lower and upper pointwise dimension of order } q \text{ of } \mu \text{ at point } x.$$

The families of characteristics $\dim_{q,\nu} Z$, $\dim_{q,\nu}\mu$ are called the generalized spectra for the dimensions of a set Z or of a measure μ. Usually the case $\nu = \mu$ or ν is equivalent to μ is considered. It follows from what was said above that the value $(1-q)\dim_{q,\nu} Z$ is the change-over point for the α-upper Caratheodory measure

$$m_c(\alpha, Z, q) = \liminf_{\varepsilon\to 0}\left\{\sum_i \nu(U_i)^q(\operatorname{diam} U_i)^\alpha : \bigcup_i U_i \supset Z,\ \operatorname{diam} U_i \le \varepsilon\right\}.$$

THEOREM 1. *Assume that*

1) *ν is equivalent to μ and $c^{-1} \le d\mu(x)/d\nu(x) \le c$ where $c > 0$ is a constant;*
2) *$d^l_\mu(x) = d^u_\mu(x) \stackrel{\text{def}}{=} d$ for μ-almost every $x \in X$.*

Then for μ-almost every $x \in X$

$$d^l_{q,\nu,\mu,\beta}(x) = d^u_{q,\nu,\mu,\beta}(x) = d \tag{4}$$

where $\beta = d(1-q)$.

PROOF: It is easy to verify that

$$d^{l,u}_{q,\nu,\mu,\alpha}(x) = \frac{1}{1-q}\frac{\alpha d}{\alpha + qd}. \tag{5}$$

It directly implies the desired result.

Now we formulate a result which allows us to calculate the generalized spectrum.

THEOREM 2. *In addition to the conditions of Theorem 1, assume that X is a compact smooth Riemannian finite dimensional manifold or a compact subset in a finite dimensional Euclidean space and μ is a continuous non-atomic measure. Then for all q*

$$\dim_{q,\nu}\mu = C^l_{q,\nu}(\mu) = C^u_{q,\nu}(\mu) = d = \dim_H \mu.$$

PROOF: It is easy to see that the assumptions about X and μ imply the condition 4 of Proposition 4. It follows from (4), (5) that for μ-almost every $x \in X$ the function $d_\alpha(x)$ is twice differentiable and

$$\left.\frac{d}{d\alpha}d_\alpha(x)\right|_{\alpha=\beta} = q$$

(recall that $\beta = d(1-q)$). This implies condition 2 of Proposition 2 for all q. Condition 1 of this proposition follows directly from (4). Further we have from condition 1 of Theorem 1 that for μ-almost every $x \in X$ and small enough s

$$C^{-1}(\operatorname{diam} U)^{d-s} \le \mu(U) \le C(\operatorname{diam} U)^{d+s}$$

where U is a ball of a small enough radius (depending on x), $C > 0$ is a constant independent of x, U. This implies Condition 3 of Proposition 2 because $qd + d(1-q) = d > 0$ uniformly over x and U. Now the desired result follows from Proposition 2.

It follows from Theorems 1 and 2 that for μ-almost every $x \in X$ there exist limits $\lim_{q\to 1} d^{\ell,u}_{q,\nu,\mu,\beta}(x) = d$ $(\beta = d(1-q))$, $\lim_{q\to 1} \dim_{q,\nu} \mu = d$. The value $\dim_{1,\nu} \mu$ is equal to the information dimension of μ and $\dim_{2,\nu} \mu$ is equal to the correlation dimension of μ (cf. [7]).

Consider the case when f is a C^2-diffeomorphism of a smooth compact Riemannian two-dimensional manifold preserving an ergodic non-atomic continuous Borel probability measure μ with non-zero Lyapunov characteristic exponents $\chi^l_\mu > 0 > \chi^2\mu$. The next result follows form Proposition 4 and Theorem 2.

THEOREM 3. *For all q*

$$\dim_{q,\mu} \mu = C^l_{q,\mu}(\mu) = C^u_{q,\mu}(\mu) = h_\mu(f)\left(\frac{1}{\chi^1_\mu} - \frac{2}{\chi^2_\mu}\right).$$

We describe another approach to the definition of the generalized spectrum in the case where f is a C^2-diffeomorphism of a smooth compact two-dimensional Riemannian manifold M preserving an ergodic continuous Borel probability measure μ with non-zero Lyapunov characteristic exponents χ^1_μ, χ^2_μ, $\chi^1_\mu > 0 > \chi^2_\mu$. Denote by .

$$B_n(x,\delta) = \{y \in M : \rho(f^k(x), f^k(y)) \le \delta \text{ for }$$
$$k = -m(n), -m(n)+1, \dots, n\}$$

where ρ is the distance in M induced by the Riemannian metrics and

$$m(n) = \operatorname{ent}\left(-\frac{\chi^1_\mu}{\chi^2_\mu}\right) n$$

(ent(a) is the greatest integer of a). One can show (cf [5]) that there exist a set Λ and functions $k(x) > 0$, $\delta(x) > 0$ such that

$$\mu(B_n(x,\delta)) \le k(x)\delta \tag{6}$$

for any $x \in \Lambda$, $n \ge 0$, $0 < \delta \le \delta(x)$. We set $\Lambda_t = \{x \in \Lambda : k(x) \le t, \delta(x) \ge t^{-1}\}$. It is easy to see that $\Lambda_t \subset \Lambda_{t+1}$, $\Lambda = \bigcup_{t\ge 1} \Lambda_t$. Denote by $k_t = \sup_{x\in\Lambda_t} k(x)$, $\delta_t = \inf_{x\in\Lambda_t} \delta(x)$. Fix $t \ge 1$, $0 < \delta \le \delta_t$ and choose F as the collection of sets $B_n(x,\delta)$ over all $x \in \Lambda_t$, $n \ge 0$. For a fixed real q we set

$$\eta(B_n(x,\delta)) = \mu(B_n(x,\delta))^q, \quad \phi(B_n(x,\delta)) = \operatorname{diam} B_n(x,\delta),$$
$$\psi(B_n(x,\delta)) = \frac{1}{n}.$$

It follows from (6) that the functions η, ϕ, ψ satisfy conditions A1–A3. In fact, one can show that for any $Z \subset \Lambda_t$, δ, n there exists a finite cover of Z by sets $B_n(x,\delta)$. Thus the dimensionlike characteristics constructed by these three functions are defined (they depend on q, t, δ). One can show that they are equal to 0 if $q = 1$. Therefore we will assume that $q \neq 1$. We use the notations $\dim_{q,t,\delta} Z - \frac{1}{1-q}\dim_c Z$ for the generalized dimension of order q of $Z \subset \Lambda_t$ and $C^{l,u}_{q,t,\delta}(Z) = \frac{1}{1-q}\operatorname{Cap}^{l,u}_c Z$ for the generalized lower and upper capacities of order q of $Z \subset \Lambda_t$. It follows from what was said above that the value $(1-q)\dim_{q,t,\delta} Z$ is the change-over point for the α-upper Caratheodory measure

$$m_c(\alpha, Z, q) = \liminf_{N\to\infty}\Big\{\sum_i \mu(B_n(x_i,\delta))^q(\operatorname{diam} B_n(x_i,\delta))^\alpha :$$
$$\bigcup_i B_n(x_i,\delta) \supset Z,\ x_i \in \Lambda_t, n \geq N\Big\}.$$

Further, for arbitrary $Z \subset \Lambda$, we set

$$\dim_q Z = \sup_{t\geq 1}\limsup_{\delta\to 0}\dim_{q,t,\delta} Z \cap \Lambda_t,$$
$$C^{l,u}_q(Z) = \sup_{t\geq 1}\limsup_{\delta\to 0}\dim_{q,t,\delta} Z \cap \Lambda_t.$$

and we will use the above names for these values. Now we can introduce, as above, the generalized dimension of order q of $\mu - \dim_q \mu$; the generalized lower and upper capacities of order q of $\mu - C^{l,u}_q(\mu)$; the generalized lower and upper pointwise dimension of order q of μ at point $x - d^{l,u}_{q,\alpha}(x)$. The families of characteristic $\dim_q Z$, $\dim_q \mu$ are called the generalized spectra for the dimensions of a set Z or of a measure μ. We formulate the result which allows us to calculate these dimensionlike characteristics.

THEOREM 4. 1) *For all q and μ-almost every $x \in M$*

$$d^l_{q,\beta}(x) = d^u_{q,\beta}(x) \stackrel{\text{def}}{=} d = h_\mu\left(\frac{q}{\chi^1_\mu} - \frac{1}{\chi^2_\mu}\right)$$

where $\beta = d(1-q)$.

2) *For all q*

$$\dim_q \mu = C^l_q(\mu) = C^u_q(\mu) = d = \dim_H \mu.$$

PROOF: Fix $t \geq 1$ such that $\mu(\Lambda_t) > 0$ and $\delta, 0 < \delta \leq \delta_t$. It follows from [5] and relation (6) that for arbitrary $a > 0$ there exists $C^1_t = C^1_t(a)$ for which

$$(C^1_t)^{-1}\exp(-(\chi^1_\mu - a)n)\delta \leq \operatorname{diam} B_n(x,\delta)$$
$$\leq C^1_t\exp(-(\chi^1_\mu + a)n)\delta \tag{7}$$

for $x \in \Lambda_t$, $n \geq 0$. In particular, this implies Condition 4 of Proposition 2. Using the result in [3] we have that for any $b > 0$, $c > 0$ there exist a set $A_t \subset \Lambda_t$ and a number $\delta^1_t \leq \delta_t$ having the following properties:

1) $\mu(\Lambda \setminus A_t) \le b$;

2) for μ-almost every $x \in A_t$ and any $\delta, 0 < \delta \le \delta_t^1$

$$(C_t^2)^{-1} \exp\big((-h_\mu(f) - c)(n + m(n))\big) \le \mu(B_n(x,\delta)) \le C_t^2 \exp\big((-h_\mu(f) + c)(n + m(n))\big) \tag{8}$$

where $C_t^2 > 0$ is a constant independent of x and n. It follows from (7), (8) that for μ-almost every $x \in A_t$ and real α and q

$$(C_t^3)^{-1}(a+b) \le \left| d_{q,\alpha}^{l,u}(x) - \alpha h_\mu(f) \left(q h_\mu(f) + \alpha \frac{\chi_\mu^1 \chi_\mu^2}{\chi_\mu^2 - \chi_\mu^1} \right) \right| \le C_t^3(a+b).$$

As a, b can be taken arbitrarily small and t is arbitrarily large we have from the above that for μ-almost every $x \in \Lambda$

$$d_{q,\alpha}^{l,u}(x) = \alpha h_\mu(f) \left(q h_\mu(f) + \alpha \frac{\chi_\mu^1 - \chi_\mu^2}{\chi_\mu^1 - \chi_\mu^1} \right).$$

This implies Condition 1 and also the fact that the function $d_\alpha(x)$ is twice-differentiable and $\frac{d}{d\alpha} d_\alpha(x) = q$. Conditions 1 and 2 of Proposition 2 follow from this. In order to prove Condition 3 of Proposition 2, we notice that (7), (8) imply, for μ-almost every $x \in A_t$ (with large enough t) and $\delta \le \delta_t^1$, $n \ge 0$, that

$$\mu(B_n(x,\delta))^q (\operatorname{diam} B_n(x,\delta))^{\alpha(1-q)} \sim \exp\left(-\left(1 - \frac{\chi_\mu^1}{\chi_\mu^2}\right) n\right).$$

This implies the desired result.

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Ya. B. Pesin
Department of Mathematics
The University of Chicago
Chicago, IL 60637

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Carathéodory's fundamental contribution to measure theory

Jean-Paul Pier

Abstract. At the beginning of this century, Lebesgue formalized the modern theory of integration. His work was completed via the incorporation of the Stieltjes integral, mainly realized by Riesz. These results were extended to very general situations by Radon and Fréchet.

Carathéodory's monograph *Vorlesungen über reelle Funktionen* constituted the first complete account on integration theory and has remained a classic during a long period. In this treatise, for the first time, the integral is superseded by the notion of measure. Both points of view, essentially equivalent and equally important, have been adopted. Whereas Young and Daniell concentrated on integrals, Carathéodory attributed the precedence to measures. The influence of Carathéodory's achievements may be traced in later developments of abstract measure theory.

1 INTEGRATION THEORY AT THE BEGINNING OF THE CENTURY

Following the outstanding accomplishments of Cauchy and Riemann, the long history of integration theory underwent a new revolution at the beginning of the 20^{th} century.

Up to 1900, the integral has been viewed as a limit of Riemannian sums, a concept that originated with Archimedes. This integral lacks two major traditional properties : The derivative of a function over an interval is not necessarily integrable; the passage to the limit behind the integration symbol is not always possible. That situation changed when Lebesgue succeeded in defining the integral for a larger class of functions.

Lebesgue's ideas relied on the notion of measure introduced by Borel in 1898 [2]. For a bounded open subset of the reals which is a finite or a countable union of pairwise disjoint intervals Borel defined the measure to be the sum of the lengths of these intervals. He thus approximated these subsets from inside whereas so far they had been included in a finite union of intervals. Borel then considered the subsets generated by bounded open subsets via the operations of countable unions or differences of subsets. The measure of such a *Borel set* was defined by complete additivity : The measure of a countable union of pairwise disjoint Borel sets is the sum of the measures of these sets. Borel observed that a subset of measure zero may be uncountable, but every countable set admits measure zero. He wrote :

"Les ensembles dont on peut définir la mesure en vertu des définitions précédentes seront dits par nous ensembles mesurables, sans que nous entendions impliquer par là qu'il n'est pas possible de donner une définition de la mesure d'autres ensembles; mais une telle définition nous serait inutile; elle pourrait même nous gêner, si elle ne laissait pas à la mesure les propriétés fondamentales que nous lui avons attribuées dans les définitions que nous avons données" ([2] p. 48).

Borel's demonstrations were not written out explicitely. In his thesis [24] Lebesgue filled in all details and introduced a new concept of utmost importance. Every Borel set was called measurable (*B*). The union of such a set *B* and a subset *N* of a Borel set with measure 0 was termed measurable (*J*); the measure of $B \cup N$ was taken equal to that of *B*. Later Borel stressed that in 1894 he had considered for the first time implicitely a set of measure zero [3].

Lebesgue formulated the measure problem in a finite-dimensional space :

"Nous nous proposons d'attacher à chaque ensemble borné sa mesure satisfaisant aux conditions suivantes :

1°. Il existe des ensembles dont la mesure n'est pas nulle.

2°. Deux ensembles égaux [i.e., en déplaçant l'un d'eux, on peut les amener à coïncider] ont même mesure.

3°. La mesure de la somme d'un nombre fini ou d'une infinité dénombrable d'ensembles, sans points communs, deux à deux, est la somme des mesures de ces ensembles.

Nous ne résoudrons ce problème de la mesure que pour les ensembles que nous appellerons mesurables" ([24] p. 235-236).

In 1904, in his famous book *Leçons sur l'intégration et la recherche des fonctions primitives* [25], Lebesgue was primarily interested in the determination of an invariant integral. He wanted his integral on real-valued bounded functions to satisfy the following conditions :

(1) For all a,b,h one has $\int_a^b f(x)dx = \int_{a+h}^{b+h} f(x-h)dx$;

(2) for all a,b,c one has $\int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^a f(x)dx = 0$;

(3) $\int_a^b [f(x) + \varphi(x)]dx = \int_a^b f(x)dx + \int_a^b \varphi(x)dx$;

(4) if $f \geq 0$ and $b \geq a$, then $\int_a^b f(x)dx \geq 0$;

(5) $\int_0^1 1\, dx = 1$;

(6) if $f_n(x)$ converges increasingly to $f(x)$, then $\int_a^b f_n(x)dx$ converges to $\int_a^b f(x)dx$.

Observing that it suffices to consider characteristic functions, Lebesgue described the problem as the association to any bounded subset E of the real line of a number $m(E) \geq 0$, called measure of E, which satisfies the following conditions :

(1') Two subsets coinciding via a translation admit the same measures;

(2') the measure of a finite or countable union of pairwise disjoint subsets is the sum of the measures of these subsets;

(3') the measure of [0,1] has value 1.

In this formulation (3') replaces (5), (2') stems from (3) and (6), (1') is (1). If $a < b$, the measure of [a,b] is $b - a$.

An arbitrary subset E is contained in a finite or countable union of intervals; the set constituted by the sums of the lengths of these intervals admits a greatest lower bound called outer measure $m_e(E)$ of E. Moreover, A being an interval covering E, $m(A) - m_e(A \setminus E)$ is the inner measure $m_i(E)$ of E. In case $m_i(E) = m_e(E)$, the set E is said to be measurable and the common value is the measure $m(E)$ of E verifying (1'), (2'), (3'). The Borel sets are Lebesgue measurable; but the new class is larger.

In 1905, by means of the axiom of choice, Vitali showed the existence of non Lebegue measurable subsets on the reals [33].

Interest focused on the linear functional aspect of the integral. In 1903, Hadamard [19] proved that every continuous linear functional on the space $\mathbf{C}([0,1])$ of continuous functions defined on [0,1] is given by

$$F(f) = \lim_{n \to \infty} \int_0^1 k_n(x) f(x) dx,$$

(k_n) being a sequence of functions in $\mathbf{C}([0,1])$. Riesz [31] called linear functional on $\mathbf{C}([0,1])$ every functional A on this space such that $A(f_i)$ converges uniformly to $A(f)$ whenever (f_i) converges uniformly to f. He verified that if α is a function of bounded variation on [0,1], then the mapping $f \longmapsto \int_0^1 f(x) d\alpha(x)$ constitutes a linear functional. The integral is interpreted as the limit of sums $\Sigma f(\zeta_i)\ (\alpha(x_{i+1}) - \alpha(x_i))$ corresponding to subdivisions of [0,1] consisting of a finite number of partial intervals $[x_i, x_{i+1}]$, ζ_i being an element of $[x_i, x_{i+1}]$; the passage to the limit signifies that the lengths of these intervals converge uniformly to 0. In order to establish the converse, Riesz considered a given functional A; let

$$F^{(\zeta)}(x) = x \text{ if } 0 \leq x \leq \zeta,$$
$$F^{(\zeta)}(x) = \zeta \text{ if } \zeta \leq x \leq 1.$$

The function $\alpha : \zeta \longmapsto A(F^{(\zeta)})$ admits derivatives that are of bounded variation; they give rise to a representation of A.

Dieudonné made this observation :

"Dès 1910, presque tous les théorèmes fondamentaux de la théorie avaient été démontrés par Lebesgue et ses émules" ([15] p. 270).

We should now quote Bourbaki :

"Il est bien clair qu'il ne restait plus qu'un pas à franchir pour aboutir à la notion générale de mesure que va définir J. Radon en 1912, englobant dans une même synthèse l'intégrale de Lebesgue et l'intégrale de Stieltjes" ([5] p. 120).

Stieltjes [32] had defined on $[a,b]$ a *mass distribution*, i.e., an increasing function φ for which the number of points presenting a discontinuity greater than a given number is finite. The sums $\sum_i f(\zeta_i)\,(\varphi(x_{i+1})-\varphi(x_i))$ corresponding to a subdivision $a = x_0 < x_1 < \ldots < x_n = b$, where $\zeta_i \in [x_{i-1},x_i]$ for $i=1,\ldots,n$, converge to the limit denoted by $\int_a^b f(x)d\varphi(x)$.

Exploiting ideas due to Lebesgue, Stieltjes, Riesz, Radon [29] generalized the notion of multiple integral associating it to a set function μ, defined on all bounded Lebesgue measurable subsets and satisfying complete additivity. Radon showed that all main theorems of Lebesgue's integration theory may be carried over to integrals $\int f(x)d\mu(x)$, $\int f\,d\mu$ associated to the *Radon measure* μ.

Shortly later, considering this type of integral $\int F(P)dh(P)$, Fréchet wrote :

"Cette définition résulte d'une sorte de fusion de l'intégrale de M. Lebesgue et de l'intégrale de Stieltjes. La définition de M. J. Radon se réduit à celle de M. Lebesgue quand h est une fonction linéaire et à celle de Stieltjes quand F est une fonction continue.

D'ailleurs, l'intégrale de Radon peut aussi s'écrire

$$\int_E F(P)df(e)$$

où $f(e)$ est une fonction additive du sous-ensemble variable e de E.

Or, c'est sous cette forme qu'apparaît ce qui me semble être le grand avantage de la définition de M. J. Radon, avantage que celui-ci ne paraît pas avoir remarqué. M. J. Radon avait pour but de réaliser un progrès dans la Théorie des fonctions en unifiant les définitions de Stieltjes et de M. Lebesgue. Mais, en fait, on remarque que, moyennant quelques légères modifications, la définition et les propriétés de l'intégrale de M. Radon s'étendent bien au-delà du Calcul intégral classique, elles sont presque immédiatement applicables au domaine infiniment plus vaste du calcul fonctionnel.

En d'autres termes, on peut conserver la majeure partie des définitions et des raisonnements de M. J. Radon en négligeant l'hypothèse faite sur la nature de l'argument P à savoir que P est un point de l'espace à n dimensions" ([17] p. 248-249).

Fréchet developed his theory for an abstract measure that is not necessarily associated to Lebesgue measurable subsets. The family of subsets he considered must be closed for countable unions and differences of subsets; the measure has to satisfy complete additivity.

2 CARATHEODORY'S FIRST RESULTS

Bourbaki gave a description of the situation :

"Avec la mémoire de Radon, la théorie générale de l'intégration pouvait être considérée comme achevée dans ses grandes lignes; comme acquisitions ultérieures substantielles, on ne peut guère mentionner que la définition du produit infini de mesures, due à Daniell, et celle de l'intégrale d'une fonction à valeurs dans un espace de Banach, donnée par Bochner en 1933, et qui préludait à l'étude plus générale de l' 'intégrale faible' développée quelques années plus tard par Gelfand, Dunford et Pettis. Mais il restait à populariser la nouvelle théorie, et à en faire un instrument mathématique d'usage courant, alors que la majorité des mathématiciens, vers 1910, ne voyait encore dans l' 'intégrale de Lebesgue' qu'un instrument de haute précision, de maniement délicat, destiné seulement à des recherches d'une extrême subtilité et d'une extrême abstraction. Ce fut là l'oeuvre de Carathéodory, dans un livre longtemps resté classique et qui enrichit d'ailleurs la théorie de Radon de nombreuses remarques originales.

Mais c'est avec ce livre aussi que la notion d'intégrale ... cède le pas pour la première fois à celle de mesure, qui avait été chez Lebesgue (comme avant lui chez Jordan) un moyen technique auxiliaire. Ce changement de point de vue était dû sans doute, chez Carathéodory, à l'excessive importance qu'il semble avoir attachée aux 'mesures p-dimensionnelles'. Depuis lors, les auteurs qui ont traité d'intégration se sont partagés entre ces deux points de vue, non sans entrer dans des débats qui ont fait couler beaucoup d'encre sinon beaucoup de sang. Les uns ont suivi Carathéodory; dans leurs exposés sans cesse plus abstraits et plus axiomatisés, la mesure, avec tous les raffinements techniques auxquels elle se prête, non seulement joue le rôle dominant, mais encore elle tend à perdre contact avec les structures topologiques auxquelles en fait elle est liée dans la plupart des problèmes où elle intervient. D'autres exposés suivent de plus ou moins près une méthode déjà indiquée en 1911 par W.H. Young, dans un mémoire malheureusement peu remarqué, et développée ensuite par Daniell" ([5] p. 122-123).

The *functional* approach of Lebesgue's integration theory was inaugurated by Young [35]. Starting off from integration of continuous functions with compact supports, by limiting processes, he defined upper integrals for functions with compact supports that are lower semicontinuous and then for arbitrary functions with compact supports. Daniell extended this theory to functions defined on an arbitrary set explaining his motivations in this way :

"The idea of an integral has been extended by Radon, Young, Riesz and others so as to include integration with respect to a function of bounded variation. These theories are based on the fundamental properties of sets of points in a space of a finite number of dimensions. In this paper a theory is developed which is independent of the nature of the elements ... It follows that, although many of the proofs given are mere translations into other language of methods already classical (particularly those due to Young), here and there, where previous proofs rested on the theory of sets of points, new methods have been devised" ([14] p. 279).

In a long communication presented to the Königlichen Gesellschaft der Wissenschaften in Göttingen on October 24, 1914, by Felix Klein, Carathéodory submitted his first results on the general measure theory. He wrote in the introductory notice of his work :

"Ich habe es ... für zweckmässig gehalten, meine Darstellung mit einer rein formalen Theorie der Messbarkeit zu beginnen. Dabei wird eine Definition der Messbarkeit zu Grunde gelegt, die einerseits allgemeiner ist, als die gewöhnliche, weil sie sich auch auf Punktmengen von unendlichem äusseren Masse erstreckt, andererseits aber scheinbar viel enger. Diese Definition ist daher viel bequemer als die ältere : sie erlaubt sämtliche in Betracht kommenden Sätze ohne tiefliegende Kunstgriffe zu beweisen; und sie ist der gewöhnlichen Definition vollständig äquivalent ..." ([6] p. 404-405).

Carathéodory defined the outer measure by five conditions.

(I) To an arbitrary subset A of the q-dimensional space $\mathbb{R}^q$ one associates a number $\mu^*(A) \in \overline{\mathbb{R}}_+$ called the outer measure of A. (II) If B is a subset of A, then $\mu^*(B) \leq \mu^*(A)$. (III) If A is the union of a finite or countable collection of subsets $A_1, A_2, \ldots$, then $\mu^*(A) \leq \mu^*(A_1) + \mu^*(A_2) + \ldots$; obviously, the right-hand side has to be convergent. By definition, the set A is said to be measurable in case

$$\mu^*(W) = \mu^*(A \cap W) + \mu^*(W \setminus (A \cap W))$$

whenever W is a set of finite outer measure; $\mu^*(A)$ is then taken to be the measure $\mu(A)$ of A. The definition makes sense also if $\mu^*(W) = +\infty$.

To this new formulation of measurability Carathéodory [12] attributed four major advantages : 1) It can be considered for linear measures. 2) It makes sense in Lebesgue's theory even if the outer measure is infinite. 3) The proofs of the principal theorems are much easier and shorter. 4) The essential advantage is independence of the definition from the notion of inner measure.

Carathéodory [6] established fundamental properties resulting directly from these conditions. The complementary subset of a measurable subset is measurable. The union and the intersection of a finite or countable collection of measurable subsets are measurable. The upper and lower limits of a sequence of measurable subsets are measurable. The measure of a union of a finite or countable collection of pairwise disjoint measurable subsets equals the sum of the measures of these subsets. If (A_n) is an increasing sequence of measurable subsets, then $\mu(\bigcup_n A_n) = \lim_{n\to\infty} \mu(A_n)$. If (A_n) is a decreasing sequence of measurable subsets and $\mu(A_1) < +\infty$, then $\mu(\bigcap_n A_n) = \lim_{n\to\infty} \mu(A_n)$.

So far it is not possible to decide whether a given subset is measurable or not. In order to insure the existence of measurable subsets, Carathéodory introduced a fourth condition.

(IV) If A_1, A_2 are subsets such that $\inf\{d(x,y) : x \in A_1, y \in A_2\} > 0$, d denoting the distance in $\mathbf{R}^q$, then

$$\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Ingenious combinations of all these properties allowed Carathéodory to prove that open subsets and closed subsets are measurable. In particular, all open *intervals* in $\mathbf{R}^q$, i.e., all cartesian products of q elementary open intervals, are measurable.

The set up is completed by a supplementary condition.

(V) The outer measure $\mu^*(A)$ of an arbitrary subset A is the lower limit of the set of all numbers $\mu(B)$, where B is measurable and contains A.

Carathéodory could then show that if A is an arbitrary subset and $\mu^*(A) < +\infty$, there exists a measurable subset B such that $B \supset A$ and $\mu(B) = \mu^*(A)$. By definition, $\mu^*(A) - \mu^*(B \setminus A)$ is the inner measure $\mu_*(A)$ of A. The subset A is measurable if and only if $\mu_*(A) = \mu^*(A)$.

Carathéodory observed that for arbitrary disjoint subsets A and B such that $\mu^*(A) < +\infty$, $\mu^*(B) < +\infty$ and $S = A \cup B$,

$$\mu_*(S) \leq \mu^*(A) + \mu_*(B) \leq \mu^*(S).$$

In particular, in case S is measurable and $\mu(S) < +\infty$,

$$\mu^*(A) + \mu_*(B) = \mu(S),$$

hence necessarily $\mu_*(B) = \mu^*(B)$; B is measurable and, analogously, A is measurable.

Moreover, if (A_n) is an increasing sequence of subsets, one has

$$\mu^*(\bigcup_n A_n) = \lim_{n\to\infty} \mu^*(A_n).$$

This general formalism being established, Carathéodoroy proceeded to the study of *linear measures*.

Consider an arbitrary subset A in $\mathbf{R}^q$ $(q > 1)$. Let (U_n) be any finite or countable sequence of subsets with diameters d_n, all less than a given number $\rho > 0$, such that $A \subset \bigcup_n U_n$. The infimum of the sums $\sum_n d_n$ for all such sequences is denoted by $L_\rho(A)$. Then

$$L^*(A) = \lim_{\rho \to 0} L_\rho(A)$$

defines an outer measure satisfying conditions (I)-(V); it is called the linear outer measure. The subsets (U_n) may be supposed to be convex or open.

Carathéodory verified that if γ is any curve lacking multiple points, its linear measure is the upper limit of the lengths of the inner polygons admitting their summits on γ.

Carathéodory stressed other remarkable properties of this particular outer measure. If $L^*(A) < +\infty$, A is of Lebesgue measure zero. As a matter of fact, one may choose $\Sigma\, d_n \leq L^*(A) + 1$; m_e denoting the ordinary outer Lebesgue measure,

$$m_e(A) \leq \Sigma\, d_n^q \leq \rho^{q-1}\, \Sigma\, d_n \leq \rho^{q-1}(L^*(A)+1).$$

As $\rho > 0$ is arbitrary, $m_e(A) = 0$.

Finally, Carathéodory mentioned *p-dimensional measures* in $\mathbb{R}^q$. Let C_k be the convex hull of U_k; d_k is the least upper bound of the lengths of the orthogonal projections of C_k on the axes. Considering orthogonal projections of C_k on p-dimensional manifolds, one may define a p-dimensional diameter $d_k^{(p)}$. These numbers are substituted to the d_k's in the definition of L_p.

Five years later Hausdorff produced an extension of the theory due to Carathéodory with whom he had corresponded by letters. He stated :

"Herr Carathéodory hat eine hervorragend einfache und allgemeine die Lebesguesche als Spezialfall enthaltene Masstheorie entwickelt und damit insbesondere das p-dimensionale Mass einer Punktmenge im q-dimensionalen Raume definiert" ([21] p. 157).

He described a p-dimensional measure for an arbitrary positive p and compared his investigations with Fréchet's interpolation procedure for dimensions. Hausdorff's general outer measure L_p admits the following interpretation in cases p=1,2,... : The subet A is included in a finite or a countable union of balls K_n with diameters $d_n < \rho$;

$$L_p(A) = \liminf_{\rho \to 0} c_p \sum_n d_n^p,$$

where c_p denotes the volume of the p-dimensional ball of diameter 1.

Cantor had been interested in the classification of continua by dimension properties. As a matter of fact, this topic, investigated by Carathéodory and Hausdorff, had rapidly been neglected by Cantor; one may agree with Hawkins' opinion :

"The reason for this is probably that Cantor soon became completely absorbed with the theory of transfinite numbers" ([22] p. 63).

3 CARATHEODORY'S FUNDAMENTAL WORK

In 1918 Carathéodory published his global treatise entitled *Vorlesungen über reelle Funktionen* dedicated to his friends Erhard Schmidt and Ernst Zermelo. He explained his motivations :

"Die Umwälzung, welche die Theorie der reellen Funktionen durch die Untersuchungen von H. Lebesgue erfahren hat, ist ein Prozess, der heute in seinen Hauptzügen als abgeschlossen gelten kann. Ein Versuch diese Theorie von Grund aus und systematisch aufzubauen scheint mir daher notwendig geworden zu sein; dies hat mich bewogen die Vorlesung, die ich im Sommersemester 1914 an der Universität Göttingen gehalten habe, auszuarbeiten, und mit manchen Erweiterungen und Zusätzen versehen, der Öffentlichkeit vorzulegen ...

In einigen ... Lehrbüchern ... erscheint [die Lebesguesche Theorie] meistens neben den älteren Integrationstheorien und ist dadurch ihres grössten Vorzugs beraubt, der darin besteht, dass sie den kürzeren und bequemeren Weg darstellt, da wo die alte Fahrstrasse oft unnötige Umwege macht" ([7] p. V).

The outer measure of the subset A in $\mathbf{R}^q$ is defined to be the greatest lower bound $m^*(A)$ of all finite or countable sums of the volumes of *intervals* covering A. The supremum of the diameters of these *intervals* may be chosen arbitray small. In particular, if A is an interval, $m^*(A)$ coincides with its volume. The novelty consisted in the interpretation of m^* as a set function. The first step concerned the determination of the class of all set functions satisfying the fundamental properties of m^*.

A priori, a set function μ^* on $\mathbb{R}^q$ is called measure function (Massfunktion) or outer measure if it admits the following properties :

I. For every $A \subset \mathbb{R}^q$, $\mu^*(A) \in \bar{\mathbb{R}}_+$; $\mu^*(\phi) = 0$; $\mu^* \neq 0$.

II. If $B \subset A$, then $\mu^*(B) \leq \mu^*(A)$.

III. If (A_n) is a finite or countable sequence of subsets, then $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$.

IV. If A and B are subsets of positive distance, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Another nontrivial example of an outer measure is provided by the point measure δ_a (at the point a); $\delta_a(A) = 1$ if $a \in A$, $\delta_a(A) = 0$ if $a \notin A$. For a fixed subset S, an induced outer measure is defined by

$$v^*(A) = \mu^*(A \cap S)$$

in case $v^* \neq 0$.

Carathéodory observed that if B is contained in an open subset H and A is contained in the closed complement K of H,

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B);$$

the situation is a particular case of IV. Putting $W = A \cup B$, one has $A = W \cap K = W \setminus (W \cap H)$, $B = W \cap H$,

$$\mu^*(W) = \mu^*(W \cap H) + \mu^*(W \setminus (W \cap H)).$$

The next purpose was the realization of the most general formulation for the latter equality. Carathéodory established the relation

$$\mu^*(W) = \mu^*(W \cap H) + \mu^*(W \setminus (W \cap H))$$

for an arbitrary subset W and an arbitrary open subset H, in case $\mu^*(W) < +\infty$.

The subset A is called measurable for μ^* if

$$\mu^*(W) = \mu^*(W \cap A) + \mu^*(W \setminus (W \cap A))$$

whenever $\mu^*(W) < +\infty$; $\mu^*(A) = \mu(A)$ is termed measure of the measurable subset A. In particular, for the point measure every subset is measurable.

As in his first version, Carathéodory verified the stability properties of the class of measurable subsets. He also indicated that if A is a measurable subset and B is an arbitrary subset,

$$\mu^*(A \cup B) = \mu(A) + \mu^*(B) - \mu^*(A \cap B).$$

Carathéodory noticed that all open, all closed, more generally all Borel subsets are measurable; every subset of outer measure zero is measurable.

Generalizing the approximation property of the outer measure of a subset by means of the outer measures of open sets, Carathéodory called the outer measure μ^* regular if for every subset A, $\mu^*(A)$ is the greatest lower bound of the numbers $\mu(B)$, B being measurable and containing A.

If μ^* is a regular outer measure, the inner measure $\mu_*(A)$ of the subset A is defined to be the least upper bound of the numbers $\mu(B)$ for all measurable subsets contained in A. If $\mu^*(A) = \mu_*(A) < +\infty$, the subset A is measurable.

For an increasing sequence (A_n),

$$\mu^*(\lim_n A_n) = \lim_n \mu^*(A_n);$$

for a decreasing sequence,

$$\mu_*(\lim_n A_n) = \lim_n \mu_*(A_n)$$

if this number is finite.

Measurable functions had been introduced by Lebesgue. Carathéodory also operated the transfer of measurability properties to real-valued functions. If (α_n) is a dense sequence of the real axis and (A_n) is a sequence of subsets in E, does there exist a function $f : E \to \mathbb{R}$ such that

$$\{x \in E : \alpha_n < f(x)\} \subset A_n \subset \{x \in E : \alpha_n \leq f(x)\}$$

whenever $n \in \mathbb{N}^*$? If E and all the sets A_n are measurable, such a function f is called measurable function. The fundamental properties of measurable subsets imply that if f is a measurable function and $\alpha \in \mathbb{R}$, the sets $\{x \in E : \alpha \leq f(x)\}$, $\{x \in E : \alpha < f(x)\}$,$\{x \in E : \alpha \geq f(x)\}$,$\{x \in E : \alpha > f(x)\}$ are measurable, and so are $\{x \in E : f(x) = -\infty\}$, $\{x \in E : f(x) = +\infty\}$.

The existence of measurable functions is insured by the theorem stating that any semicontinuous function on an everywhere dense measurable subset is measurable. Carathéodory proved the existence of nonmeasurable functions. Let A be a nonmeasurable subset of $\mathbb{R}$ and let $f(x) = x$ if $x \in A$, $f(x) = -x$ if $x \notin A$. For every $\alpha \in \mathbb{R}$, $\{x \in \mathbb{R} : f(x)$

$= \alpha\}$ is measurable; but $\{x \in \mathbb{R} : f(x) > 0\} = (A \cap \mathbb{R}_+^*) \cup (\complement A \cap \mathbb{R}_-^*)$ is nonmeasurable.

Carathéodory verified that if f is a measurable function, so is $|f|$. If f_1, f_2 are measurable functions, $f_1+f_2, f_1 f_2$, and also $\frac{f_1}{f_2}$ in case $f_2 \neq 0$, are measurable functions. If (f_n) is a sequence of measurable functions, $\sup f_n$ and $\inf f_n$ are measurable.

The notion of functions coinciding *almost everywhere* was studied by Carathéodory; he called two functions equivalent if they differ on a subset of measure zero at most.

Carathéodory gave the interpretation of a definite integral in his theory. Let E be a measurable subset of $\mathbb{R}^n$ and consider a function $f : E \to \mathbb{R}_+$. If $\{(x_1,...,x_n,y) \in \mathbb{R}^{n+1} : P = (x_1,...,x_n) \in E, 0 \leq y < f(P)\}$ is measurable in $\mathbb{R}^{n+1}$ and of finite measure, the latter is denoted by

$$\int_E f(P) dw;$$

the function is said to be summable. Carathéodory established the measurability of any summable function. He then extended the definition of summability to real-valued functions; he verified the additivity and homogeneity of the definite integral.

Carathéodory established the fundamental properties of summable functions. The function f is summable if and only if it is mesurable and $|f|$ is summable. If f is a summable function over E,

$$\left| \int_E f(P) dw \right| \leq \int_E |f(P)| dw.$$

Let (f_n) be a monotonous sequence of summable functions converging to f over E; then f is summable if and only if the sequence is bounded. Moreover,

$$\int_E f(P) dw = \lim_{n \to \infty} \int_E f_n(P) dw.$$

Carathéodory also gave a version of Fatou's lemma.

In case a function is upper semicontinuous and bounded, the definite integral may be approximated by *Darboux sums*. An interpretation of

Riemann integration was given in Carathéodory's theory. The indefinite integral was defined by Carathéodory for functions f on $\mathbf{R}^n$ that are summable over every measurable subset e of finite measure. Following Lebesgue, he considered

$$F(e) = \int_e f(P)dw;$$

F is an additive function.

4 LATER INFLUENCES OF CARATHEODORY'S IDEAS

Nearly all later books on integration theory stressed the importance of Carathéodory's constructions and incorporated at least parts of them in their developments.

Bourbaki follows a functional procedure [4] [5]. Noticing that the prominent role played by continuous functions in this method may lead to think that the topological structure is essential, Bourbaki points out that the technique can be transcribed on an arbitrary set, but justifies the choice made :

"Toutefois, cette plus grande généralité est en partie illusoire : on a pu en effet montrer que toute 'mesure abstraite' est, en un certain sens, 'isomorphe' à une mesure définie (à partir des fonctions continues) sur un espace localement compact convenable; d'autre part, dans l'immense majorité des applications, il s'agit d'ensembles E munis d'une topologie intervenant naturellement dans la question; et dans les rares exemples qui ne rentrent pas dans cette catégorie, il est souvent utile d'introduire une topologie qui en facilite l'étude" ([4] p. 7).

In his study of product measures μ on a Cartesian product, Halmos [20] quotes Carathéodory's results in order to justify the following assertion : If T is the linear transformation defined on $\mathbf{R}^n$ by $y_i = \sum_{i=1}^{n} a_{ij}x_i + b_i$ (i=1,...,n), then for every subset E in $\mathbf{R}^n$, $\mu^*(T(E)) = |\det a_{ij}|\mu^*(E)$, $\mu_*(T(E)) = |\det a_{ij}|\mu_*(E)$.

The measure theory developed by Dunford and Schwartz [16] relies on fundamental theorems due to Carathéodory.

Weir provides details showing the essential equivalence of Daniell's integration theory and Carathéodory's integration theory. As a first major problem he considers the extention of a measure from a ring of subsets to a σ-ring or a σ-algebra containing the ring, i.e., the extension for countable unions. Having achieved this result by means of the Daniell construction, he estimates natural to ask whether or not any other method of extension would lead to the same measure; he concludes :

"It is comforting to know that the most frequently used general method of Carathéodory does in fact give the same measure as the Daniell construction" ([34] p. 113).

In his recent monograph Rao makes the following introductory observation :

"Generally the subject is approached from two points of view as evidenced from the standard works. Traditionaly one starts with measure, then defines the integral and develops the subject following Lebesgue's work. Alternatively one can introduce the integral as a positive linear functional on a vector space of functions and get a measure from it, following the method of Daniell's. Both approaches have their advantages, and eventually one needs to learn both methods. As the preponderance of existing texts indicates, the latter approach does not easily lead to a full appreciation of the distinctions between the (sigma) finite, localizable, and general measures, or their impact on the subject. On the other hand, too often the former approach appears to have little motivation, rendering the subject somewhat dry" ([30] p. vii).

Rao's text provides an account of the efficiency of the Carathéodory process.

Abstract harmonic analysis could develop after the introduction of a one-sided invariant measure on a locally compact group [28]. For Haar [18] compactness of a metrizable subset signified that every sequence of points in the subset admits a limit point. He considered a locally compact metrizable separable group G. Two subsets A and B were said to be congruent if there exists $a \in G$ such that $Aa = B$. For two nonvoid open compact subsets A and B, $h(\bar{B},\bar{A})$ is the minimal number of subsets congruent to $\overline{A}$ covering $\overline{B}$. Let E be a nonvoid open conpact subset and let (K_n) be a sequence of open balls of diameters $1/n$ with common center. For every nonvoid open compact subset B and every $n \in \mathbb{N}^*$, Haar defined

$$\ell_n(\overline{B}) = \frac{h(\overline{B},\overline{K}_n)}{h(\overline{E},\overline{K}_n)} \in \mathbb{Q}_+^*.$$

Haar called A null set if for every $\varepsilon > 0$ there exists an open compact subset U containing A such that

$$\limsup_{n\to\infty} \frac{h(\overline{U},\overline{K}_n)}{h(\overline{E},\overline{K}_n)} < \varepsilon.$$

He proved that for every nonvoid open compact subset B for which the boundary is a null set, $I(B) = \lim_{n\to\infty} \ell_n(\overline{B})$ exists; $I(B) > 0$. For nonvoid open compact congruent subsets B_1 and B_2, $I(B_1) = I(B_2)$. If B_1,B_2 are open compact disjoint, $I(B_1 \cup B_2) = I(B_1) + I(B_2)$. Adapting Lebesgue's definitions of inner and outer measures, Haar determined the measurable subsets corresponding to a right invariant measure. At the end of his article, Haar acknowledged comments made by von Neumann and Riesz after having studied his text; they observed that Carathéodory's measure theory would allow to bypass all the technical developments in Haar's paper. Nowadays, invariant measures continue to be studied intensively [27].

The measure approach, as emphasized by Carathéodory, constitutes a basic step in probabiliby theory. The latter became a major part of mathematics after Kolmogoroff [23], in 1933, had produced the axioms by which a probability is interpreted as a positive measure of total value 1.

In order to show further algebraization possibilities for the general integration theory, Carathéodory [9] considered the Riesz-Fischer theorem and ergodicity. The framework was constituted by a Boolean σ-algebra, the elements of which were called somas. Carathéodory's proof is an adaptation of Weyl's method showing that any sequence of functions converging in measure admits an almost everywhere convergent subsequence.

Let $(\mathcal{A},\Delta,\cap)$ be a Boolean σ-algebra. A *measure function* $\varphi : \mathcal{A} \to \mathbb{R}_+$ admits the following properties : $\varphi(\phi) = 0$, if $X \in \mathcal{A}$ is included in finite or countable symmetric differences of somas X_j, then

$$\varphi(X) \leq \sum_j \varphi(X_j).$$

The soma U is called measurable with respect to φ if

$$\varphi(X) = \varphi(X \cap U) + \varphi(X \Delta (X \cap U))$$

whenever $X \in \mathcal{A}$. Let $\mathcal{F}$ be the family of all functions that are measurable with respect to φ, and of the form

$$f = \sum_{i=1}^{n} \alpha_i \, 1_{A_i}$$

where $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$, $A_1,\ldots,A_n \in \mathcal{A}$, $n \in \mathbb{N}^*$. Carathéodory defined

$$\int_A f \, d\varphi = \sum_{i=1}^{n} \alpha_i \, \varphi(A_i \cap A),$$

$A \in \mathcal{A}$.

Choose $A \in \mathcal{A}$ such that $0 < \varphi(A) < +\infty$ and a sequence (g_n) in $\mathcal{F}$ such that $\int_A |g_n| d\varphi < +\infty$ and $\int_A |g_{m+n} - g_n| d\varphi < \frac{1}{4^n}$ whenever $m,n \in \mathbb{N}^*$.

If $n \in \mathbb{N}^*$, let T_n be the element of $\mathcal{A}$ such that $|g_{n+1} - g_n|(t) < \frac{1}{2^n}$ whenever $t \in T_n$; let $U_n = A \backslash T_n$ and $V_n = A \backslash (U_n \Delta U_{n+1} \Delta \ldots)$. Then

$$\frac{1}{2^n} \varphi(U_n) \leq \int_{U_n} |g_{n+1} - g_n| d\varphi \leq \frac{1}{4^n};$$

hence

$$\varphi(U_n) \leq \frac{1}{2^n}$$

and

$$\varphi(A \backslash V_n) \leq \sum_{m=0}^{\infty} \varphi(U_{m+n}) \leq \frac{1}{2^{n-1}}.$$

Let $V = V_1 \Delta V_2 \Delta \ldots$; $\varphi(A \backslash V) = 0$.

Consider $k \in \mathbb{N}^*$ such that for one $m \in \mathbb{N}^*$, $V_k \cap U_{m+k} = \phi$ holds. Then $V_k \cap U_{m+k} = 0$ for every $m \in \mathbb{N}^*$; $V_k \subset T_{m+k}$.
If $t \in V_k$,

$$|g_{k+m+1} - g_{k+m}|(t) \leq 1/2^{m+k};$$

also for $p \in \mathbb{N}^*$,

$$|g_{k+p} - g_k|(t) \leq |g_{k+1} - g_k|(t) + |g_{k+2} - g_{k+1}|(t) + \ldots + |g_{k+p} - g_{k+p-1}|(t) \leq \frac{1}{2^{k-1}},$$

$$g_k(t) - \frac{1}{2^{k-1}} \leq g_{k+p}(t) \leq g_k(t) + \frac{1}{2^k}.$$

Except possibly on a subset N_k of measure 0, for $t \in V_k$,

$$0 \leq \limsup_q g_q(t) - \liminf_q g_q(t) \leq \frac{1}{2^{k-2}};$$

as (V_n) is increasing,

$$0 \leq \limsup_q g_q(t) - \liminf_q g_q(t) \leq \frac{1}{2^{k+p}}.$$

Let $N = (A \setminus V) \Delta N_1 \Delta N_2 \Delta \ldots$. On $A \setminus N$, $g = \lim_{n \to \infty} g_n$ exists; one puts $g(t) = 0$ for $t \in N$.

By Lebesgue's dominated convergence theorem

$$\lim_{n \to \infty} \int_{V_k} |g - g_n| d\varphi = 0.$$

For all choices of $k, m, n \in \mathbb{N}^*$,

$$\int_{V_k} |g - g_n| d\varphi \leq \int_{V_k} |g - g_{m+n}| d\varphi + \int_{V_k} |g_{m+n} - g_n| d\varphi.$$

Thus

$$\int_A |g - g_n| d\varphi = \lim_{k \to \infty} \int_{V_k} |g - g_n| d\varphi \leq \int_A |g_{m+n} - g_n| d\varphi < \frac{1}{4^n};$$

$$\lim_{n \to \infty} \int_A |g - g_n| d\varphi = 0.$$

Let now (f_n) be a sequence in $\mathfrak{F}$ such that $\int_A |f_n| < +\infty$, $\int_A |f_{m+n}-f_n| d\varphi \leq \varepsilon_n$ $(m,n \in \mathbb{N}^*)$ and $\lim_{n\to\infty} \varepsilon_n = 0$. One may choose a subsequence (ε_{n_k}) of (ε_n) such that $\varepsilon_{n_k} \leq \frac{1}{4^k}$ whenever $k \in \mathbb{N}^*$ and define $g_k = f_{n+k}$. By the preceding general statement, there exists $g \in \mathfrak{F}$ such that $\lim_{n\to\infty} \int_A |g-g_n| d\varphi = 0$; so also

$$\lim_{n\to\infty} \int_A |g-f_n| d\varphi = 0.$$

It suffices then to make use of Schwarz's inequality to obtain the theorem : Let (f_n) be a sequence in $\mathfrak{F}$ such that $\int_A f_n^2 \, d\varphi < +\infty$, $\int_A (f_{m+n}-f_n)^2 \, d\varphi \leq \delta_n^2$ $(m,n \in \mathbb{N}^*)$ and $\lim_{n\to\infty} \delta_n = 0$. Then there exists $g \in \mathfrak{F}$ such that $\int_A g^2 \, d\varphi < +\infty$ and $\lim_{n\to\infty} \int_A (f_n-g)^2 \, d\varphi = 0$.

Carathéodory gave the following description of ergodicity :

"Die Ergodentheorie ist aus der statistischen Mechanik entsprungen, als man aus dem statistischen Verhalten einer Schar von Bahnkurven über das asymptotische Verhalten der einzelnen Bahnkurven Schlüsse ziehen wollte. Es hat sich aber mehr und mehr gezeigt, dass diese Sätze, welche man in dieser Hinsicht aufgestellt hatte, für die ganze Integralrechnung von grundlegender Bedeutung sind" ([9] p. 368-369).

Ergodicity concerns a set S equipped with a finite measure μ. One considers a transformation T of S associating a measurable subset to any measurable subset and for which T^{-1} has the same property. Von Neumann [26] proved that given $f \in L^2(S,\mu)$ there exists $g \in L^2(S,\mu)$ such that

$$\lim_{N\to\infty} \int_S \left| g(P) - \frac{1}{N+1} \sum_{n=0}^{N} f(T^{(n)}P) \right| d\mu\,(P) = 0.$$

Birkhoff [1] showed that if f is a measurable function on S, for μ-almost every point P,

$$g(P) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{i=0}^{N} f(T^{(i)}P)$$

exists; g is a T-invariant measurable function.

Carathéodory obtained the following ergodic theorem : If $f \in L^1(M,\varphi)$, there exists a *soma* N, possibly empty, such that $\varphi(N) = 0$ and $\lim_{n\to\infty} \sigma_n f$ exists on $M \backslash N$ where

$$\sigma_n f = \frac{f + Tf + \ldots + T^{(n-1)}f}{n}.$$

Carathéodory interpreted this result as a direct generalization of Birkhoff's theorem, but wondered whether it may still be carried over to more general situations.

In his later note [10] Carathéodory used a method due to Hopf in order to further formalize ergodicity in his measure theory over Boolean algebras.

5 CARATHEODORY'S ALGEBRAIZATION PROCEDURES

As soon as 1938 Carathéodory had enhanced a further algebraization of his measure theory [8]. The editors of [11] claim that from the preface of [12] it is evident that two more volumes had been planned :

"Nach Vereitlung dieses Planes durch den Krieg und dessen Auswirkungen entschloss sich der Verfasser, aus dem für diese beiden Bände vorgesehenen Material durch geeignete Sichtung und weitgehende Umarbeitung ein selbstständiges, in sich abgeschlossenes Buch zu formen" ([11] p. 6).

In the preface to volume II of the planned books, Carathéodory characterized the generalization of Lebesgue's integration theory to abstract spaces, over a period of fifty years, as the identification with the theory of completely additive set functions. He explained his reluctance to a simple adaptation of Lebesgue's theory :

"Bei der gewöhnlichen Lebesgueschen Theorie [ist] der 'Inhalt' nur dann eine totaladditive Mengenfunktion, wenn man von allen Punktmengen des betrachtenden Euklidischen Raumes absieht, die man nicht als Summe einer Borelschen Menge und einer Nullmenge darstellen kann. Ich habe mich deshalb mit der oben erwähnten Behandlung des Integrals nie recht befreunden können, umso mehr als bei dieser Behandlung mit einer Tradition gebrochen wird, welche seit mehr als 2000 Jahren besteht und zu den schönsten Errungenschaften der Analysis geführt hat" ([13] p. 290).

From among the papers left by Carathéodory the editors of [11] quote the following lines summarizing his attitude towards measure theory :

"Das einfachste Beispiel einer [Booleschen Algebra] erhält man, wenn man die Operationen der Vereinigung, des Durchschnitts und der Differenz (oder den Übergang von einer Menge zu ihrer Komplementarmenge) auf Mengen anwendet.

Daraus erklärt sich, dass die Theorie des Masses, die ja auf Mengen von beliebigen Elementen aufgebaut werden kann, auch für Ringe von Elementen einer Booleschen Algebra ihre Bedeutung nicht zu verlieren braucht.

Vor etwa zehn Jahren habe ich bemerkt, dass man auch das Analogon einer gewöhnlichen Punktfunktion auf Booleschen Ringen bilden kann, wodurch auch die Algebraisierung des Integrals ermöglicht wird.

Die Durchführung dieses Programms hat nicht nur theoretisches Interesse. Die Sätze und Beweismethoden, die man, bei näherer Einsicht in die neuen Verhältnisse, aufzustellen veranlasst wird, sind nicht derart, dass sie in einem Raritätenkabinett ihren Platz finden sollten. Decken sie doch zwischen Resultaten, die man schon längst auf dem gewöhnlichen Wege fortschreitend erforscht hat, Zusammenhänge auf, welche sont unbemerkt geblieben wären. Sie führen ausserdem zu einem organischen, sehr einfachen und einheitlichen Aufbau der Theorie.

Freilich könnte man auf dem klassischen Wege, vom Borel-Lebesgueschen Mass ausgehend, diese Erfahrungen benutzen und die Eigenschaften des Masses und des Integrals auf eine Weise ableiten, die von der in diesem Buche gebotenen Darstellung prinzipiell nicht verschieden ist. Ein solches Verfahren wäre aber in mehr als einer Hinsicht unnatürlich, und es scheint mir deshalb vorteilhafter, die Theorie in ihrer ganzen Allgemeinheit zu entwickeln" ([11] p. 5).

This formal treatment involves measurability, measure functions, integration theory; it includes all standard topics such as Egoroff's theorem, convergence in the mean, Jordan's decomposition. Some concepts, introduced earlier by Carathéodory, such as regularity, become less relevant.

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Jean-Paul Pier
Séminaire de mathématique,
Centre universitaire de Luxembourg,
162A, Avenue de la Faïencerie,
L-1511 Luxembourg

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THE ISOPERIMETRIC INEQUALITY AND EIGENVALUES OF THE LAPLACIAN

Themistocles M. Rassias

This paper gives an account of classical proofs that have been given to the isoperimetric inequality as well as of a few properties of the eigenvalues of the Laplacian with their meaning and some of their applications to problems of Mathematical Analysis.

1. Introduction

In the following pages, I will try to present a fundamental study towards a better understanding of the classical isoperimetric problem. The real questions are: *What shape must a closed curve C in the plane have if, with a given length L it should enclose the greatest possible area? Or: When has the curve C enclosing a given area A the least possible length?* Both questions turn out to be equivalent. The answer is that the curve C has to be a circle. This is the so-called classical *isoperimetric problem*. Carathéodory had discovered that Calculus of Variations began not with the quarrel of the Bernoulli brothers in June 1696 and not with the beautiful "Traité de la lumière" of Huygens (printed in 1690 but written twelve years before) but it was Zenodoros, who lived sometime between 200 B. C. and 100 B. C. The isoperimetric problem was first approached by Zenodoros. As Carathéodory writes: "The proofs he gives are excellent and even superior in elegance to those we find ... in the geometry of Legendre". It should prove to be very useful if everybody reads this masterpiece of Carathéodory "The beginning of research in the calculus of variations" [(1937), now in Carathéodory's *Gesammelte Mathematische Schriften*, Vol. II]. This problem has been approached since the time of Zenodoros, also by the Bernoullis

(1697), Euler (1744), and Lagrange (1762), who treated it as an example of the calculus of variations. Their investigations show that if there is a curve C of given length L such that the enclosed area A has the maximum value, then C is a circle. *However they evade the question as to the actual existence of the maximum curve* (*Weierstrass*). By establishing sufficient conditions for the existence of actual maximum or minimum solutions of a large class of problems of the calculus of variations, Weierstrass has settled this question also for the isoperimetric problem. Steiner [32] proposed several ingenius ways for proving that the circle is the only curve of given length which encloses maximal area. Later the problem has been solved by various methods. A. Hurwitz (1902) has applied the so-called *completeness theorem of the theory of Fourier series*; H. Minkowski obtained a general inequality in the theory of convex domains implying as a special case the solution of the isoperimetric problem. These methods with an analysis will be followed here (see [31], and also [7], [8], [27]). Carathéodory and Study [9] proved the existence of such an extremal curve in 1909. Since then this subject was taken up in a series of papers using different methods which led to numerous generalizations and extensions of the isoperimetric problem.

2. Hurwitz's Proof

An analytic expression of the isoperimetric inequality can be formulated as follows: If C is a circle of radius r one has

$$L^2 = (2r\pi)^2 = 4r^2\pi^2 = 4\pi A\,, \tag{2.1}$$

and the statement is that the area A enclosed by a curve C of length L is smaller than that of the circle:

$$L^2 \geq 4\pi A\,. \tag{2.2}$$

The equality sign holds if and only if C is a circle. The nonnegative difference $\frac{L^2}{4\pi} - A$ is called the *isoperimetric deficit* of the curve C. In a parametric representation the curve C may be expressed by $x = f(t), y = g(t)$ where the parameter t is such that

$$t = \frac{2\pi}{L}s\,, \tag{2.3}$$

where s is the length of arc measured on C from a fixed point on C. Thus t varies continuously from 0 to 2π as s varies from 0 to L. All values of s beyond the limits of this interval may be taken into consideration; for the point (x, y) to remain on C it is then sufficient to assume the two functions $f(t), g(t)$ to be periodic with period 2π. Suppose that the derived functions $f'(t), g'(t)$ are sectionally continuous for all real values of t. With such restrictions the isoperimetric inequality (2.2) is a general statement concerning such *arbitrary functions* that can be expressed by their Fourier series:

$$\left.\begin{aligned} f(t) &= \tfrac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \\ g(t) &= \tfrac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos nt + \beta_n \sin nt) \end{aligned}\right\} \tag{2.4}$$

where a_n, b_n are the Fourier coefficients of $f(t)$, and α_n, β_n those of $g(t)$:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt dt\,, \alpha_n = \frac{1}{\pi}\int_0^{2\pi} g(t)\cos nt dt\,,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt dt\,, \beta_n = \frac{1}{\pi}\int_0^{2\pi} g(t)\sin nt dt\,.$$

Using the Hurwitz equivalence notation we may write

$$\left.\begin{aligned} f'(t) &\sim \sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt) \\ g'(t) &\sim \sum_{n=1}^{\infty}(n\beta_n \cos nt - n\alpha_n \sin nt) \end{aligned}\right\} \tag{2.5}$$

The proof of Hurwitz [16] of the isoperimetric inequality uses the *completeness relation* which for the function $f(t)$, also if it is not actually represented by its Fourier series, states

$$\frac{1}{\pi}\int_0^{2\pi} (f(t))^2 dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,, \tag{2.6}$$

and upon *Parseval's identity*

$$\frac{1}{\pi}\int_0^{2\pi} f(t)g(t)dt = \frac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n)\,, \tag{2.7}$$

which follows from (2.6) by substituting there the function $\lambda f(t) + \mu g(t)$ instead of $f(t)$ and comparing the coefficients of $\lambda\mu$ on both sides of this equality.

The proof of Hurwitz goes in the following way. From (2.3) it follows that

$$s = \frac{L}{2\pi}t \text{, which implies } \quad \dot{s} = \frac{ds}{dt} = \frac{L}{2\pi}$$

and thus by (2.6) one can write

$$\frac{1}{\pi}\int_0^{2\pi} (\dot{s})^2 dt = \frac{L^2}{2\pi^2} = \frac{1}{\pi}\int_0^{2\pi} [(f'(t))^2 + (g'(t))^2]dt$$
$$= \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2 + \alpha_n^2 + \beta_n^2).$$

On the other hand, the area A of the domain bounded by C can be written as the integral

$$A = \int_C x dy = \int_0^{2\pi} f(t)g'(t)dt$$

and therefore (2.7), with $g'(t)$ instead of $g(t)$, gives

$$A = \pi \sum_{n=1}^{\infty} n(a_n\beta_n - b_n\alpha_n).$$

Therefore the isoperimetric deficit of the curve C is equal to

$$\frac{L^2}{4\pi} - A = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2 + \alpha_n^2 + \beta_n^2) - \pi\sum_{n=1}^{\infty} n(a_n\beta_n - b_n\alpha_n)$$
$$= \frac{\pi}{2}\left[\sum_{n=1}^{\infty}((na_n - \beta_n)^2 + (n\alpha_n + b_n)^2) + \sum_{n=1}^{\infty}(n^2-1)(b_n^2 + \beta_n^2)\right],$$

which is expressed as a sum of two convergent series with nonnegative terms. Necessary and sufficient condition for $\frac{L^2}{4\pi} - A$ to be equal to zero is that all terms to be equal to zero, i.e.,

$$(n^2-1)b_n^2 = 0, (n^2-1)\beta_n^2 = 0 \text{ which imply } b_n = \beta_n = 0 \text{ if } n > 1$$

and thus from

$$na_n - \beta_n = 0 \text{ it follows that } a_n = 0 \text{ if } n > 1 \text{ and } \beta_1 = a_1,$$
$$n\alpha_n + b_n = 0 \text{ it also follows that } \alpha_n = 0 \text{ if } n > 1 \text{ and } b_1 = -\alpha_1.$$

Therefore the isoperimetric deficit is never negative and equal to zero if and only if C is given by the parametric representation

$$x = f(t) = \frac{1}{2}a_0 + a_1 \cos t - \alpha_1 \sin t$$

and

$$y = g(t) = \frac{1}{2}\alpha_0 + \alpha_1 \cos t + a_1 \sin t\,.$$

Then

$$(x - \frac{1}{2}a_0)^2 + (y - \frac{1}{2}\alpha_0)^2 = a_1^2 + \alpha_1^2\,,$$

which means that C is the circle.

QED

Remark. The above is one of the proofs given by Hurwitz [16] in which no assumption is made as to the convexity of the curve C.

3. Minkowski's Approach

Let C be a convex (simply) closed curve (it is also called an *oval*) and C_n be an n-sided convex polygon inscribed to the oval C. Suppose that $s_1, \ldots, s_n$ are its sides (and their lengths), and h_ν the distance of the side s_ν from a fixed point 0 inside the polygon. Then the area of the polygon is given by the sum

$$A_n = \frac{1}{2}\sum_{\nu=1}^{n} h_\nu s_\nu$$

and as $n \to \infty$ while the lengths of all sides of C_n tend to zero, the sequence A_n tends to the area A of the domain bounded by the curve C, and according to the definition of a curvilinear integral the limit is given by

$$A = \frac{1}{2}\int_C \bar{h}(s)ds$$

if $h = \bar{h}(s)$ denotes the distance of the tangent to C at the point s, from the fixed point O inside of C. This function h is called the *function of support* of the convex curve C. We shall note that after having fixed the point O and a starting point for measuring the arc s on C, this function not only is defined by the curve C, but also defines C uniquely. Because of the convexity of C we may choose the polar angle θ at the point O of the normal to C in the point s as independent variable varying from 0 to 2π. Therefore we have $h = \bar{h}(s) = h(\theta)$ and again we may assume that the function $h(\theta)$ is continued beyond the interval $0 \leq \theta \leq 2\pi$ as a function of period 2π. Suppose that the derived function $h'(\theta)$ is continuous and $h''(\theta)$ sectionally continuous. To prove that the curve C is determined by its function of support we introduce rectangular Cartesian co-ordinates with O as origin. We consider the curve C as the envelope of the family of straight lines, vertical to the ray through O which forms the angle θ with the x-axis, having the distance $h = h(\theta)$ from O. The equation of the general line of the family, in running co-ordinates ξ, η, therefore is

$$\xi \cos\theta + \eta \sin\theta = h(\theta) . \tag{3.1}$$

If we differentiate this equation with respect to θ we obtain

$$-\xi \sin\theta + \eta \cos\theta = h'(\theta) . \tag{3.2}$$

From (3.1) and (3.2) we obtain the parametric representation of C to be given by

$$x = h\cos\theta - h'\sin\theta \,,\; y = h\sin\theta + h'\cos\theta \,, \tag{3.3}$$

which is uniquely defined by the function $h(\theta)$.

From (3.3) we derive the radius of curvature, $r = r(\theta)$, in terms of the function of support to be equal to

$$r = \frac{ds}{d\theta} = \sqrt{x'^2 + y'^2} = h + h'' . \tag{3.4}$$

Then

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} h(\theta)r(\theta)d\theta \\ &= \frac{1}{2}\int_0^{2\pi} h(h + h'')d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (h^2 - h'^2)d\theta . \end{aligned} \tag{3.5}$$

Now, consider the curve $C^{(\delta)}$ which is parallel to C in the distance $\delta > 0$. Then its function of support is $h(\theta) + \delta$, and its radius of curvature is $r(\theta) + \delta$. Therefore its area is given by

$$A(\delta) = \frac{1}{2}\int_0^{2\pi} (h + \delta)(r + \delta)d\theta = A + B\delta + \pi\delta^2 \tag{3.6}$$

where

$$B = \frac{1}{2}\int_0^{2\pi}(h+r)d\theta = \lim_{\delta\to 0}\frac{A(\delta)-A}{\delta}.$$

The difference $A(\delta) - A$ measures the ring area of width δ between C and $C^{(\delta)}$, which for small δ is approximately equal to $L\delta$. It is clear that $B = L$. Also

$$\int_0^{2\pi} r(\theta)d\theta = \int_C ds = L\,,$$

hence

$$L = \int_0^{2\pi} h d\theta\,. \tag{3.7}$$

We now represent the function

$$\frac{h(\theta)}{L} - \frac{1}{2\pi} = f(\theta) \tag{3.8}$$

by its Fourier series

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\,.$$

From (3.7) we obtain

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{h(\theta)}{L} - \frac{1}{2\pi}\right)d\theta = 0\,.$$

Thus

$$\frac{1}{\pi}\int_0^{2\pi} f(\theta)^2 d\theta = \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,,$$
$$\frac{1}{\pi}\int_0^{2\pi} f'(\theta)^2 d\theta = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)\,.$$

Thus

$$\frac{1}{2}\int_0^{2\pi}(f(\theta)^2 - f'(\theta)^2)d\theta \le 0 \tag{3.9}$$

with the sign of equality valid if and only if

$$a_n = 0, b_n = 0 \text{ for all } n > 1\,. \tag{3.10}$$

From (3.8) and (3.9) we have

$$\frac{1}{2}\int_0^{2\pi}\left(\frac{h(\theta)^2}{L^2}-\frac{h'(\theta)^2}{L^2}-\frac{1}{\pi}\frac{h(\theta)}{L}+\frac{1}{4\pi^2}\right)d\theta \le 0\,,$$

which because of (3.5) and (3.7) means that

$$\frac{A}{L^2}-\frac{1}{2\pi}+\frac{1}{4\pi}\le 0\,, \tag{3.11}$$

i.e., the isoperimetric inequality (2.2). The sign of equality in (3.11) holds if (3.10) is valid and therefore

$$h(\theta)=\frac{L}{2\pi}+L(a_1\cos\theta+b_1\sin\theta)\,. \tag{3.12}$$

From (3.3) and (3.12) we get

$$x=La_1+\frac{L}{2\pi}\cos\theta\,,\ y=Lb_1+\frac{L}{2\pi}\sin\theta$$

which represent a circle.

QED

Minkowski's Generalization of the Isoperimetric Inequality

Let C_1 and C_2 be two ovals, L_1, L_2 their lengths, A_1, A_2 the areas of the domains bounded by C_1, C_2 respectively, and $h_1(\theta), h_2(\theta)$ their functions of support taken with respect to a point O inside both ovals. Then, the function

$$f(\theta)=\frac{h_1(\theta)}{L_1}-\frac{h_2(\theta)}{L_2}$$

again satisfies the property

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta=0$$

and therefore the inequality (3.9) is also valid.
Then

$$A_1\frac{1}{L_1^2}-2A_{12}\frac{1}{L_1}\frac{1}{L_2}+A_2\frac{1}{L_2^2}\le 0 \tag{3.13}$$

where

$$A_{12} = \frac{1}{2}\int_0^{2\pi} (h_1 h_2 - h_1' h_2')d\theta \tag{3.14}$$

is the Minkowskian "mixed area" of the two ovals C_1, C_2. The inequality (3.13) expresses that the quadratic form

$$A(\lambda_1, \lambda_2) = A_1\lambda_1^2 - 2A_{12}\lambda_1\lambda_2 + A_2\lambda_2^2$$

representing the area enclosed by the oval with the function of support $\lambda_1 h_1 + \lambda_2 h_2$ $(\lambda_1, \lambda_2 \geq 0)$, has in general a negative value for $\lambda_1 = \frac{1}{L_1}, \lambda_2 = \frac{1}{L_2}$, while for $\lambda_1 = 1, \lambda_2 = 0$ it is positive. Therefore $A(\lambda_1, \lambda_2)$ is not a definite form and therefore its discriminant

$$A_1A_2 - A_{12}^2 \leq 0\,, \tag{3.15}$$

with the sign of equality holding if and only if

$$\frac{h_1(\theta)}{L_1} - \frac{h_2(\theta)}{L_2} = a_1\cos\theta + b_1\sin\theta\,.$$

The inequality (3.15) contains the isoperimetric inequality as a special case.

Remarks. For various extensions of the isoperimetric inequality to three and more dimensions a lot of new research has been undertaken by several mathematicians and applied scientists. The geometers developed various types of symmetrizations (cf. [7], [13]), whereas the analysts applied techniques of the calculus of variations (cf. [4], [20], [25]). Very beautiful results have been obtained on several different generalizations of the isoperimetric problem in Euclidean and non-Euclidean spaces (W. Blaschke, L. A. Santaló, E. Schmidt, and others cf. the references in [4], [20]). It is clear that the spheres in higher dimensions should be characterized by a similar extremal property. In fact H. Schwarz [30] proved that *among all domains of given volume the sphere has the smallest surface area.* H. Liebmann in 1900 [18] proved that if a compact, strictly convex surface in R^3 has constant mean curvature, then it must be a sphere. H. Hopf in 1951 [15] proved a much stronger version of Liebmann's theorem in which no convexity assumptions were needed, and in fact the surface could even be allowed to have self-intersections. The only hypothesis was that the surface be defined by a regular map of a 2-sphere into R^3. A. D. Aleksandrov in 1958 [2] using an ingenious geometric argument generalized Liebmann's theorem for any surface of constant mean curvature with no assumptions on its topological type, to be a sphere. However, the surface was not allowed to have self-intersections. Aleksandrov in 1962 [3] generalized his result including the case when certain surfaces admit self-intersections.

4. Eigenvalues of the Laplacian

Let D be a simply-connected domain in $R^m, m > 1$, with a smooth boundary ∂D. Let u be a solution of the equation

$$\Delta u + \lambda u = 0 \quad \text{in} \quad D, \tag{4.1}$$

subject to the homogeneous boundary condition

$$u = 0 \quad \text{on} \quad \partial D. \tag{4.2}$$

For $m = 2$, (4.1) is also known as the *Helmholtz equation*. Someone reduces to it from separating the time variable out of the wave equation. Equations (4.1), (4.2) may then represent the vibration of a *fixed membrane*, with the eigenvalue $\lambda = k^2$, where k is proportional to a *principal frequency of vibration*. F. Pockels [24] first proved that (4.1) and (4.2) has a spectrum of infinitely many positive eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots \tag{4.3}$$

with no finite accumulation point. We can normalize the corresponding eigenfunctions $u_1, u_2, u_3, \ldots$ so they form a complete *orthonormal set* for $L_2(D)$, i.e.,

$$\int_D u_i u_j \, dx dy = \delta_{ij} \tag{4.4}$$

where δ_{ij} is Kronecker's delta and $i, j = 1, 2, 3, \ldots$. It follows that the eigenvalues satisfy the *mimimax principle*

$$\lambda_n = \min \max \frac{\int_D [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2] dx dy}{\int_D u^2 dx dy}, \tag{4.5}$$

where the maximum is over all linear combinations of the form

$$u = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \ldots + \alpha_n \phi_n, \tag{4.6}$$

and the minimum is over all choices of n linearly independent continuous and piece-wise-differentiable functions $\phi_1, \phi_2, \dots, \phi_n$, vanishing on C. The ratio of quadratic forms on the right side of (4.5) is called the *Rayleigh quotient* (cf. [27], [28]). In 1877 Lord Rayleigh [29] conjectured that *among all domains of a given area the circle has the lowest principal frequency.* This statement can again be expressed as an inequality relating the area A of D and λ_1, i.e.,

$$\lambda_1 \geq \frac{\pi j_0^2}{A} \quad (j_0 = 2.4048\dots, \text{first zero of the Bessel function } J_0)\,. \tag{4.7}$$

Equality holds only for the circle (cf. [33]). Lord Rayleigh was led to this conjecture after having computed λ_1 for a number of special cases like the square, the equilateral triangle, the semicircle, ... He also applied a very special perturbation method to approximate the value of λ_1 for a nearly circular domain (cf. [4]). C. Faber [12] and E. Krahn [17] independently, proved (4.7) using a special system of curvilinear coordinates.

Consider now a membrane with inhomogeneous mass density p,

$$\left.\begin{aligned} \Delta u + \lambda p u = 0 \quad & \text{in } D \\ u = 0 \quad & \text{on } \partial D \end{aligned}\right\} \tag{4.8}$$

L. Nehari [19] extended the Rayleigh-Faber-Krahn inequality for mass densities satisfying $\Delta \log p \geq 0$. He proved that the following inequality holds

$$\lambda_1 \geq \frac{\pi j_0^2}{\int_D p dx} \tag{4.9}$$

where the equality holds for example for the circle with constant p. The extremal case is not uniquely determined. Let $\Delta_S = \Delta/p$. Then Δ_S can be interpreted as the *Laplace-Beltrami operator* on a surface with Riemannian metric $d\sigma^2 = pds^2$. Then (4.8) can be reformulated as

$$\left.\begin{aligned} \Delta_S u + \lambda u = 0 \quad & \text{in } D \subseteq S \\ u = 0 \quad & \text{on } \partial D \end{aligned}\right\} \tag{4.10}$$

Nehari's condition means that the Gaussian curvature of S is non-positive. J. Peetre [23] proved that the first eigenvalue of Δ_S wih Dirichlet boundary values satisfies

$$\lambda_1 \geq \frac{j_0^2}{2A_\sigma(D)} \left(2\pi - \int_D K^+ dx\right). \tag{4.11}$$

This inequality includes Nehari's result. J. Peetre [22] derived an inequality of the type

$$\lambda_1 \geq \frac{\pi j_0^2(1-\varepsilon)}{A_\sigma} \tag{4.12}$$

for every general surfaces. J. Hersch [14] proved that for convex domains with inradius ρ_0, the first eigenvalue of the homogeneous membrane satisfies

$$\lambda_1 \geq \left(\frac{\pi}{2\rho_0}\right)^2 \tag{4.13}$$

where the right-hand side is the limiting value for long thin rectangles.

In the paper of R. Osserman [20] one can find other very interesting results relating λ_1 and ρ_0. There are also several variational characterizations of the eigenvalues and some very elegant upper bounds. For this and related results one can see the very interesting book of C. Bandle [4].

Little is known for the *free membrane* described by the eigenvalue problem

$$\left.\begin{aligned} \Delta u + \nu u &= 0 \quad \text{in } D \subset R^2\,, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \end{aligned}\right\} \tag{4.14}$$

($\frac{\partial}{\partial n}$ denotes the outer normal derivative). There exists a countable number of eigenvalues

$$0 = \nu_1 < \nu_2 \leq \nu_3 \leq \ldots\,.$$

G. Szegö [33] proved the following beautiful extremal property of a circle. *Among all domains of a given area the circle yields the highest second eigenvalue* ν_2. This property can be expressed in the following inequality form

$$\nu_2 \leq \frac{\pi p_1^2}{A} \quad (p_1 = 1.841\ldots \text{ zero of the Bessel function} J_1) \tag{4.15}$$

L. Nehari [19] has also considered membranes with *mixed boundary conditions*

$$\left.\begin{aligned} \Delta u + \mu u &= 0 \quad \text{in } D\,, \\ u &= 0 \quad \text{on } \Gamma\,, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on } \gamma \end{aligned}\right\} \tag{4.16}$$

where $\Gamma \cup \gamma = \partial D$ and $\Gamma \cap \gamma = \emptyset$. Nehari proved the following inequality for the lowest eigenvalue μ_1. *If* γ *is a concave arc, then*

$$\mu_1 \geq \frac{\pi j_0^2}{2A}\,.$$

Equality holds for semi-circles with Γ *as circular arc and* γ *as the straight segment.*

This inequality has been generalized by C. Bandle [4] in several ways.

It is a standard problem used to introduce variational properties of eigenvalues in mechanics to determine the equilibrium shape of a soap film suspended between two parallel coaxial circular rings. The solution to the problem relates the radius of the film r to the displacement z along the axis of symmetry by the equation of the catenary

$$r = a \cos h \frac{z-b}{a} .$$

The constants a and b are to be determined by requiring that r be equal to the fixed radii of the rings for $z = 0$ and h, where h is the separation of the rings. If the rings are of equal radius r_0, the surface is symmetrical about $z = \frac{h}{2}$, b is equal to $\frac{h}{2}$, and a, the minimum radius of the film, is to be found by solving the equation

$$r_0 = a \cosh \frac{h}{2a} .$$

There are two solutions for $\frac{h}{2r_0} < 0.66274\ldots$, only one of which is stable, and no solutions at all for $\frac{h}{2r_0} > 0.66274\ldots$. In the second case, the tubular configuration of the soap film is unstable. From the experimental point of view this can be demonstrated as follows ([11], [27]): We start with a stable tubular film with $\frac{h}{2r_0} < 0.66274\ldots$, and gradually increasing the separation between the rings until $\frac{h}{2r_0}$ approaches and then exceeds the critical value. For $\frac{h}{2r_0}$ greater than the critical value, the film collapses in the center and splits into two planar films, one on each ring. As $\frac{h}{2r_0}$ approaches the critical value, any perturbation results in a characteristic low-frequency oscillation of the film.

A mathematical analysis of this equilibrium problem using eigenfunction methods can be given to prove that the dynamical stability of the film is determined by the sign of the lowest eigenvalue λ_1 of an associated Sturm-Liouville problem, with the film stable for $\lambda_1 > 0$ and unstable for $\lambda_1 < 0$. For this analysis as well as for a number of related results one can follow [11], [27].

Applications of the Isoperimetric Inequality

We shall present a few illustrations of ways that isoperimetric inequalities have been applied to some specific problems in analysis and geometry.

I. On the unit disk, $|z| < 1$, one has the *hyperbolic metric*

$$ds^2 = \frac{2}{1-|z|^2}|dz|^2 \,. \tag{5.1}$$

This metric has constant Gauss curvature $K \equiv -1$. Thus the unit disc becomes a model for the hyperbolic plane. Then one can prove ([20]) that the isoperimetric inequality becomes

$$L^2 \geq 4\pi A + A^2 \tag{5.2}$$

for simply-connected domains, and hence, for all such domains, it follows that

$$L > A \,. \tag{5.3}$$

This property is very essential to characterize hyperbolic Riemann surfaces. On an arbitrary Riemann surface one may consider *conformal metrics*, which are Riemannian metrics of the form

$$ds = \rho(z)|dz|$$

with respect to any local conformal parameter z.

Theorem ([20]). A simply-connected Riemann surface S is of hyperbolic type if and only if there exists a conformal metric on S such that (5.3) holds for every simply-connected domain on S.

Remark. This result is a very special case of a theorem of L. Ahlfors [1] describing relations between L and A that are compatible with the existence of a quasi-conformal map of a surface onto the entire plane.

II. The following is a theorem on conformal mapping of doubly-connected domains due to T. Carleman [10]. The proof of Carleman was based upon Laurent expansions. However G. Szegö [34] gave an elegant proof based on the isoperimetric inequality.

Theorem. Consider the family of all doubly-connected plane domains bounded by an outer curve C_1 and an inner curve C_0. For each domain

D, let A_i be the area bounded by $C_i, i = 0, 1$. Then among all domains conformally equivalent to a given one, the minimum of A_1/A_0 is attained by a circular annulus.

Szegö's argument goes in the following way (cf. [20]). Let $r_0 < |z| < r_1$, be a given annulus, and let D be its image under a conformal map $f(z)$. Denote by $L(r)$ the length of the image of $|z| = r$, and $A(r)$ the area enclosed. It follows that

$$4\pi A(r) \le L(r)^2 = \left(\int_0^{2\pi} |f'(re^{i\theta})|\, r d\theta\right)^2$$
$$\le \int_0^{2\pi} |f'(re^{i\theta})|^2\, r d\theta \,.\int_0^{2\pi} r d\theta = 2\pi r A'(r)\,.$$

Thus $\frac{2}{r} \le \frac{A'(r)}{A(r)}$ for $r_0 < r < r_1$, and integrating from r_0 to r_1, one obtains

$$\log \frac{r_1^2}{r_0^2} = 2 \log \frac{r_1}{r_0} \le \log \frac{A(r_1)}{A(r_0)} = \log \frac{A_1}{A_0}\,.$$

Therefore

$$\frac{\pi r_1^2}{\pi r_0^2} \le \frac{A_1}{A_0}\,.$$

QED

III. *Maximal conformal radius. Pólya and Schiffer's inequality.*

Consider D to be a simply-connected domain in the complex z-plane, $z_0 \in D$ an arbitrary point and

$$f(z) = (z - z_0) + a_2(z - z_0)^2 + \ldots$$

a complex one-to-one function mapping D conformally onto the circle $\{w : |w| < R_{z_0}\}$. It is a consequence of the Riemann mapping theorem that such a function exists and that R_{z_0} is uniquely defined. R_{z_0} is defined to be the *conformal radius* of D with respect to z_0 and

$$\dot{R} := \sup\{R_{z_0} : z_0 \in D\}$$

is called the *maximal conformal radius* of D. Consider in D a Riemannian metric $d\sigma^2 = p ds^2$ of bounded Gaussian curvature K_0 and let A_σ be the total area of D with respect to this metric. Then (cf. [4], [5])

$$R_z^2 \le \frac{4A_\sigma}{p(z)(4\pi - K_0 A_\sigma)}\,, \text{ if } K_0 A_\sigma < 4\pi\,.$$

Pólya and Schiffer's inequality ([26]) connects the maximal conformal radius with the sum of the reciprocal first n eigenvalues. It is stated as follows: Let $\lambda_1, \ldots, \lambda_n$ be the first n eigenvalues of the fixed membrane equation in a simply connected domain D and let $\lambda_{1c}, \ldots, \lambda_{nc}$ be the corresponding eigenvalues of the circle of radius 1. Then

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_n} \geq \dot{R}^2 \left(\frac{1}{\lambda_{1c}} + \frac{1}{\lambda_{2c}} + \ldots + \frac{1}{\lambda_{nc}} \right), \tag{5.4}$$

where $\dot{R}$ denotes maximal conformal radius of D. C. Bandle [4] obtained an elegant extension to non-homogeneous membranes in the following form.

Theorem. Consider D to be a simply connected domain, $z_0 \in D$ an arbitrary point and p a mass density satisfying $\Delta \log p + 2K_0 p \geq 0$ and $K_0 \int_D p dx \leq 2\pi$. Set

$$\beta := p(z_0) R_{z_0}^2 \text{ , and } e^{u_c(r,\beta;K_0)} := \frac{\beta}{(1 + \frac{\beta K_0 r^2}{4})^2} .$$

Observe that β is a conformal invariant. Let λ_{ic} be the ith eigenvalue of

$$\left.\begin{aligned} \Delta \phi_c + \lambda_c e^{u_c(r,\beta;K_0)} \phi_c &= 0 \quad \text{in } \{x : |x| < 1 , \\ \phi_c &= 0 \quad \text{on } \{x : |x| = 1\} . \end{aligned}\right\} \tag{5.5}$$

Then

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_n} \geq \frac{1}{\lambda_{1c}} + \frac{1}{\lambda_{2c}} + \ldots + \frac{1}{\lambda_{nc}} .$$

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Themistocles M. Rassias
Department of Mathematics
University of La Verne
P O Box 51105, Kifissia
Athens, Greece 14510

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On the Minimum of $Re[f(z)/z]$ for Univalent Functions

M. O. Reade and H. Silverman

1. Introduction

Denote by S the family consisting of functions of the form $f(z) = z + \ldots$ that are analytic and univalent in $\Delta = \{z: |z| < 1\}$. In [8], R. Singh applied the area theorem to show that if $f \varepsilon S$ then $Re\ f(z)/z > 1/2$ in a disk $|z| < \rho$, where the number $\rho(> 0.41)$ is the unique positive root of the transcendental equation

$$K(r) = r\left[\ln \frac{1}{(1-r)^2}\right]^{1/2} + 2r - 1 = 0\ .$$

While not sharp, this result is close to sharp since the largest disk in which the Koebe function $k(z) = z/(1-z)^2$ satisfies $Re\ k(z)/z > 1/2$ is $|z| < \sqrt{2} - 1 = 0.414\ldots$.

In Section 2, we will prove that the Koebe function is indeed extremal. In fact, we will show that the Koebe function is extremal for $Re\ f(z)/z > \beta$ if and only if $0.468\ldots = \left(\frac{e+1}{2e}\right)^2 \leq \beta < 1$. This will be established by applying Schiffer's boundary variation [7] to find $\min_{|z|=r} Re\ \frac{f(z)}{z}$, $f \varepsilon S$, from which we determine the largest disk $|z| < \rho(\beta)$ that satisfies $Re\ f(z)/z > \beta$ over all $f \varepsilon S$ and $\beta < 1$. The case $\beta = 0$ gives the disk $|z| < \tanh(\pi/4) = 0.655\ldots$, a classical result first proved by Grunsky [4] using Loewner theory. Our variational approach is very similar to that employed by J. Brown [3] while investigating the support points of S for some point-evaluation functionals.

In Section 3, we study the same extremal problem for some subclasses of univalent functions. Using extreme point theory we find the largest disk in which $Re\ f(z)/z > \beta$ when f is starlike ($f \epsilon S^*$) as well as when f is starlike of order $\gamma(f \epsilon S^*(\gamma))$, $1/2 \leq \gamma < 1$. This improves on a result of Obradović [6].

2. Extremal properties for S.

Given a point $z_0 \epsilon \Delta$, we wish to find a function $f \epsilon S$ for which $Re\ f(z_0)/z_0 \leq Re\ g(z_0)/z_0$ for all $g \epsilon S$. If f is an extremal function, Schiffer's variational approach [7] may be used to construct a family of neighboring functions $f^* \epsilon S$ such that

$$f^*(z) = f(z) + \lambda_r \left[\frac{(f(z))^2}{w_0^2(f(z) - w_0)} \right] + 0(r^3),$$

where $w_0 \epsilon \Gamma = \partial f(\Delta)$ and $\lambda_r = 0(r^2)$ as $r \to 0$. Because f is extremal, we have

$$Re \left[\lambda_r \frac{(f(z_0))^2}{z_0 W_0^2(f(z_0) - w_0)} + 0(r^3) \right] \geq 0$$

for all sufficiently small r. From Schiffer's basic lemma [7], we know that Γ is an analytic arc satisfying

(1) $$\frac{(f(z_0))^2}{z_0 w^2(f(z_0) - w)} \left(\frac{dw}{dt} \right)^2 < 0$$

for a real parametrization $w = w(t)$. Representing Γ by $w = f(e^{it})$, we may conclude from (1) that

$$F(z) = \left(\frac{zf'(z)}{f(z)} \right)^2 \frac{(f(z_0))^2}{z_0(f(z) - f(z_0))} \leq 0$$

on $|z| = 1$. Since F is analytic in Δ aside from a simple pole at $z = z_0$, Schwarz's reflection principle allows us to continue F analytically to give a rational function in the plane with another simple pole at $z = 1/\overline{z}_0$. Consequently, F must have a zero of order two at a point $e^{i\alpha}$, where $F'(e^{i\alpha}) = 0$. Thus we can also represent F by

$$F(z) = \frac{A(z - e^{i\alpha})^2}{(z - z_0)(1 - \overline{z}_0 z)}, \quad A \text{ a constant.}$$

Letting $z \to 0$ and equating both expressions for $F(z)$, we get $A = f(z_0)e^{-2i\alpha}$. Since

$$F(e^{i\theta}) = \frac{-4Ae^{i\alpha}\sin^2(\frac{\theta-\alpha}{2})}{|e^{i\theta} - z_0|^2} < 0,\ \theta \neq \alpha\ ,$$

we also see that $Ae^{i\alpha} = f(z_0)e^{-i\alpha} > 0$ and hence $e^{i\alpha} = \frac{f(z_0)}{|f(z_0)|}$. Again equating both expressions for $F(z)$ leads to

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \frac{(f(z_0))^2}{z_0(f(z) - f(z_0))} = \frac{f(z_0)e^{-2i\alpha}(z - e^{i\alpha})^2}{(z - z_0)(1 - \overline{z}_0 z)}$$

or, equivalently,

(2) $$\left(\frac{zf'(z)}{f(z)}\right)^2 \frac{f(z_0)}{f(z_0) - f(z)} = \frac{(1 - e^{-i\alpha}z)^2}{(1 - z/z_0)(1 - \overline{z}_0 z)}\ .$$

Now (2) is identical to the expression found by Brown [3] in determining support points of S for certain point-evaluation functionals. Following Brown, we set $w = f(z)$, $P(z) = (1 - z/z_0)(1 - \overline{z}_0 z)$, and integrate (2) to obtain

(3) $$\int_{f(z_0)}^{w} \frac{dw}{w\sqrt{1 - w/f(z_0)}} = \int_{z_0}^{z} \frac{(1 - e^{-i\alpha}z)}{\sqrt{P(z)}}\frac{dz}{z}\ .$$

Letting $|z_0| = r$, $M(z) = \left[\frac{(1-r^2z/z_0)+r\sqrt{P(z)}}{(1-r^2z/z_0)-r\sqrt{P(z)}}\right]$, and $Q(z) = 2\sqrt{P(z)} + 2 - (1 + r^2)z/z_0$, we may then express (3) as

(4) $$\ln\left[\frac{1 + \sqrt{1 - f(z)/f(z_0)}}{1 - \sqrt{1 - f(z)/f(z_0)}}\right] = \ln\left[\frac{Q(z)z_0}{(1 - r^2)z}\right] + e^{-i\alpha}\ln M(z)\ .$$

Adding $\ln z$ to both sides of (4) and letting $z \to 0$, we get

$$\ln(4f(z_0)) = \ln\left(\frac{4z_0}{1 - r^2}\right) + e^{-i\alpha}\ln\left(\frac{1 + r}{1 - r}\right)$$

or, equivalently,

(5) $$\frac{f(z_0)}{z_0} = \frac{\exp\{e^{-i\alpha}\ln(\frac{1+r}{1-r})\}}{1 - r^2}\ ,\ \alpha = \arg\ f(z_0)\ .$$

In particular, for $f \epsilon S$ and $|z| = r < 1$ we find the sharp lower bound

(6) $$Re\frac{f(z)}{z} \geq \min_{\alpha\epsilon[0,2\pi)} \frac{\exp\{e^{-i\alpha}\ln(\frac{1+r}{1-r})\}}{1 - r^2}\ .$$

The extremal function associated with (5) is the one for which equality holds in (4). Exponentiating (4) and noting that $e^{-i\alpha} = |f(z_0)|/f(z_0)$, we have

$$\frac{1+\sqrt{1-f(z)/f(z_0)}}{1-\sqrt{1-f(z)/f(z_0)}} = \frac{Q(z)z_0}{(1-r^2)z}\left(M(z)\right)^{|f(z_0)|/f(z_0)} = T(z)\ ,$$

which leads to $1 - f(z)/f(z_0) = \Big((T(z)-1)/(T(z)+1)\Big)^2$ or

$$f(z) = 4f(z_0)T(z)/(1+T(z))^2\ .$$

Our problem of minimizing $Re\ f(z)/z(|z| = r, f\epsilon Z)$ is reduced to finding an appropriate α for which $\exp\{e^{-i\alpha}\ln(\frac{1+r}{1-r})\}$ is a minimum. To that end, we need

Lemma 1. If $G(\alpha) = e^{B\cos}\cos(B\sin\alpha)$, $B > 0$, then

$$G(\alpha) \geq G(\pi) = e^{-B} \quad (0 < B \leq 1)\ ,$$

$$G(\alpha) \geq e^{-\alpha_0\cot\alpha_0}\cos\alpha_0 \quad (B > 1)$$

where α_0 is the unique value in $(0,\pi)$ that satisfies $B = \alpha_0/\sin\alpha_0$. The result is sharp for all $B > 0$.

Proof. To minimize $G(\alpha)$, $0 \leq \alpha < 2\pi$, we note that $G'(\alpha) = -Be^{B\cos\alpha}\sin(\alpha + B\sin\alpha) = 0$ when

$$\alpha + B\sin\alpha = k\pi \tag{7}$$

for admissible integers k.

If $0 < B \leq 1$, then $\alpha + B\sin\alpha$ increases with α, $0 \leq \alpha < 2\pi$, and $k = 0$ or 1 in (7). In this case, $G'(\alpha) = 0$ only for $\alpha = 0$ and $\alpha = \pi$. Therefore, $G(\alpha) \geq G(\pi) = e^{-\beta}$.

If $B > 1$, set $\alpha^* = \alpha^*(k) = k\pi - \alpha$ in (7). Assuming $\alpha \neq \pi$, we see from (7) that $B = (-1)^{k+1}\alpha^*/\sin\alpha^*$, $\cos\alpha = (-1)^k\cos\alpha^*$, and $\sin\alpha = (-1)^{k+1}\sin\alpha^*$. So if a satisfies (7), then

$$e^{B\cos\alpha}\cos(B\sin\alpha) = \exp\left[(-1)^{k+1}\left(\frac{\alpha^*}{\sin\alpha^*}\right)(-1)^k\cos\alpha^*\right]\cos\alpha^*$$
$$= e^{-\alpha^*\cot\alpha^*}\cos\alpha^*\ .$$

Thus, $\min G(\alpha) = e^{-\alpha_0 \cot \alpha_0} \cos \alpha_0$ for some $\alpha_0 = \alpha^*(k)$ as long as $\min G(\alpha) \neq G(\pi)$. Now if $1 < B \leq \pi/2$, then $G(\alpha) \geq 0$ and $\cos \alpha_0 \geq 0$. We may then choose $\alpha_0 \epsilon (0, \pi/2]$. Similarly, if $B > \pi/2$ then $G(\alpha)$ can be negative and $\cos \alpha_0 < 0$. We then choose $\alpha_0 \epsilon (\pi/2, \pi)$.

Finally, it remains to show for $\alpha = \alpha_0$ satisfying $\alpha/\sin\alpha = B$, $0 < \alpha < \pi$, that $e^{-\alpha \cot \alpha} \cos \alpha < G(\pi) = e^{-B} = e^{-\alpha/\sin\alpha}$, or equivalently,

$$s(\alpha) = \cos\alpha \exp\left[\frac{\alpha}{\sin\alpha}(1 - \cos\alpha)\right] < 1 .$$

This is clearly the case when $\pi/2 \leq \alpha < \pi$. Since $\lim_{\alpha \to 0+} s(\alpha) = 1$, it suffices to show that $s(\alpha)$ is decreasing for $0 < \alpha < \pi/2$. Setting $t(\alpha) = \ln s(\alpha)$ and noting that $t'(\alpha) = -\left(\frac{1-\cos\alpha}{\sin\alpha}\right)\left[\frac{1}{\cos\alpha} - \frac{\alpha}{\sin\alpha}\right] < 0$, we see that $s(\alpha) < 1$. This completes the proof.

Theorem 1. For $|z| = r < 1$,

$$\min_{f \epsilon S} Re\, \frac{f(z)}{z} = \frac{1}{(1+r)^2} \quad \left(r \leq \frac{e-1}{e+1}\right),$$

$$\min_{f \epsilon S} Re\, \frac{f(z)}{z} = C(\alpha) := \frac{\cos\alpha}{1-r^2}\left(\frac{1-r}{1+r}\right)^{\cos\alpha} \quad \left(\frac{e-1}{e+1} < r < 1\right),$$

where $\alpha = \alpha(r)$ is the unique value in $(0, \pi)$ that satisfies $\ln\left(\frac{1+r}{1-r}\right) = \frac{\alpha}{\sin\alpha}$.

Remark. As r increases from $\frac{e-1}{e+1}$ to $\frac{e^{\pi/2}-1}{e^{\pi/2}+1}$, α increases from 0 to $\pi/2$ and $C(\alpha)$ decreases from $\left(\frac{e+1}{2e}\right)^2$ to 0. As r increases from $\frac{e^{\pi/2}-1}{e^{\pi/2}+1}$ to 1, α increases from $\pi/2$ to π and $C(\alpha)$ decreases from 0 to $-\infty$. See Appendix for specific values of α and $C(\alpha)$ as functions of r.

Proof. In view of (6), we have $\min_{|z|=r} Re\, \frac{f(z)}{z} = \frac{1}{1-r^2} \min G(\alpha)$, where $G(\alpha)$ is defined in Lemma 1 and $B = \ln\left(\frac{1+r}{1-r}\right)$. Now $r \leq (e-1)(e+1)$ if an only if $B \leq 1$, in which case $\min G(\alpha) = (1-r)/(1+r)$. Setting $B = \alpha/\sin\alpha$ when $B > 1$ and noting that $e^{-\alpha\cot\alpha} = \left(\frac{1-r}{1+r}\right)^{\cos\alpha}$, the result follows from Lemma 1.

Theorem 2. Suppose $f \epsilon S$ and $C(\alpha)$ is defined by Theorem 1.

(i) If $\left(\frac{e+1}{2e}\right)^2 \leq B < 1$, then $Re\, \frac{f(z)}{z} > B$ for $|z| < B^{-1/2-1}$.

(ii) If $\beta < \left(\frac{e+1}{2e}\right)^2$, then $Re\ \frac{f(z)}{z} > \beta$ for $|z| < \rho(\alpha) = \frac{e^{\alpha/\sin\alpha}-1}{e^{\alpha/\sin\alpha}+1}$, where $\alpha = \alpha(\beta, r)$ is the unique value in $(0, \pi)$ that satisfies $C(\alpha) = \beta$. The result is sharp for all real $\beta < 1$.

Remark. As β increases from $-\infty$ to 0, α decreases from π to $\pi/2$ and hence $\rho(\alpha)$ decreases from 1 to $\frac{e^{\pi/2}-1}{e^{\pi/2}+1} = \tanh(\pi/4) = 0.655\ldots$, the classical result of Grunsky [4]. As β increases from 0 to $\left(\frac{e+1}{2e}\right)^2$, α decreases from $\pi/2$ to 0 and hence $\rho(\alpha)$ decreases from $\tanh(\pi/4)$ to $(e-1)/(e+1)$.

Proof (of i). From Theorem 1, we see for $r \le (e-1)/e+1)$ that

$$\min_{|z|=r} Re\ \frac{f(z)}{z} = \frac{1}{(1+r)^2} = \beta < 1 \text{ when } |z| = \beta^{-1/2} - 1\ .$$

But $\beta^{-1/2} - 1 \le (e-1)/(e+1)$ if and only if $\beta \ge \left(\frac{e+1}{2e}\right)^2$.

(of ii). For $(e-1)/(e+1) < r < 1$, we have $\min_{|z|=r} Re\ f(z)/z = C(\alpha)$. In particular, the r for which $\min_{|z|=r} Re\ f(z)/z = \beta\left(< \left(\frac{e+1}{2e}\right)^2\right)$ is the one for which $C(\alpha) = \beta$. Solving $\ln\left(\frac{1+r}{1-r}\right) = \frac{\alpha}{\sin\alpha}$ as required by Theorem 1, we get $r = \rho(\alpha)$. This completes the proof.

3. Extremal properties for subclasses of S.

In [2], Brickman, MacGregor, and Wilken found the extreme points of the closed convex hull of S^* to be $z/(1-xz)^2$, $|x| = 1$, and the extreme points of the closed convex hull of the convex functions, K, to be $z/(1-xz)$, $|x| = 1$. Since the maximum or minimum of the real part of any continuous linear functional defined over a compact family H occurs at an extreme point of the closed convex hull of $H, \overline{cl}H$, the largest disk in which $Re\ f(z)/z > \beta$ for all $f \epsilon \overline{cl} S^*$ can be found by examining the extreme points of $\overline{cl} S^*$.

Theorem 3. For $|z| = r < 1$,

$$\min_{f\epsilon \overline{cl} S^*} Re\left(\frac{f(z)}{z}\right) = \begin{cases} 1/(1+r)^2 & , \ 0 < r \le 1/2, \\ (1-2r^2)/2(1-r^2)^2 & , \ 1/2 < r < 1. \end{cases}$$

Equality holds for $f(z) = z/(1-z)^2$ at $z = -r$ when $0 < r \le 1/2$ and at $z = re^{i\theta}$, $\cos\theta = (3r^2-1)/2r^3$, when $1/2 < r < 1$.

Proof. We know for $|z| = r$ and $|x| = 1$ that $\min Re\ f(z)/z = \min Re\ 1/(1-z)^2$. Setting $z = re^{i\theta}$, we have

$$Re\frac{1}{(1-z)^2} = Re\frac{(1-\bar{z})^2}{|1-z|^4} = \frac{1-r^2-2r\cos\theta+2r^2\cos^2\theta}{(1-2r\cos\theta+r^2)^2} := g(\theta)\ .$$

To minimize $g(\theta)$, we differentiate with respect to θ and simplify to get $(1-2r\cos\theta+r^2)^3g'(\theta) = 2r\sin\theta(3r^2-1-2r^3\cos\theta)$, which vanishes when $\theta = 0$, $\theta = \pi$ and $\theta_0 = \theta_0(r)$, where $\cos\theta_0 = (3r^2-1)/2r^3(1/2 \le r\ 1)$. Now $g(0) = 1/(1-r)^2, g(\pi) = 1/(1+r)^2$, and $g(\theta_0) = (1-2r^2)/2(1-r^2)^2$. In particular, $g(\pi) < g(0)$ for all r and $g(\theta_0) < g(\pi)$ for $r > 1/2$. This completes the proof.

From Theorem 3 we can give the starlike analog to Theorem 2.

Theorem 4. If $f\varepsilon \overline{cl}S^*$, then $Re\ f(z)/z > \beta$ in the disk $|z| < \rho(\beta)$, where

$$\rho(\beta) = \begin{cases} \left(1+(1-2\beta)^{-1/2}\right)^{-1/2} & ,\ \beta \le 4/9 \\ \beta^{-1/2}-1 & ,\ 4/9 < \beta < 1\ . \end{cases}$$

The result is sharp for all real $\beta < 1$.

Proof. We have

$$\frac{1}{(1+r)^2} = \beta \text{ for } r = \beta^{-1/2}-1 = s(\beta)$$

and

$$\frac{1-2r^2}{2(1-r^2)^2} = \beta \text{ for } r = \left(1+(1-2\beta)^{-1/2}\right)^{-1/2} = t(\beta)\ .$$

Now $h(\beta) = s(\beta) - t(\beta)$ is a decreasing function of β, with $h(4/9) = 0$. Since $t(\beta) \ge 1/2$ when $\beta \le 4/9$, the result follows from Theorem 3.

Remark. The extremal function for S in Theorem 1 agrees with that for S^* in Theorem 3 only when $r \le (e-1)/(e+1)$. Similarly, the extremal functions of Theorem 2 and Theorem 4 agree only for $\beta \ge \left((e+1)/2e\right)^2$. Thus the extremal functions of theorem 1 and Theorem 2 are not in S^*, respectively, when $r > (e-1)/(e+1)$ and $\beta > \left((e+1)/2e\right)^2$.

Since the extreme points of $\overline{cl}K$ are $z/(1-xz)$, we see that $\min_{|z|=r} Re\ f'$, $f\varepsilon\overline{cl}K$, is the same as $\min_{|z|=r} Re\ f(z)/z$, $f\varepsilon\overline{cl}S^*$. This produces the following consequence of Theorems 3 and 4.

Corollary 1 (i) For $|z| = r < 1$,

$$\min_{f \epsilon \overline{cl}K} Re\, f'(z) = \begin{cases} 1/(1+r)^2 & , 0 < r \le 1/2 \\ (1-2r^2)/2(1-r^2)^2 & , 1/2 < r < 1 . \end{cases}$$

(ii) If $f \epsilon \overline{cl}K$, then $Re\, f'(z) > \beta$ in the disk $|z| < \rho(\beta)$, where

$$\rho(\beta) = \begin{cases} \left(1 + (1-2\beta)^{-1/2}\right)^{-1/2} & , \beta \le 4/9 , \\ \beta^{-1/2} - 1 & , 4/9 < \beta < 1 . \end{cases}$$

For $f \epsilon \overline{cl}K$, $\min_{|z|=r} Re \frac{1}{1-xz} = \frac{1}{1+r}$. But $(1+r)^{-1} > \beta$ is equivalent to $r < (1-\beta)/\beta$, which leads to

Corollary 2. If $f \epsilon \overline{cl}K$ and $1/2 < \beta < 1$, then $Re\, f(z)/z > \beta$ for $|z| < (1-\beta)\beta$. The result is sharp.

It is shown in [1] that the extreme points of $\overline{cl}S^*(\gamma)$, $0 \le \gamma < 1$, are $z/(1-xz)^{2(1-\gamma)}$, $|x| = 1$. Thus for $|z| = r < 1$ we have

$$\min_{f \epsilon \overline{cl}S^*(\gamma)} Re \frac{f(z)}{z} = \min Re \frac{1}{(1-z)^{2(1-\gamma)}} .$$

We will use this to find the largest disk in which $Re\, f(z)/z > \beta$, $1/2 \le \beta < 1$, for $f \epsilon \overline{cl}S^*(\gamma)$. But first we need a well-known result on hypergeometric functions that can be found in [5, p. 206].

Lemma 2. For $c > b > 0$ and $z \not\epsilon [1, \infty)$,

$$\frac{1}{(1-z)^b} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-c} dt .$$

Lemma 3. If $0 < b \le 1$, then $Re\, 1/(1-z)^b \ge 1/(1+r)^b$.

Proof. The result is clear for $b = 1$. If $b < 1$, we set $c = 1$ in Lemma 2 to obtain

(8) $$\frac{1}{(1-z)^b} = \frac{1}{\Gamma(b)\Gamma(1-b)} \int_0^1 \frac{t^{b-1}}{(1-t)^b} \frac{1}{1-tz} dt .$$

Since $Re\, 1/(1-tz) \ge 1/(1+tr)$, the real part of (8) is minimized at $z = -r$.

We are now ready to prove

Theorem 5. If $f\epsilon \overline{cl S^*}$, $1/2 \leq \gamma < 1$, then $\Re\, f(z)/z > \beta$, $1/2 \geq \beta < 1$, for $|z| < \min\{\beta^{-1/2(1-\gamma)} - 1, 1\}$. The result is sharp.

Proof. Setting $b = 2(1-\gamma)$ in Lemma 3, we see for $|z| = r < 1$ that

$$\min_{f\epsilon \overline{cl S^*}(\gamma)} Re\, \frac{f(z)}{z} = \min Re\, \frac{1}{(1-z)^{2(1-\gamma)}} = \frac{1}{(1+r)^{2(1-\gamma)}}\ .$$

But $\frac{1}{(1+r)^{2(1-\gamma)}} > \beta$ is equivalent to $r < \beta^{-1/2(1-\gamma)} - 1$, and the proof is complete.

In [6] Obradović found the non-sharp result for $z\epsilon\Delta$ and $f\epsilon S^*(\gamma)$, $1/2 \leq \gamma < 1$, that $Re\, f(z)/z > 1/(3-2\gamma)$. The sharp result is a consequence of letting β in Theorem 5 be the value for which $\beta^{-1/2(1-\gamma)} - 1 = 1$. This give us

Corollary 1. If $f\epsilon \overline{cl S^*}(\gamma)$, $1/2 \leq \gamma < 1$, then $Re\, f(z)/z > 1/2^{2(1-\gamma)}$ for all $z\epsilon\Delta$.

Since $f\epsilon K(\gamma)$, the family of functions convex of order γ, if and only if $zf'\epsilon S^*(\gamma)$, we also have

Corollary 2. If $f\epsilon \overline{cl K}(\gamma)$, $1/2 \leq \gamma < 1$, then $Re\, f'(z) > \beta$, $1/2 \leq \beta < 1$, for $|z| < \min\{\beta^{-1/2(1-\gamma)} - 1, 1\}$. The result is sharp.

Remark. Our proof of Theorem 5 does not extend to $0 < \gamma < 1/2$ because we cannot choose $c = 1$ in Lemma 3. If we set $c = 2$ and $b = 2(1-\gamma)$, then

$$\frac{f(z)}{z} = \frac{1}{\Gamma(2-2\gamma)\Gamma(2\gamma)} \int_0^1 \left(\frac{t}{1-t}\right)^{1-2\gamma} \frac{1}{(1-tz)^2} dt\ . \tag{9}$$

Since $\min_{z=re^{i\theta}} Re \left(\frac{1}{(1-tz)^2}\right)$ will be attained for different values of θ as t varies, it is not clear how to minimize the real part of (9). It seems that another method is needed to find $\min_{|z|=r} Re\, f(z)/z$ over $f\epsilon S^*(\gamma)$, $0 < \gamma < 1/2$.

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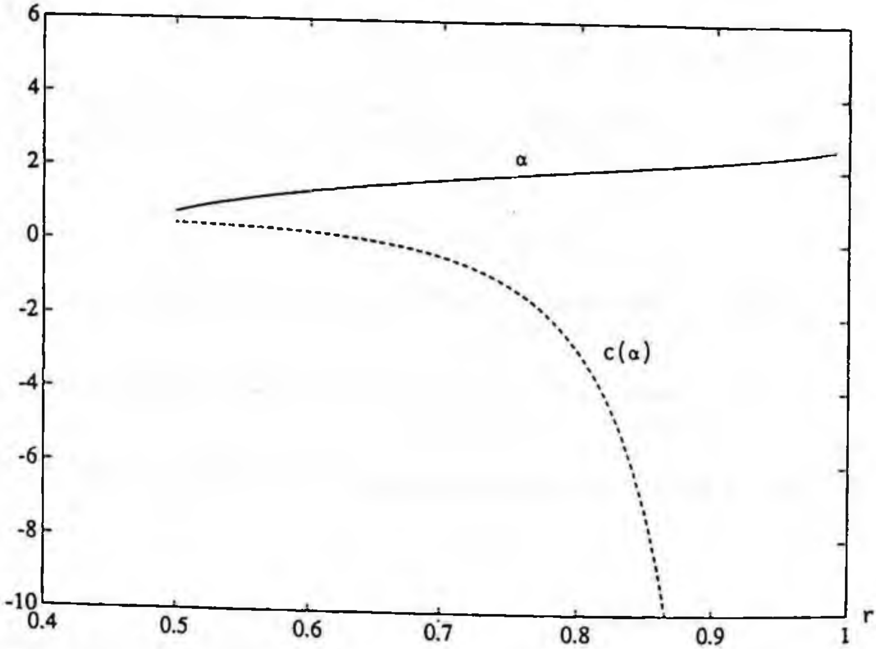
M. O. Reade
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

H. Silverman
Department of Mathematics
College of Charleston
Charleston, SC 29424

APPENDIX



r	α	c(α)
.50	0.74	0.44
.55	1.11	0.37
.60	1.35	0.25
.65	1.55	0.03
.70	1.72	-0.37
.75	1.86	-1.15
.80	2.00	-2.87
.85	2.13	-7.24
.90	2.26	-22.10
.95	2.42	-120.50
.99	2.62	-4331.98

Calculations by Professor B. A. Taylor.

Constantin Carathéodory: An International Tribute (pp. 1175–1209)
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CONVEXITY THEORIES I.

Γ-CONVEX SPACES

Helmut Röhrl

Abstract: A general notion of convexity theory is introduced which leads to the definition of the category of Γ-convex spaces. Various properties of Γ-convex spaces are obtained, and the spread of a convexity theory as well as the semi-norm of a Γ-convex space are discussed.

0. Introduction

In several previous papers, beginning with[8] and[9], the author jointly with D. Pumplün investigated the category of totally convex spaces as well as several of its subcategories. Subsequently D. Pumplün[6] and A. Wickenhäuser[13] introduced and studied the category of positively convex spaces. Convex sets which are certain objects in the category of convex spaces have been an integral part of mathematics for many years. Superconvex spaces and their category were introduced by G. Rodé[11] a decade ago and play a significant role in certain investigations into totally convex spaces. This list of what might be called "convexity theories" is by no means complete.

Since these "convexity theories" are dealt with individually and separately it seems appropriate to define a general notion of *convexity theory* that encompasses all listed ones (and more) and to develop a general theory of them. Precisely this is the purpose of the following paper.

Broadly speaking, convexity deals with sets X – perhaps imbedded in some vector space – equipped with certain abstractly or concretely defined linear combinations $\sum \alpha_i x_i$, with $x_i \in X$ and $\alpha_i \in \mathbb{C}$; these linear combinations, which can be finite or infinite, are again contained in X. It is convenient (and justifiable[8], §2) to consider the sequence $(\alpha_1, \alpha_2, \dots)$ to be infinite by adding zeros to it, should it be finite. The totality Γ of all $\alpha_* = (\alpha_1, \alpha_2, \dots)$ that occur in these linear combinations is the set of operators (or operations) and determines the category of Γ-convex sets (or better: spaces). What conditions should be satisfied by these Γ-convex spaces?

A feature that is common to all "convexity theories" is a condition on the operators:

$$\|\alpha_*\| := \sum_{i=1}^{\infty} |\alpha_i| \leq 1 \quad , \ \textit{for all} \ \ \alpha_* \in \Gamma. \tag{$\Gamma 0$}$$

Furthermore it is convenient to assume that all unit vectors $\delta_*^j := (\delta_{1,j}, \delta_{2,j}, \cdots)$, with $\delta_{i,j}$ the Kronecker symbol, satisfy

$$\delta_*^j \in \Gamma \quad , \ \text{for all} \ \ j = 1, 2, \dots . \tag{$\Gamma 1$}$$

Then one must insist on the intuitively clear and unobtrusive rule

$$\sum_{i=1}^{\infty} \delta_{i,j} x_i = x_j \quad , \ \text{for all} \ \ x_i \in X \ \ \text{and all} \ \ X. \tag{$\Gamma C1$}$$

It is usually called the PROJECTION AXIOM. Finally, if $\alpha_*, \beta_*^1, \beta_*^2, \cdots$ are in Γ and X is a Γ-convex space then we can form

$$\sum_{i=1}^{\infty} \alpha_i \left(\sum_{j=1}^{\infty} \beta_j^i x_j \right) .$$

This expression is again in X and equals in all listed cases – in particular if X is imbedded in some vector space and the linear combinations are concretely given as the ones in the vector space –

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_i\beta_j^i\right)x_j.$$

This, of course, requires that

$$\alpha_* \circ \langle \beta_*^1, \beta_*^2, \dots \rangle := \alpha_1\beta_*^1 + \alpha_2\beta_*^2 + \cdots \in \Gamma, \quad \text{for all} \quad \alpha_*, \beta_*^1, \beta_*^2, \dots \in \Gamma \qquad (\Gamma 2)$$

and that

$$\sum_{i=1}^{\infty}\alpha_i\left(\sum_{j=1}^{\infty}\beta_j^i x_j\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_i\beta_j^i\right)x_j \qquad (\Gamma C2)$$

for all $\alpha_*, \beta_*^1, \beta_*^2, \dots \in \Gamma$, all $x_j \in X$, and all X.

The last rule is usually referred to as the BIG BARYCENTER AXIOM.

At this point we have a good definition for a *convexity theory* Γ: it is a set of infinite sequences of complex numbers satisfying (Γ0), (Γ1), and (Γ2). Then a set X, equipped with abstractly defined "linear" combinations $\sum_{i=1}^{\infty}\alpha_i x_i$ where $\alpha_* \in \Gamma$ and $x_* \in X^{\boldsymbol{N}}$, is called a Γ-*convex space* if ($\Gamma C1$) and ($\Gamma C2$) are satisfied. The category ΓC of Γ-convex spaces has as its objects the just defined Γ-convex spaces and as its morphisms the obvious maps.

In §1 convexity theories Γ are defined; they turn out to be precisely the subalgebras of the algebra $\Omega := \Omega_{\mathbb{C}}$ (see[8], §2) of total convexity. An important object is the set $S_\Gamma := \left\{\sum_{i=1}^{\infty}\alpha_i : \alpha_* \in \Gamma\right\}$. It is in fact the free Γ-object on the one-point set $\{1\} \subseteq S_\Gamma$. Various properties of S_Γ are obtained and their interplay with properties of Γ itself are illuminated. Two important invariants of Γ are introduced, namely

$$\rho_\Gamma := \sup\left\{\left|\sum_{i=1}^{\infty}\alpha_i\right| : \alpha_* \in \Gamma \ \text{ and } \ \operatorname{card}(\operatorname{supp}(\alpha_*)) > 1\right\}$$

and

$$\tau_\Gamma := \sup\left\{\sum_{i=1}^{\infty}|\alpha_i| : \alpha_* \in \Gamma \ \text{ and } \ \operatorname{card}(\operatorname{supp}(\alpha_*)) > 1\right\}.$$

Obviously, $\rho_\Gamma \le \tau_\Gamma \le 1$ holds, and it is shown that there are convexity theories with $\rho_\Gamma < \tau_\Gamma$. At the close of this section the notion of convexity theory with ***pseudogroup condition*** (PG) resp. ***strong pseudogroup condition*** (SPG) is introduced.

§2 starts with the definition of Γ-***convex space***, leading to the category ΓC of Γ-***convex spaces and their morphisms.*** A brief construction shows that for any set S there is a free Γ-convex set $F_\Gamma(S)$ on that set; in other words, ΓC has sufficiently many free, and hence projective, objects. A Metatheorem implies that the computational rules of[8], §2, hold for all Γ-convex spaces, with Γ arbitrary or Γ with zero whenever a zero shows up in that rule. As in[8], §5, one can show that ΓC is an autonomous category in the sense of[3].

§3 is a short section that contains various examples. Their main purpose is to show how abtruse matters can be for certain convexity theories Γ as judged from Ω.

The spread σ_Γ of a convexity theory Γ is the subject of §4. σ_Γ takes value in the set $\{-\infty\} \cup \{t : 0 < t \le 1\}$. $\sigma_\Gamma > 0$ means if X is any Γ-convex space, $\sim$ is any Γ-congruence relation on X, and $x, y \in X$ any two elements such that $\alpha x \sim \alpha y$ for some $\alpha \in S_\Gamma$ with $|\alpha| < \sigma_\Gamma$ then $\rho x \sim \rho y$ for all $\rho \in S_\Gamma$ with $|\rho| < \sigma_\Gamma$. $\sigma_\Gamma \le \tau_\Gamma$ is true for all convexity theories, and $\rho_\Gamma \le \sigma_\Gamma$ holds under quite weak assumptions on Γ. Convexity theories Γ with (PG) plus additional weak hypotheses satisfy $\sigma_\Gamma = 1$.

In §5 we discuss the semi-norm $\|\ \|$ of a Γ-convex space. Its definition appears already in[8], §6. However, in this much more general situation things are considerably more complicated. For instance, the two possible definitions for the semi-norm as given in[8], §6, (see (6.1)) agree only under additional, although mild assumptions on Γ. The same is true for most of the statements of[8], §6; for details we refer to the body of this paper. The final result of this section gives conditions on Γ such that for all Γ-convex spaces X, $x \in X$ with $\|x\|_\Gamma = 0$ implies $x = 0$. The fact that this is not true for all infinite convexity theories was shown in[13], §1 and §4.

1. Convexity Theories

Let $\alpha_* := (\alpha_1, \alpha_2, ...)$ be an infinite sequence of complex numbers such that the sum of the absolute values of the α_i converges. We set

$$S(\alpha_*) := \sum_{i=1}^{\infty} \alpha_i \quad \text{and} \quad \|\alpha_*\| := \sum_{i=1}^{\infty} |\alpha_i|.$$

Obviously, $|S(\alpha_*)| \leq \|\alpha_*\|$ holds.

As in[8] we denote by Ω the set of all α_* for which $\|\alpha_*\| \leq 1$ holds. Clearly, the zero sequence 0_* is in Ω as are the $\delta_*^j, j = 1, 2, \ldots$, where the entry δ_i^j is the Kronecker symbol. It is customary to denote the set $\{\mathbb{N} : \alpha_i \neq 0\}$ by supp α_*, the support of α_*.

Definition 1.1. A *convexity theory* is a subset Γ of Ω satisfying

$$\delta_*^j \in \Gamma, \text{for all } \ j = 1, 2, ... \tag{$\Gamma1$}$$

$$\text{if } \ \alpha_*, \beta_*^1, \beta_*^2, ..., \ \text{ are in } \ \Gamma, \ \text{ so is } \ \sum_{i=1}^{\infty} \alpha_i \beta_*^i = \left(\sum_{i=1}^{\infty} \alpha_i \beta_1^i, \sum_{i=1}^{\infty} \alpha_i \beta_2^i, ... \right). \tag{$\Gamma2$}$$

Note, if Ω is viewed as a general Ω-algebra by letting Ω operate on itself in accordance with Definition 1.1, (ii), then the convexity theories are precisely the subalgebras of Ω.

A convexity theory Γ is called *finite* resp. *infinite*, if $\Gamma \subseteq \Omega_{fin} := \{\alpha_* \in \Omega : supp\, \alpha_* \text{ is } finite\}$ resp. $\Gamma \not\subseteq \Omega_{fin}$; Γ is called *proper*, if $\Gamma \supsetneq \Delta := \{\delta_*^j : j = 1, 2, ...\}$; it is said to be *with zero*, if $0_* \in \Gamma$ holds; it is called *real*, if $\alpha_* \in \Gamma$ implies $\alpha_i \in \mathbb{R}$, for all $i = 1, 2, \cdots$.

The set CT of all convexity theories is a complete lattice with

$$\wedge\{\Gamma_\lambda : \lambda \in \Lambda\} := \cap \{\Gamma_\lambda : \lambda \in \Lambda\} \qquad \text{and}$$

$$\vee\{\Gamma_\lambda : \lambda \in \Lambda\} := \cap \{\Gamma \in CT : \Gamma \supseteq \cup \{\Gamma_\lambda : \lambda \in \Lambda\}\}.$$

CT has a smallest element, Δ, and a largest element, Ω. $\Omega_{\mathbb{R}} := \{\alpha_* \in \Omega : \alpha_i \in \mathbb{R}$, for all $i = 1, 2, ...\}$ is the largest real convexity theory. Additional convexity theories that appear in various contexts are:

Superconvexity, given by (see[11])

$$\Omega_{sc} := \{\alpha_* \in \Omega : \alpha_i \geq 0, \text{ for all } i = 1, 2, ..., \text{ and } S(\alpha_*) = 1\},$$

Convexity, given by (see[6],[13])

$$\Omega_c := \Omega_{\text{fin}} \wedge \Omega_{sc} \ ,$$

Positive Convexity, given by (see[6],[13])

$$\mathcal{P} := \{\alpha_* \in \Omega : \alpha_i \geq 0, \text{ for all } i = 1, 2, \cdots\}$$

Strictly Positive Convexity

$$\mathcal{P}^+ := \{\alpha_* \in \mathcal{P} : S(\alpha_*) > 1\}.$$

These convexity theories are by no means the only ones that appear in the literature.

With each convexity theory Γ one can associate the subring R_Γ of $\mathbb{C}$ that is generated by the set $\{\alpha_i : \alpha_* \in \Gamma \text{ and } i = 1, 2, \dots\}$. Conversely, let R be a subring of $\mathbb{C}$ with $1 \in R$ such that for each $r \in R$ there is a $r' \in R$ and a $n \in N$ satisfying $|r| \leq n$ and $r = nr'$, then

$$\Omega_{\text{fin},R} := \{\alpha_* \in \Omega : \operatorname{supp} \alpha_* \text{ is finite and } \alpha_i \in R, \text{ for all } i = 1, 2, \dots\}$$

is a convexity theory with $R = R_{\Omega_{\text{fin},R}}$. Since there is a large number of such rings, we see that the set of finite convexity theories is distressingly large. However, infinite convexity theories seem to be more amenable to classification.

An important invariant of a convexity theory Γ is the set

$$S_\Gamma := \{S(\alpha_*) : \alpha_* \in \Gamma\}.$$

Lemma 1.2.

(i) $S_\Gamma \subseteq O(\mathbb{C}) := \{z \in \mathbb{C} : |z| \leq 1\}$

(ii) $1 \in S_\Gamma$

(iii) *$\rho \in S_\Gamma$ if and only if $\rho\delta_*^i \in \Gamma$ (for some and hence all $j = 1, 2, \cdots$)*

(iv) *for all $\alpha_* \in \Gamma$ and all $\rho^* \in (S_\Gamma)^{\mathbb{N}}$, $\sum_{i=1}^{\infty} \alpha_i \rho^i \in S_\Gamma$; in particular, S_Γ is a multiplicative monoid.*

Proof. Straight forward.

Proposition 1.3. *$S_\Gamma \subseteq bdy\ O(\mathbb{C})$ if and only if*

either: $\|\alpha_*\| = 1$ *and* $\text{card}(\text{supp}(\alpha_*)) = 1$, *for all* $\alpha_* \in \Gamma$

or: $\Gamma \subseteq \Omega_{sc}$.

Proof. Clearly, either of the two alternatives implies $S_\Gamma \subseteq bdy\ O(\mathbb{C})$. Conversely, if $S_\Gamma \subseteq bdy\ O(\mathbb{C})$ then $\alpha_* \in \Gamma$ with $\text{card}(\text{supp}(\alpha_*)) = 1$ satisfies $\|\alpha_*\| = 1$. Assume now that there is a $\alpha_* \in \Gamma$ with $\text{card}(\text{supp}(\alpha_*) > 1$. For any $\beta_* \in \Gamma$ we have $1 \leq |S(\beta_*)| \leq \|\beta_*\| \leq 1$, whence $\beta_* = e^{i\varphi_{\beta_*}} \cdot \tilde{\beta}_*$ and $\tilde{\beta}_* \in \Omega_{sc}$. Suppose there were a $\beta_* \in \Gamma$ with $\varphi_{\beta_*} \not\equiv \varphi_{\alpha_*} \bmod 2\pi$. Since, for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots) \in \Gamma$ holds, we may assume that $\tilde{\alpha}_1 \neq 0$ and $\tilde{\alpha}_2 \neq 0$. Hence Definition 1.1, (ii), applied to $\alpha_*, \beta_*^1 := \alpha_*, \beta_*^2 := \beta$, and $\beta_*^j := \delta_*^1$ for $j > 2$, leads to $\gamma_* \in \Gamma$ where

$$\gamma_* := \sum_{j=1}^{\infty} \alpha_j \beta_*^j = e^{i\varphi_{\alpha_*}} \left(\tilde{\alpha}_1 \alpha_* + \tilde{\alpha}_2 \beta_* + \left(\sum_{j=3}^{\infty} \tilde{\alpha}_j \right) \cdot \delta_*^1 \right).$$

Since

$$\begin{aligned} S(\gamma_*) &= e^{i\varphi_{\alpha_*}} \left(\tilde{\alpha}_1 e^{i\varphi_{\alpha_*}} S(\tilde{\alpha}_*) + \tilde{\alpha}_2 e^{i\varphi_{\beta_*}} S(\tilde{\beta}_*) + \sum_{j=3}^{\infty} \tilde{\alpha}_j \right) \\ &= e^{i\varphi_{\alpha_*}} \left(\tilde{\alpha}_1 e^{i\varphi_{\alpha_*}} + \tilde{\alpha}_2 e^{i\varphi_{\beta_*}} + \sum_{j=3}^{\infty} \tilde{\alpha}_j \right), \end{aligned}$$

we have $|S(\gamma_*)| < 1$, contrary to our assumption. Hence $\varphi_{\beta_*} = \varphi_{\alpha_*}$, for all $\beta_* \in \Gamma$. Since $\delta_*^1 \in \Gamma$ and $\varphi_{\delta_*^1} = 0$ holds, our assertion is proved.

A convexity theory Γ is called *normable* if there is a $\rho \in S_\Gamma$ with $0 < |\rho| < 1$.

Corollary 1.4. *The non-normable convexity theories are precisely the following:*

(i) *There is a finite subgroup G of bdy $O(\mathbb{C})$ such that*
either: $\Gamma = \{\rho\delta_*^j : \rho \in G \text{ and } j = 1, 2, \dots\}$
or: $\Gamma = \{0_*\} \cup \{\rho\delta_*^j : \rho \in G \text{ and } j = 1, 2, \dots\}$,

(ii) *There is a dense submonoid M of bdy $O(\mathbb{C})$ such that*
either: $\Gamma = \{\rho\delta_*^j : \rho \in M \textit{ and } j = 1, 2 \cdots\}$
or: $\Gamma = \{0_*\} \cup \{\rho\delta_*^j : \rho \in M \textit{ and } j = 1, 2, \dots\}$,

(iii) $\Gamma = \Omega_{sc}$,

(iv) $\Gamma = \Omega_c$.

All G and M as specified occur in (i) and (ii).

Proof. Immediate from Proposition 1.3 and from Kuhn's Theorem (see[2], p. 87).

For a convexity theory Γ we define the length of Γ by

$$lg(\Gamma) := \sup\{\text{card}(\text{supp}(\alpha_*)) : \alpha_* \in \Gamma\}.$$

It is easy to see that for any convexity theory $\Gamma, \ell g\,\Gamma > 1$ implies $\ell g\,\Gamma = \infty$.

Proposition 1.5. *Let Γ be a convexity theory with $lg(\Gamma) > 1$ such that $\{1\} \subsetneq S_\Gamma$. Then Γ is normable.*

Proof. Since the length of Γ is at least two, there is a $(\alpha_1, \alpha_2, 0, \dots) \in \Gamma$ with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. By assumption there is a $\rho \in S_\Gamma$ with $\rho \neq 1$. If $0 < |\rho| < 1$ we are done. If $\rho = 0$ then $\alpha_1\delta_*^1 + \alpha_2 \cdot 0\delta_*^1 = (\alpha_1, 0, \dots) \in \Gamma$, whence $\alpha_1 \in S_\Gamma$, making Γ normable. If $|\rho| = 1$ and $\rho \neq 1$, then either $\alpha_1\delta_*^1 + \alpha_2\rho\delta_*^1$ or $\alpha_1\delta_*^1 + \alpha_2\rho^2\delta_*^1$ has first entry of absolute value strictly between 0 and 1, again making Γ normable.

Proposition 1.6. *Let Γ be a normable convexity theory of length > 1. Then, for every $\alpha_* \in \Gamma$ with* $\text{card}(\text{supp}(\alpha_*)) > 1$ *and for every $i \in \mathbb{N}$, α_i is an accumulation point of S_Γ.*

Proof. For a choice of α_* and i as specified there is a $j \neq i$ with $\alpha_j \neq 0$ and $(\alpha_i, \alpha_j, \beta, 0, \dots) \in \Gamma$, where $\beta = \sum_{k \neq i,j} \alpha_k$. Since 0 is an accumulation point of S_Γ there are $\rho_1, \rho_2 \in S_\Gamma$ arbitrarily close to 0. Since $\alpha_i\delta_*^1 + \alpha_j\rho_1\delta_*^1 + \beta\rho_2\delta_*^1 \in \Gamma$, we have $\alpha_i + \rho_1\alpha_j + \rho_2\beta \in S_\Gamma$ and our claim follows.

Remark. If Γ is a convexity theory with zero then, for every $\alpha_* \in \Gamma$ and every $i \in \boldsymbol{N}$, $\alpha_i \in S_\Gamma$ as $(\alpha_i, 0, \dots) = \sum_j \alpha_j\delta_{ij}\delta_*^1 \in \Gamma$. Hence Proposition 1.6 is of interest only for convexity theories without zero.

For a convexity theory Γ we define, in $[-\infty, 1]$,

$$\rho_\Gamma := \sup\{|S(\alpha_*)| : \alpha_* \in \Gamma \text{ and } \operatorname{card}(\operatorname{supp}(\alpha_*)) > 1\} \quad \text{and}$$

$$\tau_\Gamma := \sup\{\|\alpha_*\| : \alpha_* \in \Gamma \text{ and } \operatorname{card}(\operatorname{supp}(\alpha_*)) > 1\}.$$

Additionally we call $S_\Gamma \cap bdy\, O(\mathbb{C})$ the *monoid* M_Γ *associated with* Γ.

Lemma 1.7. *For any convexity theory* Γ,

(i) $lg(\Gamma) = 1$ *implies* $\rho_\Gamma = \tau_\Gamma = -\infty$,

(ii) $lg(\Gamma) > 1$ *implies* $0 < \rho_\Gamma \leq \tau_\Gamma \leq 1$.

Proof. Obvious.

Proposition 1.8. *Let* Γ *be a convexity theory of length* > 1. *If* $\operatorname{card} M_\Gamma =: n > 2$, *where* $n \in \mathbb{N} \cup \{\infty\}$, *then* $\cos\left(\frac{\pi}{n}\right)\tau_\Gamma \leq \rho_\Gamma$.

Proof. Assume $n < \infty$. Let $\alpha_* = (e^{i\varphi_1}|\alpha_1|, e^{i\varphi_2}|\alpha_2|, \dots) \in \Gamma$. With $\rho_i \in M_\Gamma, i = 2, 3, \dots$, we have

$$(e^{i\varphi_1}|\alpha_1|, e^{i\varphi_2}\rho_2|\alpha_2|, e^{i\varphi_3}\rho_3|\alpha_3|, \dots) = \alpha_1\delta_*^1 + \alpha_2\rho_2\delta_*^2 + \alpha_3\rho_3\delta_*^3 + \dots \in \Gamma.$$

Since each ρ_i can be chosen to bring $\varphi_i + arg\rho_i$ to within $\frac{\pi}{n}$ of φ_1, an elementary estimate shows that $\cos\left(\frac{\pi}{n}\right) \cdot \|\alpha_*\| \leq |S(\beta_*)|$ where β_* is the stated modification of α_*. If $n = \infty$ then the modification of α_* can be made to bring $\varphi_i + arg\,\rho_i$ to within ε of φ_1, for any choice of $\varepsilon > 0$. Hence the assertion follows by an obvious limit argument.

Scholium 1.9. *There are convexity theories Γ such that $\rho_\Gamma < \tau_\Gamma$.*

Proof. Let $G \subseteq \text{bdy}\, O(\mathbb{C})$ be the set $\{t^\lambda : t = e^{\frac{2\pi i}{N}}$ and $\lambda = 0, \ldots, N-1\}$ where N is a given integer ≥ 2. Put $\gamma_* := (\gamma_2, \gamma_2, 0, \ldots)$ where $\gamma_1 = \frac{1}{4} e^{\frac{\pi i}{N}}$ and $\gamma_2 = \frac{1}{4}$. Let furthermore Γ be the convexity theory generated by the set

$$H_0 := \{\gamma_*\} \cup \{t^\lambda \delta_*^j : \lambda = 0, \ldots, N-1; j = 1, 2, \ldots\}.$$

We define inductively H_{r+1}, given H_r, $r \geq 0$ by

$$H_{r+1} := \left\{ \sum_{k=1}^{\infty} \alpha_k \beta_*^k : \alpha_*, \beta_*^1, \beta_*^2, \ldots \in H_r \right\}.$$

Evidently, $H_r \subseteq H_{r+1}$ for all $r \geq 0$. We claim that $\Gamma = \cup\{H_r : r \geq 0\}$. Obviously, $\Gamma \supseteq \cup\{H_r : r \geq 0\}$. On the other hand, $\cup\{H_r : r \geq 0\}$ is a convexity theory as $\alpha_* \in H_r$, $r \geq 0$, implies that card $(\text{supp}\,(\alpha_*))$ is finite. An easy inductive argument (on r) shows that

(i) $\alpha_* \in H_r, r \geq 0$, and $\|\alpha_*\| = 1$ implies $\alpha_* = t^\lambda \delta_*^j$, for some λ and some j,

(ii) $\alpha_* \in H_r, r \geq 0$, and $\|\alpha_*\| < 1$ implies $\|\alpha_*\| \leq \frac{1}{2}$; in particular, if card $(\text{supp}\,(\alpha_*)) \geq 2$ then $\|\alpha_*\| \leq \frac{1}{2}$.

Next we prove

(iii) $\alpha_* \in H_r, r \geq 0$, and $\|\alpha_*\| = \frac{1}{2}$ implies card $(\text{supp}\,(\alpha_*)) = 2$ and α_* has as its non-zero entries $t^{\lambda_1}\gamma_1$ and $t^{\lambda_2}\gamma_2$.

The statement is obviously true for $r = 0$. We assume that it is true for r and proceed to prove it for $r+1$. Let $\alpha_*, \beta_*^1, \beta_*^2, \ldots \in H_r$. Since

$$\frac{1}{2} = \|\sum_{k=1}^{\infty} \alpha_k \beta_*^k\| \leq \|\alpha_*\| \cdot \sup\{\|\beta_*^k\| : k = 1, 2, \ldots\}$$

we have $\|\alpha_*\| \geq \frac{1}{2}$. Hence (ii) implies $\|\alpha_*\| = 1$ or $\|\alpha_*\| = \frac{1}{2}$. Suppose $\|\alpha_*\| = 1$. Then, due to (i), $\sum_{k=1}^{\infty} \alpha_k \beta_*^k = t^\lambda \beta_*^j$ and thus $\|\beta_*^j\| = \frac{1}{2}$. By induction hypothesis we are done. Suppose now $\|\alpha_*\| = \frac{1}{2}$. Then, for some $m_1 \neq m_2, \alpha_{m_1} = t^{\lambda_2}\gamma_1$ and $\alpha_{m_2} = t^{\lambda_2}\gamma_2$, while $\alpha_i = 0$, for $i \neq m_1, m_2$. Hence $\sum_{k=1}^{\infty} \alpha_k\beta_*^k = t^{\lambda_1}\gamma_1\beta_*^{m_1} + t^{\lambda_2}\gamma_2\beta_*^{m_2}$. The above inequality now implies $\|\beta_*^{m_1}\| = \|\beta_*^{m_2}\| = 1$, and we are done using (i).

(iv) $\alpha_* \in H_r, r \geq 0$, and $\|\alpha_*\| = \frac{1}{2}$ implies $|S(\alpha_*)| \leq \frac{1}{2\sqrt{2}}\sqrt{1+\cos\left(\frac{\pi}{N}\right)}$. This is straightforward as (iii) implies $S(\alpha_*) = \frac{1}{4}\left(e^{\frac{\pi i}{N}+\frac{2\pi i\lambda_1}{N}} + e^{\frac{2\pi i\lambda_2}{N}}\right)$.

(v) There is a $M < \frac{1}{2}$ such that $\alpha_* \in H_r, r \geq 0$, and $\|\alpha_*\| < \frac{1}{2}$ imply $\|\alpha_*\| \leq M$. All elements of H_1 with norm $< \frac{1}{2}$ are of the form $\eta_* = \gamma_1\beta_*^{m_1} + \gamma_2\beta_*^{m_2}$. If $\beta_*^{m_1} = t^{\lambda_1}\delta_*^{j_1}, \beta_*^{m_2} = t^{\lambda_2}\delta_*^{j_2}$ then $\|\eta_*\| = \frac{1}{2}$, if $j_1 \neq j_2$, or $\|\eta_*\| = \frac{1}{4}\left|e^{\frac{\pi i}{N}+\frac{2\pi i\lambda_1}{N}} + e^{\frac{2\pi i\lambda_2}{N}}\right| \leq \frac{1}{2\sqrt{2}}\sqrt{1+\cos\left(\frac{\pi}{N}\right)}$. Otherwise at least one of $\|\beta_*^{m_1}\|$ or $\|\beta_*^{m_2}\|$ is $\leq \frac{1}{2}$, and hence $\|\eta_*\| \leq \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$. So, if we choose $M = \max\left(\frac{3}{8}, \frac{1}{2\sqrt{2}}\sqrt{1+\cos(\frac{\pi}{N})}\right)$ then the assertion is true for $r = 1$. Assume that assertion (v) is satisfied for r. Let $\alpha_*, \beta_*^1, \beta_*^2, \ldots \in H_r$ and put $\eta_* = \sum_{k=1}^{\infty} \alpha_k\beta_*^k$. If $\|\alpha_*\| = 1$, then (i) shows that $\eta_* = t^{\lambda}\beta_*^j$. Thus $\|\beta_*^j\| < \frac{1}{2}$, and the induction hypothesis implies $\|\beta_*^j\| \leq M$, whence $\|\eta_*\| = \|\beta_*^j\| \leq M$. If $\|\alpha_*\| = \frac{1}{2}$, then (ii) implies that $\eta_* = t^{\lambda_1}\gamma_1\beta_*^{m_1} + t^{\lambda_2}\gamma_2\beta_*^{m_2}$. If $\|\beta_*^{m_1}\| = \|\beta_*^{m_1}\| = 1$ then the discussion of $r = 1$ shows that $\|\eta_*\| \leq M$ holds. If $\|\beta_*^{m_1}\| \leq 1, \|\beta_*^{m_2}\| \leq \frac{1}{2}$ we have $\|\eta_*\| \leq \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \leq M$; similarly for $\|\beta_*^{m_2}\| \leq 1, \|\beta_*^{m_1}\| \leq \frac{1}{2}$. Finally, if $\|\alpha_*\| < \frac{1}{2}$ then indeed $\|\alpha_*\| \leq M$ and hence $\|\eta_*\| \leq M$.

(i), (ii), and (iii) imply $\tau_\Gamma = \frac{1}{2}$, while (iv) and (v) show $\rho_\Gamma < \frac{1}{2}$.

A convexity theory Γ is said to satisfy the *pseudo-group condition* if

$$\text{for all } \lambda, \rho \in S_\Gamma \text{ with } 0 < |\lambda| < |\rho|,\ \lambda\rho^{-1} \in S_\Gamma \text{ holds.} \tag{PG}$$

Obviously we have

Lemma 1.10. *If the convexity theory Γ satisfies* (PG) *then M_Γ is a group and S_Γ equals $M_\Gamma S_\Gamma$.*

Proposition 1.11. *Let Γ be a convexity theory with* (PG). *If $0 \neq \rho \in S_\Gamma$ is an accumulation point of S_Γ then so is any other point $\neq 0$ of S_Γ.*

Proof. Suppose there is a sequence $\rho_i \in S_\Gamma, i = 1,2,\dots$, with $|\rho_i| \leq |\rho|$, $\rho_i \neq \rho$, and $\lim_{i\to\infty} \rho_i = \rho$. Then, by (PG), $\rho_i\rho^{-1} \in S_\Gamma, \rho_i\rho^{-1} \neq 1$, and $\lim_{i\to\infty} \rho_i\rho^{-1} = 1$. Similarly, if there is a sequence $\rho_i \in S_\Gamma, i = 1,2,\dots$, with $|\rho_i| \geq |\rho|, \rho_i \neq \rho$, and $\lim_{i\to\infty} \rho_i = \rho$, then $\rho\rho_i^{-1} \in S_\Gamma, \rho\rho_i^{-1} \neq 1$, and $\lim_{i\to\infty} \rho\rho_i^{-1} = 1$. But, whenever $1 \in S_\Gamma$ is an accumulation point of S_Γ, so is any other point $\neq 0$ of S_Γ.

At some point we will need the following strong pseudo-group condition

$$\text{for all } \rho \in S_\Gamma \text{ and all } \alpha_* \in \Gamma \text{ with } 0 < \|\alpha_*\| < |\rho|, \rho^{-1}\alpha_* := \left(\alpha_1\rho^{-1}, \alpha_2\rho^{-1}, \dots\right) \in \Gamma \text{ holds.} \tag{SPG}$$

Obviously, (SPG) implies (PG).

2. The Category of Γ-convex Spaces

Let Γ be a convexity theory and let X be a set. A Γ-structure on X (in the sense of[1], p. 48, or[5], p. 16) is a map $\Gamma \times X^{\boldsymbol{N}} \to X$. It is convenient to write this map as $(\alpha_*, \xi^*) \mapsto \sum_{i=1}^{\infty} \alpha_i \xi^i$. A set X equipped with a Γ-structure is also called a Γ-algebra.

Definition 2.1. A Γ-algebra X is called a *Γ-convex space*, if for all $\alpha_*, \beta_*^1, \beta_*^2, \dots$ in Γ and all ξ^* in $X^{\boldsymbol{N}}$ the following axioms are satisfied

$$\sum_{i=1}^{\infty} \delta_i^j \xi^i = \xi^j \tag{$\Gamma C1$}$$

$$\sum_{i=1}^{\infty} \alpha_i \left(\sum_{j=1}^{\infty} \beta_j^i \xi^j \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_i \beta_j^i \right) \xi^j. \tag{$\Gamma C2$}$$

($\Gamma C1$) is usually called the Projection Axiom, while ($\Gamma C2$) is called the Big Barycenter Axiom.

Definition 2.2. A map $f : X \to Y$ between Γ-convex spaces is said to be a *morphism of Γ-convex spaces* if for all $\alpha_* \in \Gamma$ and all $\xi^* \in X^{\boldsymbol{N}}$

$$f\left(\sum_{i=1}^{\infty} \alpha_i \xi^i \right) = \sum_{i=1}^{\infty} \alpha_i f(\xi^i)$$

holds.

The class of all Γ-convex spaces together with their morphisms (under set-theoretical composition) forms the category ΓC of Γ-*convex spaces*. Similarly one defines finitely Γ-convex space as Γ_{fin}-convex space, where $\Gamma_{\mathrm{fin}} = \Gamma \wedge \Omega_{\mathrm{fin}}$, and denotes the corresponding category by $\Gamma_{\mathrm{fin}}C$. These notations compare to the one used in[8] as follows:

$$TC_{\mathbb{C}} = \Omega C, TC_{\mathbb{R}} = \Omega_{\mathbb{R}}C, TC_{\mathrm{fin}} = \Omega_{\mathrm{fin}}C, TC_{\mathbb{R},\mathrm{fin}} = (\Omega_{\mathbb{R}} \wedge \Omega_{\mathrm{fin}})C.$$

Clearly, ΔC is canonically isomorphic with the category of sets, while $\Delta_o := \{0_*\} \cup \Delta$ furnishes the category $\Delta_o C$ that is canonically isomorphic with the category of pointed sets. $\Omega_{sc}C$ is called the category of superconvex spaces (see[11]), and $\Omega_c C$ is variously referred to as the category of barycentric spaces (see[4],[12]) or the category of convex spaces (see[11]). Finally, $\mathcal{P}C$ is called the category of positively convex spaces (see[6]).

The set $\Gamma C(X, Y)$ of morphisms of Γ-convex spaces from X to Y becomes a Γ-algebra by setting

$$\left(\sum_{i=1}^{\infty} \alpha_i f_i\right)(x) := \sum_{i=1}^{\infty} \alpha_i f_i(x).$$

It follows from Definition 2.2 and[8], (2.4), (ix), that this Γ-algebra is in fact a Γ-convex space $\mathrm{Hom}_{\Gamma}(X, Y)$.

The category ΓC is equationally defined and hence is an algebraic category. As such it has free objects on any set S. Such an object, $F_{\Gamma}(S)$, is obtained by the following construction (see[8], p. 959):

$$F_{\Gamma}(S) := \{f \in \hat{O}(\ell_1(S)) : \alpha(f)_* \in \Gamma\},$$

where $\alpha(f)_* \in \Omega$ is such that for some injection $\varphi : \operatorname{supp} f \to \boldsymbol{N}$,

$$\alpha(f)_i = \begin{cases} f(x), & \text{if } i = \varphi(x) \\ 0, & \text{otherwise.} \end{cases}$$

Note, that the condition $\alpha(f)_* \in \Gamma$ does not depend on the choice of $\varphi : \operatorname{supp} f \to \boldsymbol{N}$. We define an operation of Γ on $F_\Gamma(S)$ by restricting the operation of Ω on $\hat{O}(\ell_1(S))$ (see[8], p. 595) to Γ and $F_\Gamma(S)$. Clearly, this makes $F_\Gamma(S)$ a Γ-convex space. The map $S \to F_\Gamma(S)$ required for having a free object is the obvious one, namely $S \ni s \mapsto \delta^s \in F_\Gamma(S)$. Thus, if S is a finite set having n elements, the free object on S can be taken to be

$$F_\Gamma(n) := \{(\alpha_1, \dots, \alpha_n) : (\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in \Gamma\},$$

with the map from S to this set the obvious one and the operation of Γ the obvious one. From this one concludes easily

Corollary 2.3. (i) *The free Γ-convex space on the one-point set $\{1\}$ is the Γ-convex space S_Γ* (see (1.2), (iv)) *together with the map $\{1\} \ni 1 \mapsto 1 \in S_\Gamma$.*

(ii) *The free Γ-convex space on the countably infinite set $\boldsymbol{N}$ is the Γ-convex space Γ* (see (1.1), (ii)) *together with the map $\boldsymbol{N} \ni n \mapsto \delta^n_* \in \Gamma$.*

(iii) *The free Γ-convex space on the finite set $\{1, \dots, n\}$ is the Γ-convex subspace $\{(\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in \Gamma\}$ of Γ together with the map $\{1, \dots, n\} \ni i \mapsto \delta^i_* \in \{(\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in \Gamma\}$.*

Given an convexity theory Γ it is important to establish computational rules that are valid for all Γ-convex spaces. For this purpose we choose a countable set $\{u_1, u_2, \dots\}$ of variables. We want to define inductively the notion of n^{th} level term over Γ in the variables $\{u_1, u_2, \dots\}$. The 0^{th} level terms are, by definition, the variables. Given what the n^{th} level terms over Γ are, the $(n+1)^{st}$ level terms are defined to be the formal expressions

$$\tau^{(n+1)} := \sum_{i=1}^{\infty} \alpha_i \tau_i^{(\ell_i)}$$

where $\alpha_* \in \Gamma$ and $\tau_i^{(\ell_i)}$ is a ℓ_i^{th} level term over Γ in the variables $\{u_1, u_2, \dots\}$, with $\ell_i \leq n$. By an equation over Γ in the variables $\{u_1, u_2, \dots\}$ we mean a formal equation $\tau = \tau'$, where τ and τ' are finite-level terms over Γ in the variables $\{u_1, u_2, \ldots\}$. We say that an equation $\tau = \tau'$ over Γ in the variables $\{u_1, u_2, \ldots\}$ is an identity (or computational rule) for all Γ-convex spaces, if for every Γ-convex space X and every map $\varphi : \{u_1, u_2, \ldots\} \to X$ the equation obtained from $\tau = \tau'$ by substituting for each u_i the element $\varphi(u_i) \in X$ is valid in X. For example, $\sum_{i=1}^{\infty} \delta_i^j u_i = u_j$ is such an equation (i.e. computational rule), while $\sum_{i=1}^{\infty} \delta_i^j u_i = u_{j+1}$ is not.

<u>Metatheorem</u> 2.4. *Let $\tau = \tau'$ be an equation over Γ in the variables $\{u_1, u_2, \ldots\}$ such that the equations obtained from $\tau = \tau'$ by substitution for each u_i the element $\delta_*^i \in \Omega$ is valid in Ω. Then $\tau = \tau'$ is a computational rule for all Γ-convex spaces.*

<u>Proof</u>. By substituting, in τ and τ', for each u_i the element $\delta_*^i \in \Gamma \subseteq \Omega$ we obtain two elements $\overline{\tau}$ and $\overline{\tau}'$ of Γ that are equal. Since Γ is the free ΓC-object on the set $\boldsymbol{N}$ by Corollary 2.3, (ii), there is, for every map φ from $\{u_1, u_2, \ldots\}$ to the Γ-convex space X, a unique ΓC-morphism $\psi : \Gamma \to X$ with $\psi(\delta_*^i) = \varphi(u_i)$.$\psi$ maps $\overline{\tau}$ (resp. $\overline{\tau}'$) to the element $\tilde{\tau}$ (resp. $\tilde{\tau}'$) of X that is obtained from τ (resp. τ') by substituting for each u_i the element $\varphi(u_i) \in X$. Since $\overline{\tau} = \overline{\tau}'$ we have $\tilde{\tau} = \tilde{\tau}'$ as had to be shown.

As an immediate consequence of Metatheorem 2.4 we have

<u>Corollary</u> 2.5. Γ *be a convexity theory. Then the computational rules* (2.4) *and* (2.12) *of*[8] *are also computational rules for all* Γ*-convex spaces, except that* (2.4), (v)-(vii), *and* (2.12), (ii), *require a convexity theory with zero.*

To the rules (2.4) and (2.12) of[8] we add another one – dubbed (2.4), (x) – which also falls under the purview of the current (2.5):

(2.4), (x) (see[13], (7.1)):

Let $f, g : \boldsymbol{N} \to \boldsymbol{N}$ be set maps such that $(f, g) : \boldsymbol{N} \to \boldsymbol{N} \times \boldsymbol{N}$ is a bijection. Then, for all $\alpha_*, \beta^i_* \in \Omega$ and $\zeta^i_* \in X^N$

$$\sum_{i=1}^{\infty} \alpha_i \left(\sum_{j=1}^{\infty} \beta^i_j \zeta^i_j \right) = \sum_{t=1}^{\infty} \alpha_{f(t)} \beta^{f(t)}_{g(t)} \zeta^{f(t)}_{g(t)}.$$

Proof.

$$\sum_{i=1}^{\infty} \alpha_i \left(\sum_{j=1}^{\infty} \beta^i_j \zeta^i_j \right) = \sum_{i=1}^{\infty} \alpha_i \left(\sum_{t=1}^{\infty} \delta^i_{f(t)} \beta^{f(t)}_{g(t)} \zeta^{f(t)}_{g(t)} \right)$$
$$= \sum_{t=1}^{\infty} \left(\sum_{i=1}^{\infty} \delta^i_{f(t)} \alpha_i \beta^{f(t)}_{g(t)} \zeta^{f(t)}_{g(t)} \right) = \sum_{t=1}^{\infty} \alpha_{f(t)} \beta^{f(t)}_{g(t)} \zeta^{f(t)}_{g(t)}.$$

As in[8], §5, we obtain

Proposition 2.6. ΓC *is an autonomous category in the sense of*[3], *i.e. it possesses a tensor product –* $\otimes_\Gamma$ *– which, together with the coherent morphisms, makes* ΓC *into a symmetric monoidal closed category.*

It should be noted that the explicit construction of the tensor product in $TC = \Omega C$, as given in[8], §5, works in ΓC just as well.

Let $\Gamma' \subseteq \Gamma$ be convexity theories. Then there is an obvious functor $U^{\Gamma}_{\Gamma'} : \Gamma C \to \Gamma' C$. It assigns to each Γ-convex space X the Γ'-convex space $U^{\Gamma}_{\Gamma'}(X)$ obtained from X by restricting the operators from Γ to Γ' and by leaving unchanged as set map each morphism of Γ-convex spaces. Apparently we have

$$U^{\Gamma'}_{\Gamma''} \circ U^{\Gamma}_{\Gamma'} = U^{\Gamma}_{\Gamma''} \quad , \text{ for } \quad \Gamma'' \subseteq \Gamma' \subseteq \Gamma. \tag{2.7}$$

$U^{\Gamma}_{\Gamma'}$ is called the *forgetful functor* from ΓC to $\Gamma' C$. An easy verification leads to

Theorem 2.8. $U^{\Gamma}_{\Gamma'}$ *has a left adjoint* $F^{\Gamma'}_{\Gamma}$, *for all* $\Gamma' \subseteq \Gamma$.

Here we want to provide an explicit construction for the *completion functor* $F^{\Gamma'}_{\Gamma}$ from $\Gamma' C$ to ΓC. For this, let X' be a Γ'-convex space with underlying set $|X'|$.

Form $F_\Gamma(|X'|)$ and denote by $\sim$ the smallest Γ-congruence relation on $F_\Gamma(|X'|)$ that contains the elements

$$\left(\delta \sum_i \alpha_i \xi'^i, \; \sum_i \alpha_i \delta^{\xi'^i}\right) \quad , \; \alpha_\bullet \in \Gamma \; \text{ and } \; \xi^\bullet \in |X'|^N.$$

Then $F_\Gamma^{\Gamma'}(X') := F_\Gamma(|X'|)/\sim$ is a Γ-convex space and the map

$$\eta_{X'} : X' \xrightarrow{\delta} F_\Gamma(|X'|) \xrightarrow{\text{can}} F_\Gamma^{\Gamma'}(X') \tag{2.9}$$

is a morphism of Γ'-convex spaces from X' to $U_{\Gamma'}^{\Gamma}\left(F_\Gamma^{\Gamma'}(X')\right)$. A routine argument shows that, for each $X' \in \Gamma' C$, $\eta_{X'} : X' \to F_\Gamma^{\Gamma'}(X')$ is a universal arrow whence we have an explicit construction for $F_\Gamma^{\Gamma'}$.

The obvious forgetful functor from ΓC to *Sets* is denoted by U^Γ, or simply U if the particular choice of Γ is clear. Note that via the isomorphism $\Delta C \approx$ *Sets*, U_Δ^Γ and U^Γ are isomorphic.

3. Examples of Congruence Relations

1. Let Γ be a convexity theory of length 1. Let $\sim$ be a congruence relation on the Γ-space S_Γ and denote the associated partition of S_Γ by $\{P_i : i \in I\}$. Then there is a map $q : S_\Gamma \times I \to I$ such that

$$\lambda \cdot P_i \subset P_{q(\lambda,i)}, \quad \text{for all} \quad \lambda \in S_\Gamma, i \in I. \tag{3.1}$$

Conversely, if a partition $\{P_i : i \in I\}$ of S_Γ and a map $q : S_\Gamma \times I \to I$ are given such that (3.1) is satisfied, then the equivalence relation determined by this partition is in fact a congruence relation. It is easy to exhibit examples of such partitions in case Γ is normable. Let $0 < \eta < 1$ and put

$$P_o := S_\Gamma \cap \{z : |z| \leq \eta\}, \quad P_\rho := \{\rho\} \quad \text{where} \quad \rho \in S_\Gamma \cap \{z : |z| > \eta\};$$

the obvious choice of $q : S_\Gamma \times (\{0\} \cup \{\rho \in S_\Gamma : |\rho| > \eta\}) \to \{0\} \cup \{\rho \in S_\Gamma : |\rho| > \eta\}$ will satisfy (3.1).

2. There are also partitions that do not fit into the previous example. In order to produce one, assume that the length of Γ is 1 and that $S_\Gamma := \{n^{-1} : n \in \boldsymbol{N}\}$. Let $\mathcal{P}$ be any set of prime numbers and put

$$P_{\mathcal{P}} := \left\{\prod p^{-e(p)} : p \in \mathcal{P}, e(p) \in \boldsymbol{N}\right\};$$

in particular, $P_\phi = \{1\}$. Then the map $q : S_\Gamma \times \{\mathcal{P}\} \to \{\mathcal{P}\}$ given by

$$q(n^{-1}, \mathcal{P}) := \mathcal{P} \cup \{p : p|n \text{ and } p \text{ prime}\}$$

satisfies (3.1).

Next, let Γ be a convexity theory of length > 1. Let $\sim$ be a congruence relation on the Γ-convex space S_Γ and denote the associated partition of S_Γ by $\{P_i : i \in I\}$. Then there is a map $q : \Gamma \times I^{\boldsymbol{N}} \to I$ such that

$$\sum_k \alpha_k \rho_{\varphi(k)} \in P_{q(\alpha_*, \varphi)}, \text{ for all } \alpha_* \in \Gamma, \varphi \in I^{\boldsymbol{N}}, \rho_{\varphi(k)} \in P_{\varphi(k)}. \tag{3.2}$$

And, again, the converse is true. But it is more complicated to give examples of such partitions. However, if $\tau_\Gamma < 1$ holds, then for any choice of η with $\tau_\Gamma \leq \eta < 1$, the partition of S_Γ given by

$$P_o := S_\Gamma \cap \{z : |z| \leq \eta\}, \quad P_\rho := \{\rho\} \text{ where } \rho \in S_\Gamma \cap \{z : |z| > \eta\}$$

satisfies (3.2) with the following choice of q :

$$q(\alpha_*, \varphi) := \begin{cases} \sum_k \alpha_k \rho_{\varphi(k)} & \text{if card (supp}(0_*)) = 1 \text{ and } \left|\sum_k \alpha_k \rho_{\varphi(k)}\right| \geq \eta \\ 0 & \text{if card (supp}(0_*)) = 1 \text{ and } \left|\sum_k \alpha_k \rho_{\varphi(k)}\right| \leq \eta \\ 0 & \text{if card (supp}(\alpha_*)) > 1 , \end{cases}$$

as is easily checked. Of course, there are convexity theories with length > 1 that have the above property. For instance,

$$\Gamma := \{z\delta_*^j : |z| \leq 1 \text{ and } j = 1, 2, \dots\} \cup \{\alpha_* \in \Omega : \|\alpha_*\| \leq \eta\}$$

does, as is easily checked. Note also, that we obtain examples for (3.1) and (3.2) by defining P_o through " $<$ " rather than " $\leq$ " and by delimiting the remaining P_ρ by " $\geq$ " rather than " $>$ ".

3. Equip S_Γ with the stated congruence relation $\sim$, and choose $x \in P_o$ and $y = 1$. Then a simple argument shows

$$\{\rho \in S_\Gamma : \rho x \sim \rho y\} = S_\Gamma \cap \{z : |z| \leq \eta\}.$$

Of course, in this equation the disk $\{z : |z| \leq \eta\}$ can be replaced by the larger disk $\{z : |z| \leq \eta'\}$ resp. $\{z : |z| < \eta'\}$ provided that

$$S_\Gamma \cap \{z : \eta < |z| \leq \eta'\} = \phi \text{ resp. } S_\Gamma \cap \{z : \eta < |z| < \eta'\} = \phi.$$

Hence, if we assume that our convexity theory has 1 as an accumulation point of $\{|\rho| : \rho \in S_\Gamma\}$, then there are infinitely many $0 < \eta' < 1$ such that for some congruence relation $\sim$ on S_Γ and some x, $y \in S_\Gamma$,

$$\{\rho \in S_\Gamma : \rho x \sim \rho y\} = S_\Gamma \cap \{z : |z| \leq \eta'\}$$

and that any two such sets are mutually distinct. In fact, if the cardinality of $\{|\rho| : \rho \in S_\Gamma \text{ and } \eta < |\rho| < 1\}$ is K then there are K distinct sets $S_\Gamma \cap \{z : |z| \leq \eta'\}$ that are equal to $\{\rho \in S_\Gamma : \rho x \sim \rho y\}$ for an appropriate congruence relation $\sim$ on S_Γ and suitable x, $y \in S_\Gamma$. These congruence relations occur if either $lg\Gamma = 1$ or $lg\Gamma > 1$ and $S_\Gamma \cap \{z : \tau_\Gamma < |z| < 1\} \neq \phi$. The example presented at the end of §1, by comparison, satisfies $lg\Gamma = \infty$ and $S_\Gamma \cap \{z : \tau_\Gamma < |z| < 1\} = \phi$. For this particular choice of Γ, partition S_Γ by

$$P_0 := S_\Gamma \cap \{z : |z| \leq \rho_\Gamma\} \ , \ P_\rho := \{\rho\} \text{ where } \rho \in S_\Gamma \cap \{z : |z| > \rho_\Gamma\}.$$

By defining

$$q(\alpha_*, \varphi) := \begin{cases} \alpha_k \rho_{\varphi(k)} & \text{if } \operatorname{supp}(\alpha_*) = \{k\} \text{ and } |\rho_{\varphi(k)}| > \rho_\Gamma \\ 0 & \text{otherwise.} \end{cases}$$

Again it is easy to check that (3.2) is satisfied. Choosing $x \in P_0$ and $y = 1$ we get

$$\{\alpha : \alpha x \sim \alpha y\} = \{\rho \in S_\Gamma : |\rho| \leq \rho_\Gamma\}.$$

4. Let $0 \leq b \leq 1$ and put

$$\Omega_{b)} := \{\alpha_* \in \Omega : \|\alpha_*\| < b\} \cup \Delta,$$
$$\Omega_{b]} := \{\alpha_* \in \Omega : \|\alpha_*\| \leq b\} \cup \Delta.$$

An easy computation shows that both $\Omega_{b)}$ and $\Omega_{b]}$ are convexity theories. Clearly,

$$S_{\Omega_{b)}} = \{z \in \mathbb{C} : |z| < b\} \cup \{1\},$$
$$S_{\Omega_{b]}} = \{z \in \mathbb{C} : |z| \leq b\} \cup \{1\}.$$

A simple application of definition (5.1) shows that

$$\|\rho\|_{\Omega_{b)}} = |\rho| \quad , \text{ for all } \rho \in S_{\Omega_{b)}},$$
$$\|\rho\|_{\Omega_{b]}} = |\rho| \quad , \text{ for all } \rho \in S_{\Omega_{b]}}.$$

4. The Spread Of A Convexity Theory

Let Γ be a convexity theory and let $\alpha \in S_\Gamma$. Denote by I_α the class of all quadruples $(X, \sim, x, y)$ where $X \in \Gamma C$, $\sim$ is a Γ-congruence relation on X, and $x, y \in X$ such that $\alpha x \sim \alpha y$. I_α is always non-empty as we can choose $x = y$. We put

$$\sigma_\Gamma(\alpha) := \cap \{\{\rho \in S_\Gamma : \ \rho x \sim \rho y\} : \ (X, \sim, x, y) \in I_\alpha \ \text{ for some } \ X \in \Gamma C\}.$$

Let $F_\Gamma(\{x_0, y_0\})$ be the free Γ-convex space on the two-point set $\{x_0, y_0\}$. Given any $(X, \sim, x, y) \in I_\alpha$ there is a unique Γ-morphism $\pi : F_\Gamma(\{x_0, y_0\}) \to X$ satisfying $\pi(\delta_{x_0}) = x$ and $\pi(\delta_{y_0}) = y$. Define, for $f, g \in F_\Gamma(\{x_0, y_0\})$ the relation "$f \underset{\pi}{\sim} g$" by

"$\pi(f) \sim \pi(g)$". An easy verification shows that "$f \underset{\pi}{\sim} g$" is a Γ-congruence relation on $F_\Gamma(\{x_0, y_0\})$ and that

$$\left\{\rho \in S_\Gamma : \rho\delta_{x_0} \underset{\pi}{\sim} \rho\delta_{y_0}\right\} = \left\{\rho \in S_\Gamma : \rho x \sim \rho y\right\}$$

holds. Hence we have

Lemma 4.1. *For all* $\alpha \in S_\Gamma$

$$\sigma_\Gamma(\alpha) = \cap\left\{\{\rho \in S_\Gamma : \rho\delta_{x_0} \sim \rho\delta_{y_0}\right\} : (F_\Gamma(\{x_0, y_0\}), \sim, \delta_{x_0}, \delta_{y_0}) \in I_\alpha\right\}.$$

Furthermore, a simple argument shows

Lemma 4.2. *The following assertions are satisfied for all* $\alpha \in S_\Gamma$

(i) $S_\Gamma \cdot \sigma_\Gamma(\alpha) \subseteq \sigma_\Gamma(\alpha)$ *and* $\sigma_\Gamma(1) = S_\Gamma$,

(ii) *if* Γ *is a convexity theory with zero, then* $0 \in \sigma_\Gamma(\alpha)$ *and* $\sigma_\Gamma(0) = \{0\}$,

(iii) *if* Γ *is a convexity theory with* (PG) *then*

$$S_\Gamma \cap \{z : |z| \leq |\alpha|\} \subseteq \sigma_\Gamma(\alpha)$$

and

$$\sigma_\Gamma(\alpha) \subseteq \sigma_\Gamma(\beta) \quad \textit{whenever} \quad |\alpha| \leq |\beta|.$$

The *spread* σ_Γ of a convexity theory is defined as the supremum in $[-\infty, +1]$ of all μ, with $0 < \mu < 1$, such that

$$\left\{\rho \in S_\Gamma : |\rho| \leq \mu\right\} \subseteq \cap\left\{\sigma_\Gamma(\alpha) : 0 \neq \alpha \in S_\Gamma\right\}. \tag{4.3}$$

Proposition 4.4. *The following statements are valid for all convexity theories*

(0) $\ell g\Gamma = 1$ *implies* $\rho_\Gamma = \sigma_\Gamma = \tau_\Gamma = -\infty$,

(i) $\sigma_\Gamma \leq \tau_\Gamma$,

(ii) $\Gamma' \subseteq \Gamma$ *implies* $\rho_{\Gamma'} \leq \rho_\Gamma$, $\sigma_{\Gamma'} \leq \sigma_\Gamma$, $\tau_{\Gamma'} \leq \tau_\Gamma$,

(iii) $\rho_{\wedge\{\Gamma_i:i\in I\}} \leq \inf\{\rho_{\Gamma_i} : i \in I\}$
$\sigma_{\wedge\{\Gamma_i:i\in I\}} \leq \inf\{\sigma_{\Gamma_i} : i \in I\}$
$\tau_{\wedge\{\Gamma_i:i\in I\}} \leq \inf\{\tau_{\Gamma_i} : i \in I\}$,

(iv) $\sup\{\rho_{\Gamma_i} : i \in I\} \leq \rho_{\vee\{\Gamma_i:i\in I\}}$
$\sup\{\sigma_{\Gamma_i} : i \in I\} \leq \sigma_{\vee\{\Gamma_i:i\in I\}}$
$\sup\{\tau_{\Gamma_i} : i \in I\} \leq \tau_{\vee\{\Gamma_i:i\in I\}}$.

Proof. (0) and (i) are true for $\ell g\Gamma = 1$ as the first example of §3 shows. (i) is evidently true for $\ell g\Gamma > 1$ and $\tau_\Gamma = 1$; it follows for $\ell g\Gamma > 1$ and $\tau_\Gamma < 1$ from the third example in §3. (ii) is a consequence of (4.3) and the fact that each Γ-convex space is a Γ'-convex space via restriction of operators. (ii), in turn, implies (iii) and (iv).

Theorem 4.5. *Let Γ be a convexity theory with $\ell g\Gamma > 1$ and* (PG) *such that for all $\rho_0 \in \{|\rho| : \rho \in S_\Gamma\}$ with $0 < \rho_0 < \rho_\Gamma$ there is a $\alpha_* \in \Omega$ satisfying*

(i) $\mathrm{card}(\mathrm{supp}\,\alpha_*) < \infty$,

(ii) *for all $i \in \mathrm{supp}\,\alpha_*$ and all $\rho \in S_\Gamma$ with $|\rho|$ sufficiently close to* ρ_0, $(\alpha_1, \dots \alpha_{i-1}, \alpha_i\rho^{-1}, \alpha_{i+1}, \dots) \in \Gamma$,

(iii) $\rho_0 < |S(\alpha_*)|$.

Then $\rho_\Gamma \leq \sigma_\Gamma$.

Proof. It follows from (PG) that $\rho x \sim \rho y$, for some $0 \neq \rho \in S_\Gamma$, implies

$$\{\rho' \in S_\Gamma : |\rho'| \leq |\rho|\} \subseteq \{\sigma \in S_\Gamma : \sigma x \sim \sigma y\}.$$

Let $\rho_0 := \sup\{|\sigma| : \sigma \in S_\Gamma \text{ and } \sigma x \sim \sigma y\}$. If $\rho_0 < \rho_\Gamma$ then (ii) implies, that for all $\rho \in S_\Gamma$ with $|\rho|$ sufficiently close to ρ_0 and all $i \in \mathrm{supp}\,\alpha_*$,

$$\alpha_* = \alpha_1\delta_*^1 + \cdots + \alpha_{i-1}\delta_*^{i-1} + \alpha_i\rho^{-1} \cdot \rho\delta_*^i + \alpha_{i+2} \cdot \delta_*^{i+1} + \cdots \in \Gamma$$

holds. Hence

$$\begin{aligned} S(\alpha_*)x &= \alpha_1 x + \cdots + \alpha_{i-1}x + \alpha_i x + \alpha_{i+1}x + \cdots \\ &= \alpha_1 x + \cdots + \alpha_{i-1}x + \alpha_i \rho^{-1} \cdot \rho x + \alpha_{i+1}x + \cdots \\ &\sim \alpha_1 x + \cdots + \alpha_{i-1}x + \alpha_i \rho^{-1} \cdot \rho y + \alpha_{i+1}x + \cdots \\ &= \alpha_1 x + \cdots + \alpha_{i-1}x + \alpha_i y + \alpha_{i+1}x + \cdots \quad . \end{aligned}$$

By assumption, $\operatorname{supp}\alpha_*$ is finite. Thus, by repeating this argument for the various elements of $\operatorname{supp}\alpha_*$, we obtain finally $S(\alpha_*)x \sim S(\alpha_*)y$, contradicting the assumption $\rho_0 < \rho_\Gamma$. Hence Theorem 4.5 is proven.

Corollary 4.6. *Let* Γ *be a convexity theory with* (PG) *such that* $\Omega^+_{\mathrm{fin}\,\mathbb{Q}} \subseteq \Gamma$. *Then* $0 < \sigma_\Gamma$ *implies* $\sigma_\Gamma = 1$.

Proof. Since $S^+_{\Omega_{\mathrm{fin},\mathbb{Q}}} = \{\rho \in \mathbb{Q} : 0 \leq \rho \leq 1\}$, the ρ_0 in the proof of Theorem 4.5 is only subject to $\rho_0 \in \mathbb{Q}$, $0 < \rho_0 < 1$. Given such a ρ_0, choose $\beta \in \mathbb{Q}$ with $\frac{\rho_0}{2-\rho_0} < \beta < \rho_0$, and put $\alpha_* = \left(\frac{\beta}{1+\beta}, \frac{\beta}{1+\beta}, 0, 0, \ldots\right)$. An easy computation shows that α_* satisfies the conditions of Theorem 4.5. Hence the assertion follows from Proposition 4.4 (ii), and Theorem 4.5.

Corollary 4.7. *Let* Γ *be a convexity theory with zero and* (PG) *such that*

(i) *there exists a* $\bar\beta_* \in \Gamma$ *with* $|S(\bar\beta_*)| = 1$ *and* $\operatorname{card}(\operatorname{supp}\bar\beta_*) \geq 2$,

(ii) *for* $\gamma_* \in \Gamma$ *and* $\sigma \in S_\Gamma$ *with* $\|\gamma_*\| + |\sigma| \leq 1, (\sigma, \gamma_1, \gamma_2, \ldots) \in \Gamma$ *holds.*

Then $0 < \sigma_\Gamma$ *implies* $\sigma_\Gamma = 1$.

Proof. $|S(\bar\beta_*)| = 1$ implies that $\bar\beta_*$ is a scalar multiple of some element of $\mathcal{P}$. Since Γ satisfies (PG), we may assume that $\bar\beta_* \in \Gamma \cap \mathcal{P}$ holds. We also may assume $\bar\beta_1 \neq 0$. Now replace $\bar\beta_*$ by $\beta'_* := \bar\beta_1 \delta^1_* + \bar\beta_2 \delta^2_* + \cdots + \bar\beta_n \delta^2_* \cdots = (b, 1-b, 0, 0, \ldots)$, with $0 < b < 1$. Define inductively $\beta^{(1)}_* := \beta^1_*$, and

$$\beta^{(n)}_* := b\beta^{(n-1)}_* + (1-b)\left(\underbrace{0, \ldots, 0}_{2^{n-1}-\text{times}}, \beta^{(n-1)}_1, \ldots, \beta^{(n-1)}_{2^{n-1}}, 0, 0, \ldots \right).$$

Then $\beta_*^{(n)} \in \Gamma \cap \mathcal{P}$ and

$$2 \le \operatorname{card}(\operatorname{supp}(\beta_*^{(n)})) < \infty, \quad S(\beta_*^{(n)}) = 1, \quad |\beta_i^{(n)}| \le (\max(b, 1-b))^i \text{ for all } i.$$

In particular, if $\varepsilon > 0$ is given, there is a $\tilde{\alpha}_* \in \Gamma \cap \mathcal{P}$ with

$$2 \le \operatorname{card}(\operatorname{supp}(\tilde{\alpha}_*)) < \infty, \quad S(\tilde{\alpha}_*) = 1, \quad |\tilde{\alpha}_i| < \varepsilon \text{ for all } i.$$

Hence, given $0 < \sigma < 1$ and $\varepsilon > 0$, there is an index j such that

$$\alpha'_* := \tilde{\alpha}_1 \delta_*^1 + \cdots + \tilde{\alpha}_j \cdot \delta_*^1 + \tilde{\alpha}_{j+1} \delta_*^2 + \cdots = (a, 1-a, 0, 0, \dots) \in \Gamma \cap \mathcal{P}$$

satisfies

$$S(\alpha'_*) = 1, \qquad 0 < \sigma - \varepsilon \le a < \sigma.$$

Now we are ready to construct the sequences α_* required in Theorem 4.5. Let $0 < \rho_0 \le \frac{1}{2}$. Choose n such that $1 - (1-\rho_0)^n > \rho_0$ – indeed $n = 2$ will do – and find $\varepsilon > 0$ such that for all $0 \le x \le \varepsilon$, $1 - (1 - (\rho_0 - x))^n > \rho_0$ holds. With this ε and with $\sigma - \rho_0$, find α'_* as outlined above. Then obtain, as above, $\alpha_*^{(n)}$ and set $\alpha_* = \alpha_1^{(n)} \delta_*^1 + \cdots + \alpha_{2^n-1}^{(n)} \delta_*^{2^n-1} + \alpha_{2^n}^{(n)} 0_* + \alpha_{2^n+1}^{(n)} 0_* + \cdots \in \Gamma \cap \mathcal{P}$. Then we have

$$S(\alpha_*) = 1 - (1-a)^n \ge 1 - (1-\rho_0)^n > \rho_0$$

and

$$\|(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots)\| + |\alpha_i \rho_0^{-1}| = 1 - (1-a)^n + a^j (1-a)^{n-j} \left(\rho_0^{-1} - 1\right) \quad (4.8)$$

for some $0 < j \le n$. Since

$$-(1-a)^n + a^j (1-a)^{n-j} \left(\rho_0^{-1} - 1\right) = -(1-a)^n \left(1 - \left(\frac{a}{1-a}\right)^j \left(\frac{1-\rho_0}{\rho_0}\right)\right)$$

and since

$$\frac{1-\rho_0}{\rho_0} < \left(\frac{1-a}{a}\right)^j, \qquad 0 < j \le n,$$

we have that (4.8) is less than one. This remains true if we replace ρ_0 by ρ with $\rho \in S_\Gamma$ and $|\rho|$ sufficiently close to ρ_0. Therefore (ii) implies that $(\alpha_1, \dots, \alpha_{i-1}, \alpha_i\rho^{-1}, \alpha_{i+1}, \dots)$ is in Γ. Thus, conditions (i), (ii), (iii) of Theorem 4.5 are verified for all $0 < \rho_0 \leq \frac{1}{2}$. Finally assume $\frac{1}{2} < \rho_0 < 1$. Choose n such that $1-(1-\frac{1}{2})^n > \rho_0$ and find $\varepsilon > 0$ such that for all $0 \leq x \leq \varepsilon$, $1-\left(1-\left(\frac{1}{2}-x\right)\right)^n > \rho_0$ holds. With this ε and with $\sigma = \frac{1}{2}$ repeat the construction given in the case $0 < \rho_0 \leq \frac{1}{2}$. Then the previous estimates remain in force and the conditions (i), (ii), (iii) of Theorem 4.5 are verified for $\frac{1}{2} < \rho_0 < 1$. Thus, Corollary 4.7 follows from Theorem 4.5.

5. The Semi – Norm of a Γ–convex Space

Definition 5.1. Let Γ be a convexity theory and let X be a Γ-convex space. For $x \in X$ we put

$$\|x\|_\Gamma := \inf\{|\lambda| : x = \lambda x' \ \text{ where } \ \lambda \in S_\Gamma \ \text{ and } \ x' \in X\}.$$

$\|x\|_\Gamma$ is called the Γ-semi-norm (or simply: semi-norm) of x in X, and $\|x\|_\Omega$ is denoted by $\|x\|$. Evidently, $\|x\|_\Gamma \leq 1$. Moreover $\|0\|_\Gamma = 0$ in case Γ is a convexity theory with zero. Evidently, Γ is non-normable if and only if for all Γ-convex spaces X and all $x \in X$, $\|x\|_\Gamma = 0$ or $= 1$; clearly, $\|0\|_\Gamma = 0$.

Let $\Gamma' \subseteq \Gamma$ be convexity theories and let X be a Γ-convex space. Then X and $U^\Gamma_{\Gamma'}(X)$ have the same underlying set. Hence an element $x \in X$ may be regarded as an element of $U^\Gamma_{\Gamma'}(X)$.

Proposition 5.2. *For all $\Gamma' \subseteq \Gamma$, and all $X \in \Gamma C$ and $x \in X$, $\|x\|_\Gamma \leq \|x\|_{\Gamma'}$. In particular, if X is a totally convex space and $x \in X$, then $\|x\| \leq \|x\|_\Gamma$, for all convexity theories Γ.*

Proof. Obvious.

Proposition 5.3. *Let X be a Γ-convex space and let $x \in X$. Then*

$$\|\alpha x\|_\Gamma \leq |\alpha| \, \|x\|_\Gamma \quad , \ \textit{for all} \ \ \alpha \in S_\Gamma.$$

If, additionally, M_Γ *is a group then*

$$\|\alpha x\|_\Gamma = |\alpha|\,\|x\|_\Gamma = \|x\|_\Gamma \quad , \; \textit{for all} \;\; \alpha \in M_\Gamma .$$

Proof. Obvious.

Corollary 5.4. *Let* Γ *be a convexity theory with* (PG), *let* X *be a* Γ*-convex space and* $x \in X$. *Then*

$$\frac{\|\beta x\|_\Gamma}{|\beta|} \le \frac{\|\alpha x\|_\Gamma}{|\alpha|} \quad , \; \textit{for all} \;\; 0 \ne \alpha \in S_\Gamma \;\; \textit{and} \;\; 0 \ne \beta \in \sigma_\Gamma(\alpha).$$

If, in addition, $\sigma_\Gamma > 0$ *then there is a* $0 \le s(x) \le 1$ *such that*

$$\|\alpha x\|_\Gamma = s(x)|\alpha|\,\|x\|_\Gamma \quad , \; \textit{for all} \;\; \alpha \in S_\Gamma \cap \{z : |z| < \sigma_\Gamma\}.$$

If $\|x\|_\Gamma \ne 0$ *then* x *determines* $s(x)$ *uniquely; if* $0 < \|x\|_\Gamma < \sigma_\Gamma$ *then* $s(x) = 1$.

Proof. If $\|\alpha x\|_\Gamma = |\alpha|\,\|x\|_\Gamma$, then for all $\beta \in S_\Gamma$

$$\|\beta x\|_\Gamma \le |\beta|\,\|x\|_\Gamma = |\beta|\frac{\|\alpha x\|_\Gamma}{|\alpha|}$$

whence the stated inequality is satisfied. If $\|\alpha x\|_\Gamma < |\alpha|\,\|x\|_\Gamma$, then $\alpha \ne 0$ and $\|x\|_\Gamma \ne 0$. By definition of $\|\;\|_\Gamma$ there is a $\lambda \in S_\Gamma$ such that

$$\|\alpha x\|_\Gamma < |\lambda| < |\alpha|\,\|x\|_\Gamma \le |\alpha| \;\; \text{and} \;\; \alpha x = \lambda y, \;\; \text{for some} \;\; y \in X.$$

Since Γ satisfies (PG) and since $|\lambda| < |\alpha|$ we have $\lambda\alpha^{-1} \in S_\Gamma$. Therefore $\alpha x = \lambda y = \alpha \cdot \lambda\alpha^{-1} y$. Since $\beta \in \sigma_\Gamma(\alpha)$ we have $\beta x = \beta \cdot \lambda\alpha^{-1} y$ and thus

$$\|\beta x\|_\Gamma = \|\beta \cdot \lambda\alpha^{-1} y\|_\Gamma \le |\beta|\frac{|\lambda|}{|\alpha|}.$$

This implies the stated inequality. If $\sigma_\Gamma > 0$ and both α and β are in $S_\Gamma \cap \{z : |z| < \sigma_\Gamma\}$ then the roles of α and β can be reversed and we obtain

$$\frac{\|\alpha x\|_\Gamma}{|\alpha|} = \frac{\|\beta x\|_\Gamma}{|\beta|} ,$$

which implies the second assertion. Obviously, $\|x\|_\Gamma \neq 0$ implies the uniqueness of $s(x)$. Finally, assume $0 < \|x\|_\Gamma < \sigma_\Gamma$. Then we have $x = \lambda y$, for some $\|x\|_\Gamma \leq |\lambda| < \sigma_\Gamma$ and $y \in X$. Let $\alpha \in S_\Gamma \cap \{z : |z| < \sigma_\Gamma\}$. Then

$$\|\alpha x\|_\Gamma = \|\alpha\lambda y\|_\Gamma = |\alpha|\,|\lambda| s(y)\|y\|_\Gamma = |\alpha|\,\|\lambda y\|_\Gamma = |\alpha|\,\|x\|_\Gamma.$$

Proposition 5.5. *Let $f : X \longrightarrow Y$ be a morphism of Γ-convex spaces. Then*

$$\|f(x)\|_\Gamma \leq \|x\|_\Gamma \quad , \quad \textit{for all} \quad x \in X.$$

Proof. $x = \lambda x'$, with $\lambda \in S_\Gamma$ and $x' \in X$, implies $f(x) = \lambda f(x')$.

Proposition 5.6. *Let $f : X \longrightarrow Y$ be a surjective morphism of Γ-convex spaces. Then*

$$\|y\|_\Gamma = \inf\left\{\|x\|_\Gamma : x \in f^{-1}(y)\right\} \quad , \quad \textit{for all} \quad y \in Y.$$

Proof. Proposition 5.5 implies $\|y\|_\Gamma \leq \inf\left\{\|x\|_\Gamma : x \in f^{-1}(y)\right\}$. Suppose that for some $y \in Y$ this inequality would be strict. Then there would be a $\lambda \in S_\Gamma$ and a $y' \in Y$ with $y = \lambda y'$ and $\|y\|_\Gamma \leq |\lambda| < \inf\left\{\|x\|_\Gamma : x \in f^{-1}(y)\right\}$. Since f is surjective there is a $x' \in X$ with $y' = f(x')$. Hence $y = f(\lambda x')$ and thus $x := \lambda x' \in f^{-1}(y)$. Hence we have the contradiction $\|x\|_\Gamma \leq \|y\|_\Gamma$.

Proposition 5.7. *Let X_1 and X_2 be two Γ-convex spaces and let $x_i \in X_i$, $i = 1, 2$. Then $\|x_1 \otimes_\Gamma x_2\|_\Gamma \leq \|x_1\|_\Gamma \|x_2\|_\Gamma$.*

Proof[8]. Proof of (6.4).

Note that Proposition 5.7 is crucial for the study of Γ-convex algebras (see[10]).

Proposition 5.8. *Let Γ be a convexity theory. Then every Γ-convex space X satisfies*

$$\Big\|\sum_{i=1}^{\infty} \alpha_i x_i\Big\|_\Gamma \leq \sum_{i=1}^{\infty} |\alpha_i|\,\|x_i\|_\Gamma \quad , \quad \textit{for all} \quad \alpha_* \in \Gamma \ \text{and} \ x_* \in X^{N},$$

if and only if Γ *itself does.*

Proof. Assume that the formula holds for $X := \Gamma$. Choose for each x_i a sequence $\lambda_i^{(n)} \in S_\Gamma n \in \boldsymbol{N}$, and a sequence $x_i^{(n)} \in X, n \in \boldsymbol{N}$, such that $x_i = \lambda_i^{(n)} x_i^{(n)}$ and $\lim_{n\to\infty} |\lambda_i^{(n)}| = \|x_i\|_\Gamma$. Then the Γ-convex subspace X' of X that is generated by the $x_i^{(n)}, i, n = 1, 2, \dots$, is countably generated and each x_i satisfies $\|x_i\|_\Gamma = \|x_i\|'_\Gamma$ where the superscript indicates the space in which the norm is taken. Since X' is countably generated there is a surjective morphism $f : \Gamma \to X'$. Let $t_i \in f^{-1}(x_i), i = 1, 2, \dots$. Then Proposition 5.5 implies

$$\left\|\sum_{i=1}^{\infty} \alpha_i x_i\right\|_\Gamma \le \left\|\sum_{i=1}^{\infty} \alpha_i x_i\right\|'_\Gamma = \left\|\sum_{i=1}^{\infty} \alpha_i f(t_i)\right\|'_\Gamma = \left\|f\left(\sum_{i=1}^{\infty} \alpha_i t_i\right)\right\|'_\Gamma$$

$$\le \left\|\sum_{i=1}^{\infty} \alpha_i t_i\right\|_\Gamma \le \sum_{i=1}^{\infty} \alpha_i \|t_i\|_\Gamma.$$

By taking the infimum of the right side with respect to all t_i as specified we obtain from Proposition 5.6

$$\left\|\sum_{i=1}^{\infty} \alpha_i x_i\right\|_\Gamma \le \sum_{i=1}^{\infty} |\alpha_i| \, \|x_i\|'_\Gamma = \sum_{i=1}^{\infty} |\alpha_i| \, \|x_i\|_\Gamma.$$

Literally the same proof furnishes

Proposition 5.9. *Let* Γ *be a convexity theory. Then every* Γ*-convex space* X *satisfies, for all* $x \in X$,

$$\|x\|_\Gamma \ge \inf\left\{\|\alpha_*\| : x = \sum_{i=1}^{\infty} \alpha_i x_i \ \textit{with} \ \alpha_* \in \Gamma \ \textit{and} \ x_* \in X^{\boldsymbol{N}}\right\}.$$

Moreover, this inequality becomes an equality for every Γ*-convex space* X *if and only if it is an equality for* Γ *itself.*

Clearly we want an addition to Proposition 5.8 and Proposition 5.9 that specifies conditions under which the inequality of Proposition 5.8 and the equality of Proposition 5.9 is satisfied in case $X = \Gamma$.

Addendum 5.10. Suppose that the convexity theory Γ satisfies (SPG) and that $\alpha_* \in \Gamma$ implies $\|\alpha_*\| \in S_\Gamma$. Then the inequality of Proposition 5.8 and the equality of Proposition 5.9 is valid for all $\alpha_* \in \Gamma$ and $x_* \in \Gamma^N$.

Proof. As stated in Proposition 5.2 we have $\|\alpha_*\| \leq \|\alpha_*\|_\Gamma$. However, under the current assumption it is easy to see that $\|\alpha_*\|_\Gamma \leq \|\alpha_*\|$ whence the last inequality is in fact an equality. Thus[8], (6.2), finishes the proof.

Proposition 5.11. *Let Γ be a convexity theory. Then for any family $\{X_i : i \in I\}$ of Γ-convex spaces*

$$\|\{x_i : i \in I\}\|_\Gamma = \sup\{\|x_i\|_\Gamma : i \in I\} \quad , \textit{ for all } \{x_i : i \in I\} \in \Pi\{X_i : i \in I\},$$

if and only if this is true for the family $\{\Gamma : i \in I\}$.

Proof. By applying Proposition 5.5 to the canonical projections $\Pi\{X_i : i \in I\} \to X_j$ we obtain $\sup\{\|x_i\|_\Gamma : i \in I\} \leq \|\{x_i : i \in I\}\|_\Gamma$ without needing any condition on Γ. Conversely, as shown in the proof of Proposition 5.8 we may assume that each X_i is countably generated. Hence there are surjective morphisms $f_i : \Gamma \to X_i, i \in I$, giving rise to the surjective morphism $F := \{f_i : i \in I\}$ from $\Gamma^I := \Pi\{\Gamma : i \in I\}$ to $\Pi\{X_i : i \in I\}$. Let $t_i \in f_i^{-1}(x_i), i \in I$.

Since $\{t_i : i \in I\} \in F^{-1}(\{x_i : i \in I\})$ and since every element of $F^{-1}(\{x_i : i \in I\})$ is of this nature, (5.6) leads to

$$\begin{aligned} \|\{x_i : i \in I\}\|_\Gamma &= \inf\{\|\{t_i : i \in I\|_\Gamma : t_i \in f_i^{-1}(x_i)\} \\ &= \inf\{\sup\{\|t_i\|_\Gamma : i \in I\} : t_i \in f_i^{-1}(x_i)\} \\ &\leq \sup\{\|\bar{t}_i\|_\Gamma : i \in I\} \quad , \text{ for all } \bar{t}_i \in f_i^{-1}(x_i). \end{aligned}$$

Given $\varepsilon > 0$, Proposition 5.6 shows that $\bar{t}_i, i \in I$, can be chosen to satisfy $\|x_i\|_\Gamma \leq \|\bar{t}_i\|_\Gamma \leq \|x_i\|_\Gamma + \varepsilon$. Hence

$$\sup\{\|\bar{t}_i\|_\Gamma : i \in I\} \leq \sup\{\|x_i\|_\Gamma : i \in I\} + \varepsilon$$

and thus

$$\|\{x_i : i \in I\}\|_\Gamma \leq \sup\{\|x_i\|_\Gamma : i \in I\}. \tag{5.12}$$

In view of Proposition 5.11 it is interesting to have conditions on Γ which imply the assumption in Proposition 5.11.

Addendum 5.13. Suppose that Γ satisfies (PG) and the condition that for every $\varepsilon > 0$ and $\Lambda \subseteq S_\Gamma$ with $\sup\{|\lambda| : \lambda \in \Lambda|\} < 1$ there is a $\rho \in S_\Gamma$ with $\sup\{|\lambda| : \lambda \in \Lambda\} < |\rho| < \sup\{|\lambda| : \lambda \in \Lambda\} + \varepsilon$. Then Proposition 5.11 is valid without further assumptions on Γ.

Proof. We need to verify (5.12) for $\alpha_*^{(i)} := x_i \in \Gamma, i \in I$. Let $\kappa := \sup\left\{\|\alpha_*^{(i)}\|_\Gamma : i \in I\right\}$. If $\kappa = 1$ we are finished. Hence let $\kappa < 1$. Given any sufficiently small $\varepsilon > 0$ there are $\lambda_i \in S_\Gamma$ and $\beta_*^{(i)} \in \Gamma$ such that $\alpha_*^{(i)} = \lambda_i \beta_*^{(i)}, i \in I$, and $|\lambda_i| \leq \kappa + \varepsilon$. By assumption there is a $\lambda \in S_\Gamma$ with $\sup\{|\lambda_i| : i \in I\} < |\lambda| < \kappa + 2\varepsilon$. Due to (PG) we have $\alpha_*^{(i)} = \lambda \cdot \lambda_i \lambda^{-1} \beta_*^{(i)} = \lambda \gamma_*^{(i)}$, where $\gamma_*^{(i)} = \lambda_i \lambda^{-1} \beta_*^{(i)}$. Hence

$$\|\left\{\alpha_*^{(i)} : i \in I\right\}\|_\Gamma \leq |\lambda| \leq \kappa + 2\varepsilon \ ,$$

which implies (5.12).

Let $f : X \longrightarrow Y$ be a morphism of Γ-convex spaces. Then the semi-norm of f as an element of $\mathrm{Hom}_\Gamma(X, Y)$ is denoted by $\|f\|_\Gamma$. Moreover we define

$$|||f|||_\Gamma := \inf\{\lambda : \|f(x)\|_\Gamma \leq \lambda \|x\|_\Gamma, \ \text{for all} \ \lambda \in I\!R \ \text{and} \ x \in X\}.$$

Proposition 5.14. *Let* $f \in \mathrm{Hom}_\Gamma(X, Y)$. *Then*

(i) $\|f(x)\|_\Gamma \leq |||f|||_\Gamma' \|x\|_\Gamma \leq |||f|||_\Gamma$, *for all* $x \in X$

(ii) $|||f|||_\Gamma = \sup\{\|f(x)\|_\Gamma : x \in X\}$

(iii) $|||f|||_\Gamma = \sup\{\|f(x)\|_\Gamma \cdot \|x\|_\Gamma^{-1} : x \in X$ *and* $\|x\|_\Gamma \neq 0\}$

(iv) *if* Γ *is such that the equality in Proposition 5.11 always holds, then* $|||f|||_\Gamma \leq \|f\|_\Gamma$.

Proof. (i) Obvious.

(ii) Let $\kappa := \sup\{\|f(x)\|_\Gamma : x \in X\}$. (i) shows that $\kappa \le |||f|||_\Gamma$. If $\kappa = 0$ then $\|f(x)\|_\Gamma = 0$, for all $x \in X$, and hence $|||f|||_\Gamma = 0$, proving the desired equality in this case. If $\kappa > 0$, let $x = \rho x'$, with $\rho \in S_\Gamma$ and $x' \in X$. By Proposition 5.3

$$\|f(x)\|_\Gamma = \|f(\rho x')\|_\Gamma = \|\rho f(x')\|_\Gamma \le |\rho|\,\|f(x')\|_\Gamma \le |\rho|\kappa.$$

But the definition of $\|x\|_\Gamma$ shows

$$\|f(x)\|_\Gamma \le \kappa\|x\|_\Gamma$$

and thus $|||f|||_\Gamma \le \kappa$.

(iii) Clearly, the claim equality is satisfied if $|||f|||_\Gamma = 0$. Hence we may assume $|||f|||_\Gamma > 0$. This, however, shows that there is a $x_0 \in X$ with $0 < \|f(x_0)\|_\Gamma \le \|x_0\|_\Gamma$, the latter inequality coming from Proposition 5.5. Hence $\kappa' := \sup\left\{\|f(x)\|_\Gamma \cdot \|x\|_\Gamma^{-1} : x \in X \text{ and } \|x\|_\Gamma \neq 0\right\}$ is a real number between 0 and 1. By (i), $\|f(x)\|_\Gamma \cdot \|x\|_\Gamma^{-1} \le |||f|||_\Gamma$, whenever $\|x\|_\Gamma \neq 0$, and thus $\kappa' \le |||f|||_\Gamma$. Conversely, if $\|x\|_\Gamma = 0$ then $\|f(x)\|_\Gamma = 0$ by Proposition 5.5, and hence $\|f(x)\|_\Gamma \le \kappa'\|x\|_\Gamma$; if $\|x\|_\Gamma > 0$ then by definition of κ', $\|f(x)\|_\Gamma \le \kappa'\|x\|_\Gamma$ whence the definition of $|||f|||_\Gamma$ implies $|||f|||_\Gamma \le \kappa'$.

(iv) There is a canonical imbedding of $\mathrm{Hom}_\Gamma(X, Y)$ into the product $Y^{U(X)}$ of $U(X)$ copies of Y; it is given by $f \to \{f(x) : x \in X\}$. By definition this imbedding is a morphism of Γ-convex spaces. Hence Proposition 5.5 shows that

$$\|\{f(x) : x \in X\}\|_\Gamma \le \|f\|_\Gamma,$$

and Proposition 5.5 together with Proposition 5.11 and (ii) finishes the argument.

The last question we want to discuss in this section is conditions on Γ that imply $x = 0$ for any $x \in X$ with $\|x\|_\Gamma = 0$, where X is an arbitrary Γ-convex space. Quite obviously, Γ must be with zero. It is known that $\Gamma = \Omega$ satisfies this condition while $\Gamma = \Omega_{\text{fin}}$ is in violation of it (see[8], (6.9) a.s.o.). Examples in[13], (1.12), shows that there are infinite convexity theories which do not satisfy the stated condition. The examples are as follows. Let $\Gamma = \mathcal{P}$. Then

[13], (1.12), (d):

Let X be $[0, +\infty]$ with the canonical meaning of $\sum \alpha_i x_i$; then $\|x\|_\Gamma = 0$, for all $x \in X$.

[13], (1.12), (e):

Let X be $\{0, 1, 2, \dots, +\infty\}$ with $\sum_{i=1}^{\infty} \alpha_i x_i$ equal $\sup\{x_i : \alpha_i \neq 0\}$ if $\alpha_* \neq 0_*$, and 0 if $\alpha_* = 0_*$; then $\|x\|_\Gamma = 0$, for all $x \in X$.

We need the following technical

Lemma 5.15. *Let Γ be an infinite convexity theory with zero and* (PG). *Then there are $\rho \in S_\Gamma$ with arbitrarily small $|\rho|$ such that $(\rho, \rho^2, \rho^3, \dots) \in \Gamma$ holds.*

Proof. First we remark that such a convexity theory is normable due to Proposition 1.5. Denote the sequence described in Definition 1.1, (ii), by $\alpha_* \circ \langle \beta_*^i \rangle$. Let $\alpha_* = (\alpha_1, \alpha_2, \dots) \in \Gamma$ satisfy $\operatorname{card}(\operatorname{supp} \alpha_*) = \infty$. By choosing for β_*^i appropriate $\delta_*^{n_i}$ we can obtain a new $\alpha_* \in \Gamma$ with $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_n| \geq \dots > 0$. Set $\alpha := \alpha_1$ and choose inductively k_i such that

$$k_1 = 1 < k_2 = 2k < k_3 = \ell_3 k_2 < \dots < k_n = \ell_n k_2 < \dots$$

and

$$|\alpha_i| > |\alpha|^{k_i} \qquad , \quad i = 1, 2, \dots \ .$$

Since Γ satisfies (PG) we have $\alpha^{k_i} \alpha_i^{-1} \in S_\Gamma$ and hence $\alpha^{k_i} \alpha_i^{-1} \delta_*^i \in \Gamma$. Therefore

$$\alpha_* \circ \langle \alpha^{k_i} \alpha_i^{-1} \delta_*^i \rangle = (\alpha, \alpha^{k_2}, \alpha^{k_3}, \dots) \in \Gamma.$$

Since $0_* \in \Gamma$ holds we have

$$(\alpha, \alpha^{k_2}, \alpha^{k_3}, \dots) \circ \langle \alpha^{k-1}\delta^1_*, \delta^2_*, 0_*, \dots \rangle = (\alpha^k, \alpha^{2k}, 0, \dots) \in \Gamma.$$

Set $\lambda := \alpha^k$ to obtain $(\lambda, \lambda^2, 0, \dots) \in \Gamma$. Hence, with $\beta^2_* = (0, \lambda, \lambda^2, 0, \dots)$ and $\beta^i_* = \delta^i_*$ otherwise

$$(\lambda, \lambda^2, 0, \dots) \circ \langle \beta^i_* \rangle = (\lambda, \lambda^3, \lambda^4, 0, \dots) \in \Gamma$$

and an obvious induction shows that

$$(\lambda, \lambda^3, \lambda^5, \dots, \lambda^{2n-1}, \lambda^{2n}, 0, \dots) \in \Gamma \quad , n = 1, 2, \cdots .$$

Choosing $\beta^i_* = \lambda\delta^i_*, i = 1, \dots, n$, and $\beta^i_* = 0_*, i \geq n+1$, leads to

$$(\lambda^2, \lambda^4, \dots, \lambda^{2n}, 0, \dots) \in \Gamma \quad , \quad n = 1, 2, \dots .$$

Set $\mu := \lambda^2$ to obtain $(\mu, \mu^2, \dots, \mu^n, 0, \dots) \in \Gamma, n = 1, 2, \dots$. Since

$$(\alpha^{k_2}, \alpha^{k_3}, \alpha^{k_4}, \dots) = (\lambda^2, (\lambda^2)^{\ell_3}, (\lambda^2)^{\ell_4}, \dots) = (\mu, \mu^{\ell_3}, \mu^{\ell_4}, \dots) \in \Gamma$$

we have, with $\beta^i_* = \left(\underbrace{0, \dots, 0}_{\ell_{i+1}-1}, \mu, \mu^2, \dots, \mu^{\ell_{i+2}-1}, 0, \dots \right) \in \Gamma$,

$$(\mu, \mu^{\ell_3}, \mu^{\ell_4}, \dots) \circ \langle \beta^i_* \rangle = (\mu^2, \mu^3, \mu^4, \dots) \in \Gamma.$$

Since $0_* \in \Gamma$, putting $\rho := \mu^2$ leads to

$$(\rho, \rho^2, \rho^3, \dots) \in \Gamma.$$

Obviously, $\rho \in S_\Gamma$. Since $\alpha_* \in \Gamma$ implies $\alpha_1^\ell \cdot \alpha_* = (\alpha_1^{\ell+1}, \alpha_1^\ell \alpha_2, \alpha_1^\ell \alpha_3, \dots) \in \Gamma$, we can find $(\rho, \rho^2, \dots) \in \Gamma$ with arbitrarily small $|\rho|$.

<u>Theorem</u> 5.16. *Let Γ be an infinite convexity theory with zero,* (PG) *and $\sigma_\Gamma > 0$ such that for all sufficiently small $\rho \in S_\Gamma, -\rho \in S_\Gamma$ holds. Then, for all Γ-convex spaces X and all $x \in X, \|x\|_\Gamma = 0$ implies $x = 0$.*

Proof. (see proof of[8], (6.9)). By Lemma 5.15 there is a $\rho \in S_\Gamma$ such that $|\rho| < \sigma_\Gamma, (\rho, \rho^2, \rho^3, \dots) \in \Gamma$, and that for all $\sigma \in S_\Gamma$ with $|\sigma| \leq |\rho|, -\sigma \in S_\Gamma$ holds. Let $x \in X$ satisfy $\|x\|_\Gamma = 0$. Then, for every $n = 0, 1, 2, \cdots$, there is a $\rho_n \in S_\Gamma$ and $x'_n \in X$ such that $|\rho_n| \leq |\rho^n|$ and $x = \rho_n x'_n$. Since Γ satisfies (PG) we have $\rho_n \rho^{-n} \in S_\Gamma$ and hence $x = \rho^n \cdot \rho_n \rho^{-n} x'_n = \rho^n x_n$, with $x_n = \rho_n \rho^{-n} x'_n$. Since $\rho^n x_n = \rho^{n+1} x_{n+1} = \rho^n \cdot \rho x_{n+1}$ and $|\rho^n| < |\rho| < \sigma_\Gamma$ we have $\rho x_n = \rho^2 x_{n+1}$. Note that the last condition imposed on Γ implies

$$(\rho, \rho^2, \rho^3, \dots) \circ \langle \pm\rho\delta^1_*, \delta^2_*, 0_*, \dots \rangle = (\pm\rho^2, \rho^2, 0, \dots) \in \Gamma.$$

Hence, putting

$$z := \sum_{n=1}^{\infty} \rho^{n+1} x_{n-1} \quad ,$$

we obtain

$$\begin{aligned}
\rho^2 z &= \sum_{n=1}^{\infty} \rho^{n+3} x_{n-1} = \rho^2 \left(\rho^2 x_0\right) + \rho^2 \left(\sum_{n=1}^{\infty} \rho^{n+2} x_n\right) \\
&= \rho^2(\rho^2 x_0) + \rho^2 \left(\sum_{n=1}^{\infty} \rho^n \cdot \rho^2 x_n\right) = \rho^2(\rho^2 x_0) + \rho^2 \left(\sum_{n=1}^{\infty} \rho^n \cdot \rho x_{n-1}\right) \\
&= \rho^2(\rho^2 x_0) + \rho^2 z.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0 &= \rho^2(\rho^2 z) - \rho^2(\rho^2 z) = \rho^2(\rho^2(\rho^2 x_0) + \rho^2 z) - \rho^2(\rho^2 z) \\
&= \rho^6 x_0.
\end{aligned}$$

Since $|\rho^6| < |\rho| < \sigma_\Gamma$ we get $\rho x_0 = 0$. Thus $0 = \rho x_0 = \rho^2 x_1$, and $|\rho^2| < |\rho| < \sigma_\Gamma$ shows that $0 = \rho x_1 = x$ as had to be shown.

Remark 5.17. Of the conditions imposed on Γ in Theorem 5.16, $0_* \in \Gamma$ is needed to even formulate Theorem 5.16. The last condition on Γ cannot be dropped as Wickenhäuser's examples show. The requirement that Γ be infinite is also necessary as all finite convexity theories violate the conclusion of (5.16) (see[8], remark following the proof of (6.9)).

If Γ satisfies the conclusion of Theorem 5.16, then the semi-norm $\| \ \|_\Gamma$ is said to be a *norm*.

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Helmut Röhrl
9322 La Jolla Farms Rd.
La Jolla, CA 92037, USA

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ADAPTED CONTACT STRUCTURES AND PARAMETER-DEPENDENT CANONICAL TRANSFORMATIONS

Hanno Rund

ABSTRACT: It is supposed that a set of 2n independent 1-forms $\{\pi_h, \pi^h : h = 1, \cdots, n\}$ is given on a product space $N = M \times \mathbf{R}$, where M is an orientable manifold of dimension 2n. The imposition of certain conditions on these 1-forms gives rise to a Cartan form π and a local contact structure on N, together with a local symplectic structure on M. A similar geometrical configuration results from the introduction of an alternative set of 1-forms $\{\bar{\pi}_j, \bar{\pi}^j : j = 1, \cdots, n\}$. A relationship between these two configurations is established by the imposition of a single invariance postulate, namely $\bar{\pi}_j \wedge \bar{\pi}^j = \pi_h \wedge \pi^h$, and it is found that this is tantamount to the introduction of a parameter-dependent canonical transformation whose functional determinant is nonvanishing on any region of N on which the Cartan form π has class $2n + 1$.

0. Introduction

This article is concerned with the construction of certain contact structures and the implications thereof with regard to the theory of nonconservative dynamical systems and parameter-dependent canonical transformations. It is supposed that a set of 2n smooth 1-forms $\{\pi_h, \pi^h : h = 1, \cdots, n\}$ is given on a product space $N = M \times \mathbf{R}$, where M is a 2n-dimensional orientable manifold. These 1-forms are subjected to two conditions which together imply the existence of local coordinates $\{p_h, q^h : h = 1, \cdots n\}$ on M, and a function H on N, such that $\pi_h \wedge \pi^h = d\pi$, where π has the structure of a Cartan form on N. In this context the function H depends on (p_h, q^h, t), where t denotes the single coordinate on $\mathbf{R}$, and it is shown that if H is *not* homogeneous of the first degree in p_h on a region D, then π has class $2n + 1$ on D and thus defines a local contact structure on D, together with a local symplectic structure on M. (Since this construction differs from the standard description of contact structures in terms of a single 1-form of maximal class on N, the terminology *adapted* contact structure is used here.) A central role is played by a vector field Z on N whose integral curves satisfy a system of differential equations that coincides with the canonical equations associated with the function H. A definition of Hamiltonian vector fields on the $(2n + 1)$-dimensional manifold N is proposed; however, it is found that, in contrast to such fields on symplectic manifolds, only certain classes of functions on N are capable of generating locally Hamiltonian vector fields. The introduction of an alternative set of 2n smooth 1-forms $\{\bar{\pi}_j,\bar{\pi}^j:j = 1, \cdots, n\}$ on N subject to similar conditions gives rise to a second set of functions $\bar{p}$, $\bar{q}^j$, $\bar{K}$, $\bar{t}$ on N such that $\{\bar{p}_j,\bar{q}^j:j = 1, \cdots, n\}$ are locally symplectic coordinates on a hypersurface $\bar{M}$ of N on which $\bar{t}$ is constant. The two configurations are related by a single invariance postulate, namely by the requirement that $\bar{\pi}_j \wedge \bar{\pi}^j = \pi_h \wedge \pi^h$. This gives rise to a set of relationships that represent $\{\bar{p}_j,\bar{q}^j,\bar{t}\}$ as functions of $\{p_h,q^h,t\}$ subject to an exactness condition that had been stipulated by Carathéodory [2] as being characteristic of t-dependent canonical transformations. However, the complete definition of the latter as given in [2] also includes the condition that the $(2n + 1) \times (2n + 1)$ functional determinant of the transformation be nonvanishing. This requirement can actually be avoided in the present treatment in which an explicit evaluation of this determinant leads to the conclusion that it cannot vanish on the aforementioned region D of N on which the 1-form π has class $2n + 1$. A brief description of the properties of t-dependent canonical transformations is given within the context of this geometrical background, with emphasis on the associated Poisson bracket and reciprocity relations. The latter are used to show that the requirement that the Poisson bracket of *any* pair of functions on N be invariant under a canonical transformation can be met if and only if the latter is independent of the parameter t.

1. The Development of a Local Contact Structure

Our considerations are based on a product space $N = M \times \mathbf{R}$, where M is an orientable manifold of dimension 2n. The single coordinates on $\mathbf{R}$ is denoted by t, the imbedding of M in N being such that M can be represented as a hypersurface $t = t_0 =$ const. of N. Thus, for the inclusion map $i:M \to N$ the resulting induced maps $i^*:\Lambda^1(N) \to \Lambda^1(M)$ are such that $i^*(dt) = 0$, where $\Lambda^1(N)$ denotes the space of 1-forms on N.

It is supposed that N is endowed with a set of 2n independent smooth 1-forms $\{\pi_h,\pi^h:h = 1, \cdots, n\}$, these being such that the set $\{\pi_h,\pi^h,dt\}$ constitutes a basis in the cotangent space $\Lambda^1_p(N)$ at each point $p \in N$. The following conditions are now imposed on these 1-forms.

Condition I: The pull-backs $i^*\pi_h$, $i^*\pi^h$ are closed 1-forms on M.

This implies the existence, at least locally, of a set of 0-forms $\{p_h,q^h:h = 1, \cdots, n\}$ on M in terms of which one has $i^*\pi_h = dp_h$, $i^*\pi^h = dq^h$. We shall regard $\{p_h,q^h\}$ as local coordinate functions on M, in terms of which we shall write

$$\pi_h = dp_h - f_h dt, \qquad \pi^h = dq^h - f^h dt, \tag{1.1}$$

where $\{f_h, f^h : h = 1, \cdots, n\}$ denotes a set of $2n$ differentiable functions of the variables (p_h, q^h, t). Consequently

$$\pi_h \wedge \pi^h = d(p_h dq^h) - (f^h dp_h - f_h dq^h) \wedge dt. \tag{1.2}$$

Since the rank of this 2-form is $2n$ by virtue of the independence of $\{\pi_h, \pi^h\}$, it is natural to stipulate

<u>Condition II</u>: The 2-form (1.2) is closed.

Since the class of any closed 2-form is identical with its rank, it follows that this condition implies that the 2-form (1.2) has class 2n. From the structure of (1.2) it is evident that it is closed if there exists a differentiable function H on N such that

$$f^h dp_h - f_h dq^h = dH - \frac{\partial H}{\partial t}dt = \frac{\partial H}{\partial p_h}dp_h + \frac{\partial H}{\partial q^h}dq^h, \tag{1.3}$$

since this entails that

$$(f^h dp_h - f_h dq^h) \wedge dt = dH \wedge dt = d(Hdt). \tag{1.4}$$

This demonstrates the sufficiency of (1.3); the necessity of (1.3) follows from the simple Lemma of Appendix A.

The substitution of (1.4) in (1.2) yields

$$\pi_h \wedge \pi^h = d(p_h dq^h - Hdt) = d\pi, \tag{1.5}$$

where

$$\pi = p_h dq^h - H(p,q,t)dt. \tag{1.6}$$

We shall henceforth refer to this 1-form as the *Cartan form* since its structure is formally identical with that of the Cartan form in the classical theory of integral invariants. It is moreover evident from (1.3) that

$$f^h = \frac{\partial H}{\partial p_h}, \qquad f_h = -\frac{\partial H}{\partial q^h}, \tag{1.7}$$

so that (1.1) can be expressed as

$$\pi_h = dp_h + \frac{\partial H}{\partial q^h}dt, \qquad \pi^h = dq^h - \frac{\partial H}{\partial p_h}dt. \tag{1.8}$$

We shall now investigate the class of the Cartan form. To this end we observe that the second member of (1.8) gives

$$p_h dq^h = p_h \pi^h + \frac{\partial H}{\partial p_h} p_h dt,$$

so that (1.6) is equivalent to

$$\pi = p_h \pi^h + h dt, \tag{1.9}$$

where we have put

$$h = \frac{\partial H}{\partial p_h} p_h - H. \tag{1.10}$$

As usual, we associate with any s-form μ on N the subspaces

$$A(\mu) = \{X \in T_p(N): X \rfloor \mu = 0\} \tag{1.11}$$

of the tangent spaces $T_p(N)$ of N at each $p \in N$. Thus, if $X \in A(d\pi)$, we have

$$X \rfloor d\pi = 0. \tag{1.12}$$

Because of (1.5) this is equivalent to

$$(X \rfloor \pi_h)\pi^h - (X \rfloor \pi^h)\pi_h = 0, \tag{1.13}$$

and hence, by virtue of the linear independence of $\{\pi_h, \pi^h\}$,

$$X \rfloor \pi_h = 0, \qquad X \rfloor \pi^h = 0. \tag{1.14}$$

If the coordinate presentation of X is given by

$$X = X_h \frac{\partial}{\partial p_h} + X^h \frac{\partial}{\partial q^h} + X^o \frac{\partial}{\partial t}, \tag{1.15}$$

one may express the conditions (1.14) by means of (1.8) as

$$X \rfloor \pi_h = X_h + X^o \frac{\partial H}{\partial q^h} = 0, \qquad X \rfloor \pi^h = X^h - X^o \frac{\partial H}{\partial p_h} = 0. \tag{1.16}$$

Similarly, if $X \in A(\pi)$, we have

$$X \rfloor \pi = 0, \tag{1.17}$$

which, because of (1.9), is equivalent to

$$p_h(X \rfloor \pi_h) = -hX^o. \tag{1.18}$$

Consequently, if $X \in A(\pi) \cap A(d\pi)$, the systems (1.16) and (1.18) must be satisfied simultaneously, which requires that

$$hX^o = 0. \tag{1.19}$$

Let us now suppose that the function H is such that the concomitant function h as defined by (1.10) does not vanish on a region D of N:

$$h(p_h, q^h, t) \neq 0. \tag{1.20}$$

The equations (1.16) and (1.19) then imply that the vector field X is zero on D, that is,

$$A(\pi) \cap A(d\pi) = \{0\}$$

at each $p \in D$. But, by definition, the class of the 1-form π at $p \in N$ is the codimension of this space, regarded as a subspace of $T_p(N)$. Thus the condition (1.20) implies that π has class $2n + 1$ on D. Conversely, if $h = 0$ at some point $q \in N$, the relation (1.19) is void at q, and the system (1.16) would admit a nontrivial solution $x \in$

$T_q(N)$ that is unique up to a multiplicative factor X^0. This establishes the

THEOREM: *In order that the class of the Cartan form* (1.6) *be* $(2n + 1)$ *on a region* D *of the manifold* N, *it is necessary and sufficient that the condition* (1.20) *be satisfied on* D.

Under the conditions of the theorem the class of π is maximal on D. It therefore defines a local contact structure on D, which we shall call an *adapted contact structure* in view of the fact that its construction depends on the given set of 2n smooth 1-forms $\{\pi_h, \pi^h\}$ in contrast to the usual definition of a contact structure that depends on a single 1-form of prescribed class $2n + 1$. Moreover, it is evident from (1.8) that the 1-forms $\{dp_h, dq^h\}$ are independent in consequence of the stipulated independence of $\{\pi_h, \pi^h\}$. Thus the closed 2-form

$$\omega = dp_h \wedge dq^h \tag{1.21}$$

on M has rank 2n, and is therefore nondegenerate. This 2-form therefore defines a *local symplectic structure* on M and admits the representation

$$\omega = d\pi + dH \wedge dt = \pi_h \wedge \pi^h + dH \wedge dt, \tag{1.22}$$

as is evident directly from (1.6) and (1.5). Also, since the rank of a closed s-form coincides with its class, we conclude that this 2-form has class 2n.

2. Canonical Vector Fields

Our subsequent analysis will be restricted to the region D of the manifold N on which the condition (1.20) is satisfied. Since the 1-form π has class $2n + 1$, the class of $d\pi$ is 2n ([4], Ch. 6), and being closed, the rank of $d\pi$ is also 2n. Thus, if the vector field $Z \in A(d\pi)$, that is, if

$$Z \rfloor d\pi = 0, \tag{2.1}$$

it follows that Z is determined uniquely up to a multiplicative factor since the codimension of $A(d\pi)$ is 2n. If the coordinate presentation of Z is given by

$$Z = Z_h \frac{\partial}{\partial p_h} + Z^h \frac{\partial}{\partial q^h} + Z^0 \frac{\partial}{\partial t}, \tag{2.2}$$

we deduce as in the case of (1.12) that (2.1) implies the relations

$$Z_h + Z^0 \frac{\partial H}{\partial q^h} = 0, \qquad Z^h - Z^0 \frac{\partial H}{\partial p_h} = 0. \tag{2.3}$$

In order to fix Z^0 we require in addition to (2.1) that the vector field Z be such as to satisfy the condition

$$Z \rfloor \pi = h, \tag{2.4}$$

where h is defined in (1.10). Because of (1.6) and (2.3) this is equivalent to

$$Z^h p_h - Z^0 H = h,$$

in which we substitute from the second member of (2.3) to obtain

$$Z^0 \Big(\frac{\partial H}{\partial p_h} p_h - H \Big) = h.$$

In view of (1.10) and (1.20) this is possible if and only if $Z^0 = 1$. The conditions (2.1) and (2.4) therefore determine Z uniquely, the latter being given by

$$Z = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_h} \frac{\partial}{\partial q^h} - \frac{\partial H}{\partial q^h} \frac{\partial}{\partial p_h}. \tag{2.5}$$

Consequently, for any differentiable function $F:N \rightarrow \mathbf{R}$, one has

$$ZF = \frac{\partial F}{\partial t} + (H,F) \tag{2.6}$$

where $(\, , \,)$ represents the standard notation for a Poisson bracket. Moreover, the integral curves of Z satisfy the following system of first order ordinary differential equations

$$\frac{dq^h}{dt} = \frac{\partial H}{\partial p_h}, \qquad \frac{dp_h}{dt} = -\frac{\partial H}{\partial q^h}, \tag{2.7}$$

whose structure is identical with that of the canonical equations of the classical calculus of variations. We shall therefore call Z the *canonical vector field associated with the function* whose structure is identical with that of the canonical equations of the classical calculus of variations. We shall therefore call Z the *canonical vector field associated with the function* H.

Remark 1: The construction based on (2.1) and (2.4) is very similar to that of a Reeb field E ([1], [5], p. 291). This field is determined by the conditions $E\rfloor\omega = 1$, and $E\rfloor d\omega = 0$. Our canonical field reduces to a Reeb field for the special case when $H = 1$, since in this case $\omega = d\pi$ in consequence of (1.22), and $h = -H = -1$ by (1.10).

We shall now derive a further important property of the canonical vector field. From (2.5) it follows directly that

$$Z\rfloor dp_h = -\frac{\partial H}{\partial q^h}, \qquad Z\rfloor dq^h = \frac{\partial H}{\partial p_h}, \qquad Z\rfloor dt = 1, \tag{2.8}$$

so that by (1.8)

$$Z\rfloor(dp_j \wedge dt) = -\pi_j, \qquad Z\rfloor(dq^j \wedge dt) = -\pi^j, \tag{2.9}$$

and thus

$$Z\rfloor\pi_h = 0, \qquad Z\rfloor\pi^h = 0. \tag{2.10}$$

By means of (2.9) it is now inferred from (1.8) that

$$Z\rfloor d\pi_h = -\frac{\partial^2 H}{\partial p_j \partial q^h}\pi_j - \frac{\partial^2 H}{\partial q^j \partial q^h}\pi^j, \tag{2.11}$$

and

$$Z\rfloor d\pi^h = \frac{\partial^2 H}{\partial q^j \partial p_h}\pi^j + \frac{\partial^2 H}{\partial p_j \partial p_h}\pi_j. \tag{2.12}$$

The Lie derivatives with respect to Z of the 1-forms (1.8) are defined as usual by

$$\pounds_Z\pi_h = Z\rfloor d\pi_h + d(Z\rfloor\pi_h), \qquad \pounds_Z\pi^h = Z\rfloor d\pi^h + d(Z\rfloor\pi^h),$$

and hence, by (2.10)-(2.12)

$$\pounds_Z\pi_h = -\frac{\partial^2 H}{\partial p_j \partial q^h}\pi_j - \frac{\partial^2 H}{\partial q^j \partial q^h}\pi^j, \tag{2.13}$$

and

$$\pounds_Z\pi^h = \frac{\partial^2 H}{\partial q^j \partial p_h}\pi^j + \frac{\partial^2 H}{\partial p_j \partial p_h}\pi_j. \tag{2.14}$$

Consequently

$$\pounds_Z(\pi_h \wedge \pi^h) = (\pounds_Z\pi_h) \wedge \pi^h + \pi_h \wedge (\pounds_Z\pi^h)$$

$$= -\frac{\partial^2 H}{\partial p_j \partial q^h}\pi_j \wedge \pi^h - \frac{\partial^2 H}{\partial q^j \partial q^h}\pi^j \wedge \pi^h + \frac{\partial^2 H}{\partial q^j \partial p_h}\pi_h \wedge \pi^j + \frac{\partial^2 H}{\partial p_j \partial p_h}\pi_h \wedge \pi_j.$$

In this expression the first sum is the negative of the third, while the second and fourth sums vanish separately by virtue of the symmetry of the partial derivatives. Thus the 2-form $\pi_h \wedge \pi^h$ is invariant by Z in the sense that

$$\pounds_Z(\pi_h \wedge \pi^h) = 0. \tag{2.15}$$

The results obtained thus far may be summarized in the

THEOREM: *The conditions* (2.1) *and* (2.4) *determine a unique vector field* Z *on* N *whose integral curves satisfy the canonical equations* (2.7). *Moreover, the 2-form* $\pi_h \wedge \pi^h$ *is invariant by* Z *in the sense of* (2.15).

Remark 2: The conditions that specify a Reeb field E (see Remark 1) are such as to ensure that $\pounds_E\omega = 0$. However, this is not generally true for the canonical field Z, as is immediately evident from (1.22) and (2.15), since these imply that

$$\pounds_Z\omega = \pounds_Z(dH \wedge dt) = (\pounds_Z dH) \wedge dt + dH \wedge (\pounds_Z dt) = d(Z \rfloor dH) \wedge dt + dH \wedge (Z \rfloor dt).$$

But according to (2.6) we have $Z \rfloor dH = ZH = \frac{\partial H}{\partial t}$, while $Z \rfloor dt = 1$ by (2.8). Thus

$$\pounds_Z\omega = d(\frac{\partial H}{\partial t}) \wedge dt. \tag{2.16}$$

The notion of a canonical vector field is closely related to that of a Hamiltonian vector field. This is readily seen as follows. It is evident from (2.5) that the canonical vector field Z admits the decomposition

$$Z = \frac{\partial}{\partial t} + Z_M, \tag{2.17}$$

where

$$Z_M = \frac{\partial H}{\partial p_h}\frac{\partial}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial}{\partial p_h} \tag{2.18}$$

is a vector field on the manifold M. It therefore follows from (1.21) and (1.22) with the aid of (2.1) and (2.8) that

$$Z_m \rfloor \omega - Z \rfloor \omega = Z \rfloor (dH \wedge dt) = (Z \rfloor dH)dt - (Z \rfloor dt)dH$$

$$= \frac{\partial H}{\partial t}dt - dH = -\left(\frac{\partial H}{\partial p_h}dp_h + \frac{\partial H}{\partial q^h}dq^h\right) = -i^*(dH) = -d(i^*H), \tag{2.19}$$

in which i^* refers, as before, to the pull-back of the inclusion map $i{:}M \to N$. This suggests that we define the function $H_M = i^*H$ on M, for which $H_M(q^h, p_h) = H(q^h, p_h, t_0)$, it being recalled that the (constant) value of t on M is denoted by t_0. Accordingly the relation (2.19) can then be expressed as

$$dH_M + Z_M \rfloor \omega = 0, \tag{2.20}$$

which shows that *the vector field* Z_M *on* M *is the Hamiltonian vector field generated by the function* H_M *in terms of the symplectic structure* (1.21) *on* M.

3. Hamiltonian Vector Fields on the Contact Manifold N

The conclusions of the previous section suggest the possibility of the introduction of Hamiltonian vector fields on the contact manifold N. We shall now show how this may be done in terms of the 2-form

$$\bar{\omega} = \pi_h \wedge \pi^h. \tag{3.1}$$

Let X, Y $\in \mathfrak{X}(N)$, where $\mathfrak{X}(N)$ denotes the Lie algebra of differentiable vector fields on N. Then

$$X \rfloor \bar{\omega} = (X \rfloor \pi_h)\pi^h - (X \rfloor \pi^h)\pi_h = \pi_h(X)\pi^h - \pi^h(X)\pi_h, \tag{3.2}$$

and

$$Y \rfloor X \rfloor \bar{\omega} = \pi_h(X)\pi^h(Y) - \pi^h(X)\pi_h(Y) = 2\pi_h \wedge \pi^h(X,Y). \tag{3.3}$$

Hence, by (3.1)

$$2\bar{\omega}(X,Y) = Y \rfloor X \rfloor \bar{\omega} = -X \rfloor Y \rfloor \bar{\omega}. \tag{3.4}$$

We also have, on the one hand,

$$\pounds_X(Y \rfloor \bar{\omega}) = X \rfloor d(Y \rfloor \bar{\omega}) + d(X \rfloor Y \rfloor \bar{\omega}),$$

while on the other

$$\pounds_X(Y \rfloor \bar{\omega}) = (\pounds_X Y) \rfloor \bar{\omega} + Y \rfloor \pounds_X \bar{\omega} = [X,Y] \rfloor \bar{\omega} + Y \rfloor \pounds_X \bar{\omega},$$

which are combined to yield

$$[X,Y] \rfloor \bar{\omega} = X \rfloor d(Y \rfloor \bar{\omega}) + d(X \rfloor Y \rfloor \bar{\omega}) - Y \rfloor \pounds_X \bar{\omega}. \tag{3.5}$$

This relation is valid for any X, Y $\in \mathfrak{X}(N)$. Now let us suppose that these vector fields are such that the 2-form $\bar{\omega}$ is invariant by each of them in the sense that

$$\pounds_X \bar{\omega} = 0, \qquad \pounds_Y \bar{\omega} = 0. \tag{3.6}$$

But according to (1.5) and (3.1) the 2-form $\bar{\omega}$ is exact; thus (3.6) requires that

$$d(X \rfloor \bar{\omega}) = 0, \qquad d(Y \rfloor \bar{\omega}) = 0, \tag{3.7}$$

which in turn implies the existence, at least locally, of two functions f and g on N such that

$$df + X\rfloor\bar{\omega} = 0, \tag{3.8}$$

and

$$dg + Y\rfloor\bar{\omega} = 0. \tag{3.9}$$

Consequently we shall regard X and Y as (locally) *Hamiltonian vector fields on* N *with respect to the 2-form* $\bar{\omega}$, *these fields being generated by* f *and* g *respectively.*

When (3.6) and (3.7) are substituted in the identity (3.5), the letter reduces to

$$[X,Y]\rfloor\bar{\omega} = d(X\rfloor Y\rfloor\bar{\omega}), \tag{3.10}$$

and hence

$$\pounds_{[X,Y]}\bar{\omega} = [X,Y]\rfloor d\bar{\omega} + d([X,Y]\rfloor\omega) = 0. \tag{3.11}$$

This shows that $\bar{\omega}$ is invariant by [X,Y] whenever it is invariant by both X and Y. Moreover, it follows from (3.4) and (3.10) that

$$2d\{\bar{\omega}(X,Y)\} + [X,Y]\rfloor\bar{\omega} = 0, \tag{3.12}$$

which indicates that [X,Y] is a locally Hamiltonian vector field on N with respect to $\bar{\omega}$, being generated by the function $2\bar{\omega}(X,Y)$ on N. This conclusion can be stated in a more illuminating manner as follows.

From (3.2) and (1.8) we deduce that

$$X\rfloor\bar{\omega} = -\pi^h(X)dp_h + \pi_h(X)dq^h - \left[\pi_h(X)\frac{\partial H}{\partial p_h} + \pi^h(X)\frac{\partial H}{\partial q^h}\right]dt, \tag{3.13}$$

and

$$Y\rfloor\bar{\omega} = -\pi^h(Y)dp_h + \pi_h(Y)dq^h - \left[\pi_h(Y)\frac{\partial H}{\partial p_h} + \pi^h(Y)\frac{\partial H}{\partial q^h}\right]dt. \tag{3.14}$$

These relations hold for any pair of vector fields X, Y. However, if the latter are such that the equations (3.6) hold, the expressions (3.13) and (3.14) may be substituted in (3.8) and (3.9) respectively, which gives

$$\frac{\partial f}{\partial p_h} = \pi^h(X), \quad \frac{\partial f}{\partial q^h} = -\pi_h(X), \quad \frac{\partial f}{\partial t} = \pi_h(X)\frac{\partial H}{\partial p_h} + \pi^h(X)\frac{\partial H}{\partial q^h}, \tag{3.15}$$

and

$$\frac{\partial g}{\partial p_h} = \pi^h(Y), \quad \frac{\partial g}{\partial q^h} = -\pi_h(Y), \quad \frac{\partial g}{\partial t} = \pi_h(Y)\frac{\partial H}{\partial p_h} + \pi^h(Y)\frac{\partial H}{\partial q^h}. \tag{3.16}$$

This yields the following expression for the Poisson bracket of f, g:

$$(f,g) = -\pi^h(X)\pi_h(Y) + \pi_h(X)\pi^h(Y) = 2\bar{\omega}(X,Y), \tag{3.17}$$

where, in the last step, we have used (3.3) and (3.4). Thus (3.12) can be expressed as

$$d(f,g) + [X,Y]\rfloor\bar{\omega} = 0. \tag{3.18}$$

At first sight this appears to be nothing other than a simple extension of a basic result of symplectic geometry according to which [X,Y] is the Hamiltonian vector field generated by the Poisson bracket (f,g) whenever X, Y are Hamiltonian vector fields generated by f and g respectively with respect to a symplectic 2-form.

There is, however, an important difference: whereas *any* differentiable function on the symplectic manifold generates a Hamiltonian vector field on M, the above analysis shows that this is not true in the present context. For, if the first two relations in (3.15) are substituted in the third, it is
found that

$$\frac{\partial f}{\partial t} = -\left(\frac{\partial H}{\partial p_h}\frac{\partial f}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial f}{\partial p_h}\right) = -(H,f), \tag{3.19}$$

which, by (2.6), is equivalent to

$$Zf = 0. \tag{3.20}$$

Thus, in order that f be such as to allow for the existences of a vector field X for which (3.8) is valid, it is necessary that f be a solution of the first order partial differential equation (3.19). Conversely, let us suppose that we are given a function f on N that satisfies this condition. By means of this function we construct the vector field

$$X = X_h \frac{\partial}{\partial p_h} + X^h \frac{\partial}{\partial q^h} + X^o \frac{\partial}{\partial t}, \tag{3.21}$$

whose components are given by

$$X_h = -\frac{\partial f}{\partial q^h} - X^o \frac{\partial H}{\partial q^h}, \qquad X^h = \frac{\partial f}{\partial p_h} + X^o \frac{\partial H}{\partial p_h}. \tag{3.22}$$

The latter entail that

$$X^h \frac{\partial H}{\partial q^h} + X_h \frac{\partial H}{\partial p_h} = (f,H). \tag{3.23}$$

Also, according to (1.8),

$$\pi_h(X) = X_h + X^o \frac{\partial H}{\partial q^h}, \qquad \pi^h(X) = X^h - X^o \frac{\partial H}{\partial p_h}, \tag{3.24}$$

so that (3.2) gives

$$X \rfloor \tilde{\omega} = \left(X_h + X^o \frac{\partial H}{\partial q^h}\right) dq^h - \left(X^h - X^o \frac{\partial H}{\partial p_h}\right) dp_h - \left(X^h \frac{\partial H}{\partial q^h} + X^h \frac{\partial H}{\partial p_h}\right) dt, \tag{3.25}$$

in which we substitute from (3.22) and (3.23) to obtain

$$X \rfloor \tilde{\omega} = -\frac{\partial f}{\partial q^h} dq^h - \frac{\partial f}{\partial p_h} dp_h + (H,f) dt.$$

Clearly this equation, taken in conjunction with the condition (3.19), implies the required relation (3.8). This state of affairs is summarized in the following

THEOREM: *In order that a differentiable function* f *on* N *be capable of generating a locally Hamiltonian vector field with respect to* $\tilde{\omega}$ *in the sense of relation*

(3.8), *it is necessary and sufficient that* f *be invariant by the canonical field* Z *in the sense of* (3.20).

The relation (3.18) thus yields the

COROLLARY: *If two functions* f, g *on* N are invariant by Z, *then so is their Poisson bracket* (f,g).

Let us denote by $\mathcal{F}_Z(N)$ the set of all differentiable functions on N that are invariant by Z. Because of the Corollary we can define a composition on $\mathcal{F}_Z(N)$ by means of Poisson brackets, thus endowing $\mathcal{F}_Z(N)$ with the structure of a Lie algebra. Also, the set $\mathcal{H}(N)$ of all vector fields on N that are locally Hamiltonian with respect to $\tilde{\omega}$ has the structure of a Lie algebra by virtue of (3.12), the composition being defined by the Lie bracket. Because of (3.18) one can interpret relations such as (3.8) and (3.9) as exemplifications of a *Lie algebra homomorphism*: $\mathcal{F}_Z(N) \rightarrow \mathcal{H}(N)$.

4. Parameter-dependent Canonical Transformations

The theory of previous sections is based on a given set of 2n independent 1-forms $\{\pi_h, \pi^h\}$ on the product manifold N, these 1-forms being subject to the conditions I and II of Section 1. Because of the latter local coordinates are thus prescribed on M, in terms of which these 1-forms admit the representations (1.8) that involve the function H. This state of affairs immediately suggests the following procedure. Let us suppose that one were to begin with a different set of 2n independent 1-forms $\{\bar{\pi}_j, \bar{\pi}^j\}$ on N, together with a new variable $\bar{t}$, such that the sets $\{\bar{\pi}_j, \bar{\pi}^j, d\bar{t}\}$ constitute bases in the cotangent spaces of N. The imbedding of the hypersurface $\overline{M}$ of N on which $\bar{t} = \bar{t}_0 =$ const. is denoted by $\bar{\imath}: \overline{M} \rightarrow N$, so that $\bar{\imath}^*(d\bar{t}) = 0$, and the 1-forms $\{\bar{\pi}_j, \bar{\pi}^j\}$ are assumed to satisfy direct analogues of conditions I and II. Again, this gives rise to local coordinates $\{\bar{p}_j, \bar{q}^j : j = 1, \cdots, n\}$ on $\overline{M}$, in terms of which these 1-forms admit the representations

$$\bar{\pi}_j = d\bar{p}_j + \frac{\partial \overline{K}}{\partial \bar{q}^j} d\bar{t}, \qquad \bar{\pi}^j = d\bar{q}^j - \frac{\partial \overline{K}}{\partial \bar{p}_j} d\bar{t}, \tag{4.1}$$

as counterparts of (1.8) for some differentiable function $\overline{K}$ of the new variables $\{\bar{p}_j, \bar{q}^j, \bar{t}\}$. In order to establish some relationship between the resulting theory and the developments described above, one must prescribe a common invariant. In standard symplectic geometry the fundamental invariant is the symplectic 2-form (1.21) on M; however, in the present context this would not be appropriate. Instead, guided by the relation (2.15), we shall stipulate *that the 2-form* (3.1) *on* N *is to be regarded as the fundamental invariant*: that is

$$\bar{\omega} = \pi_h \wedge \pi^h = \bar{\pi}_j \wedge \bar{\pi}^j. \tag{4.2}$$

If this 2-form is expressed as in (1.5), we see that (4.2) is equivalent to

$$\bar{\omega} = d\pi = d\bar{\pi}, \tag{4.3}$$

where π is given by (1.6), and

$$\bar{\pi} = \bar{p}_j d\bar{q}^j - \bar{K}(\bar{p},\bar{q},\bar{t})d\bar{t}. \tag{4.4}$$

The condition (4.3) can be integrated, at least locally, to yield $\bar{\pi} - \pi = dS$ for some function S on N, that is,

$$\bar{p}_j d\bar{q}^j - \bar{K}(\bar{p}_l,\bar{q}^l,\bar{t})d\bar{t} - \left(p_h dq^h - H(p_m,q^m,t)dt\right) = dS. \tag{4.5}$$

Thus, if the transition from the coordinates (p_h,q^h,t) to the new coordinates $(\bar{p}_j,\bar{q}^j,\bar{t})$ is described by a set of $(2n + 1)$ equations such as

$$\bar{p}_j = \bar{p}_j(p_h,q^h,t), \qquad \bar{q}^j = \bar{q}^j(p_h,q^h,t), \qquad \bar{t} = \bar{t}(p_h,q^h,t), \tag{4.6}$$

this system exemplifies a *t-dependent canonical transformation on* N by virtue of the restriction (4.5) ([2], p. 260, [3] Ch. 6, [7], Ch. 2).

We shall now derive some properties of such transformations. To this end it is recalled that in the theory of the t-independent canonical transformations a fundamentally important role is played by the Liouville 2n-form on M, namely

$$\mu = dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge \cdots \wedge dq^n. \tag{4.7}$$

This form is related to the symplectic 2-form ω on M according to the formula

$$\mu = \frac{1}{n!}(-1)^N \omega^n, \qquad N = \tfrac{1}{2}n(n-1), \tag{4.8}$$

in which ω^n denotes the exterior product $\omega \wedge \cdots \wedge \omega$ with n factors, so that in this context the invariance of μ would be guaranteed by the invariance of ω. However, in the present more general setting we must construct a suitable Liouville $(2n + 1)$-form on N, namely

$$\tilde{\mu} = \mu \wedge dt = \frac{1}{n!}(-1)^N \omega^n \wedge dt, \tag{4.9}$$

and it is this form that must be evaluated in terms of $\tilde{\omega}$, since the latter is supposed to be the fundamental invariant. In order to do this, we write (1.5) in terms of (1.21) and (3.1) as .

$$\tilde{\omega} = \omega - dH \wedge dt, \tag{4.10}$$

from which it is immediately evident that

$$\tilde{\omega}^m = \omega^m - m\omega^{m-1} \wedge dH \wedge dt \tag{4.11}$$

for any positive integer $m \leq n$. Thus

$$\tilde{\omega}^m \wedge dt = \omega^m \wedge dt,$$

which is substituted in (4.9) to yield

$$\tilde{\mu} = \frac{1}{n!}(-1)^N \tilde{\omega}^n \wedge dt. \tag{4.12}$$

The corresponding Liouville $(2n + 1)$-form on N in the new system is defined by analogy with (4.7) and (4.9) as

$$\bar{\mu} = d\bar{p}_1 \wedge \cdots \wedge d\bar{p}_n \wedge d\bar{q}^1 \wedge \cdots \wedge d\bar{q}^n \wedge d\bar{t}, \tag{4.13}$$

which, as before, is equivalent to

$$\bar{\bar{\mu}} = \frac{1}{n!}(-1)^N \bar{\omega}^n \wedge d\bar{t} \tag{4.14}$$

in consequence of the invariance of $\bar{\omega}$. This is related to (4.12) by

$$\bar{\bar{\mu}} = \bar{\mu}\frac{\partial \bar{t}}{\partial t} + \frac{1}{n!}(-1)^N\left(\frac{\partial \bar{t}}{\partial q^h}\bar{\omega}^n \wedge dq^h + \frac{\partial \bar{t}}{\partial p_h}\bar{\omega}^n \wedge dp_h\right). \tag{4.15}$$

Clearly $\omega^n \wedge dq^h = 0$, $\omega^n \wedge dp_h = 0$, so that, by (4.11) with $m = n$,

$$\bar{\omega}^n \wedge dq^h = -n\omega^{n-1} \wedge dq^h \wedge dH \wedge dt, \qquad \bar{\omega}^n \wedge dp_h = -n\omega^{n-1} dp_h \wedge dH \wedge dt,$$

so that (4.15) can be expressed as

$$\bar{\bar{\mu}} = \bar{\mu}\frac{\partial \bar{t}}{\partial t} - \frac{1}{n!}(-1)^N n\omega^{n-1} \wedge \left(\frac{\partial \bar{t}}{\partial q^h}dq^h + \frac{\partial \bar{t}}{\partial p_h}dp_h\right) \wedge dH \wedge dt. \tag{4.16}$$

To this expression we now apply the formula (B.11) of Appendix B to obtain

$$\bar{\bar{\mu}} = \left[\frac{\partial \bar{t}}{\partial t} + (H,\bar{t})\right]\bar{\mu}, \tag{4.17}$$

which gives an explicit representation of the relation between the two generalized Louiville forms on N. This result can be expressed in terms of the canonical vector field (2.5) as

$$\bar{\bar{\mu}} = Z(\bar{t})\bar{\mu}. \tag{4.18}$$

Moreover, it follows from (4.13), (4.6), (4.7) and (4.9) that

$$\begin{aligned} \bar{\bar{\mu}} &= d\bar{p}_1 \wedge \cdots \wedge d\bar{p}_n \wedge d\bar{q}^1 \wedge \cdots \wedge d\bar{q}^n \wedge d\bar{t} \\ &= \frac{\partial(\bar{p}_j, \bar{q}^j, \bar{t})}{\partial(p_h, q^h, t)} dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge \cdots \wedge dq^n \wedge dt = \frac{\partial(\bar{p}_j, \bar{q}^j, \bar{t})}{\partial(p_h, q^h, t)}\bar{\mu}. \end{aligned} \tag{4.19}$$

A comparison of this result with (4.18) yields the

THEOREM: *The functional determinant of the* t*-dependent canonical transformation* (4.6) *is given by*

$$\frac{\partial(\bar{p}_j, \bar{q}^j, \bar{t})}{\partial(p_h, q^h, t)} = Z(\bar{t}), \tag{4.20}$$

where Z *denotes the canonical vector field* (2.5).

Remark. For a t-independent canonical transformation one has $\bar{t} = t$, and (2.5) gives $Z(t) = 1$. Thus for such transformations the Jacobian (4.20) has the value unity; this is a well-known theorem ([3], p. 92) that is fundamental to Liouville's theorem in the theory of conservative dynamical systems.

The relation (4.20) allows us to examine the conditions under which the functional determinant of the canonical transformation (4.6) can vanish. To this end we note that

$$Z \rfloor d\pi = Z \rfloor d\bar{\pi} = Z \rfloor \bar{\omega} = 0$$

by virtue of (2.1) and (4.3), and hence, by (4.2)

$$Z \rfloor (\bar{\pi}_j \wedge \bar{\pi}^j) = (Z \rfloor \bar{\pi}_j)\bar{\pi}^j - (Z \rfloor \bar{\pi}^j)\bar{\pi}_j = 0.$$

Since the 1-forms $\{\bar{\pi}^j, \bar{\pi}_j\}$ are independent, it follows that

$$Z \rfloor \bar{\pi}_j = 0, \qquad Z \rfloor \bar{\pi}^j = 0. \tag{4.21}$$

By means of (2.5) and (4.1) this can be expressed as

$$Z(\bar{p}_j) = -\frac{\partial \bar{K}}{\partial \bar{q}^j} Z(\bar{t}), \qquad Z(\bar{q}^j) = \frac{\partial \bar{K}}{\partial \bar{p}_j} Z(\bar{t}). \tag{4.22}$$

Moreover, according to (2.5) we have

$$Z = \left(\frac{\partial \bar{t}}{\partial t} + \frac{\partial H}{\partial p_h}\frac{\partial \bar{t}}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial \bar{t}}{\partial p_h}\right)\frac{\partial}{\partial \bar{t}} + \left(\frac{\partial \bar{q}^j}{\partial t} + \frac{\partial H}{\partial p_h}\frac{\partial \bar{q}^j}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial \bar{q}^j}{\partial p_h}\right)\frac{\partial}{\partial \bar{q}_j}$$

$$+ \left(\frac{\partial \bar{p}_j}{\partial t} + \frac{\partial H}{\partial p_h}\frac{\partial \bar{p}_j}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial \bar{p}_j}{\partial p^h}\right)\frac{\partial}{\partial \bar{p}_j},$$

that is

$$Z = Z(\bar{t})\frac{\partial}{\partial \bar{t}} + Z(\bar{q}^j)\frac{\partial}{\partial \bar{q}^j} + Z(\bar{p}_j)\frac{\partial}{\partial \bar{p}_j}. \tag{4.23}$$

Because of (4.22) this is equivalent to

$$Z = Z(\bar{t})\left(\frac{\partial}{\partial \bar{t}} + \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial}{\partial \bar{q}^j} - \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial}{\partial \bar{p}_j}\right). \tag{4.24}$$

Now let us suppose for the moment that the functional determinant of (4.6) vanishes at some point $q \in N$. By (4.24) and (4.20) this requires that $Z = 0$ at q, and hence, by (2.4), that $h = 0$ at q. This obviously contradicts the assumption (1.20) unless $q \notin D$, it being recalled that D denotes a region of N on which $h \neq 0$. We therefore infer that the determinant (4.20) cannot vanish on D. The statement can be combined with the theorem of Section 1 to yield the following

COROLLARY: *The functional determinant of the parameter-dependent canonical transformation* (4.6) *does not vanish on any region* D *of* N *on which the Cartan form* (1.6) *has class* $2n + 1$.

The relation (4.24) suggests the definition of the vector field

$$\bar{Z} = \frac{\partial}{\partial \bar{t}} + \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial}{\partial \bar{q}^j} - \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial}{\partial \bar{p}_j} \tag{4.25}$$

as the obvious counterpart of (2.5). Thus (4.24) can be expressed as

$$Z = Z(\bar{t})\bar{Z}. \tag{4.26}$$

In particular, since $Z(t) = 1$, we have

$$Z(\bar{t})\bar{Z}(t) = 1. \tag{4.27}$$

5. Poisson Bracket Relations

Let us consider once more an arbitrary vector field $X \in \mathfrak{X}(N)$, whose coordinate presentation is given by (3.21), and for which (3.25) is expressed as

$$-X\rfloor\bar{\omega} = \xi^h dp_h - \xi_h dq^h + \xi^0 dt, \tag{5.1}$$

where

$$\xi^h = X\rfloor\pi^h = X^h - X^0\frac{\partial H}{\partial p_h}, \quad \xi_h = X\rfloor\pi_h = X_h + X^0\frac{\partial H}{\partial q^h}, \quad \xi^0 = \frac{\partial H}{\partial p_h}X_h + \frac{\partial H}{\partial q^h}X^h. \tag{5.2}$$

For a second vector field Y with

$$Y = Y_h\frac{\partial}{\partial p_h} + Y^h\frac{\partial}{\partial q^h} + Y^0\frac{\partial}{\partial t}, \tag{5.3}$$

we then have in consequence of (5.1) and (5.2)

$$X\rfloor Y\rfloor\bar{\omega} = \xi^h\eta_h - \xi_h\eta^h, \tag{5.4}$$

where we have put

$$\eta^h = Y\rfloor\pi^h = Y^h - Y^0\frac{\partial H}{\partial p_h}, \quad \eta_h = Y\rfloor\pi_h = Y_h + Y^0\frac{\partial H}{\partial q^h}, \quad \eta^0 = \frac{\partial H}{\partial p_h}Y_h + \frac{\partial H}{\partial q^h}Y^h. \tag{5.5}$$

We now turn to the $(\bar{p},\bar{q},\bar{t})$-coordinates, in terms of which we write

$$X = \bar{X}_j\frac{\partial}{\partial\bar{p}_j} + \bar{X}^j\frac{\partial}{\partial\bar{q}^j} + \bar{X}^0\frac{\partial}{\partial\bar{t}}, \qquad Y = \bar{Y}_j\frac{\partial}{\partial\bar{p}_j} + \bar{Y}^j\frac{\partial}{\partial\bar{q}^j} + \bar{Y}^0\frac{\partial}{\partial\bar{t}}, \tag{5.6}$$

together with

$$\bar{\xi}^j = X\rfloor\bar{\pi}^j = \bar{X}^j - \bar{X}^0\frac{\partial\bar{K}}{\partial\bar{p}_j}, \quad \bar{\xi}_j = X\rfloor\bar{\pi}_j = \bar{X}_j + \bar{X}^0\frac{\partial\bar{K}}{\partial\bar{q}^j}, \qquad \bar{\xi}^0 = \frac{\partial\bar{K}}{\partial\bar{p}_j}\bar{X}_j + \frac{\partial\bar{K}}{\partial\bar{q}^j}\bar{X}^j, \tag{5.7}$$

and

$$\bar{\eta}^j = Y\rfloor\bar{\pi}^j = \bar{Y}^j - \bar{Y}^0\frac{\partial\bar{K}}{\partial\bar{p}_j}, \quad \bar{\eta}_j = Y\rfloor\bar{\pi}_j = \bar{Y}_j + \bar{Y}^0\frac{\partial\bar{K}}{\partial\bar{q}^j}, \qquad \bar{\eta}^0 = \frac{\partial\bar{K}}{\partial\bar{p}_j}\bar{Y}_j + \frac{\partial\bar{K}}{\partial\bar{q}^j}\bar{Y}^j. \tag{5.8}$$

In view of the invariance condition (4.2) it is then inferred from (5.4) that

$$X\rfloor Y\rfloor\bar{\omega} = \xi^h\eta_h - \xi_h\eta^h = \bar{\xi}^j\bar{\eta}_j - \bar{\xi}_j\bar{\eta}^j. \tag{5.9}$$

This relation will be used to determine the behavior of Poisson brackets under the transformation (4.6).

To this end we return to (5.1), in which we express dp_h, dq^h, and dt in terms of $d\bar{p}_j$, $d\bar{q}^j$, $d\bar{t}$ in accordance with the inverse of (4.6). The expression thus obtained is compared with the counterpart of (5.1), namely

$$-X\rfloor\bar{\omega} = \bar{\xi}^j d\bar{p}_j - \bar{\xi}_j d\bar{q}^j + \bar{\xi}^0 d\bar{t}, \tag{5.10}$$

which yields the following relations

$$\begin{aligned}
\bar{\xi}^j &= \xi^h \frac{\partial p_h}{\partial \bar{p}_j} - \xi_h \frac{\partial q^h}{\partial \bar{p}_j} + \xi^0 \frac{\partial t}{\partial \bar{p}_j}, \\
-\bar{\xi}_j &= \xi^h \frac{\partial p_h}{\partial \bar{q}^j} - \xi_h \frac{\partial q^h}{\partial \bar{q}^j} + \xi^0 \frac{\partial t}{\partial \bar{q}^j}, \\
\bar{\xi}^0 &= \xi^h \frac{\partial p_h}{\partial \bar{t}} - \xi_h \frac{\partial q^h}{\partial \bar{t}} + \xi^0 \frac{\partial t}{\partial \bar{t}}.
\end{aligned} \tag{5.11}$$

Now let us supplse that we are given an arbitrary differentiable function F on N, for which we write

$$\bar{F}(\bar{p}_j,\bar{q}^j,\bar{t}) = F\Big(p_h(\bar{p}_j,\bar{q}^j,\bar{t}),q^h(\bar{p}_j,\bar{q}^j,\bar{t}),t(\bar{p}_j,\bar{q}^j,\bar{t})\Big) \tag{5.12}$$

by means of the inverse of (4.6). According to (5.11) we then have

$$\begin{aligned}
\bar{\xi}^j - \frac{\partial \bar{F}}{\partial \bar{p}_j} &= \left(\xi^h - \frac{\partial F}{\partial p_h}\right)\frac{\partial p_h}{\partial \bar{p}_j} - \left(\xi_h + \frac{\partial F}{\partial q^h}\right)\frac{\partial q^h}{\partial \bar{p}^j} + \left(\xi^0 - \frac{\partial F}{\partial t}\right)\frac{\partial t}{\partial \bar{p}_j}, \\
-\left(\bar{\xi}_j + \frac{\partial \bar{F}}{\partial \bar{q}^j}\right) &= \left(\xi^h - \frac{\partial F}{\partial p_h}\right)\frac{\partial p_h}{\partial \bar{q}^j} - \left(\xi_h + \frac{\partial F}{\partial q^h}\right)\frac{\partial q^h}{\partial \bar{q}^j} + \left(\xi^0 - \frac{\partial F}{\partial t}\right)\frac{\partial t}{\partial \bar{q}^j}, \\
\left(\bar{\xi}^0 - \frac{\partial \bar{F}}{\partial \bar{t}}\right) &= \left(\xi^h - \frac{\partial F}{\partial p_h}\right)\frac{\partial p_h}{\partial \bar{t}} - \left(\xi_h + \frac{\partial F}{\partial q^h}\right)\frac{\partial q^h}{\partial \bar{t}} + \left(\xi^0 - \frac{\partial F}{\partial t}\right)\frac{\partial t}{\partial \bar{t}}.
\end{aligned} \tag{5.13}$$

These relations are valid for *any* vector field $X \in \mathfrak{X}(N)$ and *any* differentiable function F on N. Now let us suppose that X is determined by the following conditions in the (p,q)-system:

$$X^h = \frac{\partial F}{\partial p_h}, \qquad X_h = -\frac{\partial F}{\partial q^h}, \qquad X^0 = 0. \tag{5.14}$$

It then follows from the third member of (5.27) that

$$\xi^0 - \frac{\partial F}{\partial t} = -\frac{\partial H}{\partial p_h}\frac{\partial F}{\partial q^h} + \frac{\partial H}{\partial q^h}\frac{\partial F}{\partial p_h} - \frac{\partial F}{\partial t} = -Z(F),$$

where, in the second step, we have invoked (2.5). Thus the substitution of (5.14) in (5.12) and (5.13) yields

$$\bar{\xi}^j = \frac{\partial \bar{F}}{\partial \bar{p}_j} - Z(F)\frac{\partial t}{\partial \bar{p}_j}, \qquad \bar{\xi}_j = -\frac{\partial \bar{F}}{\partial \bar{q}^j} + Z(F)\frac{\partial t}{\partial \bar{q}^j}, \tag{5.15}$$

together with a third relation that does not contain any further information since it may be reduced to (4.28).

Similarly, if it is supposed that the components of the vector field Y are determined in the (p,q)-coordinate system by some function G on N as

$$Y^h = \frac{\partial G}{\partial p_h}, \qquad Y_h = -\frac{\partial G}{\partial q^h}, \qquad Y^0 = 0, \tag{5.16}$$

it is found that the components (5.8) are given by

$$\bar{\eta}^j = \frac{\partial \bar{G}}{\partial \bar{p}_j} - Z(G)\frac{\partial t}{\partial \bar{p}_j}, \qquad \bar{\eta}_j = -\frac{\partial \bar{G}}{\partial \bar{q}^j} + Z(G)\frac{\partial t}{\partial \bar{q}^j}. \tag{5.17}$$

The relations (5.15) and (5.17) are now substituted in (5.9). In terms of the notation

$$\{\bar{F},\bar{G}\} = \left(\frac{\partial \bar{F}}{\partial \bar{p}_j}\frac{\partial \bar{G}}{\partial \bar{q}^j} - \frac{\partial \bar{F}}{\partial \bar{q}^j}\frac{\partial \bar{G}}{\partial \bar{p}_j}\right) \tag{5.18}$$

for Poisson brackets in $(\bar{p},\bar{q})$-coordinates it is thus found after some simplification that

$$\{\bar{F},\bar{G}\} = (F,G) + \{\bar{F},t\}Z(G) - \{\bar{G},t\}Z(F). \tag{5.19}$$

This is the relation that we have been seeking: it represents *the transformation law for the Poisson brackets of a pair of arbitrary functions on* N *under a parameter-dependent canonical transformation.* (The same formula had been derived previously ([8], p. 227) in a somewhat different context by entirely "non-symplectic" techniques.)

A slightly more symmetric form of (5.19) may be obtained as follows. As a special case let us put $\bar{F} = \bar{t}$, noting that according to (5.18) one has $\{\bar{t},\bar{G}\} = 0$ for any function $\bar{G}$. Thus

$$(\bar{t},G) = Z(\bar{t})\{\bar{G},t\}, \tag{5.20}$$

and with the aid of (4.26) and (4.27) it follows that

$$\{\bar{G},t\}Z(F) = (\bar{t},G)\bar{Z}(F) = -(G,\bar{t})\bar{Z}(F), \tag{5.21}$$

together with a similar formula obtained by an interchange of F and G. When these are substituted in (5.19) the latter assumes the required form

$$2\{\bar{F},\bar{G}\} + \{\bar{G},t\}Z(F) - \{\bar{F},t\}Z(G) = 2(F,G) + (G,\bar{t})\bar{Z}(F) - (F,\bar{t})\bar{Z}(G). \tag{5.22}$$

Additional useful Poisson bracket relations may be obtained as follows. As a special case of (5.22) we have

$$(\bar{t},\bar{p}_j) = Z(\bar{t})\frac{\partial t}{\partial \bar{q}^j}, \qquad (\bar{t},\bar{q}^l) = -Z(\bar{t})\frac{\partial t}{\partial \bar{p}_l}.$$

Also, if we put $F = \bar{p}_j$, $G = \bar{q}^l$ in (5.19), it is seen that

$$\delta_j^l = (\bar{p}_j,\bar{q}^l) + \{\bar{p}_j,t\}Z(\bar{q}^l) - \{\bar{q}^l,t\}Z(\bar{p}_j).$$

By means of (5.20), (4.22) and (4.26) this can be reduced to

$$(\bar{p}_j,\bar{q}^l) = \delta_j^l - (\bar{t},\bar{p}_j)\frac{\partial \bar{K}}{\partial \bar{p}_l} - (\bar{t},\bar{q}^l)\frac{\partial \bar{K}}{\partial \bar{q}^j}. \tag{5.23}$$

To this we adjoin two similar Poisson-bracket relations, whose derivation is carried out in the same manner:

$$(\bar{q}^l,\bar{q}^j) = -(\bar{t},\bar{q}^l)\frac{\partial \bar{K}}{\partial \bar{p}_j} + (\bar{t},\bar{q}^j)\frac{\partial \bar{K}}{\partial \bar{p}_l}, \tag{5.24}$$

and

$$(\bar{p}_j,\bar{p}_l) = -(\bar{t},\bar{p}_l)\frac{\partial\bar{K}}{\partial\bar{q}^j} + (\bar{t},\bar{p}_j)\frac{\partial\bar{K}}{\partial\bar{q}^l}, \tag{5.25}$$

(These formula also occur in [8] (p. 224), together with the corresponding Lagrange bracket relations.)

6. Further Properties of Parameter-dependent Canonical Transformations

According to our construction the sets $\{\pi^h,\pi_h,dt\}$ and $\{\bar{\pi}^j,\bar{\pi}_j,d\bar{t}\}$ constitute distinct bases of the cotangent spaces of N. The relations (4.21), together with their counterparts

$$Z\rfloor\pi_h = 0, \qquad Z\rfloor\pi^h = 0, \tag{6.1}$$

indicate that these basis elements must be related according to the scheme

$$\begin{pmatrix} \bar{\pi}^j \\ \bar{\pi}_j \end{pmatrix} = \begin{pmatrix} \bar{Q}^j{}_h & \bar{Q}^{jh} \\ \bar{P}_{jh} & \bar{P}_j{}^h \end{pmatrix}\begin{pmatrix} \pi^h \\ \pi_h \end{pmatrix} \tag{6.2}$$

for suitable coefficients P, Q, whose explicit expressions will be derived presently. We shall write the inverse of (6.2) as

$$\begin{pmatrix} \pi^h \\ \pi_h \end{pmatrix} = \begin{pmatrix} Q^h{}_j & Q^{hj} \\ P_{hj} & P_h{}^j \end{pmatrix}\begin{pmatrix} \bar{\pi}^j \\ \bar{\pi}_j \end{pmatrix}, \tag{6.3}$$

which requires that

$$\begin{pmatrix} Q^h{}_j & Q^{hj} \\ P_{hj} & P_h{}^j \end{pmatrix}\begin{pmatrix} \bar{Q}^j{}_k & \bar{Q}^{jk} \\ \bar{P}_{jk} & \bar{P}_j{}^k \end{pmatrix} = \begin{pmatrix} \delta^h_k & 0 \\ 0 & \delta^h_k \end{pmatrix}. \tag{6.4}$$

From (6.2) it follows directly that

$$\bar{\pi}_j \wedge \bar{\pi}^j = \tfrac{1}{2}(\bar{P}_{jh}\bar{Q}^j{}_k - \bar{P}_{jk}\bar{Q}^j{}_h)\pi^h \wedge \pi^k + \tfrac{1}{2}(\bar{P}_j{}^h\bar{Q}^{jk} - \bar{P}_j{}^k\bar{Q}^{jh})\pi_h \wedge \pi_k$$

$$+ (\bar{P}_j{}^h\bar{Q}^j{}_k - \bar{P}_{jk}\bar{Q}^{jh})\pi_h \wedge \pi^k. \tag{6.5}$$

Consequently the invariance condition (5.2) is tantamount to the relations

$$\bar{P}_j{}^h\bar{Q}^j{}_k - \bar{Q}^{jh}\bar{P}_{jk} = \delta^h_k, \quad \bar{P}_j{}^h\bar{Q}^{jk} - \bar{Q}^{jh}\bar{P}_j{}^k = 0, \quad \bar{P}_{jh}\bar{Q}^j{}_k - \bar{Q}^j{}_h\bar{P}_{jk} = 0,$$

which can be expressed as

$$\begin{pmatrix} \bar{P}_j{}^h & -\bar{Q}^{hj} \\ -\bar{P}_{hj} & \bar{Q}^j{}_h \end{pmatrix}\begin{pmatrix} \bar{Q}^j{}_k & \bar{Q}^{jk} \\ \bar{P}_{jk} & \bar{P}_j{}^k \end{pmatrix} = \begin{pmatrix} \delta^h_k & 0 \\ 0 & \delta^h_k \end{pmatrix}. \tag{6.6}$$

By construction, the $2n \times 2n$ matrices that appear above are all nonsingular. Thus a comparison of (6.6) with (6.4) yields

$$\begin{pmatrix} \bar{P}_j{}^h & -\bar{Q}^{jh} \\ -\bar{P}_{jh} & \bar{Q}^j{}_h \end{pmatrix} = \begin{pmatrix} Q^h{}_j & Q^{hj} \\ P_{hj} & P_h{}^j \end{pmatrix}. \tag{6.7}$$

In order to determine the explicit form of the entries in these matrices, we note that, as an immediate consequence of (1.8),

$$\frac{\partial}{\partial p_h}\rfloor\pi_k = \delta^h_k, \quad \frac{\partial}{\partial q^h}\rfloor\pi_k = 0, \quad \frac{\partial}{\partial q^h}\rfloor\pi^k = \delta^h_k, \quad \frac{\partial}{\partial p_h}\rfloor\pi^k = 0. \tag{6.8}$$

It therefore follows from (6.2) and (4.1) that

$$\bar{P}_j{}^h = \frac{\partial}{\partial p_h}\rfloor\bar{\pi}_j = \frac{\partial \bar{p}_j}{\partial p_h} + \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial \bar{t}}{\partial p_h}, \qquad \bar{P}^{jh} = \frac{\partial}{\partial q^h}\rfloor\bar{\pi}_j = \frac{\partial \bar{p}_j}{\partial q^j} + \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial \bar{t}}{\partial q^h}, \tag{6.9}$$

and

$$\bar{Q}^{jh} = \frac{\partial}{\partial p_h}\rfloor\bar{\pi}^j = \frac{\partial \bar{q}^j}{\partial p_h} - \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial \bar{t}}{\partial p_h}, \qquad \bar{Q}^j{}_h = \frac{\partial}{\partial q^h}\rfloor\bar{\pi}^j = \frac{\partial \bar{q}^j}{\partial q^h} - \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial \bar{t}}{\partial q^h}. \tag{6.10}$$

Similiarly

$$P_h{}^j = \frac{\partial}{\partial \bar{p}_j} \rfloor \pi_h = \frac{\partial p_h}{\partial \bar{p}_j} + \frac{\partial H}{\partial q^h} \frac{\partial t}{\partial \bar{p}_j}, \qquad P_{hj} = \frac{\partial}{\partial \bar{q}^j} \rfloor \pi_h = \frac{\partial p_h}{\partial \bar{q}^j} + \frac{\partial H}{\partial q^h} \frac{\partial t}{\partial \bar{q}^j}, \tag{6.11}$$

and

$$Q^{hj} = \frac{\partial}{\partial \bar{p}_j} \rfloor \pi^h = \frac{\partial q^h}{\partial \bar{p}_j} - \frac{\partial H}{\partial p_h} \frac{\partial t}{\partial \bar{p}_j}, \qquad Q^h{}_j = \frac{\partial}{\partial \bar{q}^j} \rfloor \pi^h = \frac{\partial q^h}{\partial \bar{q}^j} - \frac{\partial H}{\partial p_h} \frac{\partial t}{\partial \bar{q}^j}. \tag{6.12}$$

The substitution of (6.9)-(6.12) in (6.7) gives rise to the so-called reciprocity relations. Before listing these explicitly we should derive further identities in order to obtain a complete set of such relationships. To this end we note that, according to (2.5) and (6.9),

$$Z(\bar{p}_j) = \frac{\partial \bar{p}_j}{\partial t} + \frac{\partial H}{\partial p_h} \frac{\partial \bar{p}_j}{\partial q^h} - \frac{\partial H}{\partial q^h} \frac{\partial \bar{p}_j}{\partial p_h}$$

$$= \frac{\partial \bar{p}_j}{\partial t} + \frac{\partial H}{\partial p_h} \frac{\partial \bar{p}_j}{\partial q^h}\Big(\bar{P}_{jh} - \frac{\partial \bar{K}}{\partial \bar{q}^j} \frac{\partial \bar{t}}{\partial q^h}\Big) - \frac{\partial H}{\partial q^h}\Big(\bar{P}_j{}^h - \frac{\partial \bar{K}}{\partial \bar{q}^j} \frac{\partial \bar{t}}{\partial p_h}\Big)$$

$$= \frac{\partial \bar{p}_j}{\partial t} + \frac{\partial H}{\partial p_h} \bar{P}_{jh} - \frac{\partial H}{\partial q^h} \bar{P}_j{}^h - \frac{\partial \bar{K}}{\partial \bar{q}^j}\Big(\frac{\partial H}{\partial p_h} \frac{\partial \bar{t}}{\partial q^h} - \frac{\partial H}{\partial q^h} \frac{\partial \bar{t}}{\partial p_h}\Big).$$

We now apply (6.7) to the second and third terms on the right-hand side, while the fourth is re-written in terms of (2.5):

$$Z(\bar{p}_j) = \frac{\partial \bar{p}_j}{\partial t} - \frac{\partial H}{\partial p_h} P_{hj} - \frac{\partial H}{\partial q^h} Q^h{}_j - \frac{\partial \bar{K}}{\partial \bar{q}^j}\Big(Z(\bar{t}) - \frac{\partial \bar{t}}{\partial t}\Big),$$

which, because of (4.22), reduces to

$$\frac{\partial \bar{p}_j}{\partial t} + \frac{\partial \bar{K}}{\partial \bar{q}^j} \frac{\partial \bar{t}}{\partial t} = \frac{\partial H}{\partial p_h} P_{hj} + \frac{\partial H}{\partial q^h} Q^h{}_j.$$

By means of (6.11) and (6.12) this may be expressed as

$$\frac{\partial \bar{p}_j}{\partial t} + \frac{\partial \bar{K}}{\partial \bar{q}^j} \frac{\partial \bar{t}}{\partial t} = \frac{\partial H}{\partial p_h}\Big(\frac{\partial p_h}{\partial \bar{q}^j} + \frac{\partial H}{\partial q^h} \frac{\partial t}{\partial \bar{q}^j}\Big) + \frac{\partial H}{\partial q^h}\Big(\frac{\partial q^h}{\partial \bar{q}^j} - \frac{\partial H}{\partial p_h} \frac{\partial t}{\partial \bar{q}^j}\Big), \tag{6.13}$$

it being noted that the terms in $\partial t/\partial \bar{q}^j$ on the right-hand side cancel. A similar relation is obtained for $\partial \bar{q}^j/\partial t$. These relations are adjoined to the system (6.7) to be listed as follows:

$$\frac{\partial \bar{p}_j}{\partial p_h} + \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial \bar{t}}{\partial p_h} = \frac{\partial q^h}{\partial \bar{q}^j} - \frac{\partial H}{\partial p_h}\frac{\partial t}{\partial \bar{q}^j},$$

$$\frac{\partial \bar{p}_j}{\partial q^h} + \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial \bar{t}}{\partial q^h} = -\frac{\partial p^h}{\partial \bar{q}^j} - \frac{\partial H}{\partial q^h}\frac{\partial t}{\partial \bar{q}^j}, \tag{6.14}$$

$$\frac{\partial \bar{p}_j}{\partial t} + \frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial \bar{t}}{\partial t} = \frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial \bar{q}^j} + \frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial \bar{q}^j},$$

together with

$$\frac{\partial \bar{q}^j}{\partial p_h} - \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial \bar{t}}{\partial p_h} = -\frac{\partial q^h}{\partial \bar{p}_j} + \frac{\partial H}{\partial p_h}\frac{\partial t}{\partial \bar{p}_j},$$

$$\frac{\partial \bar{q}^j}{\partial q^h} - \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial \bar{t}}{\partial q^h} = \frac{\partial p_h}{\partial \bar{p}_j} + \frac{\partial H}{\partial q^h}\frac{\partial t}{\partial \bar{p}_j}, \tag{6.15}$$

$$\frac{\partial \bar{q}^j}{\partial t} - \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial \bar{t}}{\partial t} = -\frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial \bar{p}_j} - \frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial \bar{p}_j}.$$

These ***reciprocity relations*** are central to the entire theory of parameter-dependent canonical transformations.

As an immediate consequence we note that the third members of (6.14) and of (6.15) give

$$\frac{\partial\overline{K}}{\partial\overline{p}_j}\frac{\partial\overline{p}_j}{\partial t}+\frac{\partial\overline{K}}{\partial\overline{q}^j}\frac{\partial\overline{q}^j}{\partial t}=\frac{\partial H}{\partial p_h}\Big(\frac{\partial\overline{K}}{\partial\overline{p}_j}\frac{\partial p_h}{\partial\overline{q}^j}-\frac{\partial\overline{K}}{\partial\overline{q}^j}\frac{\partial p_h}{\partial\overline{p}_j}\Big)$$

$$+\frac{\partial H}{\partial q^h}\Big(\frac{\partial\overline{K}}{\partial\overline{p}_j}\frac{\partial q^h}{\partial\overline{q}^j}-\frac{\partial\overline{K}}{\partial\overline{q}^j}\frac{\partial q^h}{\partial\overline{p}_j}\Big)$$

$$=\frac{\partial H}{\partial p_h}\Big(\overline{Z}(p_h)-\frac{\partial p_h}{\partial\overline{t}}\Big)+\frac{\partial H}{\partial q^h}\Big(\overline{Z}(q^h)-\frac{\partial q^h}{\partial\overline{t}}\Big),\tag{6.16}$$

where, in the second step, we have used (4.25). But the counterparts of (4.22) are

$$\overline{Z}(p_h)=-\frac{\partial H}{\partial q^h}\overline{Z}(t),\qquad \overline{Z}(q^h)=\frac{\partial H}{\partial p_h}\overline{Z}(t).\tag{6.17}$$

Thus (6.16) is reduced to the useful identity

$$\frac{\partial\overline{K}}{\partial\overline{p}_j}\frac{\partial\overline{p}_j}{\partial t}+\frac{\partial\overline{K}}{\partial\overline{q}^j}\frac{\partial\overline{q}^j}{\partial t}=-\frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial\overline{t}}-\frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial\overline{t}}.\tag{6.18}$$

The third members of (6.14) and (6.15) suggest that we adjoin the following entries to (6.9)-(6.12): namely

$$\overline{P}_{jo}=\frac{\partial\overline{p}_j}{\partial t}+\frac{\partial\overline{K}}{\partial\overline{q}^j}\frac{\partial\overline{t}}{\partial t},\qquad \overline{Q}^j{}_o=\frac{\partial\overline{q}^j}{\partial t}-\frac{\partial\overline{K}}{\partial\overline{p}_j}\frac{\partial\overline{t}}{\partial t},\tag{6.19}$$

together with

$$P_{oj}=-\frac{\partial H}{\partial\overline{q}_j}+\frac{\partial H}{\partial t}\frac{\partial t}{\partial\overline{q}^j}=-\Big(\frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial\overline{q}^j}+\frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial\overline{q}^j}\Big),\tag{6.20}$$

and

$$P_o{}^j=-\frac{\partial H}{\partial\overline{p}_j}+\frac{\partial H}{\partial t}\frac{\partial t}{\partial\overline{p}_j}=-\Big(\frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial\overline{p}_j}+\frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial\overline{p}_j}\Big).\tag{6.21}$$

This allows us to express the aforementioned equations as

$$\overline{P}_{jo}=-P_{oj},\ \text{and}\ \overline{Q}^j{}_o=\overline{P}_o{}^j.\tag{6.22}$$

A geometrical interpretation of this construction may be given in terms of the 1-parameter family of hypersurfaces $\overline{M}(\bar{t}_0)$ of N as defined by the equation $\bar{t} = \bar{t}(p_h, q^h, t) = \bar{t}_0$ in which $\bar{t}_0$ denotes the parameter while the dependence of $\bar{t}$ on (p_h, q^h, t) is prescribed by (4.6). Each tangent space $T_p(\overline{M}(\bar{t}_0))$ at $p \in \overline{M}$ has a coordinate basis $\left\{\frac{\partial}{\partial \bar{p}_j}, \frac{\partial}{\partial \bar{q}^j}\right\}$, by means of which we now define a set of 2n vector fields $\{\nabla^h, \nabla_h\}$ by putting

$$\begin{pmatrix} \nabla^h \\ \nabla_h \end{pmatrix} = \begin{pmatrix} \bar{P}_j{}^h & \bar{Q}^{jh} \\ \bar{P}_{jh} & \bar{Q}^j{}_h \end{pmatrix} \begin{matrix} \frac{\partial}{\partial \bar{p}_j} \\ \frac{\partial}{\partial \bar{q}^j} \end{matrix} . \tag{6.23}$$

According to (6.6) the $2n \times 2n$ matrix that occurs on the right-hand side is non-singular. *Thus the vector fields* $\{\nabla^h, \nabla_h\}$ *also define bases in the tangent spaces* $T_p(M(t_0))$. Again, we adjoin to this set the single vector field

$$\nabla_0 = \bar{P}_{jo} \frac{\partial}{\partial \bar{p}_j} + \bar{Q}^j{}_0 \frac{\partial}{\partial \bar{q}^j}. \tag{6.24}$$

We shall also require the counterparts of (6.23) and (6.24):

$$\begin{pmatrix} \overline{\nabla}^j \\ \overline{\nabla}_j \end{pmatrix} = \begin{pmatrix} P_h{}^j & Q^{hj} \\ P_{hj} & Q^h{}_j \end{pmatrix} \begin{matrix} \frac{\partial}{\partial p_h} \\ \frac{\partial}{\partial q^h} \end{matrix} , \tag{6.25}$$

and

$$\overline{\nabla}_0 = P_{ho} \frac{\partial}{\partial p_h} + Q^h{}_0 \frac{\partial}{\partial q^h}. \tag{6.26}$$

It then follows with the aid of (6.7) and (6.20)-(6.22) that

$$\nabla^h \bar{p}_j = \overline{\nabla}_j q^h, \quad \nabla_h \bar{p}_j = -\overline{\nabla}_j p_h, \quad \nabla_0 \bar{p}_j = \left(\frac{\partial H}{\partial p_h} \frac{\partial p_h}{\partial \bar{q}^j} + \frac{\partial H}{\partial q^h} \frac{\partial q^h}{\partial \bar{q}^j}\right), \tag{6.27}$$

and

$$\nabla^h \bar{q}^j = -\overline{\nabla}^j q^h, \quad \nabla_h \bar{q}^j = \overline{\nabla}^j p_h, \quad \nabla_0 \bar{q}^j = -\left(\frac{\partial H}{\partial p_h} \frac{\partial p_h}{\partial \bar{p}_j} + \frac{\partial H}{\partial q^h} \frac{\partial q^h}{\partial \bar{p}_j}\right), \tag{6.28}$$

these two systems being equivalent to (6.14) and (6.15) respectively.

From the definitions (6.23) and (6.24) it is evident that the $2n + 1$ vector fields ∇^h, ∇_h, ∇_0 are not independent. In order to exhibit this dependence explictly, we rewrite (6.23) by means of (6.7) as

$$\begin{pmatrix}\nabla^h\\ \nabla_h\end{pmatrix} = \begin{pmatrix} Q^h{}_j & -Q^{hj}\\ -P_{hj} & P_h{}^j\end{pmatrix} \begin{matrix}\dfrac{\partial}{\partial \bar{p}_j}\\ \dfrac{\partial}{\partial \bar{q}^j}\end{matrix} . \tag{6.29}$$

Because of (6.11) and (6.12) this is equivalent to

$$\nabla^h = \left(\frac{\partial q^h}{\partial \bar{q}^j} - \frac{\partial H}{\partial p_h}\frac{\partial t}{\partial \bar{q}^j}\right)\frac{\partial}{\partial \bar{p}_j} - \left(\frac{\partial q^h}{\partial \bar{p}_j} - \frac{\partial H}{\partial p_h}\frac{\partial t}{\partial \bar{p}_j}\right)\frac{\partial}{\partial \bar{q}^j}, \tag{6.30}$$

and

$$\nabla_h = -\left(\frac{\partial p_h}{\partial \bar{q}^j} + \frac{\partial H}{\partial q^h}\frac{\partial t}{\partial \bar{q}^j}\right)\frac{\partial}{\partial \bar{p}_j} + \left(\frac{\partial p_h}{\partial \bar{p}_j} + \frac{\partial H}{\partial q^h}\frac{\partial t}{\partial \bar{p}_j}\right)\frac{\partial}{\partial \bar{q}^j}. \tag{6.31}$$

From this it follows after some simplification that

$$\frac{\partial H}{\partial q^h}\nabla^h - \frac{\partial H}{\partial p_h}\nabla_h = -P_{0j}\frac{\partial}{\partial \bar{p}_j} + P_0{}^j\frac{\partial}{\partial \bar{q}^j}$$

in terms of the notation (6.20) and (6.21). When the identities (6.22) are applied to the right-hand side, the definition (6.24) being taken into account, it is found that

$$\frac{\partial H}{\partial q^h}\nabla^h - \frac{\partial H}{\partial p_h}\nabla_h = \nabla_0, \tag{6.32}$$

which displays the above-mentioned dependence.

There are some useful alternative representations of the vector fields (6.23). If we substitute in the latter from (6.9) and (6.10) we find that

$$\nabla^h = \frac{\partial \bar{p}_j}{\partial p_h}\frac{\partial}{\partial \bar{p}_j} + \frac{\partial \bar{q}^j}{\partial p_h}\frac{\partial}{\partial \bar{q}^j} + \frac{\partial \bar{t}}{\partial p_h}\Big(\frac{\partial \bar{K}}{\partial \bar{q}^j}\frac{\partial}{\partial \bar{p}_j} - \frac{\partial \bar{K}}{\partial \bar{p}_j}\frac{\partial}{\partial \bar{q}^j}\Big).$$

By means of the definition (4.25) this may be expressed as

$$\nabla^h = \frac{\partial}{\partial p_h} - \frac{\partial \bar{t}}{\partial p_h}\bar{Z}. \tag{6.33}$$

Similarly,

$$\nabla_h = \frac{\partial}{\partial q^h} - \frac{\partial \bar{t}}{\partial q^h}\bar{Z}. \tag{6.34}$$

These relations may be used to show that for any pair of differentiable functions F, G on N

$$(\nabla^h F)(\nabla_h G) - (\nabla^h G)(\nabla_h F) = (F,G) + (G,\bar{t})\bar{Z}(F) - (F,\bar{t})\bar{Z}(G). \tag{6.35}$$

If we apply (5.21), together with the corresponding relation in which F and G are interchanged, we obtain

$$(\nabla^h F)(\nabla_h G) - (\nabla^h G)(\nabla_h F) = (F,G) - \{\bar{G},t\}Z(F) + \{F,\bar{t}\}Z(G). \tag{6.36}$$

A comparison of this expression with (5.19) shows that

$$(\nabla^h F)(\nabla_h G) - (\nabla^h G)(\nabla_h F) = \{\bar{F},\bar{G}\}, \tag{6.37}$$

where the right-hand side is the Poisson bracket (5.18). It may be shown similarly that

$$(\bar{\nabla}^j F)(\bar{\nabla}_j G) - (\bar{\nabla}^j G)\,(\bar{\nabla}_j F) = (F,G). \tag{6.38}$$

We shall now endeavor to characterize the class of parameter-dependent canonical transformations for which the Poisson bracket of any pair of functions F, G on N is invariant. From (5.19) it is evident that this is the case if and only if

$$\{\bar{F},t\} = 0, \tag{6.39}$$

for all functions $\overline{F}$ on N, which, because of (5.20), also entails that

$$(F,\bar{t}) = 0. \tag{6.40}$$

The first of these is simply

$$\frac{\partial \overline{F}}{\partial \bar{p}_j}\frac{\partial t}{\partial \bar{q}^j} - \frac{\partial \overline{F}}{\partial \bar{q}^j}\frac{\partial t}{\partial \bar{p}_j} = 0,$$

so that the special substitution $\overline{F} = \bar{p}_l$, followed by $\overline{F} = \bar{q}^l$, yields

$$\frac{\partial t}{\partial \bar{q}^j} = 0, \qquad \frac{\partial t}{\partial \bar{p}_j} = 0. \tag{6.41}$$

The relation (6.40) shows similarly that

$$\frac{\partial \bar{t}}{\partial q^h} = 0, \qquad \frac{\partial \bar{t}}{\partial \bar{p}_h} = 0. \tag{6.42}$$

From this it follows that the third member of the canonical transformation (4.6) must be of the form $\bar{t} = \bar{t}(t)$. Because (4.26) and (4.27) this implies that

$$\overline{Z} = \psi(t)Z, \text{ with } \psi(t) = dt/d\bar{t}, \tag{6.43}$$

and hence, by (2.5) and (4.25)

$$\frac{\partial \overline{K}}{\partial \bar{p}_j}\frac{\partial}{\partial \bar{q}^j} - \frac{\partial \overline{K}}{\partial \bar{q}^j}\frac{\partial}{\partial \bar{p}_j} = \psi(t)\Big(\frac{\partial H}{\partial p_h}\frac{\partial}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial}{\partial p_h}\Big). \tag{6.44}$$

Because of (6.42) one also has

$$\frac{\partial}{\partial p_h} = \frac{\partial \bar{p}_j}{\partial p_h}\frac{\partial}{\partial \bar{p}_j} + \frac{\partial \bar{q}^j}{\partial p_h}\frac{\partial}{\partial \bar{q}^j}, \quad \frac{\partial}{\partial q^h} = \frac{\partial \bar{p}_j}{\partial q^h}\frac{\partial}{\partial \bar{p}_j} + \frac{\partial \bar{q}^j}{\partial q^h}\frac{\partial}{\partial \bar{q}^j},$$

and consequently (6.44) yields

$$\frac{\partial\overline{K}}{\partial\overline{p}_j} = \psi(t)\Big(\frac{\partial H}{\partial p_h}\frac{\partial\overline{q}^j}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial\overline{q}^j}{\partial p_h}\Big), \quad \frac{\partial\overline{K}}{\partial\overline{q}^j} = -\psi(t)\Big(\frac{\partial H}{\partial p_h}\frac{\partial\overline{p}_j}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial\overline{p}_j}{\partial p_h}\Big). \tag{6.45}$$

Moreover, with the aid of (6.41) and (6.42) the first two members of each of (6.14) and (6.15) are reduced to

$$\frac{\partial\overline{p}_j}{\partial p_h} = \frac{\partial q^h}{\partial\overline{q}^j}, \quad \frac{\partial\overline{p}_j}{\partial q^h} = -\frac{\partial p_h}{\partial\overline{q}^j}, \qquad \frac{\partial\overline{q}^j}{\partial p_h} = -\frac{\partial q^h}{\partial\overline{p}_j}, \quad \frac{\partial\overline{q}^j}{\partial q^h} = \frac{\partial p_h}{\partial\overline{p}_j}. \tag{6.46}$$

This allows us to express (6.45) as

$$\frac{\partial\overline{K}}{\partial\overline{p}_j} = \psi(t)\Big(\frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial\overline{p}_j} + \frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial\overline{p}_j}\Big), \qquad \frac{\partial\overline{K}}{\partial\overline{q}^j} = \psi(t)\Big(\frac{\partial H}{\partial p_h}\frac{\partial p_h}{\partial\overline{q}^j} + \frac{\partial H}{\partial q^h}\frac{\partial q^h}{\partial\overline{q}^j}\Big), \tag{6.47}$$

from which it is deduced, again by means of (6.41), that

$$\frac{\partial}{\partial\overline{p}_j}\Big(\overline{K} - \psi(t)H\Big) = 0, \quad \frac{\partial}{\partial\overline{q}^j}\Big(\overline{K} - \psi(t)H\Big) = 0.$$

This may be integrated to yield

$$\overline{K}(\overline{p}_j,\overline{q}^j,\overline{t}) = \psi(t)H(p_h,q^h,t) + \sigma(t), \tag{6.48}$$

where σ denotes some function of the single variable t. Also, in the present context the derivatives $\partial\overline{t}/\partial t$ that occur in each of the third members of (6.14) and (6.15) are identical with $1/\psi(t)$; consequently the substitution of (6.47) in these relations reduces the latter to

$$\frac{\partial\overline{p}_j}{\partial t} = 0, \qquad \frac{\partial\overline{q}^j}{\partial t} = 0. \tag{6.49}$$

This shows that the first two members of the canonical transformation (4.6) do not involve the variable t explictly:

$$\overline{p}_j = \overline{p}_j(p_h,q^h), \qquad \overline{q}^j = \overline{q}^j(p_h,q^h). \tag{6.50}$$

These relations suggest that we are dealing with a parameter-independent canonical transformation. That this is indeed the case follows from a standard theorem ([7], p. 93) according to which, given (6.50), the relations (6.46) characterize parameter-independent canonical transformations. Conversely, it is well known ([3], p. 83) that all Poisson brackets are invariant under the latter. The above analysis therefore establishes the

THEOREM: *In order that the Poisson bracket of any pair of functions on* N *be invariant under a canonical transformation it is necessary and sufficient that the latter be parameter-independent in the sense of* (6.50). *Under these circumstances the functions* $\overline{K}$ *and* H *are related by* (6.48).

Appendix A

LEMMA: *Let* $N = M \times \mathbf{R}$, *the coordinates on* M *being collectively denoted by* $\{x^a : a = 1, \cdots, 2n\}$, *while* t *represents the single variable on* **R**. *Let*

$$\mu = \mu_a dx^a \tag{A.1}$$

be a 1*-form whose coefficients depend on* $\{x^a, t\}$. *Then, in order that* $\mu \wedge dt \in \Lambda^2(N)$ *be closed it is necessary and sufficient that there exist a differentiable function* Φ *on* N *in terms of which* μ *admits a local representation*

$$\mu = d\Phi - \frac{\partial \Phi}{\partial t} dt. \tag{A.2}$$

PROOF: Suppose that $\mu \wedge dt$ is closed, in which case $d\mu \wedge dt = 0$. But this relation represents a necessary and sufficient condition that $d\mu$ be 'divisible' by dt in the sense that there exists a 1-form ν on N such that $d\mu = \nu \wedge dt$ ([6], p. 177, Ex. 5.14). However, according to (A.1)

$$d\mu = \frac{\partial \mu_a}{\partial x^b} dx^b \wedge dx^a - \frac{\partial \mu_a}{\partial t} dx^a \wedge dt,$$

so that compatability requires that $\partial \mu_a / \partial x^b - \partial \mu_b / \partial x^a = 0$ and $\nu = -(\partial \mu_a / \partial t) dx^a$. The first of these implies the existence, at least locally, of a function Φ on N such that $\mu_a = \partial \Phi / \partial x^a$, the substitution of which in (A.1) yields (A.2). Conversely, if μ is given by (A.2), we have $\mu \wedge dt = d\Phi \wedge dt = d(\Phi \wedge dt)$, which is closed.

Appendix B

LEMMA: *Let* $T = T_h dq^h$, $S = S^h dp_h$ *denote a pair of* 1*-forms whose coefficients are functions on* N. *Then*

$$\omega^{n-1} \wedge T \wedge S = -(-1)^N (n-1)! T_h S^h \mu, \qquad N = \tfrac{1}{2}n(n-1),$$

where ω *denotes the symplectic* 2*-form* (1.21) *and* μ *is the Liouville form* (4.7).

PROOF: For the purpose of this discussion the summation convention is suspended for repeated indices that are dotted. We shall write

$$\lambda_{\dot{k}} = dp_{\dot{k}} \wedge dq^{\dot{k}}, \tag{B.1}$$

so that

$$\lambda_{\dot{k}} \wedge \lambda_{\dot{h}} = 0 \text{ if } h = k, \tag{B.2}$$

while

$$\omega = dp_h \wedge dq^h = \lambda_{\dot{1}} + \cdots + \lambda_{\dot{n}} \,. \tag{B.3}$$

Let us construct $(2n-2)$-form

$$\Lambda_{\dot{k}} = \lambda_{\dot{1}} \wedge \cdots \wedge \lambda_{\dot{k}-1} \wedge \lambda_{\dot{k}+1} \wedge \cdots \wedge \lambda_{\dot{n}}. \tag{B.4}$$

Because of (B.1) this has all of $\{dp_1, \cdots, dp_n, dq^1, \cdots, dq^n\}$ as factors with the exception of dp_k and dq^k. Thus

$$\Lambda_{\dot{k}} \wedge dp_h = 0, \text{ and } \Lambda_{\dot{k}} \wedge dq^h = 0, \text{ if } h \neq k. \tag{B.5}$$

With

$$T = T_j dq^j, \qquad S = S^j dp_j \tag{B.6}$$

we then have

$$\Lambda_{\dot{k}} \wedge T = T_{\dot{k}} \Lambda_{\dot{k}} dq^{\dot{k}} = T_{\dot{k}} \lambda_{\dot{1}} \wedge \cdots \wedge \lambda_{\dot{k}-1} \wedge \lambda_{\dot{k}+1} \wedge \cdots \wedge \lambda_{\dot{n}} \wedge dq^{\dot{k}},$$

and

$$\Lambda_{\dot{k}} \wedge T \wedge S = T_{\dot{k}} S^{\dot{k}} \lambda_{\dot{1}} \wedge \cdots \wedge \lambda_{\dot{k}-1} \wedge \lambda_{\dot{k}+1} \wedge \cdots \wedge \lambda_{\dot{n}} \wedge dq^{\dot{k}} \wedge dp_{\dot{k}},$$

that is, by (B.1) and (4.7)

$$\Lambda_{\dot{k}} \wedge T \wedge S = -T_{\dot{k}} S^{\dot{k}} \lambda_{\dot{1}} \wedge \cdots \wedge \lambda_{\dot{n}} = -(-1)^N T_{\dot{k}} S^{\dot{k}} \mu. \tag{B.7}$$

It is also readily established inductively by means of (B.1) and (B.4) that

$$\omega^{n-1} = (n-1)! \sum_{\dot{k}=1}^{n} \Lambda_{\dot{k}}. \tag{B.8}$$

This result, when taken in conjuction with (B.7), establishes the lemma.

We now observe that each term on the right-hand side of (B.8) has $(n-1)$ factors from the set $\{dq^1, \cdots, dq^n\}$. Thus

$$\omega^{n-1} \wedge \left(\frac{\partial \bar{t}}{\partial q^h} dq^h\right) \wedge dH \wedge dt = \omega^{n-1} \wedge \frac{\partial \bar{t}}{\partial q^h} dq^h \wedge \frac{\partial H}{\partial p_k} dp_k \wedge dt,$$

since the contribution from the term $(\partial H/\partial q^k) dq^k$ in dH gives rise to $(n+1)$ such factors. Thus we can apply the lemma to this expression to infer that

$$\omega^{n-1} \wedge \frac{\partial \bar{t}}{\partial q^h} dq^h \wedge dH \wedge dt = -(-1)^N (n-1)! \frac{\partial H}{\partial p_h} \frac{\partial \bar{t}}{\partial q^h} \mu \wedge dt,$$

or, in terms of (3.9)

$$n\omega^{n-1} \wedge \frac{\partial \bar{t}}{\partial q^h} dq^h \wedge dH \wedge dt = -(-1)^N n! \frac{\partial H}{\partial p_h} \frac{\partial \bar{t}}{\partial q^h} \bar{\mu}. \tag{B.9}$$

Similarly, it is found that

$$n\omega^{n-1} \wedge \frac{\partial \bar{t}}{\partial p_h} dp_h \wedge dH \wedge dt = (-1)^N n! \frac{\partial \bar{t}}{\partial p_h} \frac{\partial H}{\partial q^h} \bar{\mu}. \tag{B.10}$$

Addition of (B.9) and (B.10) then yields the formula

$$n\omega^{n-1} \wedge d\bar{t} \wedge dH \wedge dt = n\omega^{n-1} \wedge \left(\frac{\partial \bar{t}}{\partial q^h} dq^h + \frac{\partial \bar{t}}{\partial p_h} dp_h\right) \wedge dH \wedge dt$$

$$= -(-1)^N n!\left(\frac{\partial H}{\partial p_h}\frac{\partial \bar{t}}{\partial q_h} - \frac{\partial H}{\partial q^h}\frac{\partial \bar{t}}{\partial p_h}\right)\bar{\mu} = -(-1)^N n!(H,\bar{t})\bar{\mu}. \quad (B.11)$$

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Hanno Rund
Program in Applied Mathematics
University of Arizona
Tucson, AZ 85721
USA

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POTENTIAL THEORY FOR THE YUKAWA EQUATION

J.L. Schiff and W.J. Walker

1. Introduction. The potential of the strong nuclear force can be described by the solution $e^{-\mu r}/r$ of the elliptic equation

$$\Delta u = \mu^2 u \, , \quad (\mu > 0) \, .$$

This description was first proposed by the Japanese physicist Hideki Yukawa, and the equation now bears his name.

Yukawa proposed $e^{-\mu r}/r$ to be the potential of a point charge in $\mathbb{R}^3$. In this article we shall investigate Yukawan potential theory and the associated pseudo-analytic functions in the plane.

A diversity of papers have been devoted to this subject over a number of years. The story seems to begin with Bouligand [2] who was able to show that every positive solution $u(re^{i\theta})$ in the plane has a representation in the form

$$u\,(re^{i\theta}) = \int_0^{2\pi} e^{\mu r \cos(\theta - t)}\, d\lambda(t)$$

where λ is a non-decreasing function. This result was generalised by Caffarelli and Littman [6]. (We shall refer to it as the B-C-L theorem.) An important example arises if we set $\lambda(t) = t/2\pi$ to obtain the well-known representation (see [16]),

$$I_0(\mu r) = \frac{1}{2\pi}\int_0^{2\pi} e^{\mu r \cos t}\, dt ,$$

for the modified Bessel function. Using this integral, it is straightforward to show that

$$k_r(t) = \frac{e^{\mu r \cos t}}{I_0(\mu r)}, \quad 0 < r < \infty ,$$

is a *summability kernel.*

Section 2 is directed towards an H^1-space theory for real-valued solutions of the Yukawa equation. This falls within the purview of Brelot's harmonic space theory [4] and the H^p-space work of Lumer-Naim [11], but more specialised results are obtained here, using the positive solution $I_0(\mu r)$. In particular, in Section 3 the kernel $k_r(t)$ is used to study the existence of

$$U(\theta) = \lim_{r\to\infty} \frac{u(re^{i\theta})}{2\pi I_0(\mu r)}$$

which corresponds to the "far field pattern" of u.

In [8] Duffin coined the term *panharmonic* for a C^2 solution of the Yukawa equation. He turned the subject in a new direction by initiating a theory of pseudo-analytic panharmonic functions. This development fits into the framework of pseudo-analytic functions of L. Bers [1] but more detailed results can be obtained for panharmonic functions. In particular we refer to a Bieberbach type inequality (Schiff-Walker [13]) mentioned in Section 4.

Duffin used the phrase *μ-regular* to describe the pseudo-analytic functions which are characterized by Theorem 5. This leads to Theorem 9, which does not have an analogue in classical Hardy space theory.

In Section 6 we give a sampling theorem which yields an exact representation of the Fourier coefficients of a μ-regular function f, by taking a countable set of values of f on the boundary of a circle of radius r. The authors in [14] have already given a similar algorithm for the Taylor coefficients of an analytic function, but a different approach is required in the present case. Here we employ the representation of Fourier cosine coefficients developed by Bruns [5] and Wintner [17] (cf also Schiff-Walker [15]).

Finally, by letting $r \to \infty$, the representation for the Fourier coefficients of f is obtained in terms of values of the far field F of f. This requires a smoothness assumption on F and a lemma of Wintner [17].

Acknowledgement. We are indebted to Philip Quirke who undertook a computer investigation of the Bieberbach conjecture for μ-regular functions.

2. A Hardy Space of Panharmonic Functions

As mentioned above, a C^2 complex-valued solution $w(x,y)$ of the Yukawa equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \mu^2 w \quad (\mu > 0) \tag{1}$$

is called a *panharmonic* function. A potential theory for (1) was developed by Duffin [8] and will be intrumental in our development. In particular we require the following Fourier expansion of a panharmonic function, [8], p.114.

Theorem 1. *If* $w(r,\theta)$ *is panharmonic in the disk* $x^2 + y^2 < R^2$ *and continuous in* $x^2 + y^2 \leq R^2$ *then for* $0 \leq r < R$

$$w(r,\theta) = \sum_{n=-\infty}^{\infty} c_n I_{|n|}(\mu r) e^{in\theta} \tag{2}$$

where

$$c_n = \frac{1}{2\pi I_{|n|}(\mu R)} \int_0^{2\pi} w(R,\phi) e^{-in\phi}\, d\phi\,, \tag{3}$$

and I_n *is the modified Bessel function of the first kind given by*

$$I_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n \left[1 + \frac{(x/2)^2}{1\cdot(n+1)} + \frac{(x/2)^4}{1\cdot 2\cdot(n+1)\ (n+2)} + \ldots \right], \quad n = 0,1,2,\ldots\ .$$

Observe that $I_{|n|}(\mu r)e^{in\theta}$ is panharmonic for each n, and that $I_0(\mu r)$ is positive panharmonic.

Moreover, a panharmonic function w in a domain Ω also satisfies a mean value property whenever $\{ |z-z_0| \le r \} \subseteq \Omega$,

$$w(z_0) = \frac{1}{2\pi I_0(\mu r)} \int_0^{2\pi} w(z_0+re^{i\phi})\, d\phi\ .$$

Since $I_0(\mu r) > 1$ for $r > 0$ the mean value property implies:

<u>Maximum Principle</u>.

(i) Let w be complex valued panharmonic in a domain $\Omega \subseteq \mathbb{C}$. Then $|w|$ has no maximum in Ω unless $w \equiv 0$ in Ω.

(ii) Let Ω be a relatively compact subset of $\mathbb{C}$ and let $w \not\equiv 0$ be panharmonic in Ω and continuous on $\overline{\Omega}$. If $|w| \le M$ on $\partial\Omega$ then $|w| < M$ on Ω.

A subsolution u of (1) will be termed *subpanharmonic* and satisfies whenever $\{ |z-z_0| \le r \} \subseteq \Omega$

$$u(z_0) \le \frac{1}{2\pi I_0(\mu r)} \int_0^{2\pi} u(z_0+re^{i\phi})\, d\phi .$$

We now turn to the notion of a Hardy space of panharmonic functions.

Definition. *The Hardy space* $h_\mu(\mathbb{C})$ *is defined to be the space of real-valued panharmonic functions* u *in* $\mathbb{C}$ *for which the integral means*

$$M(u,R) = \frac{1}{2\pi I_0(\mu R)} \int_0^{2\pi} \left| u(Re^{i\phi}) \right| d\phi$$

are bounded for $0 \le R < \infty$.

For $u \in h_\mu(\mathbb{C})$, $|u|$ is subpanharmonic, implying $M(u,R)$ increases as $R \to \infty$. Denote

$$\| u \| = \lim_{R \to \infty} M(u,R) .$$

Substituting (3) into (2) and interchanging summation and integration, we obtain

$$w(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} w(Re^{i\phi}) \left[\sum_{n=-\infty}^{\infty} \frac{I_{|n|}(\mu r) e^{in(\theta-\phi)}}{I_{|n|}(\mu R)} \right] d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} w(Re^{i\phi}) \left[\frac{I_0(\mu r)}{I_0(\mu R)} + 2 \sum_{n=1}^{\infty} \frac{I_n(\mu r)}{I_n(\mu R)} \cos n(\theta-\phi) \right] d\phi ,$$

where the bracketed expression is a "Poisson" kernel, which we denote by $P_r^R(\theta-\phi)$.

Panharmonic functions satisfy the following (cf. Brelot [3]):

Minimum Principle. *If* u *is panharmonic in a bounded domain* Ω *and continuous on* $\overline{\Omega}$ *with* $u \geq 0$ on $\partial\Omega$ *then* $u \geq 0$ in Ω.

From the minimum principle it follows that for fixed r, θ, and R, $P_r^R(\theta-\phi) \geq 0$ on $[0,2\pi]$; for if not, suppose $P_r^R(\theta-\phi_0) < 0$. Then $P_r^R(\theta-\phi) < 0$ on some open interval I containing ϕ_0. Let F be a closed interval, $F \subset I$, and take a continuous function $f \geq 0$ such that $f > 0$ on F, $f = 0$ on I'. Then

$$u(r,\theta)= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})P_r^R(\theta-\phi)d\phi < 0,$$

a contradiction since $u \geq 0$ by the minimum principle.

There is another useful characterization of $h_\mu(\mathbb{C})$-functions the proof of which is analogous to the classical case. (For the classical proof see [9].)

Theorem 2. *A panharmonic function* u *belongs to* $h_\mu(\mathbb{C})$ *if, and only if,* $|u|$ *has a panharmonic majorant.*

Proof. If u is panharmonic and $|u|$ has a panharmonic majorant v in $\mathbb{C}$ then

$$\frac{1}{2\pi I_0(\mu R)}\int_0^{2\pi} |u(Re^{i\phi})|\, d\phi \leq \frac{1}{2\pi I_0(\mu R)}\int_0^{2\pi} v(Re^{i\phi})d\phi = v(0) < \infty$$

for $0 \leq R < \infty$, implying $u \in h_\mu(\mathbb{C})$.

On the other hand, if $u \in h_\mu(\mathbb{C})$, let $\{R_n\}_{n=1}^{\infty}$ be a sequence of radii $R_n \uparrow \infty$ as $n \to \infty$. Let

$$v_n(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \left| u(R_n e^{i\phi}) \right| P_r^{R_n}(\theta-\phi)\, d\phi\ , \quad n = 1,2,3\ \dots\ .$$

By the positivity of the kernel, $\{v_n\}_{n=1}^{\infty}$ is a sequence of positive panharmonic functions with $v_n \mid \{\, |z| = R_n \,\} = |u|$. By the maximum principle for panharmonic functions, $v_n \le v_{n+1}$, i.e. $\{v_n\}_{n=1}^{\infty}$ is increasing. But

$$v_n(0) = \frac{1}{2\pi I_0(\mu R_n)}\int_0^{2\pi} |u(R_n e^{i\phi})|\, d\phi \ \le\ K < \infty$$

for $n = 1,2,3, \dots$, so Harnack's inequality (cf. [8]) implies $v_n \uparrow v$ panharmonic in $\mathbb{C}$. Since $v_n \ge |u|$ for each n, then $v \ge |u|$, as desired. The function v so obtained is the least panharmonic majorant of $|u|$ on $\mathbb{C}$.

Corollary 1. *For* $u \in h_\mu(\mathbb{C})$,

$$\|u\| = \lim_{R\to\infty} M(u,R) = v(0)\ ,$$

where v *is the least panharmonic majorant of* $|u|$ *on* $\mathbb{C}$.

Corollary 2. $u \in h_\mu(\mathbb{C})$ *implies* $|u(r,\theta)| \le \|u\|\, e^{\mu r}$, $0 \le r < \infty$.

Proof. Let v the be least panharmonic majorant of $|u|$ in $\mathbb{C}$. By the Bouligand-Caffarelli-Littman result, $v(z) \le v(0)e^{\mu r}$, $z = re^{i\theta}$. Then

$$|u(r,\theta)| \le v(0)e^{\mu r} = \|u\|\, e^{\mu r}\ .$$

In view of the B-C-L result we define a *panharmonic-Stieltjes integral* as

$$\int_0^{2\pi} e^{\mu r\cos(\theta-t)}\, d\lambda(t)\,,$$

for $\lambda \in BV([0,2\pi])$. Then we maintain:

<u>Theorem 3</u>. *The following are equivalent in* $\mathbb{C}$:

(i) $h_\mu(\mathbb{C})$;

(ii) *the differences of two positive panharmonic functions* ;

(iii) *panharmonic-Stieltjes integrals.*

<u>Proof</u>.

(i) $\Rightarrow$ (ii). Given $u \in h_\mu(\mathbb{C})$, $|u| \le v$ for some positive panharmonic function v in $\mathbb{C}$. Then $w = v-u$ is positive panharmonic in $\mathbb{C}$, and $u = v-w$.

(ii) $\Rightarrow$ (iii). Let $u = u_1-u_2$, where u_1, u_2 are positive panharmonic in $\mathbb{C}$. By the B-C-L representation

$$u_i(r,\theta) = \int_0^{2\pi} e^{\mu r\cos(\theta-t)}\, d\lambda_i(t)\,.$$

Then $\lambda = \lambda_1-\lambda_2 \in BV([0,2\pi]$ and $u(r,\theta)$ is a panharmonic-Stieltjes integral.

(iii) $\Rightarrow$ (i). If

$$u(r,\theta) = \int_0^{2\pi} e^{\mu r\cos(\theta-t)}\, d\lambda(t)$$

for $\lambda \in BV([0,2\pi])$, then $\lambda = \lambda_1-\lambda_2$, where λ_1, λ_2 are bounded non-decreasing functions on $[0,2\pi]$. As $\int_0^{2\pi} e^{\mu r\cos(\theta-t)}\, d\lambda_i(t) = u_i(r,\theta)$ are positive panharmonic, the conclusion follows from $|u| \le u_1 + u_2$.

3. Radial Limits

We turn now to the question of radial limits of panharmonic functions in $\mathbb{C}$. For a panharmonic function $u(r,\theta)$ in $\mathbb{C}$, it is most appropriate to consider $u(r,\theta)/I_0(\mu r)$ as $r \to \infty$.

Example. Let $u(r,\theta) = e^{\mu r \cos(\theta - t)}$, for $0 \le \theta$, $t \le 2\pi$. Since (cf. [16]), $I_0(\mu r)$ is asymptotic to $e^{\mu r} / \sqrt{2\pi\mu r}$ as $r \to \infty$, we obtain

$$\lim_{r \to \infty} \frac{e^{\mu r \cos(\theta - t)}}{I_0(\mu r)} = \begin{cases} 0 & \text{if } \theta \ne t \\ \infty & \text{if } \theta = t \end{cases} .$$

Thus we can see that the radial limits of $\dfrac{u(r,\theta)}{I_0(\mu r)}$ as $r \to \infty$ do not uniquely determine the function u even up to a set of Lebesgue measure zero. However, under more stringent conditions this is possible.

A mentioned in the introduction,

$$k_r(\phi) = \frac{e^{\mu r \cos\phi}}{I_0(\mu r)}$$

is a *summability kernel*, $0 < r < \infty$ (cf. Katznelson [10]). From the preceding theorem, any $u \in h_\mu(\mathbb{C})$ can be represented by

$$u(re^{i\theta}) = \int_0^{2\pi} e^{\mu r \cos(\theta - t)}\, d\lambda(t) ,$$

where $\lambda = \lambda_1 - \lambda_2$, and λ_1, λ_2 are non-decreasing on $[0,2\pi]$.

If $d\lambda$ is AC with respect to Lebesgue measure, the Radon-Nikodym theorem implies $d\lambda = f(t)dt$, $f \in L^1([0,2\pi])$. We consider two cases:

(i) $d\lambda = f dt$, $f \in L^1([0,2\pi])$: Then

$$\frac{u(re^{i\theta})}{2\pi I_0(\mu r)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{\mu r\cos(\theta-t)}}{I_0(\mu r)} f(t)dt = (k_r * f)(\theta),$$

where $*$ is the convolution of k_r and f. Hence by Katznelson [10] p.11,

$$\| k_r * f - f \|_1 \to 0 \text{ as } r \to \infty .$$

(ii) $d\lambda = f dt$, $f \in C([0,2\pi])$:

$$\lim_{r\to\infty} \frac{u(re^{i\theta})}{2\pi I_0(\mu r)} = \lim_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} k_r(\theta-t) f(t)dt = f(\theta)$$

by [10] p.15 .

Results of this nature are summarized in the following theorem.

Theorem 4. *If* $u \in h_\mu(\mathbb{C})$ *and*

$$u(r,\theta) = \int_0^{2\pi} e^{\mu r\cos(\theta-t)}\, d\lambda(t),$$

where $d\lambda = f dt$, *then*

(i) $f \in L^1([0,2\pi])$ *implies*

$$\lim_{r\to\infty} \int_0^{2\pi} \left| \frac{u(re^{i\theta})}{2\pi I_0(\mu r)} - f(\theta) \right| d\theta = 0 .$$

(ii) $f \in C([0,2\pi])$ *implies*

$$\lim_{r \to \infty} \frac{u(re^{i\theta})}{2\pi I_0(\mu r)} = f(\theta) \; .$$

(iii) $f \in L^1([0,2\pi])$ *implies*

$$\lim_{r \to \infty} \frac{u(re^{i\theta})}{2\pi I_0(\mu r)} = f(\theta) \; , \; a.e.$$

Proof. It remains to prove (iii). This can be done using the asymptotic estimate $I_0(\mu r) \sim e^{\mu r} / \sqrt{2\pi\mu r}$ as $r \to \infty$ and the technique of [10] p.20.

4. Pseudo-Analytic Functions

Let u and v be real-valued functions which satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} + \mu u$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} - \mu v \; . \tag{4}$$

Then u and v satisfy the Yukawa equation. Hence the equations (4) are the Cauchy Riemann equations for the Yukawa equation and a function $f = u+iv$ is pseudo-analytic. Following Duffin [8] we call f *μ-regular*. If f is μ-regular and c is a constant then $g = cf$ is μ-regular only if c is real.

Theorem 5. *A function* f *is μ-regular if and only if it can be written in the form* $f = \mu w + \overline{Lw}$ *where* w *is a solution (possibly complex-valued) of the Yukawa equation and* $L = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$.

Proof. It is straightforward to check that $f = \mu w + \overline{Lw}$ is μ-regular. The converse follows from Theorem 21 [8] which gives the expansion of f in pseudo-powers. Each pseudo-power can itself be written in the form $z^{(n)} = \mu w + \overline{Lw}$ for an appropriate w.

For our purposes we rewrite Theorem 21 [8] in the form of (2).

Theorem 6. *If f is μ-regular in the plane then*

$$f(z) = \sum_{n=0}^{\infty} c_n I_n(\mu r)e^{in\theta} + \sum_{n=1}^{\infty} c_{-n} I_n(\mu r)e^{-in\theta} \tag{5}$$

where for $a > 0$ *and* $n \geq 0$

$$c_n = \frac{1}{2\pi I_n(\mu a)} \int_0^{2\pi} f(ae^{i\theta})\, e^{-in\theta}\, d\theta \tag{6}$$

and, by the μ-regularity, $c_{-n} = \overline{c_{n-1}}$, $n \geq 0$.

We particularly emphasise the condition $c_{-n} = \overline{c_{n-1}}$ which arises from the μ-regularity. It is proved by Duffin [8] using contour integration. We refer to [13] (Theorem 5), for a shorter proof.

With this notation we have the following example.

Example. *Let* $w(x,y) = e^{\mu(x\cos\alpha + y\sin\alpha)}$, $0 \leq \alpha < 2\pi$, *with f as in Theorem 5 given by*

$$f(x,y) = \mu(1 + \cos\alpha - i\sin\alpha)e^{\mu(x\cos\alpha + y\sin\alpha)}, \quad 0 \leq \alpha < 2\pi .$$

Then for all $n > 1$

$$| c_n - c_{-n} | \leq n\, | c_1 - c_{-1} | .$$

Proof. We first show that for all n,

$$c_n = \mu(1 + \cos\alpha - i \sin\alpha)e^{-in\alpha}.$$

By equation (6) we have

$$c_n = \frac{\mu(1 + \cos\alpha - i \sin\alpha)}{2\pi I_{|n|}(\mu a)} \int_0^{2\pi} e^{\mu a \cos(\theta-\alpha)} e^{-in\theta} d\theta .$$

The change of variable $\theta - \alpha = \phi$ gives

$$c_n = \frac{\mu(1 + \cos\alpha - i \sin\alpha)e^{-in\alpha}}{2\pi I_{|n|}(\mu a)} \int_0^{2\pi} e^{\mu a \cos\phi} e^{-in\phi} d\phi$$

$$= \mu(1 + \cos\alpha - i \sin\alpha)e^{-in\alpha} \frac{I_0(\mu a)}{I_{|n|}(\mu a)} (k_a{*}g)(0)$$

where $g(\theta) = e^{in\theta}$. The result follows by Theorem 4, since $\lim_{a \to \infty} (k_a{*}g)(0) = g(0) = 1$ and also $\lim_{a \to \infty} I_0(\mu a) / I_{|n|}(\mu a) = 1$.

To complete the proof note that $|\sin n\alpha| \le n |\sin\alpha|$. Hence

$$\begin{aligned} |c_n - c_{-n}| &= |\mu(1 + \cos\alpha - i \sin\alpha)| \, |e^{-in\alpha} - e^{in\alpha}| \\ &\le |\mu(1 + \cos\alpha - i \sin\alpha)| \, |2i \sin n\alpha| \\ &\le |\mu(1 + \cos\alpha - i \sin\alpha)| \, |2in \sin\alpha| \\ &= n \, |c_1 - c_{-1}| . \end{aligned}$$

Remark. This property is interesting in view of the following Bieberbach-de Branges type inequality which is a variant of Theorem 9 in [13].

Theorem 7. *Let* f *be μ-regular in* $\mathbb{C}$ *and suppose* f *is real on the real axis and real only there. Then for* $n > 1$,

$$|c_n - c_{-n}| \leq n\,|c_1 - c_{-1}| .$$

Computer studies of *univalent*, μ-regular functions in $\mathbb{C}$ indicate the following:

Conjecture: *If* f *is a univalent μ-regular function in* $\mathbb{C}$ *as given by* (5) *then the Fourier coefficients* c_n *satisfy*

$$|c_n - c_{-n}| \leq n\,|c_1 - c_{-1}| , \quad \text{for } n > 1 .$$

The conjecture implies (if $c_0 = 0$, $c_1 = 1$),

$$|c_n| \leq n + |c_{-n}| = n + |\overline{c_{n-1}}|$$

and by induction $|c_n| \leq \frac{n\,(n+1)}{2}$, $n = 1,2, \ldots$.

An analogous Bieberbach type conjecture has been posed for a class of univalent harmonic functions in the unit disk by J. Clunie and T. Sheil-Small [7].

5. A Hardy Space of μ-Regular Functions.

The notion of a Hardy space of panharmonic functions in Section 2 can be extended to μ-regular functions.

Firstly we mention the following, which precludes the possibility of nontrivial bounded μ-regular functions in $\mathbb{C}$.

Liouville's Theorem. *If* f *is μ-regular and bounded in* $\mathbb{C}$, *then* $f \equiv 0$.

This follows from the fact that the same holds for bounded panharmonic functions (cf. Brelot [4], Ozawa [12]).

Definition. *A μ-regular function* f *belongs to the class* $H_\mu(\mathbb{C})$ *if*

$$M(f,R) = \frac{1}{2\pi I_0(\mu R)} \int_0^{2\pi} | f(Re^{i\phi}) | \, d\phi \leq K < \infty$$

for $0 \leq R < \infty$.

Note that if $f \in H_\mu(\mathbb{C})$ and $f = u+iv$, then $u \in h_\mu(\mathbb{C})$, $v \in h_\mu(\mathbb{C})$, and conversely, if $f = u+iv$ is μ-regular in $\mathbb{C}$ with $u \in h_\mu(\mathbb{C})$, $v \in h_\mu(\mathbb{C})$, then $f \in H_\mu(\mathbb{C})$.

Consequently, by Theorem 2 we conclude:

Theorem 8. *A μ-regular function* f *belongs to* $H_\mu(\mathbb{C})$ *if, and only if,* $|f|$ *has a panharmonic majorant.*

By Theorem 5, every μ-regular function f can be written in the form $f = \mu w + \overline{Lw}$ where w is a complex-valued panharmonic function and $L = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$. This leads to the following:

Theorem 9. *Let* $w = u+iv$ *where* $u, v \in h_\mu(\mathbb{C})$. *Then the associated μ-regular function*

$$f = \mu w + \overline{Lw}$$

belongs to $H_\mu(\mathbb{C})$.

Proof. It suffices to consider $w = u$ as the proof for $w = iv$ is similar and the sum of two μ-regular functions is μ-regular. Indeed, by Theorem 3 it suffices to consider that $u > 0$ is panharmonic in $\mathbb{C}$. Then by the B-C-L representation of u,

$$f = \mu \int_0^{2\pi} e^{\mu(x \cos\alpha + y \sin\alpha)}\, d\lambda(\alpha) + L\, \overline{\int_0^{2\pi} e^{\mu(x \cos\alpha + y \sin\alpha)}\, d\lambda(\alpha)}$$

$$= \mu \int_0^{2\pi} e^{\mu(x \cos\alpha + y \sin\alpha)}\, d\lambda(\alpha) + \int_0^{2\pi} \mu(\cos\alpha - i \sin\alpha)\, e^{\mu(x \cos\alpha + y \sin\alpha)}\, d\lambda(\alpha)$$

$$= \mu \int_0^{2\pi} [\,(1 + \cos\alpha) - i \sin\alpha\,]\, e^{\mu(x \cos\alpha + y \sin\alpha)}\, d\lambda(\alpha)\,.$$

Consequently, $|f| \leq \gamma u$, where γ is a real constant, i.e. $|f|$ has a panharmonic majorant, and $f \in H_\mu(\mathbb{C})$.

Note that by the Riesz theorem (cf. Duren [8]) we should not expect that if $u \in h_\mu(\mathbb{C})$, then the conjugate v of u also belongs to $h_\mu(\mathbb{C})$. However, the algorithm does not find the conjugate.

The next result gives a sufficient condition for a μ-regular function to belong to $H_\mu(\mathbb{C})$.

Theorem 10. *Suppose* $f(z) = \sum_{n=-\infty}^{\infty} c_n I_{|n|}(\mu r)\, e^{in\theta}$ *is μ-regular in* $\mathbb{C}$ *and* $\sum_{n=0}^{\infty} |c_n| < \infty$. *Then* $f \in H_\mu(\mathbb{C})$.

Proof. Since f is μ-regular, $c_{-n} = \overline{c_{n-1}}$, implying $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. Hence

$$\frac{1}{2\pi I_0(\mu R)} \int_0^{2\pi} |f(Re^{i\phi})|\, d\phi \leq \frac{1}{2\pi I_0(\mu R)} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} |c_n|\, I_{|n|}(\mu R)\, d\phi$$

$$= \sum_{n=-\infty}^{\infty} |c_n| \frac{I_{|n|}(\mu R)}{I_0(\mu R)} \leq \sum_{n=-\infty}^{\infty} |c_n| < \infty .$$

6. <u>A Sampling Formula</u>.

In this section we discuss a sampling formula for Fourier cosine coefficients which was developed by Wintner [17], and more recently by Schiff-Walker [14, 15].

A key ingredient of the sampling algorithm is the well-known Möbius function, ν, from number theory, defined on the positive integers by:

(i) $\nu(1) = 1$;

(ii) $\nu(j) = 0$ if there is a prime p such that $p^2 \mid j$;

(iii) if $j = p_1 p_2 \ldots p_\ell$ is the prime factorization of p, and the p_i's are all distinct, then $\nu(j) = (-1)^\ell$.

The role the Möbius function plays in Fourier analysis can be seen by the following adaptation of a theorem of Wintner (cf. [17]).

<u>Theorem 11</u>. *Let ϕ be real-valued of period 2π on $|z| = 1$, and let $\omega_{kn} = e^{\frac{2\pi i}{kn}}$. If ϕ has the normalization $\int_0^{2\pi} \phi(e^{i\theta})\, d\theta = 0$ and $\phi' \in \mathrm{Lip}_1([0,2\pi])$, then the Fourier cosine coefficients of ϕ satisfy*

$$a_n = \sum_{k=1}^{\infty} \frac{\nu(k)}{kn} \sum_{m=1}^{kn} \phi(\omega_{kn}^m) .$$

The sampling theorem for μ-regular functions can now be established.

Theorem 12. *Let* f(z) *be* μ*-regular in* $|z| \leq r$. *Then the coefficients in the Fourier series representation* (5) *are given by the recursive relation*

$$c_n = \frac{1}{I_n(\mu r)} \sum_{k=1}^{\infty} \frac{\nu(k)}{kn} \sum_{m=1}^{kn} \left[f(\omega_{kn}^m r) - f(0) I_0(\mu r) \right] - \bar{c}_{n-1} \tag{7}$$

for $n = 1, 2, 3, \ldots$.

Proof. Letting $c_n = a_n + ib_n$ we have from (5)

$$f(z) = \sum_{n=-\infty}^{\infty} \left(a_n I_{|n|}(\mu r) \cos n\theta - b_n I_{|n|}(\mu r) \sin n\theta \right)$$

$$+ i \sum_{n=-\infty}^{\infty} \left(b_n I_{|n|}(\mu r) \cos n\theta + a_n I_{|n|}(\mu r) \sin n\theta \right)$$

$$= u(re^{i\theta}) + i\, v(re^{i\theta}) .$$

The cosine terms of $u(re^{i\theta})$ are:

$$a_0 I_0(\mu r) + \sum_{n=1}^{\infty} (a_n + a_{-n}) I_n(\mu r) \cos n\theta .$$

Likewise the cosine terms of $v(re^{i\theta})$ are:

$$b_0 I_0(\mu r) + \sum_{n=1}^{\infty} (b_n + b_{-n}) I_n(\mu r) \cos n\theta .$$

If we set $U(re^{i\theta}) = u(re^{i\theta}) - a_0 I_0(\mu r)$, $V(re^{i\theta}) = v(re^{i\theta}) - b_0 I_0(\mu r)$, then U and V satisfy the hypotheses of Theorem 11. As a consequence

$$(a_n + a_{-n}) I_n(\mu r) = \sum_{k=1}^{\infty} \frac{\nu(k)}{kn} \sum_{m=1}^{kn} U(\omega_{kn}^m r) \tag{8}$$

$$(b_n+b_{-n})\, I_n(\mu r) = \sum_{k=1}^{\infty} \frac{\nu(k)}{kn} \sum_{m=1}^{kn} V\left(\omega_{kn}^m r\right). \tag{9}$$

Combining (8) and (9),

$$(c_n+c_{-n})\, I_n(\mu r) = \sum_{k=1}^{\infty} \frac{\nu(k)}{kn} \sum_{m=1}^{kn} \left[\, f\left(\omega_{kn}^m r\right) - f(0) I_0(\mu r) \,\right].$$

But *by the μ-regularity of* f, $c_{-n} = \overline{c_{n-1}}$, and we obtain the required recursion relation.

We now suppose that by the B-C-L theorem, a μ-regular function $f(re^{i\theta})$ has the representation

$$f(re^{i\theta}) = \int_0^{2\pi} e^{\mu r \cos(\theta-t)} F(t)\, dt$$

where $F \in C[0,2\pi]$. Then by Theorem 4,

$$\lim_{r\to\infty} \frac{f(re^{i\theta})}{2\pi I_0(\mu r)} = F(\theta)$$

and $F(\theta)$ is the "far field pattern" of $f(re^{i\theta})$.

The next theorem shows how, under more stringent assumptions on $F(\theta)$, the Fourier coefficients of $f(re^{i\theta})$ in (5) may be obtained from sampled values of the far field $F(\theta)$.

Theorem 13. *Suppose* $F' \in Lip_1([0,2\pi])$ *and* $f(re^{i\theta}) = \int_0^{2\pi} e^{\mu r \cos(\theta-t)} F(t)\, dt$.
Then the coefficients in the Fourier series representation (5) *are given by the recursive relation*

$$c_n = \sum_{k=1}^{\infty} \frac{\nu(k)}{kn} \sum_{m=1}^{kn} \left[2\pi\, F\left(\omega_{kn}^{m}\right) - f(0) \right] - \overline{c_{n-1}}$$

for $n = 1,2,3,\ldots$. (We identify $F(e^{i\theta})$ with $F(\theta)$.)

Before proving Theorem 13 we shall establish the following lemmas.

Lemma 1. *If* ϕ *is a real valued function of period* 2π *and satisfies* $\phi' \in \mathrm{Lip}_1([0,2\pi])$, *then*

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta})\, d\theta - \frac{1}{n} \sum_{m=1}^{n} \phi\left(e^{\frac{2\pi i m}{n}}\right) \right| \le \frac{C}{n^2},$$

where C *is the Lipschitz constant.*

Proof. See Wintner [17] p. 4.

Lemma 2. *Suppose* $F' \in \mathrm{Lip}_1([0,2\pi])$ *with Lipschitz constant* C *and*

$$f(re^{i\theta}) = \int_0^{2\pi} e^{\mu r \cos(\theta - t)} F(t)\, dt .$$

Then for all $r > 0$,

$$\left| \frac{f_\theta(re^{i\theta_1})}{2\pi\, I_0(\mu r)} - \frac{f_\theta(re^{i\theta_2})}{2\pi\, I_0(\mu r)} \right| < C\, |\theta_1 - \theta_2| .$$

Proof. Rewriting the convolution, we have

$$f(re^{i\theta}) = \int_0^{2\pi} e^{\mu r \cos t} F(\theta - t)\, dt .$$

By differentiation under the integral,

$$\left|\frac{f_\theta(re^{i\theta_1})}{2\pi I_0(\mu r)} - \frac{f_\theta(re^{i\theta_2})}{2\pi I_0(\mu r)}\right| \le \int_0^{2\pi} \frac{e^{\mu r\cos t}}{2\pi I_0(\mu r)} \mid F'(\theta_1-t) - F'(\theta_2-t) \mid dt$$

$$< C\mid\theta_1-\theta_2\mid \int_0^{2\pi} \frac{e^{\mu r\cos t}}{2\pi I_0(\mu r)} dt = C\mid\theta_1-\theta_2\mid .$$

Proof of Theorem 13. Rewriting equation (7) we have from Theorem 12

$$c_n = \frac{I_0(\mu r)}{I_n(\mu r)} \sum_{k=1}^{\infty} \frac{v(k)}{kn} \sum_{m=1}^{kn} \left[\frac{f(\omega_{kn}^m r)}{I_0(\mu r)} - f(0)\right] - \overline{c_{n-1}} . \tag{10}$$

We wish to take $\lim\limits_{r\to\infty}$ of (10) using $\lim\limits_{r\to\infty} \dfrac{I_0(\mu r)}{I_n(\mu r)} = 1$ and $\lim\limits_{r\to\infty} \dfrac{f(\omega_{kn}^m r)}{I_0(\mu r)} = 2\pi F(\omega_{kn}^m)$.

First we apply Lemma 2 to the function $\phi(re^{i\theta}) = f(re^{i\theta})/I_0(\mu r)$ to show that ϕ_θ satisfies a Lipschitz condition uniformly in r. Also

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(re^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{I_0(\mu r)} d\theta = f(0) .$$

Hence by Lemma 1, for $r > 0$,

$$\left|\frac{1}{kn}\sum_{m=1}^{kn}\left[\frac{f(\omega_{kn}^m r)}{I_0(\mu r)} - f(0)\right]\right| < \frac{2\pi C}{k^2 n^2} .$$

Given $\varepsilon > 0$, it follows that for N sufficiently large and for $r > 0$,

$$\left|\sum_{k=N+1}^{\infty} \frac{v(k)}{kn} \sum_{m=1}^{kn}\left[\frac{f(\omega_{kn}^m r)}{I_0(\mu r)} - f(0)\right]\right| < \frac{\varepsilon}{2} .$$

Since the bound is uniform in r, the proof may be completed by considering the $\lim\limits_{r\to\infty}$ of the first N terms in the summation (10).

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J.L. Schiff and W.J. Walker
Department of Mathematics and Statistics
University of Auckland
Auckland, New Zealand.

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FREE BOUNDARY PROBLEM FOR A VISCOUS COMPRESSIBLE FLOW WITH A SURFACE TENSION

V. A. Solonnikov and A. Tani

1. Introduction

In this paper we are concerned with a free boundary problem governing the motion of an isolated mass of a viscous compressible barotropic fluid whose particles attract each other according to the Newton's law. The problem is formulated as follows: find a bounded domain $\Omega_t, t > 0$, the velocity vector field $\mathbf{v}(x,t) = (v_1, v_2, v_3)$ and the density $\rho(x,t) > 0$ defined for $x \in \Omega_t$ and satisfying the Navier-Stokes equations

$$\begin{gathered} \rho_t + \nabla \cdot \rho \mathbf{v} = 0\,, \\ \rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \nabla T = \rho(\mathbf{f} + \kappa \nabla U), x \in \Omega_t, t > 0 \end{gathered} \tag{1.1}$$

and the initial and boundary conditions

$$\begin{gathered} \rho|_{t=0} = \rho_0(x)\,,\ \mathbf{v}|_{t=0} = \mathbf{v}_0(x)\,, \quad x \in \Omega_0 \equiv \Omega\,, \\ T\mathbf{n} = -p_e(x,t)\mathbf{n} + \sigma H \mathbf{n}\,, \quad x \in \Gamma_t \equiv \partial\Omega_t\,. \end{gathered} \tag{1.2}$$

Here $\mathbf{f}(x,t)$ is the vector field of external forces and $p_e(x,t)$ is the external pressure prescribed for $x \in \mathbb{R}^3, t > 0, U = \int_{\Omega_t} \frac{\rho(y,t)dy}{|x-y|}$ is the newtonian potential, Ω is the given bounded domain, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), H$ is the twice mean curvature of $\Gamma_t, \mathbf{n}$ is the unit exterior normal to $\Gamma_t, T = (-p(\rho) + \mu'\nabla\cdot\mathbf{v})I + \mu S(\mathbf{v})$ is the stress tensor, $S(\mathbf{v})$ is the strain tensor with the elements $S_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$ and $p(\rho)$ is the pressure in the liquid which is a given smooth function of density. By ∇T we mean a vector field with the components $(\nabla T)_j = \sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x_i}, j = 1,2,3$. The constants $\sigma, \mu, \mu', \kappa$ satisfy the conditions $\sigma, \mu > 0, \kappa \geq 0, 2\mu + 3\mu' > 0$.

According to kinematic bondary conditions, Γ is the set of points $x = x(\xi, t)$ such that

$$\frac{\partial \mathbf{x}(\xi,\tau)}{\partial \tau} = \mathbf{v}(x(\xi,\tau),\tau),\ 0 \leq \tau \leq t\,,\ x(\xi,\theta) = \xi \in \Gamma \tag{1.3}$$

where $\mathbf{x}$ is the radius-vector corresponding to the point x. If we set $\hat{\mathbf{v}}(\xi,t) = \mathbf{v}(x(\xi,t),t)$ we easily see that

$$\mathbf{x}(\xi,t) = \boldsymbol{\xi} + \int_0^t \hat{\mathbf{v}}(\xi,\tau)d\tau \equiv X_{\hat{v}}(\xi,t)\,.$$

This formula gives the relationship between Lagrangean and Eulerian coordinates, i.e., ξ and x. The Jacobi matrix of the transformation $X_{\hat{v}}$ has the elements $a_{ij}(\xi,t) = \delta_{ij} + \int_0^t \frac{\partial \hat{v}_i}{\partial \xi_j} d\tau$ and the Jacobian $J_{\hat{v}}(\xi,t) = \det(a_{ij}(\xi,t))_{i,j=1,2,3}$ is the solution of the Cauchy problem

$$\frac{\partial J_{\hat{v}}(\xi,t)}{\partial t} = \sum_{i,j=1}^{3} \frac{\partial a_{ij}}{\partial t} A_{ij} = \sum_{i,j=1}^{3} A_{ij} \frac{\partial \hat{v}_i}{\partial \xi_j}\,,\ J_{\hat{v}}(\xi,0) = 1\,.$$

Hence,

$$J_{\hat{v}}(\xi,t) = 1 + \sum_{i,j=1}^{3} \int_0^t A_{ij} \frac{\partial \hat{v}_i}{\partial \xi_j} d\tau \equiv 1 + \int_0^t \mathcal{A}\nabla \cdot \mathbf{v} d\tau$$

where A_{ij} are algebraic adjuncts of a_{ij} and $\mathcal{A} = (A_{ij})_{i,j=1,2,3}$. Moreover, since $\sum_{i,j=1}^{3} A_{ij} \frac{\partial \hat{v}_i}{\partial \xi_j} = \sum_{i,j,k} A_{ij} a_{kj} \frac{\partial v_i}{\partial x_k} = \nabla \cdot \mathbf{v}(x,t)|_{x=X_{\hat{v}}} \times J_{\hat{v}}(\xi,t)$, it follows that

$$J_{\hat{v}}(\xi,t) = exp\left(\int_0^t \nabla \cdot \mathbf{v}|_{x=X_{\hat{v}}} d\tau\right) = \exp\left(\int_0^t \nabla_{\hat{v}} \cdot \hat{\mathbf{v}} d\tau\right)$$

where

$$\nabla_{\hat{v}} = \left(\sum_{i=1}^{3} \frac{\partial \xi_i}{\partial x_k} \frac{\partial}{\partial \xi_i}\right)_{k=1,2,3} = J_{\hat{v}}^{-1} \mathcal{A}\nabla\,.$$

The problem (1.1)–(1.3) can be written in Lagrangean coordinates as the following initial-boundary value problem in a given domain Ω:

$$\begin{aligned} &\widehat{\rho}_t + \widehat{\rho}\nabla_{\hat{v}} \cdot \hat{\mathbf{v}} = 0\,, \\ &\widehat{\rho}\mathbf{v}_t - \nabla_{\hat{v}} T_{\hat{v}}(\hat{\mathbf{v}}) = \widehat{\rho}(\hat{\mathbf{f}} + x\nabla_{\hat{v}}\widehat{U})\,, \xi \in \Omega, t > 0\,, \\ &\widehat{\rho}(\xi,0) = \rho_0(\xi), \hat{\mathbf{v}}(\xi,0) = \mathbf{v}_0(\xi)\ , \xi \in \Omega\,, \\ &T_{\hat{v}}\mathbf{n} = -\hat{p}_e\mathbf{n} + \sigma H\mathbf{n}\,,\ \xi \in \Gamma, t > 0\,. \end{aligned} \tag{1.4}$$

Here $\hat{p}_e(\xi,t) = p_l(X_{\hat{v}},t), \hat{\mathbf{f}}(\xi,t) = \mathbf{f}(X_{\hat{v}},t), \widehat{\rho}(\xi,t) = \rho(X_{\hat{v}},t), \widehat{U} = U(X_{\hat{v}},t)$, $\mathbf{n} = |J_{\hat{v}}^{-1}A\mathbf{n}_0|^{-1}J_{\hat{v}}^{-1}A\mathbf{n}_0, \mathbf{n}_0$ is a unit exterior normal to Γ at the point ξ, and

$$\widehat{T}_{\hat{v}} = (-p(\widehat{\rho}) + \mu'\nabla_{\hat{v}}\cdot\hat{\mathbf{v}})I + \mu S_{\hat{v}}(\hat{\mathbf{v}}),$$
$$(S_{\hat{v}}(\mathbf{w}))_{ij} = J_{\hat{v}}^{-1}\sum_{k=1}^{3}\left(A_{ik}\frac{\partial w_j}{\partial\xi_k} + A_{jk}\frac{\partial w_i}{\partial\xi_k}\right). \tag{1.5}$$

The function $\widehat{\rho}(\xi,t)$ can be excluded from (1.4), since

$$\widehat{\rho}(\xi,t) = \rho_0(\xi)\exp\left(-\int_0^t \nabla_{\hat{v}}\cdot\hat{\mathbf{v}}d\tau\right) = \rho_0(\xi)J_{\hat{v}}^{-1}(\xi,t).$$

Next, we can rewrite the boundary condition, making use of the formula

$$H\mathbf{n} = \nabla_{\hat{v}}(t)\mathbf{x} = \nabla_{\hat{v}}(t)X_{\hat{v}}(\xi,t)$$

where $\nabla_{\hat{v}}$ is the Laplace-Beltrami operator on Γ_t (which depends on $\hat{\mathbf{v}}$). If we project now the boundary condition $T_{\hat{v}}\mathbf{n} = -\hat{p}_e\mathbf{n} + \sigma\nabla_{\hat{v}}(t)X_{\hat{v}}$ onto the tangent plane to Γ_t, then onto the tangent plane to Γ, and to the normal to Γ, we arrive at the initial-boundary value problem for $\hat{\mathbf{v}}$

$$\begin{aligned}
&\hat{\mathbf{v}}_t - \rho_0^{-1}(\xi)A\nabla T_{\hat{v}}'(\hat{\mathbf{v}}) = \hat{\mathbf{f}} + \kappa\nabla_{\hat{v}}\widehat{U} - \rho_0^{-1}(\xi)A\nabla p(\rho_0 J_{\hat{v}}^{-1}),\\
&\hat{\mathbf{v}}|_{t=0} = \hat{\mathbf{v}}_0(\xi),\\
&\mu\textstyle\prod_0\prod S_{\hat{v}}(\hat{\mathbf{v}})\mathbf{n}|_{\xi\in\Gamma} = 0,\\
&\mathbf{n}_0\cdot T_{\hat{v}}'\mathbf{n} - \sigma\mathbf{n}_0\cdot\Delta_{\hat{v}}X_{\hat{v}}|_{\xi\in\Gamma} = (\mathbf{n}_0\cdot\mathbf{n})[p(\rho_0 J_{\hat{v}}^{-1}) - p_e(X_{\hat{v}},t)]|_{\xi\in\Gamma}
\end{aligned} \tag{1.6}$$

where

$$T_{\hat{v}}' = T_{\hat{v}} + p(\widehat{\rho})I = \mu'\nabla_{\hat{v}}\cdot\hat{\mathbf{v}}I + \mu S_{\hat{v}}(\hat{\mathbf{v}}),$$
$$\textstyle\prod_0\mathbf{w} = \mathbf{w} - \mathbf{n}_0(\mathbf{n}_0\cdot\mathbf{w}),\quad \prod\mathbf{w} = \mathbf{w} - \mathbf{n}(\mathbf{n}\cdot\mathbf{w}),$$
$$\widehat{U}(\xi,t) = \int_{\Omega_t}\frac{\rho(y,t)dy}{|X_{\hat{v}}(\xi,t) - y|} = \int_{\Omega}\frac{\rho_0(\eta)d\eta}{|X_{\hat{v}}(\xi,t) - X_{\hat{v}}(\eta,t)|}.$$

In addition to (1.6), we consider a linear problem

$$\begin{aligned}
&\mathbf{w}_t - \rho_0^{-1}(\xi)A\nabla T_u'(\mathbf{w}) = \mathbf{f}(\xi,t),\ \mathbf{w}|_{t=0} = \mathbf{w}_0(\xi),\\
&\mu\textstyle\prod_0\prod_u S_u(\mathbf{w})\mathbf{n}|_{\xi\in\Gamma} = \prod_0\mathbf{b},\\
&\mathbf{n}_0\cdot T_u'(\mathbf{w})\mathbf{n} - \sigma\mathbf{n}_0\cdot\Delta_u(t)\int_0^t\mathbf{w}d\tau|_{\xi\in\Gamma} = b
\end{aligned} \tag{1.7}$$

where all the differential operators are determined by a given vector field $\mathbf{u}$, namely, $\mathcal{A}$ is the matrix of algebraic adjuncts to

$$a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial \xi_j} d\tau, \nabla_u = J_u^{-1}\mathcal{A}\nabla,$$

$$J_u = 1 + \int_0^t \mathcal{A}\nabla \cdot \mathbf{u}(\xi,\tau)d\tau,\ T_u(\mathbf{w}) = \mu'\nabla_u \cdot \mathbf{w}I + \mu\nabla_u S_u(\mathbf{w}),$$

$$(S_u(\mathbf{w}))_{ij} = J_u^{-1}\sum_{k=1}^{3}\left(A_{ik}\frac{\partial w_j}{\partial \xi_k} + A_{jk}\frac{\partial w_i}{\partial \xi_k}\right),$$

$$\mathbf{n} = \frac{J_u^{-1}\mathcal{A}\mathbf{n}_0}{|J_u^{-1}\mathcal{A}\mathbf{n}_0|} = \mathcal{A}\mathbf{n}_0|\mathcal{A}\mathbf{n}_0|^{-1}$$

is a unit exterior normal to $\Gamma_t = \{x = X_u(\xi,t), \xi \in \Gamma\}$, $\prod_u \mathbf{w} = \mathbf{w} - \mathbf{n}(\mathbf{n}\cdot\mathbf{w})$, and $\Delta_u(t)$ is the Laplace-Beltrami operator on Γ_t. When $\mathbf{u} = 0$, (1.7) reduces to

$$\begin{aligned} &\mathbf{w}_t - \rho_0^{-1}(\xi)\nabla T'(\mathbf{w}) = \mathbf{f},\ \mathbf{w}|_{t=0} = \mathbf{w}_0, \\ &\mu\prod\nolimits_0 S(\mathbf{w})\mathbf{n}_0|_{\xi\in\Gamma} = \prod\nolimits_0 \mathbf{b} \\ &\mathbf{n}_0 \cdot T'(\mathbf{w})\mathbf{n}_0 - \sigma\mathbf{n}_0 \cdot \Delta_0 \int_0^t \mathbf{w}d\tau|_{\xi\in\Gamma} = b \end{aligned} \tag{1.8}$$

where $T'(\mathbf{w}) = \mu'\nabla\cdot\mathbf{w} + \mu S(\mathbf{w})$ and Δ_0 is the Laplace-Beltrami operator on Γ.

We consider problems (1.6)–(1.8) in S. L. Sobolev - L. N. Slobodetskii spaces. Let $\mathcal{G}$ be a domain in $\mathbb{R}^n$. By $W_2^r(\mathcal{G})$ we mean the space of functions $u(x), x \in \mathcal{G}$, equipped with the norm

$$\|u\|_{W_2^r(\mathcal{G})} = \left(\sum_{|\alpha|<r}\|D^\alpha u\|^2_{L_2(\mathcal{G})} + \|u\|^2_{\dot{W}_2^r(\mathcal{G})}\right)^{1/2}$$

where $D_u^\alpha = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ is a generalized derivative in the sense of S. L. Sobolev, and

$$\|u\|^2_{\dot{W}_2^r(\mathcal{G})} = \sum_{|\alpha|=r}\|D^\alpha u\|^2_{L_2(\mathcal{G})} = \sum_{|\alpha|=r}\int_{\mathcal{G}} |D^\alpha u(x)|^2 dx$$

in the case of integral r and

$$\|u\|^2_{\dot{W}_2^r(\mathcal{G})} = \sum_{|\alpha|=[r]}\int_{\mathcal{G}}\int_{\mathcal{G}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{r\}}}dxdy$$

in the case of non-integer $r = [r] + \{r\}, 0 < \{r\} < 1$. Now, we define an anisotropic space $W_2^{r,r/2}(\mathfrak{G}_T)$ of functions determined in $\mathfrak{G}_T = \mathcal{G} \times (0,T)$: $x \in \mathcal{G}, t \in (0,T)$ as $W_2^{r,r/2}(\mathfrak{G}_T) = L_2(0,T; W_2^r(\mathcal{G})) \cap L_2(\mathcal{G}; W_2^{r/2}(0,T))$ and introduce in this space the norm

$$\|u\|^2_{W_2^{r,r/2}(\mathfrak{G}_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^r(\mathcal{G})} dt + \int_{\mathcal{G}} \|u(x,\cdot)\|^2_{W_2^{r/2}(0,T)} dx . \quad (1.9)$$

Finally, we denote by $H_\gamma^{r,r/2}(\mathfrak{G}_T), \gamma \geq 0$ the space of functions $u(x,t)$ with a finite norm

$$\|u\|^2_{H_\gamma^{r,r/2}(\mathfrak{G}_T)} = \int_0^T e^{-2\gamma t} \left(\|u\|^2_{W_2^r(\mathcal{G})} + \gamma^r \|u\|^2_{L_2(\mathcal{G})} \right) dt + \int_{-\infty}^T e^{-2\gamma t} dt$$
$$\times \int_0^\infty \left\| \frac{\partial^k u_0(\cdot,t)}{\partial t^k} - \frac{\partial^k u_0(\cdot,t-\tau)}{\partial t^k} \right\|^2_{L_2(\mathcal{G})} \frac{d\tau}{\tau^{1+r-2k}} \quad (1.10)$$

($\frac{r}{2}$ is a non-integer, $k = \left[\frac{r}{2}\right], u_0(x,t) = u(x,t)$ for $t > 0, u_0(x,t) = 0$ for $t < 0$). In the case of integral $r/2$ the double integral in the norm should be replaced by

$$\int_{-\infty}^T e^{-2\gamma t} \left\| \frac{\partial^r u}{\partial t^r} \right\|^2_{L_2(\mathcal{G})} dt .$$

For $T < \infty$, the space $H_\gamma^{r,r/2}(\mathfrak{G}_T)$ can be identified with the subspace of $W_2^{r,r/2}(\mathfrak{G}_T)$ consisting of functions $u(x,t)$ that can be extended by zero into the domain $t < 0$ without loss of smoothness. In the case $r > 1$ this implies that

$$\frac{\partial^i u}{\partial t^i}\Big|_{t=0} = 0 , \ i = 0, \dots , \left[\frac{r-1}{2}\right] .$$

The norm $\|u\|_{H_0^{r,r/2}(\mathfrak{G}_T)}, r < 1$, is equivalent to

$$\|u\|^{(r,r/2)}_{\mathfrak{G}_T} = \left(\|u\|^2_{W_2^{r,r/2}(\mathfrak{G}_T)} + T^{-r} \|u\|^2_{L_2(\mathfrak{G}_T)} \right)^{1/2}$$

which in its term is equivalent to $\|u\|_{W_2^{r,r/2}(\mathfrak{G}_T)}$ for any fixed $T > 0$.

The space $W_2^r(\mathcal{G})$ of functions defined on a smooth manifold $\mathcal{G}$ is introduced in a standard way by means of local coordinates and partition of unity, and $W_2^{r,r/2}(\mathfrak{G}_T), \mathfrak{G}_T = \mathcal{G} \times (0,T)$ can be defined in the same way as above. The spaces of vector fields whose components belong to $W_2^r(\mathcal{G}), W_2^{r,r/2}(\mathfrak{G}_T)$ etc are denoted by the same symbols.

Let us now describe results of the paper. First of all, we consider the problem (1.8) in the spaces $H_\gamma^{l+2,l/2+1}(Q_T), Q_T = \Omega\times(0,T)$. The following theorem is proved in Sec. 3.

Theorem 1.1. Let $\Gamma \in W_2^{l+3/2}, l > 1/2, \rho_0 \in W_2^{1+l}(\Omega), \rho_0(\xi) \geq R_0 > 0$. For arbitrary $\mathbf{f} \in H_\gamma^{l,l/2}(Q_T), \mathbf{b} \in H_\gamma^{l+1/2+1/4}(G_T), G_T = \Gamma\times(0,T)$ and for $b = b' + \sigma\int_0^t B d\tau$ with $b' \in H_\gamma^{l+1/2,l/2+1/4}(G_T), B \in H_\gamma^{l-1/2,l/2-1/4}(G_T)$ the problem (1.8) has a unique solution $\mathbf{w} \in H_\gamma^{l+2,l/2+1}(Q_T)$, provided γ is large enough, and

$$\begin{aligned}\|\mathbf{w}\|_{H_\gamma^{l+2,l/2+1}(Q_T)} \leq & c\Big(\|\mathbf{f}\|_{H_\gamma^{l,l/2}(Q_T)} + \|\mathbf{b}\|_{H_\gamma^{l+1/2,l/2+1/4}(G_T)} \\ & + \|b'\|_{H_\gamma^{l+2,l/2+1}(G_T)} + \sigma\|B\|_{H_\gamma^{l-1/2,l/2-1/4}(G_T)}\Big)\end{aligned} \tag{1.11}$$

with a constant c independent of T.

The theorem is proved in the same way as in the case of incompressible liquid [1–3], first in the half-space, then in a bounded domain. In the case of the half-space we give an explicit formula for the solution, and in a bounded domain we prove *a priori* estimates and establish the solvability of the problem (1.8) by the construction of a regularizer. This method was used in the theory of general parabolic initial-boundary value problems [4]. It should be observed that our problem (1.8) is not parabolic in the sense of [4] since the complementing condition is violated because of a complicated structure of boundary operator $T\mathbf{n}-\sigma\Delta_{\hat{v}}(t)X_{\hat{v}}$ containing terms of different order none of which can be regarded as a principle one.

For the problem (1.7) with a given $\mathbf{u}$ the following theorem is proved.

Theorem 1.2. Let $\Gamma \in W_2^{3/2+l}, l \in (1/2,1), \rho_0 \in W_2^{l+1}(\Omega), \rho_0(\xi) \geq R_0 > 0$ and suppose that

$$T^{1/2}\|\mathbf{u}\|_{Q_T}^{(l+2,l/2+1)} \leq \delta \tag{1.12}$$

where δ is a small number and

$$\begin{aligned}\Big(\|\mathbf{u}\|_{Q_T}^{(l+2,l/2+1)}\Big)^2 = & \|\mathbf{u}\|^2_{W_2^{l+2,l/2+1}(Q_T)} + T^{-l}\big(\|\mathbf{u}_t\|^2_{L_2(Q_T)} \\ & + \sum_{|\alpha|=2}\|D_x^\alpha \mathbf{u}\|^2_{L_2(Q_T)}\big) + \sup_{t\leq T}\|\mathbf{u}(\cdot,t)\|^2_{W_2^{1+l}(\Omega)}\cdot\end{aligned}$$

For arbitrary $\mathbf{f} \in W_2^{l,l/2}(Q_T), \mathbf{w}_0 \in W_2^{1+l}(\Omega), \mathbf{b} \in W_2^{l+1/2,l/2+1/4}(G_T)$ and $b = b' + \sigma \int_0^t B d\tau$ with $b' \in W_2^{l+1/2,l/2+1/4}(G_T), B \in W_2^{l-1/2,l/2-1/4}(G_T)$ satisfying the compatibility conditions

$$\begin{aligned} \mu \prod\nolimits_0 S(\mathbf{w}_0)\mathbf{n}_0|_\Gamma &= \prod\nolimits_0 \mathbf{b}|_{t=0}\,, \\ \mathbf{n}_0 \cdot T'(\mathbf{w}_0)\mathbf{n}_0|_\Gamma &= b'|_{t=0} \end{aligned} \tag{1.13}$$

the problem (1.7) is uniquely solvable in $W_2^{l+2,l/2+1}(Q_T)$ and

$$\begin{aligned} \|\mathbf{w}\|_{Q_T}^{(l+2,l/2+1)} \leq & c(T)\Big(\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|\mathbf{w}_0\|_{W_2^{1+l}(\Omega)} + \|\mathbf{b}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\ & + \|b'\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \sigma\|B\|_{G_T}^{(l-1/2,l/2-1/4)}\Big) \end{aligned} \tag{1.14}$$

where $c(T)$ is a non-decreasing function of T.

The restriction $l < 1$ minimizes the number of compatibility conditions. Theorems 1.1 and 1.2 hold also for $\sigma = 0$, in which case the problems (1.7), (1.8) are parabolic and are considered in [4] under more restrictive assumptions on the data (in particular on the boundary Γ).

Theorem 1.3. Let $\Gamma \in W_2^{5/2+l}, l \in (1/2, 1), \rho_0 \in W_2^{1+l}(\Omega), \rho(\xi, t) \geq R_0 > 0, p \in c^3(\mathbb{R}_+)$ and assume that $\mathbf{f}$ has continuous derivatives of order one and two, p_e is three times continuously differentiable with respect to x_m and that $\mathbf{f}, \mathbf{f}_{x_k}$ satisfy the Hölder condition with the exponent $\beta \geq 1/2$, and $p_e, \nabla p_e$ satisfy the Lipschitz condition with respect to t. Then for arbitrary $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$ such that

$$-p(\rho_0)\mathbf{n}_0 + \mu'(\nabla \cdot \mathbf{v}_0)\mathbf{n}_0 + \mu S(\mathbf{v}_0)\mathbf{n}_0|_{\xi\in\Gamma} = \sigma H \mathbf{n}_0 - p_e \mathbf{n}_0|_{t=0}$$

the problem (1.6) has a unique solution $\mathbf{v} \in W_2^{l+1,l/2+1}(Q_{T'})$ on a finite time interval $(0, T')$ whose magnitude T' depends on the data, i.e., on the norms of $\mathbf{f}, p_e, \mathbf{v}_0, \rho_0$ and on the mean curvature of Γ (see the condition (5.25) below).

We observe that evolution free boundary problems for the compressible fluid are considered in [6–9]. The papers [10–14] are concerned with free boundary problems of a viscous incompressible flow both for $\sigma > 0$ and for $\sigma = 0$.

2. Model Problem in the Half-space

In this section we consider the problem (1.8) in the half-space $\mathbb{R}^3_+(x_3 > 0)$ with $\rho_0 = \text{const}$ and with homogeneous initial conditions

$$\begin{aligned}
&\mathbf{w}_t - [(\nu+\nu')\nabla(\nabla\cdot\mathbf{w}) + \nu\nabla^2\mathbf{w}] = \mathbf{f} \quad (x_3 > 0),\\
&\mathbf{w}|_{t=0} = 0,\\
&\mu\left(\frac{\partial w_\alpha}{\partial x_3} + \frac{\partial w_3}{\partial x_\alpha}\right)\bigg|_{x_3=0} = b_\alpha(x',t) \quad (x' = (x_1,x_2)\in\mathbb{R}^2),\\
&\mu'\nabla\cdot\mathrm{w} + 2\mu\frac{\partial w_3}{\partial x_3} + \sigma\Delta'\int_0^t w_3 d\tau = b_3(x',t)
\end{aligned} \tag{2.1}$$

where $\nu = \frac{\mu}{\rho_0}, \nu' = \frac{\mu'}{\rho_0}, \Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_1^2}$. We assume first that $\mathbf{f} = 0$. After the Fourier transform with respect to x_1, x_2 and the Laplace transform with respect to t this problem takes the form

$$\begin{aligned}
&S\tilde{w}_\alpha - (\nu+\nu')i\xi_\alpha\left(i\xi_1\hat{w}_1 + i\xi_2\hat{w}_2 + \frac{d}{dx_3}\tilde{w}_3\right) - \nu\left(\frac{d^2}{dx_3^2}\tilde{w}_\alpha - \xi^2\tilde{w}_\alpha\right) = 0,\\
&S\tilde{w}_3 - (\nu+\nu')\frac{d}{dx_3}\left(i\xi_1\tilde{w}_1 + i\xi_2\tilde{w}_2 + \frac{d}{dx_3}\tilde{w}_3\right) - \nu\left(\frac{d^2}{dx_3^2}\tilde{w}_3 - \xi^2\tilde{w}_3\right) = 0,\\
&\mu\left(\frac{\partial\tilde{w}_\alpha}{\partial x_3} + i\xi_\alpha\tilde{w}_3\right)\bigg|_{x_3=0} = \tilde{b}_\alpha, \quad \alpha = 1,2,\\
&\mu'\left(i\xi_1\tilde{w}_1 + i\xi_2\tilde{w}_2 + \frac{d\tilde{w}_3}{dx_3}\right) + 2\mu\frac{d\tilde{w}_3}{dx_3} - \frac{\sigma}{S}\xi^2\tilde{w}_3\bigg|_{x_3=0} = \tilde{b}_3,\\
&\tilde{\mathbf{w}} \to 0 \quad (x_3 \to +\infty).
\end{aligned} \tag{2.1'}$$

The solution of this system of ordinary differential equations vanishing as $x_3 \to \infty$ has the form

$$\begin{aligned}
\tilde{\mathbf{w}} &= h_1\begin{pmatrix} r\\ 0\\ i\xi_1\end{pmatrix}e^{-rx_3} + h_2\begin{pmatrix} 0\\ r\\ i\xi_2\end{pmatrix}e^{-rx_3} + h_3\begin{pmatrix} i\xi_1\\ i\xi_2\\ -r_1\end{pmatrix}e^{-r_1x_3}\\
&= h_3\begin{pmatrix} i\xi_1\\ i\xi_2\\ -r_1\end{pmatrix}(e^{-r_1x_3} - e^{-rx_3}) + \begin{pmatrix} h_1r + i\xi_1h_3\\ h_2r + i\xi_2h_3\\ H - r_1h_3\end{pmatrix}e^{-rx_3}
\end{aligned} \tag{2.2}$$

with $r = \sqrt{\frac{s}{\nu} + \xi^2}, r_1 = \sqrt{\frac{s}{2\nu+\nu'} + \xi^2}, H = i\xi_1h_1 + i\xi_2h_2$. The constants

h_α are determined by boundary conditions, which reduce to

$$\mu(-h_\alpha r^2 + i\xi_\alpha H - 2i\xi_\alpha r_1 h_3) = \tilde{b}_\alpha, \ \alpha = 1,2,$$
$$\mu'\left(rH - \xi^2 h_3\right) + (\mu' + 2\mu)(-rH + r_1^2 h_3) - \frac{\sigma\xi^2}{s}(H - r_1 h_3) = \tilde{b}_3 .$$

The first two equations imply

$$\mu(-r^2 H - \xi^2 H + 2\xi^2 r_1 h_3) = i\xi_1\tilde{b}_1 + i\xi_2\tilde{b}_2 = D .$$

We have obtained the system of two linear equations for H and h_3 that gives us

$$H = -\frac{1}{\rho_0\mathcal{P}}\{[\mu(r_1^2+\xi^2) + (\mu+\mu')(r_1^2-\xi^2) + \frac{\sigma}{s}\xi^2 r_1]D - 2\mu r_1\xi^2\tilde{b}_3\},$$
$$h_3 = -\frac{1}{\rho_0\mathcal{P}}\{(2\mu r + \frac{\sigma}{s}\xi^2)D - \mu(r^2+\xi^2)\tilde{b}_3\} \tag{2.3}$$

with

$$\mathcal{P} = \rho_0 s^2 + \frac{4\mu(\mu+\mu')}{2\mu+\mu'}s\xi^2\frac{r}{r+r_1} + \frac{\sigma\rho_0}{2\mu+\mu'}\frac{s\xi^2}{r_1+|\xi|} + \sigma|\xi|^3 . \tag{2.4}$$

Consequently,

$$H - r_1 h_3 = -\frac{1}{\rho_0\mathcal{P}}\{[\mu(r_1^2+\xi^2-2rr_1) + \frac{\mu+\mu'}{2\mu+\mu'}\rho_0 s]D + \rho_0 s r_1\tilde{b}_3\},$$
$$h_\alpha r + i\xi_\alpha h_3 = -\frac{\tilde{b}_\alpha}{\mu r} + \frac{i\xi_\alpha}{r}(H - r_1 h_3) + \frac{i\xi_\alpha}{r}(r - r_1)h_3 . \tag{2.5}$$

Thus, the solution of (2.1) is given by (2.2)–(2.5). We now pass to estimates of this solution in $H_\gamma^{l+2,l/2+1}(\mathbb{R}_+^3 \times (0,\infty))$.

Lemma 2.1. For arbitrary $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$ and $s = \gamma + i\xi_0$ with $\gamma > 0$ the following inequalities hold

$$|s|\xi^2 \le c_1^{-1}|\mathcal{P}|, \xi^2|s|^{1/2} + \sigma|\xi|^3 + |s|^2 \le c_2(1 + \frac{1}{\sqrt{\gamma}})|\mathcal{P}| . \tag{2.6}$$

The constants c_1 and c_2 are independent of γ.

Proof. Since $|\arg \frac{r}{r_1+r}|, |\arg \frac{1}{r_1+|\xi|}| \leq \frac{\pi}{4}$, we have

$$|\mathcal{P}(\xi,s)| \geq |s|\mathrm{Re}(\rho_0 s + \frac{4\mu(\mu+\mu')}{\mu'+2\mu}\xi^2 \frac{r}{r+r_1} + \frac{\sigma\rho_0\xi^2}{(\mu'+2\mu)}\frac{1}{r+|\xi|} + \frac{\sigma|\xi|^3}{s}\Big) \geq c_1|s|\xi^2$$

with $c_1 = \inf \frac{4\mu(\mu+\mu')}{2\mu+\mu'}\mathrm{Re}\frac{r}{r+r_1}$; consequently, $\xi^2|s|^{1/2} \leq \frac{|s|\xi^2}{\sqrt{\gamma}} \leq \frac{|\mathcal{P}|}{c_1\sqrt{\gamma}}$. Finally for $|s| \leq \xi^2$

$$|s|^2 \leq |s|\xi^2 \leq c_1^{-1}|\mathcal{P}|$$

$$\sigma|\xi|^3 \leq |\mathcal{P}| + \frac{4\mu(\mu+\mu')|r||s|\xi^2}{(2\mu+\mu')|r+r_1|} + \frac{\sigma\rho_0\xi^2|s|}{(2\mu+\mu')(|r_1|+|\xi|)} + \rho_0|s|^2 \leq c_3(1+\frac{1}{\sqrt{\gamma}})|\mathcal{P}|$$

and in the case $|s| \leq \xi^2$

$$\sigma|\xi|^3 \leq \sigma|\xi|^2|s|^{1/2} \leq \frac{\sigma|\mathcal{P}|}{c_1\sqrt{\gamma}},$$

$$\rho_0|s|^2 \leq |\mathcal{P}| + \frac{4\mu(\mu+\mu')|r||s|\xi^2}{(2\mu+\mu')|r_1+r|} + \frac{\sigma\rho_0\xi^2|s|}{(2\mu+\mu')|r_1+|\xi||} + \sigma|\xi|^3 \leq c_4(1+\frac{1}{\sqrt{\gamma}})|\mathcal{P}|,$$

which completes the proof.

Lemma 2.2. For all $\xi \in \mathbb{R}^2$ and $\gamma = \mathrm{Re}\, s > 0$ the vectors $\mathbf{V} = h_3(r_1 - r)\begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1 \end{pmatrix}$ and $\mathbf{W} = \begin{pmatrix} h_1 r + i\xi_1 h_3 \\ h_2 r + i\xi_2 h_3 \\ H - r h_3 \end{pmatrix}$ satisfy the inequalities

$$|\mathbf{V}| \leq c_5(|\tilde{b}_1| + |\tilde{b}_2| + |\tilde{b}_3|), \tag{2.7}$$

$$|\mathbf{W}| \leq \frac{c_6}{\sqrt{|s|+\xi^2}}(|\tilde{b}_1| + |\tilde{b}_2| + |\tilde{b}_3|). \tag{2.8}$$

Moreover, if $\tilde{b}_1 = \tilde{b}_2 = 0$ and $b_3 = \frac{\sigma}{s}\tilde{B}$ then

$$|\mathbf{V}| \leq \frac{c_7\sigma|\tilde{B}|}{\sqrt{|s|+\xi^2}}, \quad |\mathbf{W}| \leq \frac{c_8\sigma|\tilde{B}|}{|s|+\xi^2}. \tag{2.9}$$

Proof. Since

$$r_1 - r = \left(\frac{1}{2\nu+\nu'} - \frac{1}{\nu}\right)\frac{s}{r_1+r} = -\frac{\mu+\mu'}{\mu(2\mu+\mu')}\frac{\rho_0 s}{r_1+r},$$

it follows that

$$h_3(r_1-r) = \frac{\mu+\mu'}{\mu(2\mu+\mu')(r_1+r)}\Big\{\frac{s}{\mathcal{P}}\Big[2\mu r\sum_{\alpha=1}^{2} i\xi_\alpha\tilde{b}_\alpha - \mu(\xi^2+r^2)\tilde{b}_3\Big] + \frac{\sigma\xi^2}{\mathcal{P}}\sum_{\alpha=1}^{2} i\xi_\alpha\tilde{b}_\alpha\Big\}.$$

From this formula and from (2.6) we conclude that

$$|h_3(r_1-r)| \le \frac{c_9}{\sqrt{|s|+\xi^2}}(|\tilde{b}_1|+|\tilde{b}_2|+|\tilde{b}_3|)$$

which implies (2.7). Now, we can write

$$H - r_1h_3 = -\frac{s}{\mathcal{P}}\Big\{\Big[\frac{(\mu+\mu')^2}{\mu(\mu'+2\mu)^2}\frac{\rho_0 s}{(r_1+r)^2} - \frac{\mu}{\mu'+2\mu}\Big]\sum_{\alpha=1}^{2} i\xi_\alpha\tilde{b}_\alpha + r_1\tilde{b}_3\Big\},$$

hence,

$$|H - r_1h_3| \le \frac{c_{10}}{\sqrt{|s|+\xi^2}}(|\tilde{b}_1|+|\tilde{b}_2|+|\tilde{b}_3|).$$

From (2.5) we conclude that this type of estimate is true for $h_\alpha r + i\xi_\alpha h_3$, so (2.8) holds.

Assume now that $\tilde{b}_1 = \tilde{b}_2 = 0, \tilde{b}_3 = \frac{\sigma}{S}\tilde{B}$. Then

$$h_3(r_1-r) = -\frac{\sigma(\mu+\mu')}{2\mu+\mu'}\frac{r^2+\xi^2}{r_1+r}\frac{\tilde{B}}{\mathcal{P}},\quad H - r_1h_3 = -\frac{\sigma r_1\tilde{B}}{\mathcal{P}}$$

and (2.9) follows from $|\mathcal{P}| \ge c_{11}(\gamma)(|s|+\xi^2)^{3/2}$ which is a consequence of (2.6). The lemma is proved.

Theorem 2.1. Let $l > 1/2, \mathbb{D}_T = \mathbb{R}^3_+ \times (0,T), \mathbb{R}_T = \mathbb{R}^2\times(0,T), T < \infty$. For arbitrary $\mathbf{f} \in H_\gamma^{l,l/2}(\mathbb{D}_T), b_\alpha \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T), \alpha = 1,2$ and

$b_3 = b_3' + \sigma \int_0^t B d\tau$ where $b_3 \in H_\gamma^{l+1/2,l+1/4}(\mathbb{R}_T)$, $B \in H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T)$, the problem (2.1) has a unique solution $\mathbf{w} \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)$ and

$$\|\mathbf{w}\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} \leq c_{12}(\gamma)(\|\mathbf{f}\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} + \sum_{\alpha=1}^{2} \|b_\alpha\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T)} + \|b_3'\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T)} + \|B\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T)}) \tag{2.10}$$

with a constant $c_{14}(\gamma)$ independent of T.

Proof. Without restriction of generality we can assume that $T = \infty$, since we can arrive at this case after appropriate extension of $\mathbf{f}$ and b_i. First we construct the solution $\mathbf{w}' \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_\infty)$ of the system

$$\mathcal{L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \mathbf{w} = \mathbf{w}_t' - [(\nu + \nu')\nabla(\nabla \cdot \mathbf{w}') + \nu \nabla^2 \mathbf{w}'] = \mathbf{f}\,. \tag{2.11}$$

We extend $\mathbf{f}$ into the half-space $x_3 < 0$ in such a way that

$$\|\mathbf{f}\|_{H_\gamma^{l,l/2}(\mathbb{R}^3 \times (0,\infty))} \leq c_{13}\|\mathbf{f}\|_{H_\gamma^{l,l/2}(\mathbb{D}_\infty)} \tag{2.12}$$

and make the Fourier transform with respect to $x_i, i = 1,2,3$, and the Laplace transform with respect to t. The system (2.11) is transformed into $\mathcal{L}(i\xi, S)\tilde{\mathbf{w}}' = \mathbf{f}$ and $\tilde{\mathbf{w}}' = \mathcal{L}^{-1}(i\xi, s)\mathbf{f}$ satisfies the inequality

$$\int_{-\infty}^{\infty} d\xi_0 \int_{\mathbb{R}^3} |\tilde{\mathbf{w}}(\xi, \gamma + i\xi_0)|^2 (|\gamma + i\xi_0| + \xi^2)^{l+2} d\xi \leq c_{14} \int_{-\infty}^{\infty} d\xi_0 \int_{\mathbb{R}^3} |\mathbf{f}(\xi_1 \gamma + i\xi_0)|^2 (|\gamma + i\xi_0| + \xi^2)^l d\xi$$

which is equivalent to

$$\|\mathbf{w}'\|_{H_\gamma^{l+1,l/2+1}(\mathbb{R}^3 \times (0,\infty))} \leq c_{15}\|\mathbf{f}\|_{H_\gamma^{l,l/2}(\mathbb{D}_\infty)}\,. \tag{2.13}$$

The vector field $\mathbf{w}'' = \mathbf{w} - \mathbf{w}'$ should be a solution of (2.1) with $\mathbf{f} = 0$ and with the functions

$$b_\alpha - \mu\left(\frac{\partial w_\alpha'}{\partial x_3} + \frac{\partial w_3'}{\partial x_\alpha}\right)\bigg|_{x_3=0} = d_\alpha(x', t), \alpha = 1, 2,$$

$$b_3' - \mu'\nabla \cdot \mathbf{w}' - 2\mu \frac{\partial w_3'}{\partial x_3} + \sigma \int_0^t (B - \Delta' w_3') d\tau|_{x_3=0} = d_3' + \sigma \int_0^t D d\tau \tag{2.14}$$

in the boundary conditions. The functions w_i'' are given by (2.5) and it follows from Lemma 2.2 that

$$\left|\left(\frac{d}{dx_3}\right)^j \tilde{\mathbf{w}}''\right| \le c_{16}\left(|\tilde{d}_1 + |\tilde{d}_2| + |\tilde{d}_3'| + \frac{\sigma|\tilde{D}|}{\sqrt{|s|+\xi^2}}\right)\left|\frac{d^j e_1(x_3)}{dx_3^j}\right| + \frac{c_{17}}{\sqrt{|s|+\xi^2}}\left(|\tilde{d}_1| + |\tilde{d}_2| + |\tilde{d}_3'| + \frac{\sigma|\tilde{D}|}{\sqrt{|s|+\xi^2}}\right)\left|\frac{d^j e_0(x_3)}{dx_3^j}\right| \tag{2.15}$$

where $e_0(x_3) = e^{-rx_3}, e_1(x_3) = \frac{e^{-r_1x_3}-e^{-rx_3}}{r_1-r}$. It is not hard to show (see Lemma 3.1 in [2]) that

$$\int_0^\infty \left|\frac{d^j e_0}{dx_3^j}\right|^2 dx_3 \le c_{18}(|s|+\xi^2)^{j-1/2},$$

$$\int_0^\infty \left|\frac{d^j e_1}{dx_3^j}\right|^2 dx_3 \le c_{19}(|s|+\xi^2)^{j-3/2},$$

$$\int_0^\infty\int_0^\infty \left|\frac{d^j e_0(x_3+z)}{dx_3^j} - \frac{d^j e_0(x_3)}{dx_3^j}\right|^2 \frac{dx_3dz}{z^{1+2\lambda}} \le c_{20}(|s|+\xi^2)^{j+\lambda-1/2},$$

$$\int_0^\infty\int_0^\infty \left|\frac{d^j e_1(x_3+z)}{dx_3^j} - \frac{d^j e_1(x_3)}{dx_3^j}\right|^2 \frac{dx_3dz}{z^{1+2\lambda}} \le c_{21}(|s|+\xi^2)^{j+\lambda-3/2}$$

$(0<\lambda<1)$. Assuming (to be definite) that l is not integral, we can deduce from (2.15) the estimate

$$\sum_{j=0}^{2+[l]}\int_{-\infty}^\infty d\xi_0\int_{\mathbb{R}^2}(|s|+\xi^2)^{l+2-j}d\xi\int_0^\infty \left|\frac{d^j\tilde{\mathbf{w}}''}{dx_3^j}\right|^2 dx_3$$
$$+\int_{-\infty}^\infty d\xi_0\int_{\mathbb{R}^2}d\xi\int_0^\infty dx_3\int_0^\infty \left|\left(\frac{d}{dx_3}\right)^{2+[l]}(\tilde{\mathbf{w}}(x_3+z,\xi,\xi_0)\right.$$
$$\left.-\tilde{\mathbf{w}}(x_3,\xi,\xi_0))\right|^2\frac{dz}{z^{1+2(l-[l])}}$$
$$\le c_{22}\int_{-\infty}^\infty d\xi_0\int_{\mathbb{R}^2}\left(|\tilde{d}_1|^2+|\tilde{d}_2|^2+|\tilde{d}_3'|^2+\frac{\sigma^2|\tilde{D}|^2}{|s|+\xi^2}\right)(|s|+\xi_2)^{l+1/2}d\xi$$

$(s=\gamma+i\xi_0)$ which is equivalent to

$$\|\mathbf{w}''\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_\infty)} \le c_{23}\Big(\|d_1\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_\infty)} + \|d_2\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_\infty)} + \|d_3'\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_\infty)} + \sigma\|D\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_\infty)}\Big). \tag{2.16}$$

Making use of (2.14) and of trace theorems for the spaces $H_\gamma^{r,r/2}(\mathbb{D}_T)$ we can evaluate the right-hand side of (2.16) in terms of the right-hand side of (2.10). The inequality (2.10) is a consequence of (2.16) and (2.13).

The uniqueness of the solution $\mathbf{v} \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)$ of (2.1) follows from the energy inequality. Let $\mathbf{w}$ be a solution of a homogeneous problem (2.1). Then

$$0 = \int_{\mathbb{R}^3_+} \mathcal{L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \mathbf{w} \cdot \mathbf{w} dx = \frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{w}\|^2_{L_2(\mathbb{R}^3_+)} +$$

$$\rho_0^{-1}\sigma \|\nabla' \int_0^t w_3 d\tau\|^2_{L_2(\mathbb{R}^2)}\Big) + \nu' \|\nabla \cdot \mathbf{w}\|^2_{L_2(\mathbb{R}^3_+)} + \frac{\nu}{2}\|s(\mathbf{w})\|^2_{L_2(\mathbb{R}^3_+)} \tag{2.17}$$

where $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. The last two terms can be estimated from below by

$$\rho_0^{-1}\Big(2\mu \sum_{i=1}^3 \Big\|\frac{\partial w_i}{\partial x_i}\Big\|^2 - \frac{2\mu}{3}\Big\|\sum_{i=1}^3 \frac{\partial w_i}{\partial x_i}\Big\|^2_{L_2(\mathbb{R}^3_+)}\Big) \geq 0,$$

hence (2.17) implies that $\mathbf{w} = 0$. The proof is completed.

3. The Problem (1.8) in a Bounded Domain

In this section we consider the problem (1.8) in a bounded domain Ω whose boundary Γ belongs to $W_2^{l+3/2}, l > 1/2$. This means that in the neighbourhood of arbitrary point $\xi \in \Gamma$, the surface Γ is determined by the equation

$$y_3 = \varphi(y'),\ y' = (y_1, y_2) \in K_d$$

in a cartesian coordniate system (y_1, y_2, y_3) with the origin in ξ and with y_3-axis directed along $-\mathbf{n}_0(\xi)$. The function φ is defined in a disc $K_d : |y'| < d$, and it satisfies the conditions $\varphi(0) = 0, \nabla'\varphi(0) = 0$ and $\|\varphi\|_{W_2^{3/2+l}(K_d)} \leq M$. The constants d and M are independent of ξ.

It can be assumed that φ is extended into $\mathbb{R}^3_+$ (see [2], Sec. 4), belongs to $W_2^{l+2}(\mathbb{R}^3_+)$, and $\varphi(0) = 0$, $\frac{\partial \varphi}{\partial y_i}|_{y=0} = 0, i = 1, 2, 3$. In virtue of imbedding theorems,

$$\sup_{|y|\leq\lambda} |\varphi(y)| \leq c_1 M\lambda \qquad \sup_{|y|\leq\lambda} |\nabla\varphi(y)| \leq c_1 M \lambda^\beta \tag{3.1}$$

where $\beta \in (0, 1), \beta \leq l - 1/2$. The transformation $y = Y(z)$:

$$y_1 = z_1,\ y_2 = z_2,\ y_3 = z_3 + \varphi(z) \tag{3.2}$$

which is invertible if $|\varphi_{z_3}| < 1$, maps $\mathbb{R}^3_+$ onto the domain $y_3 > \varphi(y')$.

We prove Theorem 1.1 in two steps: first we obtain the estimate (1.11) and then we establish the solvability of (1.8).

Consider the neighbourhood of the point $\xi \in \Gamma$ assuming for the sake of simplicity that $\xi = 0$ and that the coordinates $\{y_i\}$ coincide with $\{x_i\}$. Let $\zeta_\lambda(x) = \zeta(\frac{x}{\lambda})$ where $\zeta \in C_0^\infty(\mathbb{R}^3), \zeta(x) = 1$ for $|x| \leq 1/2, \zeta(x) = 0$ for $|x| \geq 1$. The vector field $\mathbf{u}_\lambda = \mathbf{w}\zeta_\lambda$ satisfies the relationships

$$\begin{aligned}
&\mathbf{u}_{\lambda t} - \rho_0^{-1}(x)\nabla T'(\mathbf{u}_\lambda) = \zeta_\lambda \mathbf{f} + \mathbf{k}_1(\mathbf{w}), \quad \mathbf{w}|_{t=0} = 0, \\
&\mu \prod\nolimits_0 S(\mathbf{u}_\lambda)\mathbf{n}_0|_{x\in\Gamma} = \prod\nolimits_0 \mathbf{b}\zeta_\lambda + \mathbf{k}_2(\mathbf{w}), \\
&\mathbf{n}_0 \cdot T'(\mathbf{u}_\lambda)\mathbf{n}_0 - \sigma \mathbf{n}_0 \cdot \Delta_0 \int_0^t \mathbf{u}_\lambda d\tau|_{x\in\Gamma} = b\zeta_\lambda + k_3(\mathbf{w}) + \sigma \int_0^t k_4(\mathbf{w})d\tau
\end{aligned} \tag{3.3}$$

where $b = b' + \sigma \int_0^t B d\tau$,

$$\begin{aligned}
\mathbf{k}_1(\mathbf{w}) &= \rho_0^{-1}(\zeta_\lambda \nabla T'(\mathbf{w}) - \nabla T'(\zeta_\lambda \mathbf{w})) = -\rho_0^{-1}[(\mu + \mu')(\nabla\zeta_\lambda(\nabla \cdot \mathbf{w}) \\
&\quad + \nabla(\nabla\zeta_\lambda \cdot \mathbf{w}) + \mu(\mathbf{w}\nabla^2\zeta_\lambda + 2(\nabla\zeta_\lambda \cdot \nabla)\mathbf{w}], \\
\mathbf{k}_2(\mathbf{w}) &= \mu \prod\nolimits_0 (S(\mathbf{w}\zeta_\lambda) - \zeta_\lambda S(\mathbf{w}))\mathbf{n}_0 = \mu \prod\nolimits_0 \left(\mathbf{w}\frac{\partial\zeta_\lambda}{\partial n} + (\mathbf{w}\cdot\mathbf{n}_0)\nabla\zeta_\lambda\right), \\
\mathbf{k}_3(\mathbf{w}) &= \mathbf{n}_0(T'(\mathbf{w}\zeta_\lambda) - \zeta_\lambda T'(\mathbf{w}))\mathbf{n}_0 = \mu'\nabla\zeta_\lambda \cdot \mathbf{w} + 2\mu(\mathbf{w}\cdot\mathbf{n}_0)\frac{\partial\zeta_\lambda}{\partial n}, \\
k_4(\mathbf{w}) &= (\zeta_\lambda \Delta_0 \mathbf{w} - \Delta_0(\zeta_\lambda \mathbf{w}_0)) \cdot \mathbf{n}_0 .
\end{aligned}$$

In new coordinates $z = Y^{-1}(x)$ (3.3) takes the form

$$\begin{aligned}
&\hat{\mathbf{u}}_{\lambda t} - \hat{\rho}_0^{-1}(z)\nabla_1 \hat{T}(\hat{\mathbf{u}}_\lambda) = \hat{\zeta}_\lambda \hat{\mathbf{f}} + \hat{\mathbf{k}}_1(\mathbf{w}), \\
&\hat{\mathbf{u}}_\lambda|_{t=0} = 0 \\
&\mu \prod\nolimits_0 \hat{S}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0|_{z_3=0} = \prod\nolimits_0 \hat{\mathbf{b}}\hat{\zeta}_\lambda + \hat{\mathbf{k}}_2(\mathbf{w}), \\
&\mathbf{n}_0 \cdot \hat{T}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0 - \sigma \mathbf{n}_0 \cdot \Delta_0 \int_0^t \hat{\mathbf{u}}_\lambda d\tau|_{z_3=0} = \hat{b}\hat{\zeta}_\lambda + \hat{k}_3(\mathbf{w}) + \sigma \int_0^t \hat{k}_4(\mathbf{w})d\tau
\end{aligned} \tag{3.4}$$

where $\hat{\mathbf{u}}_\lambda(z,t) = \mathbf{u}_\lambda(Y(z),t)$ etc., $\nabla_1 = y^*\nabla, y = \left(\frac{\partial z_i}{\partial x_k}\right)_{i,k=1,2,3} \equiv (Y_{ik})_{i,k=1,2,3}$, i.e.,

$$y^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_{z_1} & \varphi_{z_2} & 1+\varphi_{z_3} \end{pmatrix},$$

$\mathbf{n}_0 = \left(\frac{\varphi_{z_1}}{\sqrt{1+|\nabla'\varphi|^2}}, \frac{\varphi_{z_2}}{\sqrt{1+|\nabla'\varphi|^2}}, -\frac{1}{\sqrt{1+|\nabla'\varphi|^2}}\right)$, and by $\hat{S}, \hat{T}, \hat{k}_i$ are denoted operators S, T', k_i written in coordinates $\{z\}$, in particular,

$$\hat{T}(\mathbf{w}) = \mu'\nabla_1 \cdot \mathbf{w} + \mu\nabla\hat{S}(\mathbf{w})\,, \hat{S}_{ij} = \sum_{m=1}^{3}\left(Y_{mi}\frac{\partial w_j}{\partial z^m} + Y_{mj}\frac{\partial w_i}{\partial z_m}\right).$$

Let us rewrite (3.4) as

$$\begin{aligned}
&\hat{\mathbf{u}}_{\lambda t} - \hat{\rho}_0^{-1}(0)\nabla T'(\hat{\mathbf{u}}_\lambda) = \hat{\zeta}_\lambda\hat{\mathbf{f}} + \hat{\mathbf{k}}_1(\mathbf{w}) + \hat{\rho}_0^{-1}(z)[\nabla_1\hat{T}'(\hat{\mathbf{u}}_\lambda) - \nabla T'(\hat{\mathbf{u}}_\lambda)] \\
&+ [\hat{\rho}_0^{-1}(z) - \hat{\rho}_0^{-1}(0)]\nabla T'(\hat{\mathbf{u}}_\lambda) \\
&\equiv \mathbf{F}(z,t)\,, \qquad\qquad \hat{\mathbf{u}}_\lambda|_{t=0} = 0\,, \\
&-\mu S_{\alpha 3}(\hat{\mathbf{u}}_\lambda \textstyle\prod_0 |_{z_3=0} = \left(\prod_0 \hat{\mathbf{b}}\hat{\zeta}_\lambda\right)_\alpha + \hat{k}_{2\alpha}(\mathbf{w}) - \mu\Big[S_{\alpha 3}(\hat{\mathbf{u}}_\lambda) \\
&+ \left(\textstyle\prod_0 \hat{S}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0\right)_\alpha\Big]\Big|_{z_3=0} \equiv g_\alpha(z',t), \alpha = 1,2, \\
&T'_{33}(\hat{\mathbf{u}}_\lambda) + \sigma\Delta'\int_0^t \hat{u}_{\lambda 3}d\tau|_{z_3=0} = \hat{b}'\hat{\zeta}_\lambda + \sigma\int_0^t \hat{\zeta}_\lambda\hat{B}d\tau \\
&+ (T'_{33}(\hat{\mathbf{u}}_\lambda) - \mathbf{n}_0\cdot\hat{T}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0) + \sigma\int_0^t (\Delta'\hat{u}_{\lambda 3} + \mathbf{n}_0\cdot\Delta_0\hat{\mathbf{u}}_\lambda)d\tau|_{z_3=0} \\
&= h + \sigma\int_0^t H d\tau
\end{aligned} \tag{3.5}$$

with $h = \hat{b}'\hat{\zeta}_\lambda + (T'_{33}(\hat{\mathbf{u}}_\lambda) - \mathbf{n}_0\cdot\hat{T}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0)|_{z_3=0}, H = \hat{B}\hat{\zeta}_\lambda + (\Delta'\hat{u}_{\lambda 3} + \mathbf{n}_0\cdot\Delta_0\hat{\mathbf{u}}_\lambda)$. As supp $\mathbf{u}_\lambda \subset \bar{\Omega}_\lambda \equiv \{\bar{\Omega}\cap(|x| \le \lambda)\}$, supp $\hat{\mathbf{u}}_\lambda \subset V_\lambda = Y^{-1}\Omega_\lambda$, supp $\mathbf{F} \subset V_\lambda$ and supp g_α, supp h, supp $H \subset V'_\lambda = V_\lambda \cap \{x_3 = 0\}$, we can extend these functions by zero into $\mathbb{R}^2\backslash V_\lambda$ and consider (3.5) as the initial-boundary value problem in $\mathbb{R}^3_+$. The function $\hat{\rho}_0$ can be extended into $\mathbb{R}^3_+$ in such a way that $\hat{\rho}_0 \in W_2^{1+l}(\mathbb{R}^3_+)$ and $\hat{\rho}_0(z) \ge R_1 > 0$. Applying (2.10) we obtain

$$\begin{aligned}
&||\hat{\mathbf{u}}_\lambda||_{H_\gamma^{2+l,1+l/2}(\mathbb{D}_T)} \le c_2\Big(||\mathbf{F}||_{H_\gamma^{l,l/2}(\mathbb{D}_T)} + \sum_{\alpha=1}^{2}||g_\alpha||_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T)} \\
&+ ||h||_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T)} + \sigma||H||_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T)}\Big).
\end{aligned} \tag{3.6}$$

The next step is the estimate of norms in the right-hand side. Consider for instance the term $\mathbf{F}_1 = (\hat{\rho}_0^{-1}(z) - \hat{\rho}_0^{-1}(0))\nabla T'(\mathbf{u}_\lambda)$ in $\mathbf{F}$. We can estimate

$\mathbf{F}_1$ by applying the corollary of Lemma 4.1 from [2]. Since

$$\max_{V_\lambda} |\hat{\rho}_0^{-1}(z) - \hat{\rho}_0^{-1}(0)| \le c_3 \|\hat{\rho}_0^{-1}\|_{W_2^{l+1}(\mathbb{R}^3_+)} \lambda^\beta \,, \ \beta \in (0,1) \,, \ \beta \le l - 1/2 \,,$$

this corollary implies

$$\|\mathbf{F}_1\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} \le (c_4\lambda^\beta + \varepsilon + c_5(\varepsilon)\gamma^{-l/2}) \|\hat{\rho}_0^{-1}\|_{W_2^{l+1}(\mathbb{R}^3_+)} \times \|\nabla T'(\hat{\mathbf{u}}_\lambda)\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)}$$

with arbitrary $\varepsilon \in (0,1)$. In the same way we can evaluate $F_2 = \hat{\rho}_0^{-1}(z) \times [\nabla_1 \hat{T} - \nabla T']$ making use of the inequality

$$\sup_{V_\lambda} |Y_{ij}(z) - \delta_{ij}| \le c_6 \lambda^\beta \,. \tag{3.7}$$

Now, since the expression $\hat{\mathbf{k}}_1(\mathbf{w})$ does not contain second derivatives of $\mathbf{w}$,

$$\|\hat{\mathbf{k}}_1(\mathbf{w})\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} \le c_7(\lambda) \|\hat{\mathbf{w}}\|_{H_\gamma^{l+1,\frac{l+1}{2}}(V_{2\lambda,T})} \,, V_{2\lambda,T} = V_{2\lambda} \times (0,T) \,.$$

Hence,

$$\|\mathbf{F}\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} \le c_8[c_4\lambda^\beta + \varepsilon + c_5(\varepsilon)\gamma^{-l/2}] \|\hat{\rho}_0^{-1}\|_{W_2^{l+1}(\mathbb{R}^3_+)} \times \|\hat{\mathbf{u}}_\lambda\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} + c_7(\lambda)\|\hat{\mathbf{w}}\|_{H_\gamma^{l+1,\frac{l+1}{2}}(V_{2\lambda},T)} + \|\hat{\zeta}_\lambda \hat{\mathbf{f}}\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} \,.$$

The estimates of g_α, h, H are also based on imbedding theorems for $H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)$. We have:

$$S_{\alpha 3} + \left(\textstyle\prod_0 \hat{S}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0\right)_\alpha = S_{\alpha 3} + \hat{S}_{\alpha 3} n_3 + \sum_{\beta=1}^{2} \hat{S}_{\alpha\beta} n_\beta - n_\alpha(\mathbf{n}_0 \cdot \hat{S}(\hat{\mathbf{u}}_\lambda)\mathbf{n}_0) \,,$$

$$T'_{33} - \mathbf{n}_0 \cdot \hat{T}\mathbf{n}_0 = T'_{33}(\hat{\mathbf{u}}_\lambda) - n_3^2 \hat{T}_{33}(\hat{\mathbf{u}}_\lambda) - \sum_{i+j<6} n_i n_j \hat{T}_{ij} \,,$$

$$\Delta' \hat{u}_{\lambda 3} + \mathbf{n}_0 \cdot \Delta_0 \hat{\mathbf{u}}_\lambda = (\Delta' - \Delta_0)\hat{u}_{\lambda 3} + (1 + n_3)\Delta_0 \hat{u}_{\lambda 3} + \sum_{\beta=1}^{2} n_\beta \Delta_0 \hat{u}_{\lambda\beta} \,.$$

In local coordinates $(z_1, z_2) \in K_d$

$$\Delta_0 = \frac{1}{\sqrt{1+|\nabla'\varphi|^2}}\Bigg[\frac{\partial}{\partial z_1}\Bigg(\frac{1+\varphi_{z_2}^2}{\sqrt{1+|\nabla'\varphi|^2}}\frac{\partial}{\partial z_1} - \frac{\varphi_{z_1}\varphi_{z_2}}{\sqrt{1+|\nabla'\varphi|^2}}\frac{\partial}{\partial z_2}\Bigg) + \frac{\partial}{\partial z_2}\Bigg(-\frac{\varphi_{z_1}\varphi_{z_2}}{\sqrt{1+|\nabla'\varphi|^2}}\frac{\partial}{\partial z_1} + \frac{1+\varphi_{z_1}^2}{\sqrt{1+|\nabla'\varphi|^2}}\frac{\partial}{\partial z_2}\Bigg)\Bigg]$$

which shows that the leading coefficients of $\Delta_0 - \Delta'$ are proportional to $\varphi_{z_\alpha}, \alpha = 1, 2$. Making use of (3.7) and of the estimate

$$|n_1(z')| + |n_2(z')| + |1 + n_3(z')| \leq c_g \lambda^\beta, \ z' \in K_{2\lambda}$$

we prove that

$$\sum_{\alpha=1}^{2} ||g_\alpha||_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T)} + ||h||_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T)} + \sigma||H||_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T)}$$
$$\leq (c_g\lambda^\beta + \varepsilon + c_{10}(\varepsilon)\gamma^{-l/2})||\hat{\mathbf{u}}_\lambda||_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} + c_{11}(\lambda)||\mathbf{w}||_{H_\gamma^{l+1,\frac{l+1}{2}}(V_{2\lambda,T})} + \sum_{\alpha=1}^{2} ||\hat{b}_\alpha\hat{\zeta}_\lambda||_{H_\gamma^{l+1/2,l/2+1/4}(K_{2\lambda,T})}$$
$$+ ||\hat{b}'\hat{\zeta}_\lambda||_{H_\gamma^{l+1/2,l/2+1/4}(K_{2\lambda,T})} + \sigma||\hat{B}\hat{\zeta}_\lambda||_{H_\gamma^{l-1/2,l/2-1/4}(K_{2\lambda,T})}$$

where $K_{2\lambda,T} = K_{2\lambda} \times (0, T)$. Choosing ε and λ small, and γ large enough we can conclude that

$$||\hat{\mathbf{u}}_\lambda||^2_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} \leq c_{13}\Bigg\{||\mathbf{w}||^2_{H_\gamma^{l+1,\frac{l+1}{2}}(V_{2\lambda,T})} + ||\hat{\mathbf{f}}\hat{\zeta}_\lambda||^2_{H_\gamma^{l,l/2}(V_{2\lambda,T})}$$
$$+ \sum_{\alpha=1}^{2} ||\hat{b}_\alpha\hat{\zeta}_\lambda||^2_{H_\gamma^{l+1/2,l/2+1/4}(K_{2\lambda,T})} + ||\hat{b}'\hat{\zeta}_\lambda||^2_{H_\gamma^{l+1/2,l/2+1/4}(K_{2\lambda,T})} \tag{3.8}$$
$$+ \sigma^2||\hat{B}\hat{\zeta}_\lambda||^2_{H_\gamma^{l-1/2,l/2-1/4}(K_{2\lambda,T})}\Bigg\}.$$

Similar inequalities hold in neighbourhoods of any point on Γ or in the interior of Ω (in the latter case functions b_α, b', B do not enter into the estimates). When we cover Ω by a finite number of such neighbourhoods and make the summation of (3.8) over all the neighbourhoods, we obtain

$$\|\mathbf{w}\|^2_{H_\gamma^{l+2,l/2+1}(Q_T)} \le c_{14}\Big\{\|\mathbf{w}\|^2_{H_\gamma^{l+1,\frac{l+1}{2}}(Q_T)} + \|\mathbf{f}\|^2_{H_\gamma^{l,l/2}(Q_T)}$$
$$+ \|\mathbf{b}\|^2_{H_\gamma^{l+1/2,l/2+1/4}(G_T)} + \|b'\|^2_{H_\gamma^{l+1/2,l/2+1/4}(G_T)}$$
$$+ \sigma^2\|B\|^2_{H_\gamma^{l-1/2,l/2-1/4}(G_T)}\Big\}.$$

It remains to make use of the estimate

$$\|\mathbf{w}\|_{H_\gamma^{l+1,\frac{l+1}{2}}(Q_T)} \le c_{15}\gamma^{-1}\|\mathbf{w}\|_{H_\gamma^{l+2,l/2+1}(Q_T)}$$

which is a consequence of the definition of the norm $\|\mathbf{w}\|_{H_\gamma^{r,r/2}(Q_T)}$ and of an interpolation inequality. Taking γ sufficiently large we immediately arrive at (1.11).

The solvability of (1.8) will be proved by the construction of a regularizer (see for instance [4]), i.e. of a linear continuous operator R defined on the space $\mathcal{H}_{\gamma,l} = H_\gamma^{l,l/2}(Q_T) \times H_{\gamma,n}^{l+1/2,l/2+1/4}(G_T) \times H_\gamma^{l+1/2,l/2+1/4}(G_T) \times H_\gamma^{l-1/2,l/2-1/4}(G_T)$ and making correspond to every $F = (\mathbf{f}, \mathbf{b}, b, B) \in \mathcal{H}_{\gamma,l} : \mathbf{f} \in H_\gamma^{l,l/2}(Q_T)$,

$$\mathbf{b} \in H_{\gamma,n}^{l+1/2,l/2+1/4}(G_T) = \left\{\mathbf{u} \in H_\gamma^{l+1/2,l/2+1/4}(G_T) : \mathbf{u}\cdot\mathbf{n}_0 = 0\right\},$$

$b' \in H_\gamma^{l+1/2,l/2+1/4}(G_T)$, $B \in H_\gamma^{l-1/2,l/2-1/4}(G_T)$ the solution $\mathbf{v} \in H_\gamma^{l+2,l/2+1}(Q_T)$ of the problem

$$\mathbf{v}_t - \rho_0^{-1}(x)\nabla T'(\mathbf{v}) = \mathbf{f} + \mathcal{M}_1 F\,,\ \mathbf{v}|_{t=0} = 0\,,$$
$$\mu\prod\nolimits_0 S(\mathbf{v})\mathbf{n}_0|_{G_T} = \mathbf{b} + \mathcal{M}_2 F\,,$$
$$\mathbf{n}_0\cdot T'(\mathbf{v})\mathbf{n}_0 - \sigma\mathbf{n}_0\cdot\Delta_0\int_0^t \mathbf{v}\,d\tau = b' + \mathcal{M}_3 F + \sigma\int_0^t (B + \mathcal{M}_4 F)\,d\tau$$

where $\mathcal{M}F = (\mathcal{M}_1 F, \mathcal{M}_2 F, \mathcal{M}_3 F, \mathcal{M}_4 F)$ is a contraction operator in $\mathcal{H}_{\gamma,l}$ for large γ. The solution of (1.8) can be expressed in terms of the regularizer as $\mathbf{w} = R(I + \mathcal{M})^{-1}F \in H_\gamma^{l+2,l/2+1}(Q_T)$.

Let $\{\zeta_j(x)\}$ be a smooth partition of unity subordinate to the covering of Ω by the balls $K_{i\lambda} = \{|x - \xi_i| \le b_i\lambda\}$. It is convenient to assume that $\xi_1, \ldots, \xi_{M_\lambda} \in \Omega, \mathrm{dist}(\xi_j, \Gamma) \ge \frac{\lambda}{2}, b_j = 1/2\,(j = 1, \ldots, M_\lambda)$ and $\xi_{M_\mu+1}, \ldots, \xi_{N_\lambda} \in \Gamma, b_{M_\lambda+1} = \ldots = b_{N_\lambda} = 1$. Assume also that

$$|D^\alpha\zeta_j| \le c_{16}(\alpha)\lambda^{-|\alpha|}$$

where c_{16} is independent of λ and j and let $\eta_j(x,\lambda)$ be smooth function with supp $\eta_j \subset K_{j,3\lambda/2}$ satisfying (3.13) and such that $\eta_i(x,\lambda)\zeta_i(x,\lambda) = \zeta_i(x,\lambda)$.

We define the vector field $\mathbf{v} = RF$ by the formula

$$\mathbf{v} = \sum_{i=1}^{N_\lambda} \eta_i \mathbf{v}_i(x,t)$$

where $\mathbf{v}_i, i \le M_\lambda$, is a solution of the Cauchy problem

$$\mathbf{v}_{it} - \rho_0^{-1}(x)\nabla T'(\mathbf{v}_i) = \mathbf{f}\zeta_i\,,\ x \in \mathbb{R}^3\,,\ \mathbf{v}_i|_{t=0} = 0\,,$$

(we suppose that $\mathbf{f}\zeta_i = 0$ for $|x-\xi_i| > b_i\lambda$) and $\mathbf{v}_i, i > M_\lambda$ is defined in terms of a certain initial boundary value problem in the half-space $D_i = \{x \in \mathbb{R}^3 : (\mathbf{x}-\xi_i)\cdot\mathbf{n}(\xi_i) < 0\}$ that is described below. Let $\{y\}$ be local cartesian coordinates at the point $\xi_i : y = C_i(x-\xi_i)(C_i$ is an orthogonal matrix), $\varphi_i(y')$ be the function defining Γ in the neighbourhood of ξ_i and let Y be the corresponding transformation (3.2). The transformation $Z_i(x) = C_i^{-1}Y_i^{-1}C_i(x-\xi_i)+\xi_i$ maps the domain $y_3 > \varphi(y')$ onto D_i and its Jacobi matrix equals I at the point ξ_i. Let

$$\mathbf{f}_i(z,t) = \mathbf{f}(Z_i^{-1}(z),t)\zeta_i(Z_i^{-1}(z)),\ \mathbf{b}_i(z,t) = \mathbf{b}(Z_i^{-1}(z),t)\zeta_i(Z_i^{-1}(z))\,,$$

$B_i(z,t) = B(Z_i^{-1}(z),t)\zeta_i(Z_i^{-1}(z))$ and let $\mathbf{w}_i$ be a solution of the problem

$$\begin{aligned}
&\mathbf{w}_{it} - \rho_0^{-1}(\xi_i)\nabla T'(\mathbf{w}_i) = \mathbf{f}_i(z,t)\,,\ z \in D_i\,,\\
&\mathbf{w}_i|_{t=0} = 0\,,\\
&\mu\textstyle\prod_i S(\mathbf{w}_i)\mathbf{n}_0(\xi_i)|_{z\in\partial D_i} = \prod_i \mathbf{b}_i\,,\\
&\mathbf{n}_0(\xi_i)\cdot T'(\mathbf{w}_i)\mathbf{n}_0(\xi_i) - \sigma\Delta_i'\int_0^t \mathbf{w}_i d\tau\cdot\mathbf{n}_0(\xi_i)|_{z\in\partial D_i} = b_i + \sigma\int_0^t B_i d\tau
\end{aligned} \tag{3.9}$$

where Δ_i' is the Laplacean on the tangent plane ∂D_i and $\prod_i \mathbf{w} = \mathbf{w} - \mathbf{n}_0(\xi_i)(\mathbf{n}_0(\xi_i)\cdot\mathbf{w})$. We set $\mathbf{v}_i(x,t) = \mathbf{w}_i(Z_i(x),t)$.

Clearly, R is a linear continuous operator from $\mathcal{H}_{\gamma,l}$ into $H_\gamma^{l+2,l/2+1}(Q_T)$. To calculate $\mathcal{M}F$, we write (3.9) in coordinates $\{x\}$ in the neighbourhood $|x-\xi_i| \le 2\lambda$ of ξ_i

$$\begin{aligned}
&\mathbf{v}_{it} - \rho_0^{-1}(\xi_i)\tilde{\nabla}_i\tilde{T}(\mathbf{v}_i) = \mathbf{f}\zeta_i\,,\ \mathbf{v}_i|_{t=0} = 0\,,\\
&\mu\textstyle\prod_i \tilde{S}_i(\mathbf{v}_i)\mathbf{n}_0(\xi_i)|_{x\in\Gamma} = \prod_i \mathbf{b}\zeta_i\,,\\
&\mathbf{n}_0(\xi_i)\tilde{T}_i(\mathbf{v}_i)\mathbf{n}_0(\xi_i) - \sigma\mathbf{n}_0(\xi_i)\cdot\Delta_i'\int_0^t \mathbf{v}_i d\tau|_{\xi\in\Gamma} = b\zeta_i + \sigma\int_0^t B\zeta_i d\tau\,.
\end{aligned} \tag{3.10}$$

Here

$$\tilde{\nabla}_i = Z_i^* \nabla, (\tilde{S}_i(\mathbf{v}))_{jk} = \sum_{m=1}^{3} \left(Z_{mk}^{(i)} \frac{\partial v_j}{\partial x_m} + Z_{mj} \frac{\partial v_k}{\partial x_m} \right),$$

$$\tilde{T}_i(\mathbf{v}) = \mu' \tilde{\nabla}_i \cdot \mathbf{v} + \mu \tilde{\nabla}_i \tilde{S}_i(\mathbf{v}), \; Z_i = \left(Z_{mk}^{(i)} \right)_{m,k=1,2,3}$$

is the matrix of Jacobi of the transformation Z_i. It is easily seen that

$$\mathcal{M}_1 F = \Big\{ \sum_{i>M_\lambda} \eta_i \Big[\rho_0^{-1}(\xi_i) \tilde{\nabla}_i \tilde{T}_i(\mathbf{v}_i) - \rho_0^{-1}(x) \nabla T'(\mathbf{v}_i) \Big]$$
$$+ \sum_{i=1}^{M_\lambda} \eta_i (\rho_0^{-1}(\xi_i) - \rho_0^{-1}(x)) \nabla T'(\mathbf{v}_i) \Big\} + \sum_i \rho_0^{-1}(x) \Big(\eta_i \nabla T'(\mathbf{v}_i)$$
$$- \nabla T'(\mathbf{v}_i \eta_i) \Big) \equiv \mathcal{M}_1' F + \mathcal{M}_1'' F,$$

$$\mathcal{M}_2 F = \sum_{i>M_\lambda} \eta_i \prod_0 \Big\{ \mu \Big(\prod_0 S(\mathbf{v}_i) \mathbf{n}_0$$
$$- \prod_i \tilde{S}_i(\mathbf{v}_i) \mathbf{n}_0(\xi_i) \Big) + \Big(\prod_i - \prod_0 \Big) \mathbf{b}_i \zeta_i \Big\}$$
$$- \mu \sum_{i>M_\lambda} \prod_0 (\eta_i S(\mathbf{v}_i) \mathbf{n}_0 - S(\eta_i \mathbf{v}_0) \mathbf{n}_0)$$
$$\equiv \mathcal{M}_2' F + \mathcal{M}_2'' F,$$

$$\mathcal{M}_3 F = \sum_{i>M_\lambda} \eta_i \Big(\mathbf{n}_0 \cdot T'(\mathbf{v}_i) \mathbf{n}_0 - \mathbf{n}_0(\xi_i) \cdot \tilde{T}_i(\mathbf{v}_i) \mathbf{n}_0(\xi_i) \Big)$$
$$- \sum_{i>M_\lambda} \mathbf{n}_0 \cdot (\eta_i T'(\mathbf{v}_i) \mathbf{n}_0 - T'(\eta_i \mathbf{v}_i) \mathbf{n}_0) \equiv \mathcal{M}_3' F + \mathcal{M}_3'' F,$$

$$\mathcal{M}_4 F = \sum_{i>M_\lambda} \eta_i (\mathbf{n}_0(\xi_i) \Delta_i' - \mathbf{n}_0(x) \Delta_0) \mathbf{v}_i + \sum_{i>M_\lambda} \mathbf{n}_0(x) (\eta_i \Delta_0 \mathbf{v}_i$$
$$- \Delta_0(\eta_i \mathbf{v}_i)) \equiv \mathcal{M}_4' F + \mathcal{M}_4'' F.$$

$\mathcal{M}'' = (\mathcal{M}_1'', \mathcal{M}_2'', \mathcal{M}_3'', \mathcal{M}_4'')$ is a smoothing operator, i.e.,

$$\|\mathcal{M}'' F\|^2_{\mathcal{H}_{\gamma,l}} = \|\mathcal{M}_1'' F\|^2_{H_\gamma^{l,l/2}(Q_T)} + \|\mathcal{M}_2'' F\|^2_{H_\gamma^{l+1/2,l/2+1/2}(G_T)}$$
$$+ \|\mathcal{M}_3'' F\|^2_{H_\gamma^{l+1/2,l/2+1/2}(G_T)} + \|\mathcal{M}_4'' F\|^2_{H_\gamma^{l-1/2,l/2-1/4}(G}$$
$$\le c_{17}(\lambda) \sum_i \|\mathbf{v}_i\|^2_{H_\gamma^{l+1,\frac{l+1}{2}}(\Omega_{i,2\lambda} \times (0,T))}, \; \Omega_{i,2\lambda} = \Omega \cap K_{i,2}$$

The right-hand side does not exceed

$$c_{18}(\lambda)\gamma^{-1}\sum_i{}' \|\mathbf{v}_i\|^2_{H_\gamma^{l+1,\frac{l+1}{2}}(\Omega_{i,2\lambda}\times(0,T))},$$

hence,

$$\|\mathcal{M}''F\|_{\mathcal{H}_{\gamma,l}} \le c_{19}(\lambda)\gamma^{-1/2}\|F\|_{\mathcal{H}_{\gamma,l}}.$$

Finally, we estimate $\mathcal{M}'F$ making use of the smallness of differences $\rho_0^{-1}(\xi)-\rho_0^{-1}(x), Z^{(i)}_{mk}-\delta_{mk}, \mathbf{n}_0(x)-\mathbf{n}_0(\xi_i)$ and of leading coefficients of $\Delta_i'-\Delta_0$ when λ is small. Repeating the above arguments we can show that

$$\|\mathcal{M}'F\|_{\mathcal{H}_{\gamma,l}} \le (c_{20}\lambda^\beta + c_{21}(\lambda)\gamma^{-1/2})\|F\|_{\mathcal{H}_{\gamma,l}}$$

with c_{20} independent of λ. Hence, for small λ and large $\gamma, \mathcal{M}$ is a contraction operator. Theorem 1.1 is proved.

4. Proof of Theorem 1.2

We suppose first that $\mathbf{u} = 0$ and construct the vector field $\mathbf{V} \in W_2^{2+l,1+l/2}(Q_\infty)$ satisfying the initial condition $\mathbf{V}|_{t=0} = \mathbf{w}_0$ and the inequality

$$\|\mathbf{V}\|_{W_2^{2+l,1+l/2}(Q_\infty)} \le c_1\|\mathbf{w}_0\|_{W_2^{1+l}(\Omega)}. \tag{4.1}$$

For the difference $\mathbf{u} = \mathbf{w} - \mathbf{V}$ we get the problem (1.8) with homogeneous initial condition and with the functions $\mathbf{g} = \mathbf{f} - \mathbf{V}_t + \rho_0^{-1}\nabla T'(\mathbf{V}), \mathbf{d} = \mathbf{b} - \mu S(\mathbf{V})\mathbf{n}_0, d = b - \mathbf{n}_0 \cdot T'(\mathbf{V})\mathbf{n}_0 + \sigma\int_0^t \mathbf{n}_0 \cdot \Delta_0\mathbf{V}d\tau$ instead of $\mathbf{f}, \mathbf{b}, b$ in the right-hand sides. The compatibility conditions reduce to $\mathbf{d}|_{t=0} = 0, d|_{t=0} = b'|_{t=0} - \mathbf{n}_0 \cdot T'(\mathbf{w}_0)\mathbf{n}_0|_\Gamma = 0$, hence $d' = b' - \mathbf{n}_0 \cdot T'(\mathbf{V})\mathbf{n}_0 \in H_\gamma^{l+1/2,l/2+1/4}(G_T) D = B + \mathbf{n}_0 \cdot \Delta_0\mathbf{V} \in H_\gamma^{l-1/2,l/2-1/4}(G_T)$ for $T < \infty$. When we apply Theorem 1.1 and take into account of (4.1), we prove Theorem 1.2 for $\mathbf{u} = 0$.

In the general case we write (1.7) in the form

$$\begin{aligned}
&\mathbf{w}_t - \rho_0^{-1}(\xi)\nabla T'(\mathbf{w}) = \mathbf{f} + \mathbf{l}_1(\mathbf{w}),\ \mathbf{w}|_{t=0} = \mathbf{w}_0(\xi),\\
&\mu\prod\nolimits_0 S(\mathbf{w})\mathbf{n}_0|_{\xi\in\Gamma} = \prod\nolimits_0 \mathbf{b} + \mathbf{l}_2(\mathbf{w}),\\
&\mathbf{n}_0 \cdot T'(\mathbf{w})\mathbf{n}_0 - \sigma\mathbf{n}_0 \cdot \Delta_0\int_0^t \mathbf{w}d\tau|_{\xi\in\Gamma} = b' + \sigma\int_0^t Bd\tau\\
&\qquad + l_3(\mathbf{w}) + \sigma\int_0^t l_4(\mathbf{w})d\tau
\end{aligned} \tag{4.2}$$

where

$$
\begin{aligned}
\mathbf{l}_1(\mathbf{w}) &= \rho_0^{-1}(\xi)\left[\mathcal{A}\nabla T_u'(\mathbf{w}) - \nabla T'(\mathbf{w})\right], \\
\mathbf{l}_2(\mathbf{w}) &= \mu \prod\nolimits_0 \left(\prod\nolimits_0 S(\mathbf{w})\mathbf{n}_0 - \prod\nolimits_u S_u(\mathbf{w})\mathbf{n} \right), \\
l_3(\mathbf{w}) &= \mathbf{n}_0 \cdot \left[T'(\mathbf{w})\mathbf{n}_0 - T_u'(\mathbf{w})\mathbf{n} \right], \\
l_4(\mathbf{w}) &= \mathbf{n}_0 \cdot \left[(\Delta_u(t) - \Delta_0)\mathbf{w} + \dot{\Delta}_u(t) \int_0^t \mathbf{w}\, d\tau \right]
\end{aligned}
$$

and $\dot{\Delta}_u(t)$ is the operator whose coefficients are derivatives of the coefficients of $\Delta_u(t)$ with respect to t. It is shown in [3] (see (3.13), (3.33)) that for small δ

$$
\|l_4(\mathbf{w})\|_{G_T}^{(l-1/2, l/2-1/4)} \le C_2 \delta \|\mathbf{w}\|_{Q_T}^{(l+2, l/2+1)} . \tag{4.3}
$$

Moreover, basing on estimates of the elements of $\mathcal{A} - I = \mathcal{B}$ obtained in [3] one can show (in the same way as in Lemmas 2.6 and 2.7 from [3]) that

$$
\begin{aligned}
&\|\mathbf{l}_1(\mathbf{w})\|_{Q_T}^{(l,l/2)} + \|\mathbf{l}_2(\mathbf{w})\|_{W_2^{l+1/2, l/2+1/4}(G_T)} \\
&+ \|l_3(\mathbf{w})\|_{W_2^{l+1/2, l/2+1/4}(G_T)} \le c_3 \delta \|\mathbf{w}\|_{Q_T}^{(l+2, l/2+1)} .
\end{aligned} \tag{4.4}
$$

Let us rewrite (4.2) as

$$
\mathbf{w} = L(\mathbf{f} + \mathbf{l}_1(\mathbf{w}), \mathbf{w}_0, \mathbf{b} + \mathbf{l}_2(\mathbf{w}), b' + l_3(\mathbf{w}), B + l_4(\mathbf{w})) \tag{4.5}
$$

where L is an operator which makes correspond to every element

$$
\begin{aligned}
&(\mathbf{f}, \mathbf{w}_0, \mathbf{b}, b, B) \in W_2^{l,l/2}(Q_T) \times W_2^{l+1}(\Omega) \times W_2^{l+1/2, l/2+1/4}(G_T) \\
&\times W_2^{l+1/2, l/2+1/4}(G_T) \times W_2^{l-1/2, l/2-1/4}(G_T)
\end{aligned}
$$

such that $\mathbf{b} \cdot \mathbf{n}_0 = 0$ and that the compatibility conditions (1.13) are satisfied, the solution of (1.8) with $b = b' + \sigma \int_0^t B d\tau$. We have already shown that L is a continuous operator. It follows from (4.4) that the operator $L(\mathbf{l}_1(\mathbf{w}), 0, \mathbf{l}_2(\mathbf{w}), l_3(\mathbf{w}), l_4(\mathbf{w}))$ is a contraction operator in $W_2^{2+l, 1+l/2}(Q_T)$, provided that δ is small enough. It follows that Eq. (4.5) is uniquely solvable which proves Theorem 1.2.

5. Proof of Theorem 1.3

We begin with auxiliary propositions. Consider the function

$$\rho(\xi,t)=\rho_0(\xi)J_u^{-1}(\xi,t)=\rho_0(\xi)\big[1+\int_0^t A\nabla\cdot \mathbf{u}\,d\tau\big]^{-1}$$

where $\mathcal{A}$ is determined by the transformation X_u.

Lemma 5.1. Suppose that $\rho_0\in W_2^{l+1}(\Omega), l\in(1/2,1)$, and that $\mathbf{u}\in W_2^{l+2,l/2+1}(Q_T)$ satisfies (1.12). Then

$$\|\rho(\cdot,t)\|_{W_2^{l+1}(\Omega)}\le c_1\|\rho_0\|_{W_2^{l+1}(\Omega)}, \tag{5.1}$$

$$\Big(\int_0^t \|\nabla\rho(\cdot,t)-\nabla\rho(\cdot,t-\tau)\|^2_{L_2(\Omega)}\frac{d\tau}{\tau^{1+l}}\Big)^{1/2}\le c_2\|\rho_0\|_{W_2^{l+1}(\Omega)}, \tag{5.2}$$

$$\Big(\int_0^T d\tau\int_0^t \|\rho(\cdot,t)-\rho(\cdot,t-\tau)\|^2_{L_2(\Gamma)}\frac{d\tau}{\tau^{3/2+l}}\Big)^{1/2}\le c_3T^{3/4-l/2}\max_{\xi\in\Omega}|\rho_0(\xi)|. \tag{5.3}$$

Proof. It is well known (see for instance [2,3]) that

$$\begin{aligned}&\|fh\|_{W_2^{l+1}(\Omega)}\le c_4\|f\|_{W_2^{l+1}(\Omega)}\|h\|_{W_2^{l+1}(\Omega)},\\ &\|fg\|_{W_2^{l}(\Omega)}\le c_5\|f\|_{W_2^{l+1}(\Omega)}\|g\|_{W_2^{l}(\Omega)}.\end{aligned} \tag{5.4}$$

In virtue of the first inequality (5.4), (5.1) reduces to the estimate for $\|J_u^{-1}\|_{W_2^{l+1}(\Omega)}$. Applying the estimates

$$|A_{ij}(\xi,t)|\le c_6\|A_{ij}(\cdot,t)\|_{W_2^{l+1}(\Omega)}\le c_7,\ \int_0^t|\mathcal{A}\nabla\cdot\mathbf{u}|d\tau\le c_8\delta \tag{5.5}$$

obtained in [3] under the condition (1.12), we see that

$$\begin{aligned}&0<1-c_8\delta\le|J_u(\xi,t)|\le 1+c_8\delta,\\ &\|\nabla J_u\|_{L_2(\Omega)}\le\int_0^t\big(\max_{i,j}|A_{ij}|\|D^2\mathbf{u}\|_{L_2(\Omega)}\\ &\qquad+\max_{i,j}\|\nabla A_{ij}\|_{L_3(\Omega)}\|D\mathbf{u}\|_{L_6(\Omega)}\big)d\tau\end{aligned}$$

where $D\mathbf{v} = \left(\frac{\partial v_i}{\partial \xi_j}\right)_{i,j=1,2,3}$, $D^2\mathbf{v} = \left(\frac{\partial^2 v_i}{\partial \xi_j \partial \xi_k}\right)_{i,j,k=1,2,3}$, $|v|_\Omega = \sup_{\xi\in\Omega} |v(\xi)|$.
Since $W_2^l(\Omega)$ is continuously imbedded into $L_3(\Omega)$ we conclude that

$$\|\nabla J_\mathbf{u}\|_{L_2(\Omega)} \le c_9 \int_0^t \|D\mathbf{u}\|_{W_2^1(\Omega)} d\tau .$$

In a similar way it can be shown that

$$\begin{aligned}\|\nabla J_\mathbf{u}\|_{W_2^l(\Omega)} &\le c_5 \int_0^t \big(\max_{i,j} \|A_{ij}\|_{W_2^{l+1}(\Omega)} \|D^2\mathbf{u}\|_{W_2^l(\Omega)} \\ &\qquad + \max_{i,j} \|\nabla A_{ij}\|_{W_2^l(\Omega)} \|D\mathbf{u}\|_{W_2^{l+1}(\Omega)}\big) d\tau \\ &\le c_{10} \int_0^t \|D\mathbf{u}\|_{W_2^{l+1}(\Omega)} d\tau \le c_{11}\delta .\end{aligned} \tag{5.6}$$

This implies

$$\|\nabla J_\mathbf{u}^{-1}\|_{W_2^l(\Omega)} = \|J_\mathbf{u}^{-2}\nabla J_\mathbf{u}\|_{W_2^l(\Omega)} \le c_{12}\|\nabla J_\mathbf{u}\|_{W_2^l(\Omega)} \le c_{13}\delta$$

which leads to (5.1). Now, from

$$\|\nabla\rho(\cdot,t) - \nabla\rho(\cdot,t-\tau)\|_{L_2(\Omega)} \le c_{14}\big\|\nabla\rho_0\|_{L_3(\Omega)} \int_{t-\tau}^t \|\mathcal{A}\nabla\mathbf{u}\|_{L_6(\Omega)} d\tau'$$

$$+ |\rho_0|_\Omega \int_{t-\tau}^t \|\nabla(\mathcal{A}\nabla\cdot\mathbf{u})\|_{L_2(\Omega)} d\tau'\big) \le c_{15}\|\rho_0\|_{W_2^{l+1}(\Omega)} \int_{t-\tau}^t \|D\mathbf{u}\|_{W_2^1(\Omega)} d\tau'$$

it follows that

$$\begin{aligned}&\left(\int_0^t \|\nabla\rho(\cdot,t) - \nabla\rho(\cdot,t-\tau)\|_{L_2(\Omega)} \frac{d\tau}{\tau^{1+l}}\right)^{1/2} \\ &\le \frac{c_{15}}{\sqrt{l}} \|\rho_0\|_{W_2^{l+1}(\Omega)} \int_0^t \|D\mathbf{u}\|_{W_2^1(\Omega)} \frac{d\tau}{(t-\tau)^{l/2}} \\ &\le \frac{c_{15} t^{\frac{1-l}{2}}}{\sqrt{l(1-l)}} \left(\int_0^t \|D\mathbf{u}\|^2_{W_2^1(\Omega)} d\tau\right)^{1/2} \|\rho_0\|_{W_2^{l+1}(\Omega)} \\ &\le c_{16}\delta \|\rho_0\|_{W_2^{l+1}(\Omega)}\end{aligned} \tag{5.7}$$

and (5.2) is proved. Similarly,

$$\begin{aligned}&\left(\int_0^t \|\rho(\cdot,t) - \rho(\cdot,t-\tau)\|^2_{L_2(\Gamma)} \frac{d\tau}{\tau^{3/2+l}}\right)^{1/2} \\ &\le c_{17} \sup_\xi |\rho_0(\xi)| \int_0^t \frac{\|D\mathbf{u}\|_{L_2(\Gamma)}}{(t-\tau)^{l/2+1/4}} d\tau \\ &\le \frac{4c_{17}}{3-2l} |\rho_0|_\Omega \sup_{\tau\le t} \|D\mathbf{u}(\cdot,\tau)\|_{L_2(\Gamma)} t^{3/4-l/2}\end{aligned} \tag{5.8}$$

which leads to (5.3) after easy calculations. The proof of the lemma is completed.

Lemma 5.2. Let $p(\rho)$ satisfy the hypotheses of Theorem 1.3, then

$$\|\nabla p(\rho)\|_{Q_T}^{(l,l/2)} \le c_{18}(T)\|\rho_0\|_{W_2^{l+1}(\Omega)}(1+\|\rho_0\|_{W_2^{l+1}(\Omega)}), \tag{5.9}$$

$$\|p(\rho)\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \le \max_{a>0}|p(a)|\sqrt{T|\Gamma|} + c_{19}(T)\|\rho_0\|_{W_2^{l+1}(\Omega)}(1+\|\rho_0\|_{W_2^{l+1}(\Omega)}) \tag{5.10}$$

where $c_i(T)$ are non-decreasing power functions of T.

Proof. Suppose that ρ is extended from Ω into $\mathbb{R}^3$ in such a way that $\|\rho\|_{W_2^{l+1}(\mathbb{R}^3)} \le c_{20}\|\rho\|_{W_2^{l+1}(\Omega)}$ for a fixed t. It is not hard to prove (see [2], Lemma 4.1) that $\nabla p(\rho) = p'(\rho)\nabla\rho$ satisfies the inequalities

$$\|\nabla \rho\|_{W_2^l(\Omega)} \le \max_a|p'(a)|\|\nabla\rho\|_{W_2^l(\Omega)} + \max_a|p''(a)|\|\nabla\rho\|_{L_3(\Omega)}\Big(\int_{\mathbb{R}^3}\|\rho(x+z)-\rho(x)\|^2_{L_6(\Omega)}\frac{dz}{|z|^{3+2l}}\Big)^{1/2}$$
$$\le c_{21}\|\nabla\rho\|_{W_2^l(\Omega)}(1+\|\rho\|_{W_2^{l+1}(\Omega)}), \tag{5.11}$$

$$T^{-l/2}\|\nabla p(\rho)\|_{L_2(Q_T)} \le T^{\frac{1-l}{2}}\max_a|p'(a)|\sup_{t\le T}\|\nabla\rho(\cdot,t)\|_{L_2(\Omega)},$$

$$\|\nabla p(\rho(\cdot,t)) - \nabla p(\rho(\cdot,t-\tau))\|_{L_2(\Omega)} \le \max_a|p'(a)|\|\nabla\rho(\cdot,t)-\nabla\rho(\cdot,t-\tau)\|_{L_2(\Omega)} + \max_a|p''(a)|\|\nabla\rho\|_{L_3(\Omega)}\|\rho(\cdot,t-\tau)-\rho(\cdot,t)\|_{L_6(\Omega)}, \tag{5.12}$$

and

$$\Big(\int_0^T dt\int_0^t\|\nabla p(\rho(\cdot,t))-\nabla p(\rho(\cdot,t-\tau))\|^2_{L_2(\Omega)}\frac{d\tau}{\tau^{1+l}}\Big)^{1/2}$$
$$\le c_{22}\Big\{\Big(\int_0^T dt\int_0^t\|\nabla\rho(\cdot,t)-\nabla\rho(\cdot,t-\tau)\|^2_{L_2(\Omega)}\frac{d\tau}{\tau^{1+l}}\Big)^{1/2} + \sup_{t\le T}\|\nabla\rho(\cdot,t)\|_{W_2^l(\Omega)}\Big(\int_0^T dt\int_0^t\|\rho(\cdot,t)-\rho(\cdot,t-\tau)\|^2_{W_2^l(\Omega)}\frac{d\tau}{\tau^{1+l}}\Big)^{1/2}\Big\}. \tag{5.13}$$

The estimates (5.11)–(5.13), (5.1), (5.2) imply (5.9), and (5.10) follows from (5.3), (5.9) and from the inequalities

$$\|p(\rho)\|_{W_2^{l+1/2}(\Gamma)} \le \max_a |p(a)||\Gamma|^{1/2} + c_{23}\|\nabla p(\rho)\|_{W_2^l(\Omega)}, \forall t \in (0,T),$$
$$\|p(\rho(\cdot,t)) - p(\rho(\cdot,t-\tau))\|_{L_2(\Gamma)} \le \max_a |p'(a)|\|\rho(\cdot,t) - \rho(\cdot,t-\tau)\|_{L_2(\Gamma)}.$$

The lemma is proved.

Let $\mathbf{u}'$ be another vector field satisfying (1.12) and generating the transformation $X_{\mathbf{u}'}$, the matrix $\mathcal{A}'$, the function $J_{\mathbf{u}'}$ etc., and let $\rho'(\xi,t) = \rho_0(\xi)J_{\mathbf{u}'}^{-1}(\xi,t)$. We estimate the differences $\rho - \rho', p(\rho) - p(\rho')$.

Lemma 5.3. If $\mathbf{u}$ and $\mathbf{u}'$ satisfy (1.12) and $p(\rho)$ satisfies the hypotheses of Theorem 1.3, then

$$\|\rho - \rho'\|_{W_2^{l+1}(\Omega)} \le c_{24}\|\rho_0\|_{W_2^{l+1}(\Omega)} \int_0^t \|\mathbf{u} - \mathbf{u}'\|_{W_2^{2+l}(\Omega)} d\tau, \tag{5.14}$$

$$\|\nabla\rho - \nabla\rho'\|_{Q_T}^{(l,l/2)} + \|\rho - \rho'\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$$
$$\le c_{25}\|\rho_0\|_{W_2^{l+1}(\Omega)}\mathcal{N}_T[\mathbf{u} - \mathbf{u}'], \tag{5.15}$$

$$\|\nabla p(\rho) - \nabla p(\rho')\|_{Q_T}^{(l,l/2)} + \|p(\rho) - p(\rho')\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$$
$$\le c_{26}\|\rho_0\|_{W_2^{l+1}(\Omega)}(1 + \|\rho_0\|_{W_2^{l+1}(\Omega)})^2\mathcal{N}_T[\mathbf{u} - \mathbf{u}'] \tag{5.16}$$

where c_i are non-decreasing power functions of T and

$$\mathcal{N}_T[\mathbf{v}] = \int_0^T \|\mathbf{v}\|_{W_2^{l+2}(\Omega)} dt + \sup_{t\in T} \int_0^t \|D\mathbf{v}\|_{W_2^l(\Omega)} \frac{d\tau}{(t-\tau)^{1/2}}$$
$$+ \sup_{t\in T} \int_0^t \|D\mathbf{v}\|_{L_2(\Gamma)} \frac{d\tau}{(t-\tau)^{1/2+1/4}}.$$

Proof. In virtue of (5.4)–(5.6) the difference

$$\tilde{\rho}(\xi,t) = \rho(\xi,t) - \rho(\xi',t) = \frac{\rho_0(\xi)}{J_{\mathbf{u}}(\xi,t)J_{\mathbf{u}'}(\xi,t)} \int_0^t [(\mathcal{A}' - \mathcal{A})\nabla\cdot\mathbf{u}' \tag{5.17}$$
$$+ \mathcal{A}\nabla\cdot(\mathbf{u}' - \mathbf{u})]d\tau$$

satisfies the inequality

$$\begin{aligned}
&\|\bar\rho(\cdot,t)\|_{W_2^{l+1}(\Omega)} \\
&\le c_{27}\|\rho_0\|_{W_2^{l+1}(\Omega)}\int_0^T \|(\mathcal{A}'-\mathcal{A})\nabla\cdot\mathbf{u}'+\mathcal{A}\nabla\cdot(\mathbf{u}'-\mathbf{u})\|_{W_2^{l+1}(\Omega)}dt \\
&\le c_{28}\|\rho_0\|_{W_2^{l+}(\Omega)}\Big\{\max_{i,j}\sup_{\tau\le t}\|A_{ij}-A'_{ij}(\cdot,\tau)\|_{W_2^{l+1}(\Omega)}\int_0^t \|D\mathbf{u}\|_{W_2^{l+1}(\Omega)}d\tau \\
&\quad+\max_{i,j}\sup_{\tau\in t}\|A_{ij}(\cdot,\tau)\|_{W_2^{l+1}(\Omega)}\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{l+1}(\Omega)}d\tau\Big\}
\end{aligned}$$

The estimates of differences $A_{ij}-A'_{ij}$ were established in [3], Lemma 2.2. Applying these estimates and the inequality

$$\int_0^t \|D\mathbf{u}'\|_{W_2^{l+1}(\Omega)}d\tau \le t^{1/2}\left(\int_0^t \|D\mathbf{u}'\|^2_{W_2^{l+1}(\Omega)}d\tau\right)^{1/2}\le\delta$$

we arrive at (5.14). In the same way we show that

$$\begin{aligned}
T^{-l/2}\|\nabla\bar\rho\|_{L_2(Q_T)} &\le c_{27}T^{-l/2}\|\rho_0\|_{W_2^{l+1}(\Omega)}\int_0^T \|(\mathcal{A}'-\mathcal{A})\nabla\cdot\mathbf{u}'+ \\
&\qquad \mathcal{A}\nabla\cdot(\mathbf{u}'-\mathbf{u})\|_{W_2^1(\Omega)}d\tau \\
&\le c_{29}\|\rho_0\|_{W_2^{l+1}(\Omega)}T^{\frac{1-l}{2}}\left(\int_0^T \|D\mathbf{u}'\|^2_{W_2^1(\Omega)}dt\right)^{1/2}\int_0^T \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{l+1}(\Omega)}dt \\
&\le c_{29}\delta\|\rho_0\|_{W_2^{l+1}(\Omega)}\int_0^T \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{l+1}(\Omega)}dt\,.
\end{aligned}$$

Now, repeating the arguments in (5.7), (5.8) we obtain

$$\begin{aligned}
&\left(\int_0^t \|\nabla\bar\rho(\cdot,t)-\nabla\bar\rho(\cdot,t-\tau)\|^2_{L_2(\Omega)}\frac{d\tau}{\tau^{1+l}}\right)^{1/2}\le c_{30}\|\rho_0\|_{W_2^{l+1}(\Omega)} \\
&\times\Big\{\delta\int_0^t |(\mathcal{A}'-\mathcal{A})\nabla\cdot\mathbf{u}'+\mathcal{A}\nabla\cdot(\mathbf{u}-\mathbf{u}')|_\Omega d\tau \\
&+\Big[\int_0^t \frac{d\tau}{\tau^{1+l}}\Big(\int_{t-\tau}^t \|(\mathcal{A}'-\mathcal{A})\cdot\nabla\mathbf{u}'+\mathcal{A}\nabla\cdot(\mathbf{u}-\mathbf{u}')\|_{W_2^1(\Omega)}d\tau'\Big)^2\Big]^{1/2}\Big\} \\
&\le c_{31}\|\rho_0\|_{W_2^{l+1}(\Omega)}\Big\{\int_0^t \|\mathbf{u}-\mathbf{u}'\|_{W_2^{l+2}(\Omega)}d\tau \\
&+\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^1(\Omega)}\frac{d\tau}{(t-\tau)^{l/2}}\Big\},
\end{aligned}$$

$$\left(\int_0^t \|\tilde\rho(\cdot,t)-\tilde\rho(\cdot,t-\tau)\|^2_{L_2(\Gamma)}\frac{d\tau}{\tau^{3/2+l}}\right)^{1/2}$$
$$\le c_{32}\|\rho_0\|_{W_2^{l+1}(\Omega)}\left\{\int_0^t \|\mathbf{u}-\mathbf{u}'\|_{W_2^{l+2}(\Omega)}d\tau\left[1+\int_0^t(\|D\mathbf{u}\|_{L_2(\Gamma)}\right.\right.$$
$$\left.\left.+\|D\mathbf{u}'\|_{L_2(\Gamma)})\frac{d\tau'}{(t-\tau')^{l/2+1/4}}\right]+\delta\int_0^t \|D(\mathbf{u}-\mathbf{u}'\|_{L_2(\Gamma)}\frac{d\tau}{(t-\tau)^{l/2+1/4}}\right\}.$$

consequently,

$$\left(\int_0^T dt\int_0^t \|\nabla\tilde\rho(\cdot,t)-\nabla\tilde\rho(\cdot,t-\tau)\|^2_{L_2(\Omega)}\frac{d\tau}{\tau^{1+l}}\right)^{1/2}+\left(\int_0^T\int_0^t \|\tilde\rho(\cdot,t)\right.$$
$$\left.-\tilde\rho(\cdot,t-\tau)\|^2_{L_2(\Gamma)}\frac{d\tau}{\tau^{3/2+l}}\right)^{1/2}$$
$$\le c_{33}(T)\|\rho_0\|_{W_2^{l+1}(\Omega)}\mathcal{N}_T[\mathbf{u}-\mathbf{u}']\left\{1+T^{5/4-l/2}(\sup_{t\le T}\|D\mathbf{u}(\cdot,t)\|_{L_2(\Gamma)}\right.$$
$$\left.+\sup_{t\in T}\|D\mathbf{u}'(\cdot,t)\|_{L_2(\Gamma)})\right\}$$
$$\le c_{33}(T)(1+2\delta T^{3/4-l/2})\|\rho_0\|_{W_2^{l+1}(\Omega)}\mathcal{N}_T[\mathbf{u}-\mathbf{u}']$$

which implies (5.15). Consider the difference

$$p(\rho)-p(\rho')=\int_0^1 p'(\rho_s)ds(p-\rho')\,,\ \ \rho_s=\rho'+s(\rho-\rho')\,.$$

Since $p'(\rho_s)$ satisfies (5.9) and (5.10), (5.16) follows immediately from this representation, and from (5.14), (5.15). The lemma is proved.

Lemma 5.4. If $\mathbf{f}$ and p_e satisfy the hypotheses of Theorem 1.3 and $\mathbf{u},\mathbf{u}'$ satisfy (2.12), then

$$\|\mathbf{f}(X_{\mathbf{u}},t)-\mathbf{f}(X_{\mathbf{u}'},t)\|^{(l,l/2)}_{Q_T}\le c_{34}(T)\int_0^T\|\mathbf{u}-\mathbf{u}'\|_{W_2^{l+1}(\Omega)}dt\,, \tag{5.18}$$

$$\|p_e(X_{\mathbf{u}},t)-p_e(X_{\mathbf{u}'},t)\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\le c_{35}(T)\mathcal{N}_T[\mathbf{u}-\mathbf{u}']\,. \tag{5.19}$$

The estimate (5.18) is proved in [3], Lemma 4.3. The proof of (5.19) is similar and is based on the representation formula

$$p_e(X_{\mathbf{u}},t)-p_e(X_{\mathbf{u}'},t)=\sum_{k=1}^3\int_0^1 p_{ex_k}(X_{\mathbf{u}_s},t)ds\int_0^t(u_k-u_k')d\tau$$

where $\mathbf{u}_s = \mathbf{u}' + s(\mathbf{u} - \mathbf{u}')$.

Lemma 5.5. For arbitrary $\varepsilon \in (0,T)$

$$\mathcal{N}_T[\mathbf{v}] \leq \varepsilon^{1/4}\|\mathbf{v}\|_{Q_T}^{(l+2,l/2+1)} + c_{35}(\varepsilon)\|\mathbf{v}\|_{Q_{T-\varepsilon}}^{(l+2,l/2+1)},$$
$$\|D\mathbf{v}\|_{G_T}^{(l-1/2,l/2-1/4)} \leq c_{37}\varepsilon^{1/4}\|\mathbf{v}\|_{Q_T}^{(l+2,l/2+1)} + c_{38}(\varepsilon)\|\mathbf{v}\|_{Q_T}^{(l+2,l/2+1)}.$$

Proof is given in [3], Lemma 4.3.

Now we proceed to prove Theorem 1.3. We make use of the formula

$$\Delta_{\hat{v}}(t)X_{\hat{v}} = \Delta_0\xi + \Delta_{\hat{v}}(t)\int_0^t \hat{\mathbf{v}}d\tau + \int_0^t \dot{\Delta}_{\hat{v}}(\tau)\xi d\tau$$

to write the boundary conditions (1.6) in the form

$$\begin{aligned}\mathbf{n}_0 \cdot T'_{\hat{v}}\mathbf{n}_0 - \sigma\mathbf{n}_0 \cdot \Delta_{\hat{v}}(t)\int_0^t \hat{\mathbf{v}}d\tau|_{\xi\in\Gamma} =& \sigma H_0(\xi) + \sigma\int_0^t \mathbf{n}_0 \cdot \dot{\Delta}(\tau)\xi d\tau|_{\xi\in\Gamma} \\ &+ (\mathbf{n}_0 \cdot \mathbf{n})[p(\rho J_{\hat{u}}^{-1}) - p_e(X_{\hat{u}},t)]|_{\xi\in\Gamma}\end{aligned}$$

where H_0 is the twice mean curvature of Γ.

We solve (1.6) by successive approximations taking $\mathbf{u}^{(0)} = 0$ and defining $\mathbf{u}^{(m+1)}, m \geq 0$, as a solution of a linear initial-boundary value problem

$$\begin{aligned}&\mathbf{u}_t^{(m+1)} - \rho_0^{-1}(\xi)\mathcal{A}_m\nabla T'_m(\mathbf{u}^{(m+1)}) = \mathbf{f}(X_m,t) - \kappa\nabla_m U_m(X_m,t) \\ &\qquad\qquad - \rho_0^{-1}(\xi)\mathcal{A}_m\nabla p(\rho_0 J_m^{-1}), \\ &\mathbf{u}^{(m+1)}|_{t=0} = \mathbf{v}_0(\xi), \\ &\mu\textstyle\prod_0\prod_m S_m(\mathbf{u}^{(m+1)})\mathbf{n}_m|_{\xi\in\Gamma} = 0, \\ &\mathbf{n}_0 \cdot T'_m(\mathbf{u}^{(m+1)})\mathbf{n}_0 - \sigma\mathbf{n}_0 \cdot \Delta_m(t)\int_0^t \mathbf{u}^{(m+1)}d\tau|_{\xi\in\Gamma} = \sigma H_0(\xi) \\ &+ \sigma\int_0^t \mathbf{n}_0 \cdot \dot{\Delta}_m(\tau)\xi d\tau + (\mathbf{n}_0 \cdot \mathbf{n}_m)[p(\rho_0 J_m^{-1}) - p_e(X_m,t)]|_{\xi\in\Gamma}.\end{aligned} \tag{5.20}$$

Here $\nabla_m = \nabla_{u^{(m)}}, X_m = X_{u^{(m)}}, J_m = J_{u^{(m)}}, \mathcal{A}_m$ is a matrix of algebraic adjuncts to the elements $a_{ij}^{(m)} = \delta_{ij} + \int_0^t \frac{\partial u_i^{(m)}}{\partial \xi_j}d\tau, S_m = S_{u^{(m)}}, T'_m = T'_{u^{(m)}}, \mathbf{n}_m$ is an exterior normal to the surface $\Gamma_m(t) = \{\mathbf{x} = X_m(\xi,t), \xi \in$

$\Gamma\}$ at the point $X_{u^{(m)}}$, $\prod_m \mathbf{w} = \mathbf{w} - \mathbf{n}_m(\mathbf{n}_m \cdot \mathbf{w})$, and $\Delta_m(t)$ is the Laplace-Beltrami operator on $\Gamma_m(t)$.

For $m = 0$ (5.20) reduces to

$$\begin{aligned}
&\mathbf{u}_t^{(1)} - \rho_0^{-1}(\xi)\nabla T'(\mathbf{u}^{(1)}) = \mathbf{f}(\xi, t) + \kappa\nabla U(\xi) - \rho_0^{-1}(\xi)\nabla p(\rho_0)\,,\\
&\mathbf{u}^{(1)}|_{t=0} = \mathbf{v}_0(\xi)\,,\\
&\mu \prod\nolimits_0 S(\mathbf{u}^{(1)})\mathbf{n}_0|_{\xi\in\Gamma} = 0\,,\\
&\mathbf{n}_0 \cdot T'(\mathbf{u}^{(1)})\mathbf{n}_0 - \sigma\mathbf{n}_0 \cdot \Delta_0 \int_0^t \mathbf{u}^{(1)}d\tau|_{\xi\in\Gamma} = \sigma H_0(\xi) + p(\rho_0) - p_e(\xi, t)|_{\xi\in\Gamma}
\end{aligned}$$

with $U(\xi) = \int_\Omega \frac{\rho_0(\eta)d\eta}{|\xi-\eta|}$. The compatibility conditions are satisfied, the solution $\mathbf{u}^{(1)}$ is defined for $t > 0$ and in virtue of (1.14), for any $T < \infty$

$$\begin{aligned}
\|\mathbf{u}^{(1)}\|_{Q_T}^{(l+2,l/2+1)} &\le c_{39}(T)\big(\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|\rho_0^{-1}\nabla p(\rho_0)\|_{W_2^l(\Omega)}\\
&+ \kappa\|\nabla U\|_{W_2^l(\Omega)} + \|\mathbf{v}_0\|_{W_2^{l+1}(\Omega)} + \sigma\|H_0\|_{W_2^{l+1/2}(\Gamma)}\\
&+ \|p(\rho_0)\|_{W_2^{l+1}(\Omega)} + \|p_e\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\big).
\end{aligned}\tag{5.21}$$

Suppose that $\mathbf{u}^{(j)}, j = 1, \dots, m$ are defined and satisfy (1.12) on the interval $(0, T_m)$. When we substract from each other the equalities (5.20) for neighbouring indices j and $j-1$, we obtain the following initial-boundary problem for $\mathbf{z}^{(j+1)} = \mathbf{u}^{(j+1)} - \mathbf{u}^{(j)}$:

$$\begin{aligned}
&\mathbf{z}_t^{(j+1)} - \rho_0^{-1}(\xi)A_j\nabla_j T_j'(\mathbf{z}^{(j+1)}) = \mathbf{l}_1^{(j)}(\mathbf{u}^{(j)}) - \mathbf{l}_1^{(j-1)}(\mathbf{u}^{(j)}) + \mathbf{f}(X_j, t)\\
&- \mathbf{f}(X_{j-1}, t) + \kappa\nabla_j[U_j(X_j, t) - U_{j-1}(X_{j-1}, t)]\\
&+ \kappa(\nabla_j - \nabla_{j-1})U_{j-1}(X_{j-1}, t) - p_0^{-1}(A_j - A_{j-1})\nabla p(\rho_0 J_j^{-1})\\
&- \rho_0^{-1}A_{j-1}\nabla(p(\rho_0 J_j^{-1}) - p(\rho_0 J_{j-1}^{-1}))\,,\\
&\mathbf{z}^{(j+1)}|_{t=0} = 0\,,\\
&\mu \prod\nolimits_0 \prod\nolimits_j S_j(\mathbf{z}^{(j+1)})\mathbf{n}_j|_{\xi\in\Gamma} = \mathbf{l}_2^{(j)}(\mathbf{u}^{(j)}) - \mathbf{l}_2^{(j-1)}(\mathbf{u}^{(j)})\,,\\
&\mathbf{n}_0 \cdot T_j'(\mathbf{z}^{(j+1)})\mathbf{n}_j - \sigma\mathbf{n}_0 \cdot \Delta_j(t)\int_0^t \mathbf{z}^{(j+1)}d\tau|_{\xi\in\Gamma} = l_3^{(j)}(\mathbf{u}^{(j)})\\
&- l_3^{(j-1)}(\mathbf{u}^{(j)}) + \sigma\int_0^t [l_4^{(j)}(\mathbf{u}^{(j)}) - l_4^{(j-1)}(\mathbf{u}^{(j)})]d\tau + \sigma\int_0^t \mathbf{n}_0 \cdot (\dot{\Delta}_j(\tau)\\
&- \dot{\Delta}_{j-1}(\tau))\xi d\tau + (\mathbf{n}_0 \cdot \mathbf{n}_j - \mathbf{n}_{j-1})[p(\rho_0 J_j^{-1}) - p_e(X_j, t)]\\
&+ (\mathbf{n}_0 \cdot \mathbf{n}_{j-1})[(p(\rho_0 J_j^{-1}) - p(\rho_0 J_{j-1}^{-1})) + (p_e(X_{j-1}, t) - p_e(X_j, t))]|_{\xi\in\Gamma}.
\end{aligned}\tag{5.22}$$

Here

$$\mathbf{l}_1^{(j)}(\mathbf{w}) = \rho_0^{-1}(\xi)[\mathcal{A}_j \cdot \nabla T_j'(\mathbf{w}) - \nabla T'(\mathbf{w})]$$
$$\mathbf{l}_2^{(j)}(\mathbf{w}) = \mu \prod\nolimits_0 \left(\prod\nolimits_0 S(\mathbf{w})\mathbf{n}_0 - \prod\nolimits_j S_j(\mathbf{w})\mathbf{n}_j \right),$$
$$\mathbf{l}_3^{(j)}(\mathbf{w}) = \mathbf{n}_0 \cdot (T'(\mathbf{w})\mathbf{n}_0 - T_j'(\mathbf{w})\mathbf{n}_j),$$
$$\mathbf{l}_4^{(j)}(\mathbf{w}) = \mathbf{n}_0 \cdot [(\Delta_j(t) - \Delta_0)\mathbf{w} + \dot{\Delta}_j(t) \int_0^t \mathbf{w}d\tau].$$

We evaluate $\mathbf{z}^{(j+1)}$ applying (1.14) and taking into account Lemmas 5.2–5.5 and the following estimates that are actually established in [3]

$$\|\mathbf{l}_1^{(j)}(\mathbf{u}^{(j)}) - \mathbf{l}_1^{(j-1)}(\mathbf{u}^{(j)})\|_{Q_T}^{(l,l/2)} + \|\mathbf{l}_2^{(j)}(\mathbf{u}^{(j)}) - \mathbf{l}_2^{(j-1)})(\mathbf{u}^{(j)})\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$$
$$+ \|l_3^{(j)}(\mathbf{u}^{(j)}) - l_3^{(j-1)}(\mathbf{u}^{(j)})\|_{W_2^{l+1/2,l/2+1/4}(G_T)}$$
$$+ \|l_4^{(j)}(\mathbf{u}^{(j)}) - l_4^{(j-1)}(\mathbf{u}^{(j)})\|_{G_T}^{(l-1/2,l/2-1/4)}$$
$$\leq c_{40}(T)\delta\|\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)}\|_{Q_T}^{(l+2,l/2+1)},$$
$$\|\nabla_j(U_j(X_j,t) - U_{j-1}(X_{j-1},t)\|_{Q_T}^{(l,l/2)} + \|(\nabla_j - \nabla_{j-1})U_{j-1}(X_{j-1},t)\|_{Q_T}^{(l,l/2)}$$
$$\leq c_{41}\mathcal{N}_T[\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)}],$$
$$\|\mathbf{n}_0 \cdot (\dot{\Delta}_{j-1}(t) - \dot{\Delta}_j(t))\xi\|_{G_T}^{(l-1/2,l/2-1/4)} \leq c_{42}(T)\|D(\mathbf{u}^{(j)}$$
$$- \mathbf{u}^{(j-1)})\|_{G_T}^{(l-1/2,l/2-1/4)}, T \leq T_m.$$

As a result, we arrive at the inequality

$$\|\mathbf{z}^{(j+1)}\|_{Q_T}^{(l+2,l/2+1)} \leq (c_{43}\delta + c_{44}\varepsilon^{1/4})\|\mathbf{z}^{(j)}\|_{Q_T}^{(l+2,l/2+1)} + c_{45}(\varepsilon)\|\mathbf{z}^{(j)}\|_{Q_{T-\varepsilon}}^{(l+2,l/2+1)} \tag{5.23}$$

which holds for $T \leq T_m, \varepsilon \in (0,1)$. If we choose δ and ε in such a way that $c_{43}\delta + c_{44}\varepsilon^{1/4} \leq 1/2$, we obtain the following estimates for $\sum_{m+1}(T) = \sum_{j=1}^{m+1} \|\mathbf{z}^{(j)}\|_{Q_T}^{(l+2,l/2+1)}$

$$\sum\nolimits_{m+1}(T) \leq 2\sum\nolimits_1(T) + 2C_{45}\sum\nolimits_{m+1}(T-\varepsilon)$$

and as a consequence

$$\sum\nolimits_{m+1}(T) \leq c_{46}\sum\nolimits_1(T) \equiv c_{46}(T)\|\mathbf{u}^{(1)}\|_{Q_T}^{(l+2,l/2+1)} \tag{5.24}$$

with a non-decreasing (exponential) function $c_{46}(T)$. Now,

$$\|\mathbf{u}^{(m+1)}\|_{Q_T}^{(l+2,l/2+1)} \le \sum\nolimits_{m+1}(T) + \|\mathbf{u}^{(1)}\|_{Q_T}^{(l+2,l/2+1)} \le (1+c_{46}(T))c_{39}(T)\phi(T)$$

where $\phi(T)$ is the sum of norms in (5.21).

The condition (1.12) for $\mathbf{u}^{(m+1)}$ is satisfied, provided that

$$T^{1/2}(1+c_{46}(T))c_{39}(T)\phi(T) \le \delta\,. \tag{5.25}$$

This holds for $T \le T'$; consequently, $\|\mathbf{u}^{(m)}\|_{Q_{T'}}^{(l+2,l/2+1)}$ are uniformly bounded, (5.24) is satisfied when $T = T'$ and $\{\mathbf{u}^{(m)}\}$ converges in $W_2^{2+l,1+l/2}(Q_T)$ to the solution of problem (1.6).

This solution is unique, since the difference $\mathbf{z} = \hat{\mathbf{v}} - \hat{\mathbf{v}}'$ of two solutions is a solution of a linear problem of the type (5.22) (with $\mathbf{u}^{(j+1)}, \mathbf{u}^{(j)}$ replaced by $\hat{\mathbf{v}}$ and $\hat{\mathbf{v}}'$) for which the analogue of (5.23) holds true:

$$\|\mathbf{z}\|_{Q_T}^{(l+2,l/2+1)} \le (c_{43}\delta + c_{44}\varepsilon^{1/4})\|\mathbf{z}\|_{Q_T}^{(l+2,l/2+1)} + c_{45}(\varepsilon)\|\mathbf{z}\|_{Q_{T-\varepsilon}}^{(l+2,l/2+1)}$$

This implies $\mathbf{z} = 0$, and Theorem 1.1 is proved.

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V. A. Solonnikov
Leningrad Branch of
V. A. Steklov Math. Institute
Leningrad, Fontanka 27
USSR 191011

A. Tani
Keyo University
Faculty of Science and Technology
14-1 Hiyoshi 3 Chome
Kohokuku Jokohama
223 Japan

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SOME LAURICELLA MULTIPLE HYPERGEOMETRIC SERIES ASSOCIATED WITH THE PRODUCT OF SEVERAL BESSEL FUNCTIONS

H.M. Srivastava

The present work is motivated essentially by some recent developments in the theory of the light changes of eclipsing variables in which frequent use is made of certain integrals and expansions associated with the product of two or more Bessel functions. Starting from some rather elementary expansions involving Bessel functions, it is shown how readily one can obtain much more general results involving, for example, Lauricella's multiple hypergeometric functions $F_A^{(r)}$ and $F_C^{(r)}$ of r variables. Further extensions of these and other similar results to hold true for the (Srivastava–Daoust) generalized Lauricella hypergeometric function are also considered. Finally, relevant connections of many of these general expansions with those available in the literature are pointed out, and a brief discussion of their basic (or q–) extensions is presented.

1. Introduction, Definitions, and Preliminaries

Certain classes of integrals and expansions associated with the product of two or more Bessel functions $J_\nu(z)$, where (*cf.*, *e.g.*, [35])

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \, (\tfrac{1}{2}z)^{\nu+2m}}{m! \; \Gamma(\nu+m+1)}, \tag{1.1}$$

are potentially useful in a wide variety of problems in several seemingly *diverse* fields of physical, astrophysical, and engineering sciences, and indeed also of statistics and operations research. For instance, in the theory of the light changes of eclipsing variables, the fractional loss of light, suffered by an eclipse of a circular disk of fractional radius r_1 (and darkened at the limb to the Nth degree) by an opaque disk of radius r_2 with their centres separated by a fractional (projected) distance δ, is represented by the associated alpha-function $\alpha_N^0(r_1,r_2,\delta)$ of order N, defined by (*cf.* [14]; see also [15, Sections I.3 and III.3])

$$\alpha_N^0(r_1,r_2,\delta) = 2^\nu \, \Gamma(\nu) \int_0^\infty (kt)^{-\nu} \, J_\nu(kt) J_1(t) J_0(ht) \; dt, \tag{1.2}$$

where, for convenience,

$$\nu = \frac{N+2}{2}, \quad h = \frac{\delta}{r_2}, \quad \text{and} \quad k = \frac{r_1}{r_2}. \tag{1.3}$$

More generally, if the transparency of the occulting disk increases with the angle of foreshortening in the same manner as the limb-darkening of the eclipsed star, that is, if the transparency function $g(\rho,\zeta)$ of the second aperture is given by [14, p. 232, Equation (3.36)]

$$g(\rho,\zeta) = g_\lambda(\rho,\zeta) = \begin{cases} [1-(\rho/r_1)^2]^\lambda & (\rho \leqq r_2), \\ 0 & (\rho > r_2), \end{cases} \tag{1.4}$$

in place of

$$g(\rho,\zeta) = g_0(\rho,\zeta) = \begin{cases} 1 & (\rho \leqq r_2), \\ 0 & (\rho > r_2), \end{cases} \tag{1.5}$$

then Equation (1.2) is to be replaced by [15, p. 34, Equation (3.38)]

$$\alpha^0_{N,\lambda}(r_1,r_2,\delta) = 2^{\nu+\lambda}\, \Gamma(\nu)\Gamma(\lambda+1)\, k^{-\nu} \cdot \int_0^\infty t^{-\nu-\lambda}\, J_\nu(kt) J_{\lambda+1}(t) J_0(ht)\, dt, \tag{1.6}$$

which, in view of Equation (1.5), reduces immediately to (1.2) when $\lambda = 0$; here ν, h, and k are given, as before, by (1.3).

In another situation of an entirely different nature, let $P_N(R;\, r_1,\cdots,r_N|\, p)$ denote the probability that the final distance of an object, after executing a generalized random walk in a space of p dimensions (with *unequal* stretches $r_1,\cdots,r_N$, say), is less than a distance R from its starting point, then it is easily found that (*cf.*, *e.g.*, [35, p. 421])

$$P_N(R;r_1,\cdots,r_N|\, p) = R\{\Gamma(q)\}^{N-1} \int_0^\infty (\tfrac{1}{2}Rt)^{q-1}\, J_q(Rt) \prod_{j=1}^{N} \left\{\frac{J_{q-1}(r_jt)}{(\tfrac{1}{2}r_jt)^{q-1}}\right\} dt, \tag{1.7}$$

where, for convenience, $q = \tfrac{1}{2}p$.

For a systematic investigation of each of the aforementioned situations, and many more in other fields, one finds a genuine need for generalizations of the widely useful (*discontinuous*) integral of Weber and Schafheitlin [35, p. 398 *et seq.*] which indeed provides different analytic expressions for the infinite integral:

$$\int_0^\infty t^{\rho-1}\, J_\mu(xt) J_\nu(yt)\, dt$$

according as x is smaller than, equal to, or larger than y. One such generalization, motivated especially by (1.7), is due to Srivastava and Exton [30] who gave the integral formula (see also [31, p. 50, Equation 1.7(12)]):

$$\int_0^\infty t^{\rho-1} \prod_{j=1}^{N} \{J_{\mu_j}(x_j t)\}\, dt$$

$$= \frac{2^{\rho-1}\, x_1^{\mu_1} \cdots x_{N-1}^{\mu_{N-1}}\, x_N^{\mu_N - M}\, \Gamma(\frac{1}{2}M)}{\Gamma(\mu_1+1)\cdots\Gamma(\mu_{N-1}+1)\ \Gamma(\mu_N - \frac{1}{2}M+1)}$$

$$\cdot\ F_C^{(N-1)}\left[\tfrac{1}{2}M,\ \tfrac{1}{2}M-\mu_N;\, \mu_1+1,\cdots,\mu_{N-1}+1;\ \frac{x_1^2}{x_N^2},\cdots,\ \frac{x_{N-1}^2}{x_N^2}\right], \tag{1.8}$$

where $x_1,\cdots,x_N$ are positive real numbers constrained by the inequality:

$$x_N > x_1 + \cdots + x_{N-1} \qquad (N = 2,3,4,\cdots), \tag{1.9}$$

$$M = \rho + \mu_1 + \cdots + \mu_N\,, \tag{1.10}$$

$$\mathrm{Re}(1+\mu_1 + \cdots + \mu_N) > \mathrm{Re}(1-\rho) - \tfrac{1}{2}N\,, \tag{1.11}$$

and the special rôle played by x_N can indeed be assumed by any of the remaining variables $x_1,\cdots,x_{N-1}$. Here, and in what follows, $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$ denote the Lauricella hypergeometric functions of r variables (*cf.* [17]; see also [31, p. 33]). For the sake of ready reference, we recall here the definition of each of these multivariable hypergeometric functions (together with the precise regions of convergence of the multiple series defining them) in terms of the Pochhammer symbol $(\lambda)_m$ given by

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad m = 0, \\ \lambda(\lambda+1)\cdots(\lambda+m-1), & \forall\, m \in \mathbb{N} = \{1,2,3,\cdots\}. \end{cases} \tag{1.12}$$

Thus we have

$$F_A^{(r)}\left[\alpha,\ \beta_1,\cdots,\beta_r;\ \gamma_1,\cdots,\gamma_r;\ z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_r}(\beta_1)_{m_1}\cdots(\beta_r)_{m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_r)_{m_r}} \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!} \tag{1.13}$$

$$(|z_1| + \cdots + |z_r| < 1),$$

$$F_B^{(r)}\left[\alpha_1,\cdots,\alpha_r,\beta_1,\cdots,\beta_r;\gamma;z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_r)_{m_r}\ (\beta_1)_{m_1}\cdots(\beta_r)_{m_r}}{(\gamma)_{m_1+\cdots+m_r}} \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!} \tag{1.14}$$

$$(\max\{|z_1|,\cdots,|z_r|\} < 1),$$

$$F_C^{(r)}\left[\alpha,\ \beta;\gamma_1,\cdots,\gamma_r;z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_r}(\beta)_{m_1+\cdots+m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_r)_{m_r}} \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!} \tag{1.15}$$

$$(\sqrt{|z_1|} + \cdots + \sqrt{|z_r|} < 1),$$

and

$$F_D^{(r)}\left[\alpha,\beta_1,\cdots,\beta_r;\gamma;z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_r}\,(\beta_1)_{m_1}\cdots(\beta_r)_{m_r}}{(\gamma)_{m_1+\cdots+m_r}} \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!} \tag{1.16}$$

$$(\max\{|z_1|,\cdots,|z_r|\} < 1).$$

For $r = 2$, these functions are precisely the two-variable hypergeometric functions of Appell (*cf.* [2]; see also [3, p. 14]) who denoted them by F_2, F_3, F_4, and F_1, respectively. More importantly, when $r = 1$ (or, alternatively, when only one variable is nonzero), each of these functions reduces at once to the familiar hypergeometric function:

$$F(\alpha,\ \beta;\gamma;z) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}z + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2 + \cdots \tag{1.17}$$

$$(z \in \mathcal{U} = \{z : |z| < 1\};\ \gamma \neq 0,-1,-2,\cdots),$$

which corresponds to a special case

$$r - 1 = s = 1$$

of the generalized hypergeometric series defined by

$${}_rF_s(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_s;z)$$

$$= {}_rF_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r; \\ \\ \beta_1,\cdots,\beta_s;\end{matrix}\ z\right]$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_r)_m}{(\beta_1)_m \cdots (\beta_s)_m} \frac{z^m}{m!} \tag{1.18}$$

$$(r \leqq s+1;\ r < s+1 \quad \text{and} \quad |z| < \infty;\quad r = s+1 \quad \text{and} \quad z \in \mathcal{U};$$

$$r = s+1,\quad z \in \partial\mathcal{U},\quad \text{and} \quad \mathrm{Re}(\omega) > 0),$$

where

$$\omega = \sum_{j=1}^{s} \beta_j - \sum_{j=1}^{r} \alpha_j , \tag{1.19}$$

provided, of course, that no zeros appear in the denominator of (1.18). It should be remarked in passing that, for the celebrated hypergeometric differential equation:

$$z(1-z) \frac{d^2 w}{dz^2} + \{\gamma - (\alpha+\beta+1)z\} \frac{dw}{dz} - \alpha\beta w = 0, \tag{1.20}$$

the study of which goes back to Leonhard Euler (1707–1783), Carl Friedrich Gauss (1777–1855), and Ernst Eduard Kummer (1810–1893), the function

$$F(\alpha, \beta; \gamma; z)$$

or, more precisely,

$${}_2F_1(\alpha, \beta; \gamma; z)$$

is the only solution that is regular at the point $z = 0$ and assumes the value 1 at this point (*cf.* [5, p. 138]).

The recent works of Kopal [16], and Srivastava and Kopal [32], were motivated by the continuing importance of the associated alpha–functions

$$\alpha_N^0(r_1, r_2, \delta) \quad \text{and} \quad \alpha_{N,\lambda}^0(r_1, r_2, \delta)$$

in, for example, an interpretation of the observed light changes of eclipsing variables. In the systematic presentation of a number of new (and computable) expressions for these associated alpha–functions, and also for their various partial derivatives, these earlier works made use of certain special cases of

(i) the Srivastava–Exton formula (1.8) (in addition, of course, to the case $N = 2$ given by the aforementioned Weber–Schafheitlin integral), and

(ii) The Bessel-function expansion:

$$(\tfrac{1}{2}z)^{\lambda-\mu_1-\cdots-\mu_r} \prod_{j=1}^{r} \{J_{\mu_j}(x_j z)\}$$

$$= \frac{x_1^{\mu_1} \cdots x_r^{\mu_r}}{\Gamma(\mu_1+1)\cdots\Gamma(\mu_r+1)} \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(z)$$

$$\cdot F_C^{(r)}\left[-n,\ \lambda+n;\ \mu_1+1,\cdots,\mu_r+1;\ x_1^2,\cdots,x_r^2\right], \tag{1.21}$$

which was given by Srivastava [21, p. 150, Equation (5.1)] who also showed similarly that [21, p. 150, Equation (5.2)]

$$(\tfrac{1}{2}z)^{\lambda-\mu_1-\cdots-\mu_r} \prod_{j=1}^{r} \{J_{\mu_j}(x_j z)\}$$

$$= \frac{x_1^{\mu_1} \cdots x_r^{\mu_r}\, \Gamma(\lambda+1)}{\Gamma(\mu_1+1) \cdots \Gamma(\mu_r+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} J_{\lambda+n}(z)$$

$$\cdot F_C^{(r)}\left[-n,\ \lambda+1;\ \mu_1+1,\cdots,\mu_r+1;\ x_1^2,\cdots,x_r^2\right]. \tag{1.22}$$

The object of the present paper is mainly to demonstrate how readily we can develop much more general expansion formulas than (1.21) and (1.22) by means of some simple techniques based (for example) upon the integral representation:

$$F_A^{(r)}\left[\alpha,\beta_1,\cdots,\beta_r;\gamma_1,\cdots,\gamma_r;z_1,\cdots,z_r\right]$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}\,t^{\alpha-1}\;{}_1F_1\left[\begin{matrix}\beta_1;\\ \\ \gamma_1;\end{matrix}\;z_1t\right]\cdots{}_1F_1\left[\begin{matrix}\beta_r;\\ \\ \gamma_r;\end{matrix}\;z_rt\right]dt \tag{1.23}$$

$$(\text{Re}(z_1+\cdots+z_r) < 1;\quad \text{Re}(\alpha) > 0),$$

which is due essentially to Erdélyi [6, p. 696, Equation (1)], and the integral representation [29, p. 40, Equation (10)]:

$$F_C^{(r)}\left[\tfrac{1}{2}\alpha,\;\tfrac{1}{2}\alpha+\tfrac{1}{2};\;\gamma_1,\cdots,\gamma_r;z_1^2,\cdots,z_r^2\right]$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}\,t^{\alpha-1}\;{}_0F_1\left[\begin{matrix}-;\\ \\ \gamma_1;\end{matrix}\;\tfrac{1}{4}z_1^2\,t\right]\cdots{}_0F_1\left[\begin{matrix}-;\\ \\ \gamma_r;\end{matrix}\;\tfrac{1}{4}z_r^2\,t\right]dt \tag{1.24}$$

$$(|\text{Re}(z_1)| + \cdots + |\text{Re}(z_r)| < 1;\quad \text{Re}(\alpha) > 0),$$

which (in a slightly modified form) was applied by Srivastava and Exton [30, p. 2] in order to prove their general result (1.8).

2. Polynomial Expansions of Lauricella Functions

We begin by rewriting the definition (1.1) in its *equivalent* forms:

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left[\begin{matrix} \text{---}; \\ \nu+1; \end{matrix}\ -\tfrac{1}{4}z^2\right]$$

$$= \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\ e^{\pm iz}\ {}_1F_1\left[\begin{matrix} \nu+\frac{1}{2}\,; \\ 2\nu+1\,; \end{matrix}\ \mp 2iz\right], \tag{2.1}$$

where the third member follows from the second by appealing to Kummer's formula [19, p. 126, Theorem 43]:

$${}_0F_1\left[\begin{matrix} \text{--}; \\ \lambda; \end{matrix}\ \tfrac{1}{4}z^2\right] = e^{-z}\ {}_1F_1\left[\begin{matrix} \lambda-\frac{1}{2}; \\ 2\lambda-1; \end{matrix}\ 2z\right]. \tag{2.2}$$

In view of (2.1), each of the integrals in (1.23) and (1.24) can be restated as an integral involving the product of r Bessel functions of different arguments. Furthermore, if we make use of (2.2) in the integrand of (1.24), and evaluate the resulting integral by appealing to Erdélyi's result (1.23), we shall arrive at the following transformation formula relating the Lauricella functions $F_A^{(r)}$ and $F_C^{(r)}$ (*cf.* [29, p. 39]):

$$F_C^{(r)}\left[\tfrac{1}{2}\alpha,\ \tfrac{1}{2}\alpha+\tfrac{1}{2};\ \gamma_1,\cdots,\gamma_r;\ z_1^2,\cdots,z_r^2\right]$$

$$= (1+z_1+\cdots+z_r)^{-\alpha}\ F_A^{(r)}\Big[\alpha,\ \gamma_1-\tfrac{1}{2},\cdots,\gamma_r-\tfrac{1}{2};$$

$$2\gamma_1-1,\cdots,2\gamma_r-1;\ \frac{2z_1}{1+z_1+\cdots+z_r},\ \cdots,\ \frac{2z_r}{1+z_1+\cdots+z_r}\Big] \tag{2.3}$$

$$\left[\sum_{j=1}^{r} |z_j/(1+z_1+\cdots+z_r)| < \tfrac{1}{2}\right]$$

or, equivalently,

$$F_A^{(r)}\left[\alpha,\ \gamma_1-\tfrac{1}{2},\cdots,\gamma_r-\tfrac{1}{2};\ 2\gamma_1-1,\cdots,2\gamma_r-1;\ 2z_1,\cdots,2z_r\right]$$

$$= (1-z_1-\cdots-z_r)^{-\alpha}\quad F_C^{(r)}\left[\tfrac{1}{2}\alpha,\ \tfrac{1}{2}\alpha+\tfrac{1}{2};\ \gamma_1,\cdots,\gamma_r;\ \frac{z_1^2}{(1-z_1-\cdots-z_r)^2},\ \cdots,\ \frac{z_r^2}{(1-z_1-\cdots-z_r)^2}\right] \tag{2.4}$$

$$(|z_1| + \cdots + |z_r| < \tfrac{1}{2}).$$

Now we turn to our expansion formula (1.21). Replacing z by zt in it, multiplying each side by

$$t^{\mu+\nu-\lambda-1}\ J_{\mu-\nu}(t)\ dt\ ,$$

and integrating over the semi-infinite interval $(0,\infty)$, if we apply the Srivastava–Exton formula (1.8) with $N = r+1$ and $N = 2$, we shall obtain the following result expressing the Lauricella function $F_C^{(r)}$ in series of multivariable polynomials associated with $F_C^{(r)}$ itself:

$$F_C^{(r)}\left[\mu,\ \nu;\ \mu_1+1,\cdots,\mu_r+1;\ x_1 z,\cdots,x_r z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(\lambda+n)_n}\ \frac{(-z)^n}{n!}\ {}_2F_1\left[\begin{matrix}\mu+n,\ \nu+n;\\ \\ \lambda+2n+1;\end{matrix}\ z\right]$$

$$\cdot\ F_C^{(r)}\left[-n,\ \lambda+n;\ \mu_1+1,\cdots,\mu_r+1;\ x_1,\cdots,x_r\right], \tag{2.5}$$

where the arguments have been adjusted conveniently, and the parametric constraints can be waived by appealing to the principle of analytic continuation, provided that each side of (2.5) exists.

In a similar manner, the expansion formula (1.22) yields

$$F_C^{(r)}\left[\mu,\ \nu;\ \mu_1+1,\cdots,\mu_r+1;\ x_1 z,\cdots,x_r z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(\lambda+1)_n} \frac{(-z)^n}{n!}\ {}_2F_1\left[\begin{matrix}\mu+n,\ \nu+n; \\ \\ \lambda+n+1;\end{matrix}\ z\right]$$

$$\cdot\ F_C^{(r)}\left[-n,\ \lambda+1;\ \mu_1+1,\cdots,\mu_r+1;\ x_1,\cdots,x_r\right], \qquad (2.6)$$

provided (as before) that each side exists.

If, in the expansion formula (1.21) with z replaced by izt, we make use of the ${}_1F_1$ representation (2.1) for each of the Bessel functions, multiply both sides by

$$e^{-t}\, t^{\mu-\lambda-1}\, dt\ ,$$

and integrate over the semi-infinite interval $(0,\infty)$ by appealing to the integral (1.23), we shall obtain

$$F_A^{(r)}\left[\mu,\ \mu_1+\tfrac{1}{2},\cdots,\mu_r+\tfrac{1}{2};\ 2\mu_1+1,\cdots,2\mu_r+1;\right.$$

$$\left.\frac{2x_1 z}{1+(x_1+\cdots+x_r)z},\cdots,\frac{2x_r z}{1+(x_1+\cdots+x_r)z}\right]$$

$$= \left\{\frac{1+z}{1+(x_1+\cdots+x_r)z}\right\}^{-\mu} \sum_{n=0}^{\infty} \frac{(\tfrac{1}{2}\mu)_n(\tfrac{1}{2}\mu+\tfrac{1}{2})_n}{(\lambda+n)_n} \frac{\{-z^2/(1+z)^2\}^n}{n!}$$

$$\cdot\ {}_2F_1\left[\begin{matrix}\mu+2n,\ \lambda+2n+\frac{1}{2};\\ 2\lambda+4n+1;\end{matrix}\ \frac{2z}{1+z}\right]$$

$$\cdot\ F_C^{(r)}\left[-n,\ \lambda+n;\ \mu_1+1,\cdots,\mu_r+1;\ x_1^2,\cdots,x_r^2\right], \tag{2.7}$$

provided that each member exists.

In a similar way, we find from the expansion formula (1.22) that

$$F_A^{(r)}\left[\mu,\ \mu_1+\tfrac{1}{2},\cdots,\mu_r+\tfrac{1}{2};\ 2\mu_1+1,\cdots,2\mu_r+1;\ \frac{2x_1z}{1+(x_1+\cdots+x_r)z},\cdots,\frac{2x_rz}{1+(x_1+\cdots+x_r)z}\right]$$

$$=\left\{\frac{1+z}{1+(x_1+\cdots+x_r)z}\right\}^{-\mu}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}\mu)_n(\frac{1}{2}\mu+\frac{1}{2})_n}{(\lambda+n)_n}\frac{\{-z^2/(1+z)^2\}^n}{n!}$$

$$\cdot\ {}_2F_1\left[\begin{matrix}\mu+2n,\ \lambda+n+\frac{1}{2};\\ 2\lambda+2n+1;\end{matrix}\ \frac{2z}{1+z}\right]$$

$$\cdot\ F_C^{(r)}\left[-n,\ \lambda+1;\ \mu_1+1,\cdots,\mu_r+1;\ x_1^2,\cdots,x_r^2\right], \tag{2.8}$$

provided that both sides exist.

Formulas (2.7) and (2.8) express the Lauricella function $F_A^{(r)}$ in series of Lauricella polynomials $F_C^{(r)}$. It should be noticed, however, that these expansion formulas can be deduced *directly* from (2.5) and (2.6), respectively, by making use of the transformation (2.3) or (2.4) in conjunction with the quadratic transformation [7, p. 111, Equation 2.11(4)]:

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \\ 2\beta;\end{matrix}\; 2z\right] = (1-z)^{-\alpha}\; {}_2F_1\left[\begin{matrix}\tfrac{1}{2}\alpha,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}; \\ \\ \beta+\tfrac{1}{2};\end{matrix}\; \left[\frac{z}{1-z}\right]^2\right] \tag{2.9} $$

for the Gaussian hypergeometric function.

3. Further Generalizations and Basic (or q–) Extensions

A closer look at the expansion formulas (2.5) and (2.6), and at their consequences (2.7) and (2.8), would suggest the existence of much more general results involving, for example, the (Srivastava–Daoust) generalized Lauricella hypergeometric function of r variables, defined by (*cf.* [27, p. 454] and [31, p. 37])

$$ F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} $$

$$ \equiv F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\left[\begin{matrix}[(a):\ \theta',\cdots,\theta^{(r)}]: \\ \\ [(c):\ \psi',\cdots,\psi^{(r)}]:\end{matrix}\right. $$

$$ \left.\begin{matrix}[(b'):\ \varphi'];\cdots;[(b^{(r)}):\ \varphi^{(r)}]; \\ \\ [(d'):\ \delta'];\cdots;[(d^{(r)}):\ \delta^{(r)}];\end{matrix}\; z_1,\cdots,z_r\right] $$

$$ = \sum_{m_1,\cdots,m_r=0}^{\infty} \Xi(m_1,\cdots,m_r)\ \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!}, \tag{3.1} $$

where, for convenience,

$$\Xi(m_1,\cdots,m_r) = \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta'_j+\cdots+m_r\theta_j^{(r)}}}{\prod_{j=1}^{C} (c_j)_{m_1\psi'_j+\cdots+m_r\psi_j^{(r)}}}$$

$$\cdot \frac{\prod_{j=1}^{B'} (b'_j)_{m_1\varphi'_j} \cdots \prod_{j=1}^{B^{(r)}} (b_j^{(r)})_{m_r\varphi_j^{(r)}}}{\prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r\delta_j^{(r)}}}, \tag{3.2}$$

the coefficients

$$\begin{cases} \theta_j^{(k)}\ (j=1,\cdots,A),\ \varphi_j^{(k)}\ (j=1,\cdots,B^{(k)}),\ \psi_j^{(k)}\ (j=1,\cdots,C), \\ \delta_j^{(k)}\ (j=1,\cdots,D^{(k)});\ \forall\ k\in\{1,\cdots,r\} \end{cases} \tag{3.3}$$

are real and non-negative, and (a) abbreviates the array of A parameters

$$a_1,\cdots,a_A\ ,$$

$(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}\quad (j = 1,\cdots,B^{(k)};\quad \forall\ k \in \{1,\cdots,r\}),$$

with similar interpretations for

$$(c)\ \text{ and }\ (d^{(k)})\quad (k \in \{1,\cdots,r\}),$$

et cetera.

The case $r = 2$ of the multiple hypergeometric series (3.1) was introduced and studied earlier by Srivastava and Daoust [26]. For the precise conditions under which the multiple series (3.1) and its special case when $r = 2$ converge absolutely, see Srivastava and Daoust [28]; see also Exton ([9, Section 3.7] and [10, Section 1.4]). In particular, when each of the real numbers listed in (3.3) is equated to 1, the generalized Lauricella function (3.1) reduces to a direct multivariable extension of the Kampé de Fériet function (*cf.*[13], see also [3, p. 150] and [31, p. 27]). We shall denote this *special* multivariable hypergeometric function simply by (*cf.* [31, p. 38])

$$F_{C:D';\cdots;D^{(r)}}^{A:B';\cdots;B^{(r)}}\left[\begin{matrix}(a):\ (b');\cdots;(b^{(r)}); \\ \\ (c):\ (d');\cdots;(d^{(r)});\end{matrix}\quad z_1,\cdots,z_r\right]. \tag{3.4}$$

Our derivations here of the aforementioned generalizations of the expansion formulas (2.5) to (2.8) involving the general multivariable hypergeometric functions (3.1) and (3.4), would employ multidimensional mathematical induction together with some elementary operational techniques which are based upon the classical Laplace transformation:

$$\mathscr{L}\{f(t):\ p\} = \int_0^\infty e^{-pt} f(t)\ dt = F(p), \tag{3.5}$$

the inverse Laplace transformation:

$$\mathscr{L}^{-1}\{F(p):t\} = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{pt}\,F(p)\ dp = f(t), \tag{3.6}$$

and the Riemann–Liouville fractional derivative operator D_z^μ defined by (*cf.* [8, Vol. II, Chapter 13]; see also [34])

$$D_z^{\mu}\{f(z)\} = \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-\zeta)^{-\mu-1} f(\zeta)\, d\zeta & (\mathrm{Re}(\mu) < 0), \\ \dfrac{d^m}{dz^m} D_z^{\mu-m}\{f(z)\} & (m-1 \leqq \mathrm{Re}(\mu) < m;\; m \in \mathbb{N}). \end{cases} \tag{3.7}$$

In what follows we shall find the need for a number of operational formulas involving the linear operators $\mathscr{L}$, $\mathscr{L}^{-1}$, and D_z^{μ}. Operational images (or operational representations) of many classes of special functions in the Laplace transformation (3.5) can be found from the Eulerian integral [*cf.* Equations (1.23) and (1.24)]:

$$\int_0^{\infty} e^{-pt}\, t^{\lambda-1}\, dt = \frac{\Gamma(\lambda)}{p^{\lambda}} \tag{3.8}$$

$$(\min\{\mathrm{Re}(\lambda),\ \mathrm{Re}(p)\} > 0).$$

On the other hand, computation of the inverse Laplace transformation (3.6) is facilitated largely by Hankel's contour integral in the *equivalent* form (see, *e.g.*, [36, p. 245, Example 1] and [18, p. 17, Equation 2.7(5)]):

$$\frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} e^{p}\, p^{-z}\, dp = \frac{1}{\Gamma(z)} \tag{3.9}$$

$$(\tau > 0;\ \ \mathrm{Re}(z) > 0).$$

Making use of the Γ–function formulas (3.8) and (3.9), we can easily find from the definition (1.18) that (*cf.* [8, Vol. I, p. 219, Equation 4.23(17)])

$$\mathscr{L}\left\{ t^{\lambda-1}\ {}_rF_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r; \\ \\ \beta_1,\cdots,\beta_s;\end{matrix}\ zt\right] : p\right\}$$

$$= \frac{\Gamma(\lambda)}{p^{\lambda}} \; {}_{r+1}F_s\left[\begin{matrix} \lambda, \; \alpha_1, \cdots, \alpha_r ; \\ \\ \beta_1, \cdots, \beta_s ; \end{matrix} \; \frac{z}{p}\right] \tag{3.10}$$

(Re(λ) > 0; Re(p) > 0 if $r < s$; Re(p) > Re(z) if $r = s$)

and [*op. cit.*, p. 297, Equation 5.21(1)]

$$\mathscr{L}^{-1}\left\{p^{-\mu} \; {}_rF_s\left[\begin{matrix} \alpha_1, \cdots, \alpha_r ; \\ \\ \beta_1, \cdots, \beta_s ; \end{matrix} \; \frac{z}{p}\right] : t\right\}$$

$$= \frac{t^{\mu-1}}{\Gamma(\mu)} \; {}_rF_{s+1}\left[\begin{matrix} \alpha_1, \cdots, \alpha_r ; \\ \\ \mu, \; \beta_1, \cdots, \beta_s ; \end{matrix} \; zt\right] \tag{3.11}$$

(Re(μ) > 0; $r \leqq s + 1$),

which incidentally follows also from (3.10) in view of (3.6).

In the case of the fractional derivative operator D_z^{μ} defined by (3.7), it is known that

$$D_z^{\mu}\{z^{\lambda-1}\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} \; z^{\lambda-\mu-1} \tag{3.12}$$

(Re(λ) > 0),

which immediately yields the operational formula:

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}\ {}_rF_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;\\ \\ \beta_1,\cdots,\beta_s;\end{matrix}\ z\right]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}\ z^{\mu-1}\ {}_{r+1}F_{s+1}\left[\begin{matrix}\lambda,\ \alpha_1,\cdots,\alpha_r;\\ \\ \mu,\ \beta_1,\cdots,\beta_s;\end{matrix}\ z\right] \qquad (3.13)$$

$(\mathrm{Re}(\lambda) > 0;\quad |z| < \infty \text{ when } r \leqq s;\ z \in \mathcal{U} \text{ when } r = s+1).$

It is not difficult to extend each of the operational formulas (3.10), (3.11), and (3.13) to hold true for such classes of generalized multivariable hypergeometric functions as those defined by (3.1). Thus, following Srivastava and Manocha [33, p. 289, Theorem 2], if we let

$$\theta(z_1,\cdots,z_r) = \sum_{m_1,\cdots,m_r=0}^{\infty} \Lambda(m_1,\cdots,m_r)\ z_1^{m_1}\cdots z_r^{m_r} \qquad (3.14)$$

for a suitably bounded multiple sequence

$$\{\Lambda(m_1,\cdots,m_r)\}\ \ (m_j \in \mathbb{N}_0;\ \ j = 1,\cdots,r),$$

then

$$\mathscr{L}\left\{t^{\lambda-1}\ \theta(z_1 t^{\rho_1},\cdots,z_r t^{\rho_r}) : p\right\}$$

$$= \frac{\Gamma(\lambda)}{p^{\lambda}} \sum_{m_1,\cdots,m_r=0}^{\infty} \Lambda(m_1,\cdots,m_r)$$

$$\cdot\ (\lambda)_{m_1\rho_1+\cdots+m_r\rho_r}\ z_1^{m_1}\cdots z_r^{m_r} \qquad (3.15)$$

$$\left[\min\{\mathrm{Re}(\lambda),\ \mathrm{Re}(p)\} > 0;\quad \rho_j > 0 \quad (j = 1,\cdots,r)\right]$$

$$\mathscr{L}^{-1}\left\{p^{-\mu}\,\theta(z_1 p^{-\sigma_1}, \cdots, z_r p^{-\sigma_r}) : t\right\}$$

$$= \frac{t^{\mu-1}}{\Gamma(\mu)} \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{\Lambda(m_1,\cdots,m_r)}{(\mu)_{m_1\sigma_1+\cdots+m_r\sigma_r}} z_1^{m_1} \cdots z_r^{m_r} \tag{3.16}$$

$$[\mathrm{Re}(\mu) > 0;\quad \sigma_j > 0 \quad (j = 1,\cdots,r)],$$

and

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}\,\theta(z_1 z^{\kappa_1},\cdots,z_r z^{\kappa_r})\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(\lambda)_{m_1\kappa_1+\cdots+m_r\kappa_r}}{(\mu)_{m_1\kappa_1+\cdots+m_r\kappa_r}}$$

$$\cdot\ \Lambda(m_1,\cdots,m_r)\,(z_1 z^{\kappa_1})^{m_1} \cdots (z_r z^{\kappa_r})^{m_r} \tag{3.17}$$

$$[\mathrm{Re}(\lambda) > 0;\quad \kappa_j > 0 \quad (j = 1,\cdots,r)],$$

provided that each of the operations in (3.15), (3.16), and (3.17) is validated by the absolute convergence of the integrals and series involved.

Employing the various notations and conventions surrounding the definitions (3.1) and (3.4), and making use of the linear operators $\mathscr{L}$, $\mathscr{L}^{-1}$, and D_z^{μ}, we shall now prove the multivariable hypergeometric expansion formula:

$$F^{A+E:B';\cdots;B^{(r)}}_{C+G:D';\cdots;D^{(r)}}\left[\begin{matrix}(a),\ (e):\ (b');\cdots;(b^{(r)});\\ \\ (c),\ (g):\ (d');\cdots;(d^{(r)});\end{matrix}\ x_1 z,\cdots,x_r z\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma_n[(e),(u);\ (g),(v)]}{(\lambda+n)_n} \frac{(-z)^n}{n!}$$

$$\cdot\ {}_{E+U}F_{G+V+1}\left[\begin{matrix}(e)+n,\ (u)+n;\\ \\ \lambda+2n+1,\ (g)+n,\ (v)+n;\end{matrix}\ z\right]$$

$$\cdot\ F^{A+V+2:B';\cdots;B^{(r)}}_{C+U:D';\cdots;D^{(r)}}\left[\begin{matrix}-n,\ \lambda+n,\ (a),(v):\\ \\ (c),(u):\end{matrix}\right.$$

$$\left.\begin{matrix}(b');\cdots;(b^{(r)});\\ \\ (d');\cdots;(d^{(r)});\end{matrix}\ x_1,\cdots,x_r\right], \qquad (3.18)$$

provided that

$$E + U < G + V + 2 \quad (\text{or} \quad E + U = G + V + 2 \quad \text{and} \quad z \in \mathcal{U}) \qquad (3.19)$$

and

$$1 + C + D^{(j)} - A - B^{(j)} \geqq E - G \quad (j = 1,\cdots,r), \qquad (3.20)$$

where the equality holds true when the variables z and $x_1,\cdots,x_r$ are constrained appropriately (*cf.* [28, p. 158]; see also [31, p. 38]), it being understood that exceptional parameter values which would render either side of (3.18) invalid or undefined are tacitly excluded. Here, and in what follows, we find it to be convenient to write

$$\Gamma_n[(e),(u);\ (g),(v)] = \frac{\prod_{j=1}^{E} (e_j)_n \prod_{j=1}^{U} (u_j)_n}{\prod_{j=1}^{G} (g_j)_n \prod_{j=1}^{V} (v_j)_n} \qquad (n \in \mathbb{N}_0). \tag{3.21}$$

Proof. We begin by rewriting the expansion formula (2.5) in the form:

$$F^{2:0;\cdots;0}_{0:1;\cdots;1}\left[\begin{matrix}\mu,\ \nu:\ \text{—};\cdots;\text{—}; \\ \text{——}:\ \mu_1;\cdots;\mu_r;\end{matrix}\ x_1 z,\cdots,x_r z\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma_n[\mu,\nu;\text{—}]}{(\lambda+n)_n} \frac{(-z)^n}{n!}\ {}_2F_1\left[\begin{matrix}\mu+n,\ \nu+n; \\ \lambda+2n+1;\end{matrix}\ z\right]$$

$$\cdot\ F^{2:0;\cdots;0}_{0:1;\cdots;1}\left[\begin{matrix}-n,\ \lambda+n:\ \text{—};\cdots;\text{—}; \\ \text{———}:\ \mu_1;\cdots;\mu_r;\end{matrix}\ x_1,\cdots,x_r\right] \tag{3.22}$$

$$\left[|z| < \min\left\{1,\ (\sqrt{|x_1|} + \cdots + \sqrt{|x_r|})^{-2}\right\}\right],$$

where we have replaced μ_j by $\mu_j - 1$ $(j = 1,\cdots,r)$.

Formula (3.22) corresponds to the general result (3.18) when

$$A = B^{(j)} = C = D^{(j)} - 1 = E - 2 = G = U = V = 0 \tag{3.23}$$

$$(j = 1,\cdots,r).$$

Thus, in order to prove the expansion formula (3.18) by appealing to the principle of multidimensional mathematical induction on the various non–negative integers involved in (3.23), we first replace x_j in (3.22) by $x_j t$ $(j = 1,\cdots,r)$, individually or collectively, multiply each side by $t^{\alpha-1}$, and operate upon both sides by $\mathscr{L}$ and $\mathscr{L}^{-1}$ (or, simply, by $D_t^{\alpha-\beta}$). Applying this procedure successively, and making use of such operational formulas as (3.15), (3.16), and (3.17), with

$$\rho_j = \sigma_j = 1 \qquad (j = 1,\cdots,r), \tag{3.24}$$

we shall thus be led eventually to an expansion like (3.18) with, of course,

$$E - 2 = G = U = V = 0. \tag{3.25}$$

Next we replace z in (3.22) by zt, multiply each side by $t^{\alpha-1}$, and operate upon both sides by $\mathscr{L}$ and $\mathscr{L}^{-1}$ (or, simply, by $D_t^{\alpha-\beta}$). Again, if we apply this procedure successively and make use of such operational formulas as (3.10), (3.11), and (3.13), we shall thus arrive eventually at an expansion formula which would yield the general result (3.18) upon trivially cancelling some of the numerator and denominator parameters.

In case we successively apply the above operational techniques *directly* to the general result (3.18), followed by an appropriate cancellation of some of the numerator and denominator parameters, each of the non–negative integers involved in (3.23) would increase by 1, and the proof of (3.18) by the principle of multidimensional mathematical induction would thus be completed.

In a similar manner, if we start from the expansion formula (2.6), we can obtain the result:

$$F^{A+E:B';\cdots;B^{(r)}}_{C+G:D';\cdots;D^{(r)}}\left[\begin{matrix}(a),(e):\ (b');\cdots;(b^{(r)});\\ \\ (c),(g):\ (d');\cdots;(d^{(r)});\end{matrix}\quad x_1 z,\cdots,x_r z\right]$$

$$= \sum_{n=0}^{\infty} \Gamma_n[(e),(u);\ (g),(v)]\ \frac{(-z)^n}{n!}$$

$$\cdot\ {}_{E+U}F_{G+V}\left[\begin{matrix}(e)+n,\ (u)+n;\\ \\ (g)+n,\ (v)+n;\end{matrix}\ z\right]$$

$$\cdot\ F^{A+V+1:B';\cdots;B^{(r)}}_{C+U:D';\cdots;D^{(r)}}\left[\begin{matrix}-n,\ (a),(v): & (b');\cdots;(b^{(r)}); \\ & & x_1,\cdots,x_r \\ (c),(u): & (d');\cdots;(d^{(r)});\end{matrix}\right], \qquad (3.26)$$

provided that

$$E + U < G + V + 1 \quad (\text{or } E + U = G + V + 1 \ \text{ and } \ z \in \mathcal{U}), \qquad (3.27)$$

and the constraints surrounding (3.20) are satisfied.

In view of the principle of confluence exhibited (for example) by

$$\lim_{\lambda \to \infty} \left\{(\lambda)_m \left[\frac{z}{\lambda}\right]^m\right\} = z^m = \lim_{\mu \to \infty} \left\{\frac{(\mu z)^m}{(\mu)_m}\right\} \qquad (3.28)$$

$$(|z| < \infty; \quad m \in \mathbb{N}_0),$$

it is not difficult to observe that the expansion formula (3.26) is, in fact, a limiting case of (3.18) when z is replaced by λz, and x_j by x_j/λ $(j = 1,\cdots,r)$, and $\lambda \to \infty$.

By applying the operational formulas (3.15), (3.16), and (3.17), *without* such constraints as (3.24), each of the expansions (3.18) and (3.26) can easily be extended to hold true for the (Srivastava–Daoust) generalized Lauricella function (3.1). A general expansion of this type, corresponding to (3.18), was proven markedly differently by Srivastava and Daoust [27, p. 456, Equation (4.3)]. More generally, for $\ell_j \in \mathbb{N}$ $(j = 1,\cdots,r)$, we have

$$\mathcal{F}\left[x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}\right] \equiv F^{A+E:B';\cdots;B^{(r)}}_{C+G:D';\cdots;D^{(r)}}\left[\begin{matrix}[(a):\ \theta',\cdots,\theta^{(r)}], [(e):\ \ell_1,\cdots,\ell_r]:\ [(b'):\ \varphi'];\cdots;[(b^{(r)}):\ \varphi^{(r)}];\\ [(c):\ \psi',\cdots,\psi^{(r)}], [(g):\ \ell_1,\cdots,\ell_r]:\ [(d'):\ \delta'];\cdots;[(d^{(r)}):\ \delta^{(r)}];\end{matrix}\ x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma_n[(e),(u);\ (g),(v)]}{(\lambda+n)_n} \frac{(-z)^n}{n!} \cdot {}_{E+U}F_{G+V+1}\left[\begin{matrix}(e)+n,\ (u)+n;\\ \lambda+2n+1,\ (g)+n,\ (v)+n;\end{matrix}\ z\right]$$

$$\cdot F^{A+V+2:B';\cdots;B^{(r)}}_{C+U:D';\cdots;D^{(r)}}\left[\begin{matrix}[-n:\ell_1,\cdots,\ell_r],[\lambda+n:\ell_1,\cdots,\ell_r],[(a):\theta',\cdots,\theta^{(r)}], [(v):\ell_1,\cdots,\ell_r]:[(b'):\varphi'];\cdots;[(b^{(r)}):\varphi^{(r)}];\\ [(c):\psi',\cdots,\psi^{(r)}], [(u):\ell_1,\cdots,\ell_r]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ x_1,\cdots,x_r\right], \tag{3.29}$$

which, in the limiting case when z is replaced by λz, and x_j by x_j/λ^{ℓ_j} $(j = 1,\cdots,r)$, and $\lambda \to \infty$, yields

$$\mathcal{F}(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} \Gamma_n[(e),(u);\ (g),(v)]\ \frac{(-z)^n}{n!}$$

$$\cdot\ {}_{E+U}F_{G+V}\left[\begin{matrix}(e)+n,\ (u)+n; \\ \\ (g)+n,\ (v)+n;\end{matrix}\ z\right]$$

$$\cdot\ F^{A+V+1:B';\cdots;B^{(r)}}_{C+U:D';\cdots;D^{(r)}}\left[\begin{matrix}[-n:\ell_1,\cdots,\ell_r],\ [(a):\theta',\cdots,\theta^{(r)}],\ [(v):\ell_1,\cdots,\ell_r]: \\ \\ [(c):\psi',\cdots,\psi^{(r)}],\ [(u):\ell_1,\cdots,\ell_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\cdots;[(b^{(r)}):\varphi^{(r)}]; \\ \\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ x_1,\cdots,x_r\right]. \tag{3.30}$$

In addition to the conditions (3.19) and (3.27), respectively, the expansion formulas (3.29) and (3.30) require for the non-negative coefficients (3.3) that

$$1 + \sum_{j=1}^{C} \psi_j^{(k)} + \sum_{j=1}^{D^{(k)}} \delta_j^{(k)} - \sum_{j=1}^{A} \theta_j^{(k)} - \sum_{j=1}^{B^{(k)}} \varphi_j^{(k)} \geqq E - G \tag{3.31}$$

$$(k = 1,\cdots,r),$$

where the equality holds true when the variables z and $x_1,\cdots,x_r$ are constrained as before.

The expansion formulas (3.29) and (3.30), together with a *mild* extension of (3.30) not contained in (3.29), were deduced elsewhere by Srivastava [22] from the following general results involving multiple power series with essentially arbitrary terms:

Theorem 1. *For bounded complex coefficients* $\Lambda(m_1,\cdots,m_r)$ *and* Ω_n $(\forall\ n,m_j \in \mathbb{N}_0;\ j=1,\cdots,r)$, *let the multivariable function* $\Phi(z_1,\cdots,z_r)$ *be defined by*

$$\Phi(z_1,\cdots,z_r) = \sum_{m_1,\cdots,m_r=0}^{\infty} \Lambda(m_1,\cdots,m_r)\,\Omega_L\, z_1^{m_1}\cdots z_r^{m_r}, \tag{3.32}$$

where, and in what follows,

$$L = \ell_1 m_1 + \cdots + \ell_r m_r \tag{3.33}$$

for arbitrary positive integers $\ell_1,\cdots,\ell_r$.

Then

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,(\lambda+n)_n} \sum_{k=0}^{\infty} \frac{\Omega_{n+k}}{(\lambda+2n+1)_k} \frac{z^k}{k!}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L\leqq n} (-n)_L(\lambda+n)_L\, \Lambda(m_1,\cdots,m_r)\, x_1^{m_1}\cdots x_r^{m_r} \tag{3.34}$$

$$(\lambda \neq 0,-1,-2,\cdots),$$

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{k=0}^{\infty} \Omega_{n+k} \frac{z^k}{k!}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L \leqq n} (-n)_L \, \Lambda(m_1,\cdots,m_r) \, x_1^{m_1} \cdots x_r^{m_r}, \tag{3.35}$$

and

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{k=0}^{\infty} \Omega_{n+k} \frac{z^k}{k!}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L \leqq n} (-n)_L \, (\beta-\alpha n+L)_{n+k-L}$$

$$\cdot \left[\frac{\beta - \alpha L + L}{\beta - \alpha n + L}\right] \Lambda(m_1,\cdots,m_r) \, x_1^{m_1} \cdots x_r^{m_r} \tag{3.36}$$

$$(\alpha \text{ arbitrary; } \beta \neq 0),$$

provided that the variables z *and* $x_1,\cdots,x_r$ *are so constrained that each member of the expansion formulas* (3.34), (3.35), *and* (3.36) *exists.*

The expansion formula (3.35) is, just as we observed above in the case of its hypergeometric form (3.3), a limiting case of (3.34) when z is replaced by λz, and x_j by x_j/λ^{ℓ_j} $(j = 1,\cdots,r)$, and $\lambda \to \infty$. Moreover, as already shown by Srivastava [22, p. 306], the expansion formula (3.35) would follow also from (3.36) in the special case when $\alpha = 0$.

All these classes of polynomial expansions were applied recently by Srivastava [24] with a view to deducing various Neumann expansions for multivariable hypergeometric functions in series of the Bessel functions $J_\nu(z)$ and $I_\nu(z)$ or of their such products as

$$J_\mu(z)J_\nu(z), \; I_\mu(z)I_\nu(z), \; \text{and} \; J_\nu(z)I_\nu(z).$$

On the other hand, by applying a terminating version of a known summation theorem for a well-poised hypergeometric ${}_5F_4$ series [20, p. 244, Equation (III.13)], Srivastava [25, pp. 257–259] gave a unification (and generalization) of the multivariable polynomial expansions (3.34) and (3.35), and hence also of (3.29) and (3.30), which is contained in

Theorem 2. *For* $\Phi(z_1,\cdots,z_r)$ *and* L *defined by* (3.32) *and* (3.33), *respectively,*

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} \frac{(-\mu)_n}{(\lambda+n)_n} \frac{z^n}{n!} \sum_{k=0}^{\infty} \Omega_{n+k} \frac{(\mu)_k}{(\lambda+2n+1)_k} \frac{z^k}{k!}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L \leqq n} \frac{(-n)_L\,(\lambda+n)_L\,(\lambda+\mu+n+k+L)_{n-L}}{(\mu-n+1)_L\,(\lambda+\mu+2L+1)_{n-L}}$$

$$\cdot\ \Lambda(m_1,\cdots,m_r)\ x_1^{m_1}\cdots x_r^{m_r}, \tag{3.37}$$

provided that the parameters λ *and* μ, *and the variables* z *and* $x_1,\cdots,x_r$, *are so constrained that each side of the expansion formula* (3.37) *exists.*

In view of (3.28), the expansion formula (3.37) would yield (3.34) if in (3.37) we replace z by z/μ, and x_j by x_j/μ^{ℓ_j} $(j = 1,\cdots,r)$, and then let $\mu \to \infty$. Furthermore, a limiting case of (3.37) when z is replaced by $\lambda z/\mu$, and x_j by $x_j(\mu/\lambda)^{\ell_j}$ $(j = 1,\cdots,r)$, and $\lambda,\mu \to \infty$ leads us to (3.35). Yet another limiting case of the expansion formula (3.37) when z is replaced by λz, and x_j by x_j/λ^{ℓ_j} $(j = 1,\cdots,r)$, and $\lambda \to \infty$ would yield the multivariable polynomial expansion:

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} (-\mu)_n \frac{z^n}{n!} \sum_{k=0}^{\infty} \Omega_{n+k}\, (\mu)_k \frac{z^k}{k!}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L \leqq n} \frac{(-n)_L}{(\mu-n+1)_L} \Lambda(m_1,\cdots,m_r)\, x_1^{m_1} \cdots x_r^{m_r}, \tag{3.38}$$

which provides a generalization of (3.35) different from (3.34) and (3.36).

Finally, we turn to some basic (or q–) extensions of the multivariable polynomial expansions considered in this section. Indeed, for real or complex q $(|q| < 1)$, we write

$$(\lambda;q)_\infty = \prod_{j=0}^{\infty} (1-\lambda\, q^j) \tag{3.39}$$

and let $(\lambda;q)_\mu$ be defined by

$$(\lambda;q)_\mu = \frac{(\lambda;q)_\infty}{(\lambda q^\mu;q)_\infty} \tag{3.40}$$

for arbitrary (real or complex) parameters λ and μ, so that [*cf.* Equation (1.12)]

$$(\lambda;q)_m = \begin{cases} 1, & \text{if } \ m = 0, \\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{m-1}), & \forall\, m \in \mathbb{N}, \end{cases} \tag{3.41}$$

and, by l'Hôpital's rule,

$$\lim_{q \to 1} \left\{ \frac{(q^\lambda;q)_m}{(q^\mu;q)_m} \right\} = \frac{(\lambda)_m}{(\mu)_m} \qquad (m \in \mathbb{N}_0). \tag{3.42}$$

(See, for details, Bailey [4, Chapter 8], Slater [20, Chapter 3], and Exton [11].)

A basic (or q–) extension of Theorem 1 was given by Srivastava [23] who also deduced the corresponding expansions for a general multivariable basic (or q–) hypergeometric function analogous to the (Srivastava–Daoust) generalized Lauricella function (3.1). For the sake of completeness, we recall here a q–extension of Theorem 2, which is given by (*cf.* [12])

Theorem 3. *For* $\Phi(z_1,\cdots,z_r)$ *and* L *defined as in Theorem* 1 *and Theorem* 2,

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} \frac{(\mu;q)_n}{(\lambda q^n;q)_n} \frac{(z/\mu)^n}{(q;q)_n}$$

$$\cdot \sum_{k=0}^{\infty} \Omega_{n+k} \frac{(1/\mu;q)_k}{(\lambda q^{2n+1};q)_k} \frac{z^k}{(q;q)_k}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L \leqq n} \frac{(q^{-n};q)_L\,(\lambda q^n;q)_L\,(\lambda q^{n+k+L}/\mu;q)_{n-L}}{(q^{1-n}/\mu;q)_L\,(\lambda q^{2L+1}/\mu;q)_{n-L}}$$

$$\cdot\ \Lambda(m_1,\cdots,m_r)\,(x_1 q^{\ell_1})^{m_1} \cdots (x_r q^{\ell_r})^{m_r}, \tag{3.43}$$

provided that the parameters λ *and* μ, *and the variables* z *and* $x_1,\cdots,x_r$, *are so constrained that each side of the expansion formula* (3.43) *exists.*

The assertion (3.43) with $\lambda = 0$ immediately yields the following q–extension of the polynomial expansion (3.38):

$$\Phi(x_1 z^{\ell_1},\cdots,x_r z^{\ell_r}) = \sum_{n=0}^{\infty} (\mu;q)_n \frac{(z/\mu)^n}{(q;q)_n}$$

$$\cdot \sum_{k=0}^{\infty} (1/\mu;q)_k \; \Omega_{n+k} \; \frac{z^k}{(q;q)_k}$$

$$\cdot \sum_{m_1,\cdots,m_r=0}^{L \leqq n} \frac{(q^{-n};q)_L}{(q^{1-n}/\mu;q)_L} \; \Lambda(m_1,\cdots,m_r)$$

$$\cdot (x_1 q^{\ell_1})^{m_1} \cdots (x_r q^{\ell_r})^{m_r}, \qquad (3.44)$$

which, for $\mu = 1/\nu$, was given by Srivastava [23, Part I, p. 323, Equation (A.2)]. Jain and Srivastava [12] applied some of the aforementioned further consequences of Theorem 3 (given by Srivastava [23]) to derive various summation (or multiplication) formulas for the q–Lauricella functions $\Phi_A^{(r)}$, $\Phi_C^{(r)}$, and $\Phi_D^{(r)}$ of r variables, where (*cf.*, *e.g.*, [12, p. 15])

$$\Phi_A^{(r)}\left[\alpha, \beta_1,\cdots,\beta_r; \gamma_1,\cdots,\gamma_r; q; z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} (\alpha;q)_{m_1+\cdots+m_r}$$

$$\cdot \prod_{j=1}^{r} \left\{\frac{(\beta_j;q)_{m_j}}{(\gamma_j;q)_{m_j}} \frac{z_j^{m_j}}{(q;q)_{m_j}}\right\}, \qquad (3.45)$$

$$\Phi_B^{(r)}\left[\alpha_1,\cdots,\alpha_r,\beta_1,\cdots,\beta_r;\gamma;q;z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{1}{(\gamma;q)_{m_1+\cdots+m_r}}$$

$$\cdot \prod_{j=1}^{r} \left\{(\alpha_j;q)_{m_j}\ (\beta_j;q)_{m_j}\ \frac{z_j^{m_j}}{(q;q)_{m_j}}\right\}, \tag{3.46}$$

$$\Phi_C^{(r)}\left[\alpha,\ \beta;\gamma_1,\cdots,\gamma_r;q;z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} (\alpha;q)_{m_1+\cdots+m_r}\ (\beta;q)_{m_1+\cdots+m_r}$$

$$\cdot \prod_{j=1}^{r} \left\{\frac{1}{(\gamma_j;q)_{m_j}}\ \frac{z_j^{m_j}}{(q;q)_{m_j}}\right\}, \tag{3.47}$$

and [1, p. 621]

$$\Phi_D^{(r)}\left[\alpha,\ \beta_1,\cdots,\beta_r;\gamma;q;z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(\alpha;q)_{m_1+\cdots+m_r}}{(\beta;q)_{m_1+\cdots+m_r}}$$

$$\cdot \prod_{j=1}^{r} \left\{(\beta_j;q)_{m_j}\ \frac{z_j^{m_j}}{(q;q)_{m_j}}\right\}. \tag{3.48}$$

Each of these q-Lauricella functions is contained in the generalized multivariable basic (or q-) hypergeometric function considered by Srivastava [23]. Furthermore, in view of the limit relationship (3.42), it is not difficult to see that the q-Lauricella functions $\Phi_A^{(r)}$, $\Phi_B^{(r)}$, $\Phi_C^{(r)}$, and $\Phi_D^{(r)}$ would reduce, when $q \to 1$ *after* suitable parametric changes, to the familiar Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$, respectively. Indeed, as already remarked by Jain and Srivastava [12, p. 23], none of the q-polynomial expansions (considered in the present context) would apply to the q-Lauricella function $\Phi_B^{(r)}$ defined by (3.46).

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H.M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 2Y2
CANADA

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ON THE ALTERNATIVE STABILITY OF THE CAUCHY EQUATION

J. Tabor

Let X be a commutative group, Y a normed space and let $0 \leq \varepsilon < 1$. We consider the following functional inequality

$$f(x+y)-f(x)-f(y) \leq \varepsilon \quad \max\{\|f(x+y)\|, \|f(x)+f(y)\|\}$$
$$\text{for } x, y \in X.$$

Similar results, as in the case when the maximum is replaced by the minimum, are obtained.

Let $(X, +)$ be a commutative group, Y a normed space and let $0 \leq \varepsilon < 1$. The following functional inequality was considered in [1], [3] and [4]:

$$\|f(x+y) - f(x) - f(y)| \leq \varepsilon \min\{\|f(x+y)\|, \|f(x) + f(y)\|\} \qquad (1)$$
$$\text{for } x, y \in X,$$

where $f : X \to Y$.

At the twenty-sixth International Symposium on Functional Equations (Sant Feliu de Guixols, 1988) S. Redhofer asked the question what we would get replacing the minimum by the maximum. The paper answers to this question.

Consider the following condition

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \max\{\|f(x+y)\|, \|f(x) + f(y)\|\} \qquad (2)$$
$$\text{for } x, y \in X,$$

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where $f : X \to Y$.
Obviously (2) is equivalent to the alternative

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \|f(x+y)\| \tag{3}$$

or

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \|f(x) + f(y)\| \tag{4}$$

for $x, y \in X$.
On the other hand, the conjunction of (3) and (4) becomes (1).

We begin with the investigations of the relations between (1) and (2).

Proposition 1. If $f : X \to Y$ satisfies (2) then

$$\|f(x+y) - f(x) - f(y)\| \le \frac{\varepsilon}{1-\varepsilon} \min\{\|f(x+y)\|, \|f(x) + f(y)\|\} \quad \text{for } x, y \in X. \tag{5}$$

Proof. Fix arbitrarily $x, y \in X$ and suppose that (3) holds true. Then

$$\|f(x+y)\| - \|f(x) + f(y)\| \le \varepsilon \|f(x+y)\|$$

and hence

$$\|f(x+y)\| \le \frac{1}{1-\varepsilon} \|f(x) + f(y)\|.$$

Since $\varepsilon \le \frac{\varepsilon}{1-\varepsilon}$ we get from (3)

$$\|f(x+y) - f(x) - f(y)\| \le \frac{\varepsilon}{1-\varepsilon} \|f(x+y)\|.$$

Thus

$$\|f(x+y) - f(x) - f(y)\| \le \frac{\varepsilon}{1-\varepsilon} \min\{\|f(x+y)\|, \|f(x) + f(y)\|\}.$$

In the case where (4) is satisfied the proof runs similarly.

Proposition 1 means that (2) implies (1), but with $\frac{\varepsilon}{1-\varepsilon}$ in place of ε. The question arises whether converse implication is true, i.e., whether (1) with $\frac{\varepsilon}{1-\varepsilon}$ in place of ε implies (2) (or equivalently whether (1) implies (2) with ε replaced by $\frac{\varepsilon}{1+\varepsilon}$). In the case where $Y = R$ the answer is positive.

Proposition 2. If $f : X \to R$ satisfies (1) then

$$|f(x+y) - f(x) - f(y)| \leq \frac{\varepsilon}{1+\varepsilon} \max\{|f(x+y)|, |f(x)+f(y)|\} \quad \text{for } x, y \in X\,. \tag{6}$$

Proof. For an indirect proof suppose that for some fixed pair $(x, y) \in X^2$

$$|f(x+y) - f(x) - f(y)| > \frac{\varepsilon}{1+\varepsilon} \max\{|f(x+y)|, |f(x)+f(y)|\}$$

i.e.,

$$|f(x+y) - f(x) - f(y)| > \frac{\varepsilon}{1+\varepsilon} |f(x+y)| \tag{7}$$

and

$$|f(x+y) - f(x) - f(y)| > \frac{\varepsilon}{1+\varepsilon} |f(x)+f(y)|\,. \tag{8}$$

Since, by Proposition 2 of [4] (cf. [3], too), f is odd, we may assume that $f(x+y) \geq 0$ (in the other case we may replace f by $-f$). Then by (3)

$$f(x+y) - f(x) - f(y) \leq \varepsilon f(x+y)$$

and so

$$f(x) + f(y) \geq (1-\varepsilon) f(x+y) \geq 0\,.$$

Since $f(x) + f(y) \geq 0$ we get from (4),

$$f(x+y) \leq (1+\varepsilon)(f(x) + f(y))\,. \tag{9}$$

Suppose that

$$f(x+y) - f(x) - f(y) \geq 0\,.$$

Then (7) becomes

$$f(x+y) - f(x) - f(y) > \frac{\varepsilon}{1+\varepsilon} f(x+y)$$

whence we obtain

$$f(x+y) > (1+\varepsilon)(f(x)+f(y)).$$

which contradicts to (9).
Suppose now that

$$f(x+y) - f(x) - f(y) < 0.$$

Then (3) becomes

$$f(x) + f(y) - f(x+y) \leq \varepsilon f(x+y)$$

i.e.,

$$f(x) + f(y) \leq (1+\varepsilon)f(x+y). \tag{10}$$

On the other hand by (8)

$$f(x) + f(y) - f(x+y) > \frac{\varepsilon}{1+\varepsilon}(f(x)+f(y))$$

and hence

$$f(x) + f(y) > (1+\varepsilon)f(x+y)$$

which contradicts to (10).

The assumption that $Y = R$ is essential for Proposition 2. It is shown by the following

Example 1. Consider functions $f_1, f_2 : R \to R$ defined as follows

$$f_1(x) = \begin{cases} \frac{2}{3}x & \text{for } x \in <0,1> \\ x - \frac{1}{3} & \text{for } x > 1 \end{cases}$$
$$f_1(x) = -f_1(-x) \quad \text{for } x < 0;$$
$$f_2(x) = \begin{cases} x & \text{for } x \in <0,3> \\ \frac{2}{3}x + 1 & \text{for } x > 3, \end{cases}$$
$$f_2(x) = -f_2(-x) \quad \text{for } x < 0.$$

Let $\varepsilon = \frac{1}{2}$. By Theorem 4 [3] f_1 and f_2 satisfy (1). Therefore the mapping $F : R^2 \to R^2$, $F(x_1, x_2) := (f_1(x_1), f_2(x_2))$ also satisfies (1). But for

$x = (10,4), y = (-9,-3)$ we have

$$\|F(x+y) - F(x) - F(y)\| = \|(-\frac{1}{3}, \frac{1}{3})\| = \frac{\sqrt{2}}{3};$$

$$\begin{aligned}\frac{\varepsilon}{1+\varepsilon}\max\{\|F(x+y)\|, \|F(x)+F(y)\|\} &= \frac{1}{3}\max\{\|(\frac{2}{3},1)\|, \|(1,\frac{1}{3})\|\}\\ &= \frac{1}{3}\max\{\frac{\sqrt{13}}{3}, \frac{\sqrt{10}}{3}\}\\ &= \frac{\sqrt{13}}{9} < \frac{\sqrt{2}}{3}.\end{aligned}$$

In the case where $0 \le \frac{\varepsilon}{1-\varepsilon} < 1$, i.e., where $0 \le \varepsilon < \frac{1}{2}$, investigation of inequality (2) can be, due to Proposition 1, reduced to investigation of inequality (1) (of course with another ε). If additionally $Y = R$, then inequality (2) is equilvalent to inequality (1) but with $\frac{\varepsilon}{1-\varepsilon}$ in place of ε. So, in the case where $0 \le \varepsilon < \frac{1}{2}$ the results of [1], [3] and [4] can be applied to inequality (2). Roughly one can say that in this case problem of inequality (2) is solved. The situation is quite different in the case where $\varepsilon \ge \frac{1}{2}$. In this case solutions of (2) need not have properties of solutions of (1). For example every solution of (1) is odd, every solution of (1) continuous at a point is continuous, but it is not true when we replace (1) by (2). It is shown by the following examples.

Example 2. Let $\varepsilon > \frac{1}{2}$ and let $f : X \to R$ be bounded. Then for sufficiently large $c, g(x) := f(x) + c$ satisfies (2). In fact, let $c > n > 0$ and

$$|f(x)| \le n \text{ for } x \in X.$$

Then

$$|g(x+y) - g(x) - g(y)| \le |f(x+y) - f(x) - f(y)| + c \le 3n + c$$

and

$$|g(x) + g(y)| \ge 2\,(c-n).$$

In order that (2) holds, it is enough to assume that

$$2\varepsilon(c-n) \ge 3n + c \text{ i.e., } c \ge \frac{3(n+2)}{2\varepsilon - 1}.$$

Example 3. Let $\varepsilon = \frac{1}{2}$. Then $f(x) = c$ satisfies (2).

As we have seen above, the results of [3] and [4] concerning inequality (1) do not hold true for inequality (2). However we can generalize these results assuming (2) and the condition $f(0) = 0$ instead of (1) or (3) respectively. We start with some preliminary lemmas.

Lemma 1. If $f : X \to Y$ satisfies (2) and $f(0) = 0$ then
(i) f is odd;
(ii) for $x, c \in X$
either

$$\|f(x+c) - f(x) - f(c)\| \le \varepsilon \|f(c)\| \tag{11}$$

or

$$\|f(x+c) - f(x) - f(c)\| \le \varepsilon \|f(x+c) - f(x)\|\,; \tag{12}$$

(iii) $(1-\varepsilon)\|f(c)\| \le \|f(x+c) - f(x)\| \le \frac{1}{1-\varepsilon}\|f(c)\|$ for $x, c \in X$.

Proof. (i) Inserting into (2) $y = -x$ we get

$$\|f(x) + f(-x)\| \le 0 \text{ or } \|f(x) + f(-x)\| \le \varepsilon \|f(x) + f(-x)\|.$$

Since $0 \le \varepsilon < 1$ we have

$$f(x) + f(-x) = 0\,.$$

(ii) Putting into (3) and (4) $y = c - x$ we obtain

$$\|f(c) - f(x) - f(c-x)\| \le \varepsilon \|f(c)\|$$

or

$$\|f(c) - f(x) - f(c-x)\| \le \varepsilon \|f(x) + f(c-x)\|\,.$$

Now changing c to $-c$ and making use of oddity of f we get (ii).
(iii) Fix arbitrarily $x, c \in X$. Making use of (ii) and the triangle inequality we obtain

$$(1-\varepsilon)\|f(c)\| \le \|f(x+c) - f(x)\| \le (1+\varepsilon)\|f(c)\|$$

or

$$\frac{1}{1+\varepsilon}\|f(c)\| \le \|f(x+c)-f(x)\| \le \frac{1}{1-\varepsilon}\|f(c)\|.$$

Since $1-\varepsilon \le \frac{1}{1+\varepsilon}$ and $1+\varepsilon \le \frac{1}{1-\varepsilon}$ we have finally

$$(1-\varepsilon)\|f(c)\| \le \|f(x+c)-f(x)\| \le \frac{1}{1-\varepsilon}\|f(c)\|.$$

Let X be a group. We say that X is 2-divisible if for each $a \in X$ the equation $2x = a$ has the unique solution.

Lemma 2. Let X be a 2-divisible abelian group and Y a pre-Hilbert space*. If $f : X \to Y$ satisfies (2) and $f(0) = 0$ then

$$\|f(x)\| \le \varepsilon_0^n \|f(2^n x)\| \quad \text{for } x \in X, n \in N \tag{13}$$

where $\varepsilon_0 = \max\left\{\frac{4-2\varepsilon^2}{4-\varepsilon^2}, \frac{1}{2-\varepsilon}\right\}$.

Proof. Consider an $x \in X$. In virtue of Lemma 1 (ii) either

$$\|f(2x)-2f(x)\| \le \varepsilon\|f(x)\| \tag{14}$$

or

$$\|f(2x)-2f(x)\| \le \varepsilon\|f(2x)-f(x)\|. \tag{15}$$

From (14) we get

$$\|f(x)\| \le \frac{1}{2-\varepsilon}\|f(2x)\|. \tag{16}$$

On the other hand (15) becomes

$$(f(2x)-2f(x))^2 \le \varepsilon^2(f(2x)-f(x))^2.$$

Hence

$$\begin{aligned}(4-\varepsilon^2)(f(x))^2 &\le (\varepsilon^2-1)(f(2x))^2 + (4-2\varepsilon^2)f(2x)f(x)\\ &\le (4-2\varepsilon^2)f(2x)f(x).\end{aligned}$$

Making use of the Schwarz inequality we obtain

$$(4-\varepsilon^2)\|f(x)\|^2 \le (4-2\varepsilon^2)\|f(2x)\|\,\|f(x)\|,$$

i.e.,

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*i.e. a complex linear space endowed with the inner product.

$$\|f(x)\| \leq \frac{4-2\varepsilon^2}{4-\varepsilon^2}\|f(2x)\|. \tag{17}$$

The alternative of (16) and (17) may be rewritten as

$$\|f(x)\| \leq \varepsilon_0\|f(2x)\| \quad \text{for } x \in X.$$

Induction completes the proof.

Theorem 1. Let X be a 2-divisible abelian topological group such that for every open set $U \subset X$ the set $\frac{1}{2}U$ is open and let Y be a pre-Hilbert space. If $f : X \to Y$ satisfies (2), $f(0) = 0$ and f is locally bounded at a point then f is continuous.

Proof. In virtue of Lemma 1 (iii) f is locally bounded at zero, i.e., there exist a neighbourhood U of zero and a constant $M > 0$ such that

$$\|f(x)\| \leq M \quad \text{for } x \in U. \tag{18}$$

Put

$$U_n := 2^{-n}U \quad \text{for } n \in N. \tag{19}$$

According to our assumption U_n is a neighbourhood of zero. Making use of Lemma 2, (18) and (19) we obtain

$$\|f(x)\| \leq \varepsilon_0^n\|f(2^n x)\| \leq \varepsilon_0^n M \quad \text{for } x \in U_n. \tag{20}$$

We may assume that $0 < \varepsilon < 1$ (in the case where $\varepsilon = 0, f$ is additive). Then $1 < \varepsilon_0 < 1$ and in consequence of (20) f is continuous at zero. By Lemma 1 (iii) f is continuous at each point.

For the next considerations we need a lemma.

Lemma 3. If $f : X \to Y$ satisfies (2) then

$$\|f(x+y)\| \leq \frac{1}{1-\varepsilon}\|f(x)+f(y)\| \quad \text{for } x, y \in X. \tag{21}$$

Proof. From the alternative of (3) and (4) we obtain directly that for $x, y \in X$ either

$$\|f(x+y)\| \leq \frac{1}{1-\varepsilon}\|f(x)+f(y)\|$$

or

$$\|f(x+y)\| \le (1+\varepsilon)\|f(x)+f(y)\|.$$

Sonce $1+\varepsilon \le \frac{1}{1-\varepsilon}$ we have (21).

Theorem 1, Lemma 3 and the theorem of Steinhaus (cf. [2] p. 69) imply the following

Corollary 1. Let Y be a pre-Hilbert space. If $f : R^n \to Y$ satisfies (2), $f(0) = 0$ and f is bounded on a set of positive inner Lebesgue measure then f is continuous.

Proof. Let $T \subset R^n$ be a set of positive inner Lebesgue measure and let

$$\|f(x)\| \le M \quad \text{for } x \in T.$$

Applying Lemma 3 we obtain

$$\|f(x+y)\| \le \frac{1}{1-\varepsilon}\|f(x)+f(y)\| \le \frac{1}{1-\varepsilon}2M \quad \text{for } x, y \in T,$$

i.e., f is bounded on $T+T$. By the theorem of Steinhaus int $(T+T) = 0$, and hence we can use Theorem 1.

Similarly, applying topological analogue of the theorem of Steinhaus, i.e., the theorem of S. Picard (cf. [2], p. 48) we obtain the next corollary.

Corollary 2. Let X be a 2-divisible abelian topological group such that for every open set $U \subset X$ the set $\frac{1}{2}U$ is open and let Y be a pre-Hilbert space. If $f : X \to Y$ satisfies (2), $f(0) = 0$ and f is bounded on a set of the second category with the Baire property then f is continuous.

Lemma 4. (i) If $f : X \to R$ satisfies (2) then

$$\operatorname{sgn} f(x+y) = \operatorname{sgn}(f(x)+f(y)) \quad \text{for } x, y \in X.$$

(ii) If $f : X \to R$ satisfies (2) and $f(0) = 0$ then

$$\operatorname{sgn}(f(x+y)-f(x)) = \operatorname{sgn} f(y) \quad \text{for } x, y \in X.$$

Proof. (i) follows directly from the alternative of (3) and (4) instead (ii) from Lemma 1 (ii).

We consider now the case when the range of f is included in R. We begin with a lemma.

Lemma 5. Let X be an abelian topological group. If $f : X \to R$ satisfies (2), $f(0) = 0$ and f is locally bounded from above (from below) at a point then f is locally bounded at each point.

Proof. We prove first that if f is locally bounded from above (from below) at a point then f is locally bounded from above (from below) at zero.

Let $x \in X$ be fixed and let

$$f(x+h) \leq M \quad \text{for } h \in U\,, \tag{22}$$

where U is a neighbourhood of zero.

Consider a $h \in U$ and suppose that $f(h) \geq 0$. We obtain from Lemma 1 (iii) and Lemma 4 (ii)

$$(1-\varepsilon)f(h) \leq f(x+h) - f(x)\,,$$

and further by (22)

$$f(h) \leq \frac{1}{1-\varepsilon}(f(x+h) - f(x)) \leq \frac{1}{1-\varepsilon}(M - f(x))\,.$$

Thus

$$f(h) \leq \max\left\{\frac{1}{1-\varepsilon}(M - f(x)), 0\right\} \quad \text{for} \quad h \in U\,.$$

This means that f is locally bounded from above at zero. But f is odd so f is locally bounded at zero. Lemma 1 (iii) completes the proof.

Lemma 4 and Theorem 1 imply directly the following theorem.

Theorem 2. Let X be a 2-divisible abelian topological group such that for every open set $U \subset X$ the set $\frac{1}{2}U$ is open. If $f : X \to R$ satisfies (2), $f(0) = 0$ and f is locally bounded from above (from below) at a point then f is continuous.

From Lemma 3, Lemma 4, Theorem 2 and the theorem of Steinhaus one can obtain the following theorem.

Theorem 3. If $f : R^n \to R$ satisfies (2), $f(0) = 0$ and f is bounded from above (from below) on a set of positive inner Lebesgue measure then f is continuous.

Proof. Let
$$f(x) \le M \quad \text{for } x \in T\,,$$
where $T \subset R^n$ is a set of positive inner Lebesgue measure. Consider $x, y \in T$ and suppose that $f(x+y) \ge 0$. Then by Lemma 3 (i) $f(x)+f(y) \ge 0$ and hence, in virtue of Lemma 2,
$$f(x+y) \le \frac{1}{1-\varepsilon}(f(x)+f(y)) \le \frac{2M}{1-\varepsilon}\,.$$
Thus
$$f(x+y) \le \max\left\{\frac{2M}{1-\varepsilon}, 0\right\} \quad \text{for } x, y \in T\,.$$

This means that f is bounded from above on $T+T$. But by the theorem of Steinhaus (cf. [2], p. 69) int $(T+T) = 0$. Hence f is locally bounded from above. Theorem 2 completes the proof.

Similarly, applying the theorem of S. Picard (cf. [2], p. 48), we get the next theorem.

Theorem 4. Let X be a 2-divisible abelian topological group such that for every open set $U \subset X$ the set $\frac{1}{2}U$ is open. If $f : X \to R$ satisfies (2) $f(0) = 0$ and f is bounded from above (from below) on a set of the second category with the Baire property then f is continuous.

Making use of Lemma 1 and Lemma 4 one can easily obtain further results for functions $f : R \to R$ ($f : X \to R$) satisfying (2) and the condition $f(0) = 0$, similar to those obtained in [3] and [4] for functions satisfying (1) or (3). Only the statements including estimations need to be changed respectively.

For example in Theorem 3 [3] and in Theorem 5 [4] $1+\varepsilon$ should be replaced by $\frac{1}{1-\varepsilon}$.

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J. Tabor
Dept. of Mathematics
Pedagogical University of Cracow
Podchorazych 2
30-084 Cracow
Poland

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THE COMPLIANCE AND THE STRENGTH DIFFERENTIAL TENSORS FOR THE DESCRIPTION OF FAILURE OF THE GENERAL ORTHOTROPIC BODY

P. S. Theocaris

1. ABSTRACT

The spectral decomposition of the compliance fourth-rank tensor related with transversely isotropic materials was developed and its characteristic values were calculated by using its components in a Cartesian frame defining the principal material directions. Imposing the eigenvalues of the 6x6 matrix associated with the contracted 4th-rank symmetric tensor to be strictly positive, as implied by the positive definiteness of the elastic potential, bounds of the values of Poisson's ratios were established restraining considerably their existing limits for the orthotropic materials.

Energy orthogonal states of stress for the transversely isotropic material were also established by decomposing the elastic potential in distinct parts associated with the deformation eigen-states of the material symmetry. Thus, the unsolved as yet problem of extension of the separation of the elastic energy to anisotropic materials was efficiently realized.

It was shown that the necessary parameters for the unvariant description of the elastic behavior of a transversely isotropic medium are the four eigenvalues of the spectral decomposition and a dimensionless parameter defined by an eigenangle ω. Thus the general orthotropic material could be equally well defined, instead of its five classical meduli and Poisson's ratios, by these equivalent five indepependent variables.

2. INTRODUCTION

The definition of energy orthogonal stress states was first anticipated by Rychlewski [1], by denoting stress tensors mutually orthogonal and at the same time colinear with their respective strain tensors. Rychlewski [2] has shown that, if a given stress tensor is decomposed in energy orthogonal tensors, then these tensors also decompose the elastic energy function. The decomposition of the elastic compliance tensor in elementary fourth-rank tensors served as a means for the energy orthogonal decomposition of the stress tensor, the appropriate decomposition being the spectral one.

Different decompositions, but not spectral, of the fourth-rank tensor were also given by Walpole [3, 4], Srinivasan and Nigam [5] and others, in order to simplify calculations with fourth-rank tensors used especially in crystallography, and to obtain invariant expressions for the components of the stiffness or compliance tensors.

Assuming the orientation of the axis of elastic symmetry of the transversely isotropic medium to be known with respect to a fixed coordinate system, the complete description of the anisotropic structure of this medium in terms of the invariant parameters emerging from the spectral decomposition of its compliance tensor, necessitates five of these parameters to be known. That is, the four eigenvalues of the compliance tensor and a dimensionless parameter, called the eigenangle ω, which was shown to determine the orientation of eigentensors associated with the eigenvalues of the compliance tensor, when represented in a stress coordinate system.

In this paper the compliance tensor for a transversely isotropic (transtropic) material, usually representing a fibrous reinforced composite, was decomposed spectrally and its characteristic values were defined. Based on the properties of this decomposition, energy-orthogonal stress states were established. It was further shown that positiveness of the eigenvalues of the 6x6 matrix associated with the respective stiffness symmetric tensor establishes more restrictive bounds for the values of Poisson's ratios, than those already existing in the literature. Moreover, the variation of the eigenangle ω was studied in detail within bounds imposed by classical thermodynamics. It was shown that the eigenangle ω can be succesfully used as a single parameter which characterizes the material anisotropy and it is phenomenologically related with quantities accounting for the fracture toughness of the medium.

3. ELASTIC INVARIANTS OF THE TRANSVERSELY ISOTROPIC MEDIUM

Consider a transversely isotropic medium with its axis of elastic symmetry parallel to the 0-33 axis of a right-handed (0-11, 22, 33) reference frame and the set of unit vectors **i, j, k** associated with this coordinate system, vector **k** being directed along the axis of elastic symmetry. The compliance tensor, **S,** of the medium, when *spectrally decomposed,* was shown to be given by the following relation [6]:

$$\mathbf{S} = \lambda_1 \mathbf{E}_1 + .. + \lambda_4 \mathbf{E}_4 , \qquad (1)$$

in which the roots of *the minimum polynomial* of **S**, λ_m, m=1, ..., 4, are given by:

$$\lambda_1 = (1 + v_T) / E_T$$

$$\lambda_1 = 1/2G_L$$

$$\lambda_3 = (1 - v_T) / 2E_T + 1/2E_L + \left\{ \left[(1 - v_T)/2E_T - 1/2E_L \right]^2 + 2v_L^2/E_L^2 \right\}^{1/2} \qquad (2)$$

$$\lambda_3 = (1 - v_T) / 2E_T + 1/2E_L - \left\{ \left[(1 - v_T) / 2E_T - 1/2E_L \right]^2 + 2v_L^2 / E_L^2 \right\}^{1/2}$$

Subscripts T and L in the engineering constants of relations (2) denote the transverse (isotropic) plane and the orthogonal (longitudinal) plane containing the axis of elastic symmetry.

Idempotent tensors $\mathbf{E}_m$ figuring in relation (1) are known to decompose the unit element, **I**, of the fourth-rank symmetric tensor space and satisfy the following set of equations [3]:

$$\mathbf{I} = \mathbf{E}_1 + ... + \mathbf{E}_4$$

$$\mathbf{E}_m . \mathbf{E}_n = 0, \quad m \neq n \qquad (3)$$

$$\mathbf{E}_m . \mathbf{E}_m = \mathbf{E}_m .$$

Moreover, tensors $\mathbf{E}_m$, m=1, ... , 4, subdivide the second-rank symmetric tensor space, **L**, into orthogonal subspaces, $\mathbf{L}_{\lambda m}$, consisting of eigentensors of the compliance tensor **S**. For, if σ is an element of **L**, by means of equation (3), one has:

$$\mathbf{I} . \sigma = \mathbf{E}_1 . \sigma + ... + \mathbf{E}_4 . \sigma = \sigma_1 + ... + \sigma_4 = \sigma \qquad (4)$$

whereas eigentensors σ_m, m=1, ... , 4 of the compliance tensor **S** satisfy the set of equations:

$$\sigma_m . \sigma_n = 0, \quad m \neq n$$

$$S . \sigma_m = \lambda_m \sigma_m . \tag{5}$$

The simplicity introduced by the spectral decomposition of **S** in the mathematical analysis of the theory of Elasticity, involving anisotropic elastic behavior, is reflected in the elementary linear form that generalized Hooke's law assumes. Indeed, if σ_m represents a stress tensor, the associated strain tensor (elastic eigendeformation) is simply expressed by:

$$\varepsilon_m = \lambda_m \sigma_m, \quad m = 1, ..., 4 . \tag{6}$$

Tensors σ_m and ε_m were called by W. Thomson [7] *orthogonal stresses and strains,* because of the property they possess, expressed by the first of relations (5).

It is a simple matter to prove by using relations (5) and (6) that the unique valid energetic decomposition of the elastic potential into distinct strain energy densities, each associated with some eigendeformation of the transversely isotropic medium, is expressed by:

$$2T(\sigma) = \sigma . S . \sigma = \lambda_1 \sigma_1 . \sigma_1 + ... + \lambda_4 \sigma_4 . \sigma_4 \tag{7}$$

Eigentensors σ_m, m=1, ... , 4, can be readily calculated once the idempotent tensors E_m were shown to be given by [6]:

$$E_1 = E^1_{ijkl} = 1/2\,(b_{ik}\,b_{jl} + b_{jk}\,b_{il} - b_{ij}\,b_{kl})$$

$$E_2 = E^2_{ijkl} = 1/2\,(b_{k}\,a_{j} + b_{ij}\,a_{jk} + b_{jk}\,a_{il})$$

$$E_3 = E^3_{ijkl} = \mathbf{f} \otimes \mathbf{f} = f_{ij}\,f_{kl} \qquad (8)$$

$$E_4 = E^4_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij}\,g_{kl}$$

Second-rank axisymmetric tensors **a, b, f** and **g** of relations (8) are defined by:

$$\mathbf{a} = \mathbf{k} \otimes \mathbf{k}$$

$$\mathbf{a} + \mathbf{b} = 1 = \delta_{ij}$$

$$\mathbf{f} = \frac{1}{\sqrt{2}} \cos\omega\mathbf{b} + \sin\omega\mathbf{a} \qquad (9)$$

$$\mathbf{g} = \frac{1}{\sqrt{2}} \sin\omega\mathbf{b} - \cos\omega\mathbf{a}\ ,$$

with

$$\cos 2\omega = \left[(1 - v_T)/2E_T - 1/2E_L \right] \left\{ \left[(1 - v_T)/2E_T - 1/2E_L \right]^2 + 2v_L^2/E_L^2 \right\}^{-1/2} \qquad (10)$$

The two first idempotent tensors of relations (8), i.e., $\mathbf{E}_1$ and $\mathbf{E}_2$, were also derived by Walpole [4] in the presentation of an invariant decomposition of the transversely isotropic fourth-rank tensor, which however, did not correspond to the spectral decomposition of this tensor.

The eigentensors of the transversely isotropic compliance tensor **S** are derived by the orthogonal projection of a second-rank symmetric tensor **σ** on the subspaces $L_{\lambda K}$, produced by the *linear operators* E_m as follows :

$$\sigma_m = E_m . \sigma , \quad m = 1, \ldots , 4. \tag{11}$$

Considering the contracted Cartesian form of the symmetric stress tensor **σ**, i.e., **σ** - σ_i, i = 1, ..., 6, eigentensors σ_m were found to be expressed by [6] :

$$\sigma_1 = \left[1/2\,(\sigma_1 - \sigma_2), 1/2\,(\sigma_2 - \sigma_1), 0, 0, 0, \sigma_6\right]^T$$

$$\sigma_2 = \left[0, 0, 0, \sigma_4, \sigma_5, 0\right]^T$$

$$\sigma_3 = \left(\frac{1}{\sqrt{2}}\cos\omega\,(\sigma_1 + \sigma_2) + \sin\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega, 0, 0, 0\right]^T \tag{12}$$

$$\sigma_4 = \left(\frac{1}{\sqrt{2}}\sin\omega\,(\sigma_1 + \sigma_2) - \cos\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\sin\omega, \frac{1}{\sqrt{2}}\sin\omega, -\cos\omega, 0, 0, 0\right]^T$$

In relations (12) the first two eigentensors σ_1 and σ_2 are independent of the specific material properties and they remain the same for all the elements of the transversely isotropic class. On the contrary, eigentensors σ_3 and σ_4 have components, which are functions of the *eigenangle* ω, given by relation (10), and depending on the engineering elastic constants of the material.

Thus, eigenangle ω, together with the four eigenvalues λ_m given by relations (2), constitute the *five invariant elastic constants* necessary for the description of the elastic behavior of the transversely isotropic media. Moreover, besides its characterization as an elastic constant, eigenangle ω controls the values of the parts in which the elastic potential is decomposed.

Furthermore, it can be readily shown by adding relations (12) that :

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 \quad . \qquad (13)$$

It may be derived from relations (12), that the characteristic states of stress, which correspond to the spectral decomposition of the compliance tensor **S** for a transtropic material, decompose the generic stress tensor in a well-defined manner. Indeed, the states σ_1 and σ_2 are *shears*, with σ_2 simple shear and σ_1 a superposition of pure and simple shear. The sum of σ_3 and σ_4 is the orthogonal supplement to the shear subspace of σ_1 and σ_2.

4. CHARACTERISTIC STATES OF THE SPACE L OF THE SECOND-RANK SYMMETRIC TENSORS **L** FOR THE TRANSTROPIC MATERIAL **S**.

We define the orthogonal subspaces of **L** in terms of which the space of the second-rank symmetric tensors, **L**, is expressed as their direct sum. These subspaces constitute characteristic states of the tensor **S** and satisfy the following relations :

$$\mathbf{S} \cdot \sigma_m = \lambda_m \sigma_m \qquad (m = 1 \text{ to } 4) \qquad (14)$$

with λ_m given by relations (2). These stress states are simply defined by equations of the form :

$$\sigma_m = E_m . \sigma \tag{15}$$

with E_m given by relations (8).

Then, the contracted stress tensor σ, expressed in the form of a 6-D vector, is given by :

$$\sigma = \left[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \right]^T \tag{16}$$

whose components σ_m (m = 1 to 4) are given by the relations (12).

For a loading σ, for which it is valid that $\sigma \in L_{\lambda K}$ (S), the corresponding *coaxial* strain tensor and elastic energy are given by :

$$\varepsilon = \lambda_K \sigma \ , \qquad 2T = \lambda_K \sigma \ . \ \sigma \ . \tag{17}$$

For a state of a generic stressing, which does not belong to any of the subspaces $L_{\lambda K}$ (S) , the strain tensor and the elastic energy are given also in simplified form, after performing the decomposition (12):

$$\varepsilon = S \ . \sigma = (\lambda_1 E_1 + ... + \lambda_m E_m) . \sigma = \lambda_1 \sigma_1 + ... + \lambda_m \sigma_m \tag{18}$$

$$2T(\sigma_1 + ... + \sigma_m) = 2T(\sigma_1) + ... + 2T(\sigma_m) =$$

$$= \lambda_1 (\mathrm{tr}\sigma_1^2) + ... + \lambda_m (\mathrm{tr}\sigma_m^2) \ . \qquad m \leq 6 \tag{19}$$

It is well known from the isotropic elasticity that the strain energy density at any given stress, $\boldsymbol{\sigma}$, can be separated into two components, *the voluminal and the distortional* parts, accounting for the recoverable elastic energy stored by dilatation and distortion of the solid respectively.

Such a separation for the anisotropic solid with axplicity identified parts, as is the case with isotropic materials, is not in general conceivable. However, by means of decompositions of the stress tensor in the form of (12), it is possible to distinguish either some loadings, or some classes of anisotropic materials, for which such a decomposition of the elastic energy in dilatational and distortional parts constitutes a well-defined process.

Consider again the transtropic solid and its characteristic stress states given by relations (12). The associated with $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ strain tensors, $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are related with pure form distortion of the solid, without any volume change. This is obvious, since the only normal strain components are those of tensor $\boldsymbol{\varepsilon}_1$, for which it is valid that :

$$\varepsilon_{(1)\,1} + \varepsilon_{(1)\,2} = 0 \quad , \varepsilon_{(1)\,3} = 0$$

whereas for the $\boldsymbol{\varepsilon}_2$ - tensor it is valid that :

$$\varepsilon_{(2)\,1} = \varepsilon_{(2)\,2} = \varepsilon_{(2)\,3} = 0$$

Thus, the following part of the elastic energy of a transversely isotropic solid:

$$2T_d = \lambda_1 \sigma_1 \cdot \sigma_1 + \lambda_2 \sigma_2 \cdot \sigma_2 \qquad (20)$$

due to the contribution of the $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ tensors creates a *purely distortional elastic energy.*

The remaining $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ parts of the decomposition (12) are associated neither solely with a pure distortional, nor with pure dilatational components of the elastic energy. Their respective tensors $\boldsymbol{\varepsilon}_3$ and $\boldsymbol{\varepsilon}_4$ produce both volume changes and shape distortions.

A useful in applications with orthotropic materials geometric interpretation arises for the energy-orthogonal stress states, if we consider the "projections" of $\boldsymbol{\sigma}_K$ on the principal 3-D stress space. Then, the characteristic state $\boldsymbol{\sigma}_2$ vanishes, whereas stress states $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ are represented by three mutually orthogonal vectors, shown in Fig.1, oriented along directions with the following associated unit vectors :

$$\begin{aligned} &\mathbf{e}_1 : \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \\ &\mathbf{e}_3 : \left(\frac{1}{\sqrt{2}}\cos\omega , \frac{1}{\sqrt{2}}\cos\omega, \sin\omega\right) \\ &\mathbf{e}_4 : \left(\frac{1}{\sqrt{2}}\sin\omega, \frac{1}{\sqrt{2}}\sin\omega, -\cos\omega\right). \end{aligned} \tag{21}$$

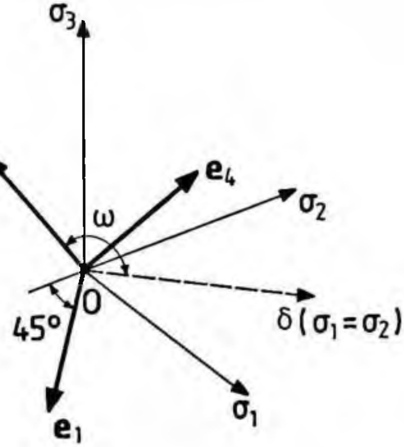

Fig.1 - The projection of σ_K on the principal three-dimensional stress space.

The angle ω defined by relation (10) is expressed by means of the components S_{ijkl} of the initial Cartesin coordinate system. We denote by $(0\ \sigma_1, \sigma_2, \sigma_3)$ the principal stress Cartesian coordinate system with the σ_3-axis parallel to the axis of material symmetry and the (σ_1, σ_2) its isotropic plane.

Then, vector $\mathbf{e}_1$ being vertical to the plane $\sigma_1 = \sigma_2$ (diagonal) and to σ_3-axis, see Fig. 2, lies on the intersection of π-plane (deviatoric) and the plane $\sigma_3 = 0$.

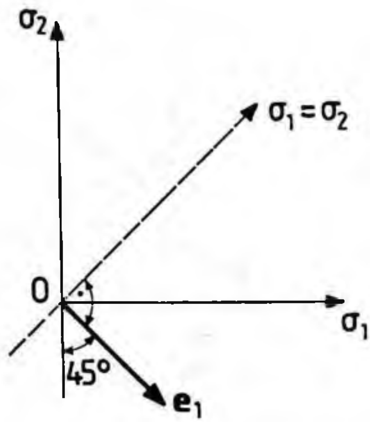

Fig.2 - Vector $\mathbf{e}_1$ lying on the intersection of the deviatoric and $\sigma_3 = 0$ planes.

This is valid for every transversely isotropic solid, as well as for the isotropic body. Its direction cosines are thus *independent of the elastic properties* of the material and retain their values as given by the first of relations (21). Vectors $\mathbf{e}_3$ and $\mathbf{e}_4$, which are mutually orthogonal, lie always on the $\sigma_1 = \sigma_2$ diagonal plane, with the vector $\mathbf{e}_4$ subtending an angle $(\pi - \omega)$ with σ_3-axis, as shown in Fig.3, but their direction cosines are functions of the components of the compliance tensor, as defined by relations (10) and (21).

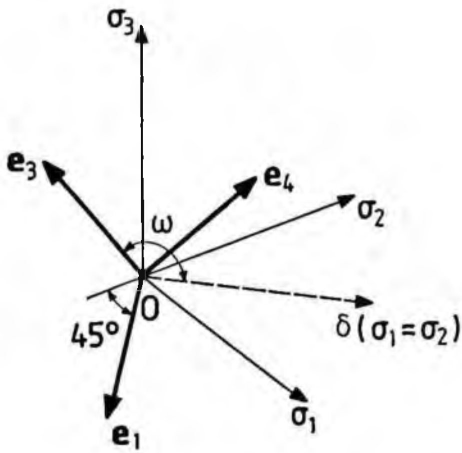

Fig. 3 - Vectors $\mathbf{e}_3$ and $\mathbf{e}_4$ lie always on the (σ_3 , δ)-plane with $\mathbf{e}_4$ subtending an angle (π-ω) with the σ_3-axis.

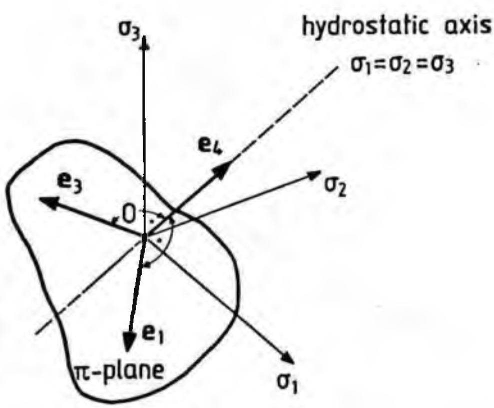

Fig. 4 - Vector $\mathbf{e}_4$ concides always with the direction of the hydrostatic axis for Isotropic solids.

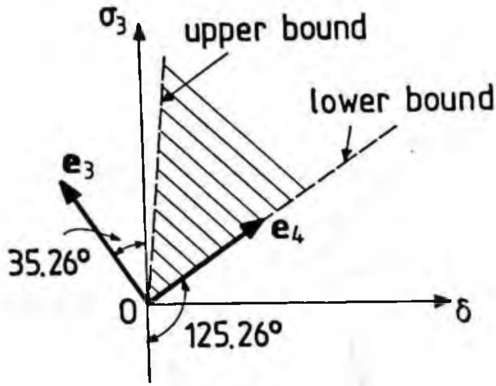

Fig. 5 - Vectors $\mathbf{e}_3$ and $\mathbf{e}_4$ remain always on the main diagonal plane ($\sigma_1 = \sigma_2$).

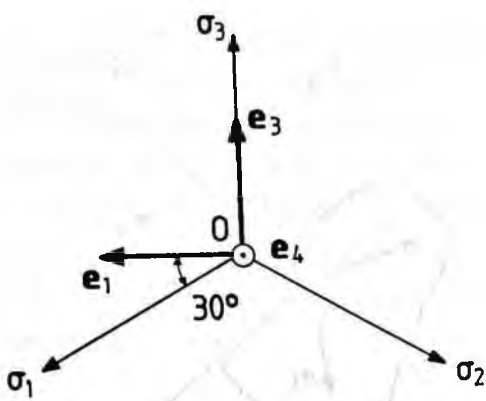

Fig. 6 - Vector $\mathbf{e}_1$ is normal to the principal diagonal plane (σ_3 , δ) as indicated in the deviatoric π-plane.

In terms of these two last relations it can be derived for the isotropic solid that vector $\mathbf{e}_4$ has the positive direction of the hydrostatic axis, Fig. 4, whereas vector $\mathbf{e}_3$ lies on the deviatoric plane. Both vectors $\mathbf{e}_3$ and $\mathbf{e}_4$ remain on the main diagonal plane $\sigma_1=\sigma_2$, as it was also shown in Fig. 5 and Fig. 6.

Let the initial coordinate system $(0 - \sigma_1\sigma_2\sigma_3)$ transform to the one dictated by the directions of $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$, with axis σ_3 having the direction of $\mathbf{e}_3$ and axis σ_1 the direction of $\mathbf{e}_1$. If we denote by $(0 - \bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_3)$ the new coordinate system, it is obvious that the expression for the elastic energy function becomes:

$$2T = \lambda_1\bar{\sigma}_1^2 + \lambda_4\bar{\sigma}_2^2 + \lambda_3\bar{\sigma}_3^2 \,. \qquad (22)$$

By giving the value, 2T = 1, equation (22) represents an *ellipsoid*, centered at the origin 0 of the coordinate system and having axes of symmetry along the directions, $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$. The lengths of the semi-axes of the ellipsoid along the axes of the coordinate system are respectively $1/\sqrt{\lambda 1}$, $1/\sqrt{\lambda 4}$ and $1/\sqrt{\lambda 3}$.

Thus, the energy orthogonal stress states, which decompose a given loading $\boldsymbol{\sigma}$, were shown also to decompose appropriately the elastic energy function, as described by relation (19).

When they are represented geometrically in the principal stress space, they lie along the directions of the semi-axes of the ellipsoid represented by relation (22), which is the geometric representation of the elastic energy function when it is normalized to 2T = 1.

5. BOUNDS FOR THE ANISOTROPIC POISSON'S RATIOS

An important consequence of the spectral decomposition analysis is the simple proof of the positiveness of the elastic potential, expressed by:

$$\lambda_K > 0 \quad ; \qquad K = 1, \ldots, 4 . \tag{23}$$

Since all the elastic moduli should be positive, i.e., E_L, E_T, G_L, $G_T > 0$, the values for the Poisson ratios v_L, v_T should be also bounded by the validity of inequalities (23), which in combination with relations (2) yield:

$$|v_T| \leq 1$$

$$|v_L| \leq \left((1 - v_T) E_L / 2E_T \right)^{1/2} . \tag{24}$$

It may be derived from relations (24) that the transverse or isotropic Poisson's ratio, v_T, has bounds which differ from the bounds for the isotropic solid, which are: $-1.0 \leq v_i \leq 1/2$.

Since it is necessary that all the inequalities of the system (24) should be satisfied in order to yield a positive value of the strain energy density (SED), bounds based on only a partial fulfilment of these inequalities should be erroneous and must be rejected. Therefore, if an experimentally established value for v_T is found to be larger than unity, then because of the validity of the first inequality of (24), this value should be rejectable.

On the other hand, a value for v_T satisfying the inequality $|v_T| \leq 1.0$ should satisfy together with the respective value for v_L the second inequality in the system (24).

A similar remark should be made for all orthotropic materials for which the bounds for their Poisson's ratios are given by the relationships [8]:

$$|v_{ij}| \leq (E_{ii} / E_{jj})^{1/2} \tag{25}$$

and

$$2v_{12}v_{23}v_{13}\frac{E_{33}}{E_{11}} < 1 - v_{12}^2\frac{E_{22}}{E_{11}} - v_{23}^2\frac{E_{33}}{E_{22}} - v_{13}^2\frac{E_{33}}{E_{11}} \tag{26}$$

where the repeated indices in (25) do not mean summations.

The inequality (26) is more stringent than the inequalities (25) and this relationship should be always checked for its validity during the evaluation of any experimental result.

Thus, the experimental results derived from measurements in boron/epoxy orthotropic plates, cited by Jones [9], yielded:

$$E_{11} = 11.86 \times 10^6 \text{psi}, \ E_{22} = 1.33 \times 10^6 \text{psi}, \ v_{12} = 1.97$$

These values satisfy the inequality (45) and therefore the author concludes that the value for v_{12} = 1.97 is a *reasonable one*. However, such value is rather biased, if one tries to satisfy the second bound expressed by the inequality (26). Indeed, the satisfaction of this bound restricts further the spectrum of the accepted values for the Poisson ratios of the composite.

An alternative method for establishing the bounds for the values of the Poisson ratios for transversely isotropic materials was followed by Christensen [10]. According to his method the values of other elastic constants are maximized in idendity relationships with the quantities of Poisson's ratios. Thus, for the transversely isotropic elastic body the following relationships were used in order to establish the bounds of Poisson's ratios [10]:

$$E_{22} = \frac{4\mu_{23} K_{23}}{K_{23} + \mu_{23} + v_{12}^2 \mu_{23} \frac{K_{23}}{E_{11}}} \tag{27}$$

and

$$v_{23} = \frac{K_{23} - \mu_{23} - 4v_{12}^2 \mu_{23} \frac{K_{23}}{E_{11}}}{K_{23} + \mu_{23} + 4v_{12}^2 \mu_{23} \frac{K_{23}}{E_{11}}} \tag{28}$$

where in this notation the (2, 3) - plane is accepted as the isotropic transverse plane and the 1-axis is the strong axis of the material, μ_{23} and K_{23} are the respective shear and plane strain bulk moduli.

Solving for v_{12} from Eq. (27) and introducing the limiting values for the plane-strain bulk modulus $K_{23} \to \infty$ and the shear modulus $\mu_{23} \to \infty$ the following values for the longitudinal Poisson's ratio $v_{12} = v_L$ were established:

$$|v_{12}| < \left(\frac{E_{11}}{E_{22}}\right)^{1/2} \tag{29}$$

However, for the limiting values of K_{23} and μ_{23} the transverse (isotropic) Poisson's ratio, $v_{23} = v_T$ becomes $v_T = -1.0$ and this is the necessary condition for the bound of relation (29) to be valid. For any other value of the transverse Poisson ration $v_T \neq 1.0$, inequality (29) overestimates the bounds for v_L, as it can be easily derived from the exact expressions (24).

A similar procedure with this followed by Christensen was used in ref. [11], where the bounds only for v_T are established. It was therefore erroneously suggested in this reference as the appropriate interval of valid values for v_T the interval [0,1].

In conclusion, it should be again pointed out that in establishing the appropriate bounds for the elastic constants of an orthotropic material *all conditions* (23) for ascertaining the positiveness of the strain energy density should be satisfied.

6. RESULTS

The energy-orthogonal decomposition of the stress tensor $\boldsymbol{\sigma}$, was obtained by means of the spectral decomposition of the symmetric fourth-rank tensor, **S**, which unambiguously defines the positiveness of the elastic energy expressed by:

$$2T = \boldsymbol{\sigma} . \mathbf{S} . \boldsymbol{\sigma}$$

The decomposition of the tensor $\boldsymbol{\sigma}$ for the transversely isotropic solid gave four energy-orthogonal stress states, which decompose in a straightforward manner the elastic energy function.

It was shown that the stress vector $\boldsymbol{\sigma}$ in the 6-D Euclidean space can be expressed by only four eigentensors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$, expressed by relations (12). It was further proved that the eigentensors of the compliance tensor **S** may be given the significance of a stress tensor and express energy-orthogonal loadings. The mathematical expression of this definition was given by:

$$\sigma_K . S . \sigma_N = \sigma_K . \varepsilon_N = 0 \qquad K \neq N \qquad (30)$$

which expresses the normality of the six-dimensional vectors of an eigentensor of stress and an eigentensor of strain, which corresponds to a different than the former stress-eigentensor.

For the transversely isotropic elastic body a generic stress tensor, which is analysed into four eigentensors, may decompose the strain energy density according to relation (19). If we project these eigentensors in the Euclidean space of the principal stresses (0, σ_1, σ_2, σ_3), this projection yields a zero value for σ_2, whereas the projections of σ_1, σ_3 and σ_4 represent three normal to each other vectors.

It was also shown that the $\mathbf{e}_3$- and $\mathbf{e}_4$-vectors are equally inclined to the axes $0\sigma_1$ and $0\sigma_2$ of this frame and therefore they lie on the main diagonal plane $\sigma_1 = \sigma_2$, whereas the $\mathbf{e}_1$-vector is normal to the $0\sigma_3$-axis and therefore it lies on the deviatoric π-plane.

It can be readily proved from relation (10) that angle ω for the isotropic solid is equal to 125.26° and generally varies between 0° and 180°. However, typical values of the angle ω for highly anisotropic fiber composites are near to the bound of 180°, whereas for metal matrix composites, which are characterized by a moderate anisotropy, values of the eigenangle ω approach the bound of the isotropic material, i.e., 125.26°.

Generalizing the above findings for the anisotropic compliance tensor **S** we may derive that the ellipsoid representing its elastic potential has as directions of its principal semi-axes the same directions with its eigentensors whose lengths are equal to $1/(\lambda_m)^{1/2}$, where λ_m is the eigenvalue corresponding to each eigentensor.

It is worthwhile pointing out that from all the polar radii ending on the surface of the strain energy density ellipsoid, which represent stress vectors, only those which are colinear with the principal axes of the ellipsoid have respective strain-vectors, which are colinear with the stress-vectors, whereas in all other cases the stress- and strain-vectors subtend some angle. The same phenomenon happens also for the isotropic elastic bodies.

It was succeeded with this analysis, based on the spectral decomposition of the compliance tensor, to establish energy-orthogonal stress- and strain-states and to separate the SED into well-defined components. Similar, but less general, decompositions were recently introduced by the author [12, 13], based on geometric properties of the stress- and strain-vectors of the transversely isotropic body.

A final important remark, which should be made, concerns relation (19). According to this relation, the elastic potential should be always positive definite, and this property is satisfied only when the tensor **S** is positive definite. It is, however, well known from the algebra of fourth-rank tensors[14] that the necessary condition of the validity of this property is that all the eigenvalues λ_K are positive, fact which constitutes the basis of the analysis of this paper.

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P.S. THEOCARIS
Secretary General of
NATIONAL ACADEMY OF ATHENS
P.O. Box 77230
Athens 175 10 Greece

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QUASIDIRECT PRODUCT GROUPS AND THE LORENTZ TRANSFORMATION GROUP

Abraham A. Ungar

ABSTRACT The direct product group and its generalization into the semidirect product group are standard in group theory. The aim of this article is to introduce a further generalization of the concept into a so called *quasidirect product group*, and to show its relevance by demonstrating that the Lorentz group is the *quasidirect* product of boosts and rotations in analogy with the Galilean group which is the *semidirect* product of boosts and rotations.

1. INTRODUCTION

The quasidirect product structure of the Lorentz transformation group of special relativity is reflected in the harmonious interplay of the Thomas rotation and the relativistically admissible velocities. Let

$$\mathbb{R}_c^3 = \left\{ \mathbf{v} \in \mathbb{R}^3 : \ |\mathbf{v}| < c \right\}$$

be the space of all the relativistically admissible velocities, where c is a positive constant which, in special relativity, represents the speed of light in empty space, and where $\mathbb{R}^3$ is the Euclidean 3-space. The relativistic velocity addition law, according to which the composition of $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$ is $\mathbf{u}*\mathbf{v}\in\mathbb{R}_c^3$, gives rise to a groupoid, $(\mathbb{R}_c^3, *)$. Furthermore, the Thomas rotation of special relativity, $\mathrm{tom}[\mathbf{u};\mathbf{v}]$, gives rise to a mapping

$$\mathrm{tom}: \ \mathbb{R}_c^3\times\mathbb{R}_c^3 \to \mathrm{Aut}(\mathbb{R}_c^3)$$

from the cartesian product $\mathbb{R}_c^3\times\mathbb{R}_c^3$ into the group $\mathrm{Aut}(\mathbb{R}_c^3)$ of the automorphisms of $\mathbb{R}_c^3$. The resulting triple, $(\mathbb{R}_c^3, *, \mathrm{tom})$, gives rise to a *weakly associative-commutative group* which turns out to be a *loop*[3] possessing interesting properties, like the following weak commutative and associative laws and the loop property,

$\mathbf{u}*\mathbf{v} = \mathrm{tom}[\mathbf{u};\mathbf{v}](\mathbf{v}*\mathbf{u})$	Weak commutative law
$\mathbf{u}*(\mathbf{v}*\mathbf{w}) = (\mathbf{u}*\mathbf{v})*\mathrm{tom}[\mathbf{u};\mathbf{v}]\mathbf{w}$	Right weak associative law
$(\mathbf{u}*\mathbf{v})*\mathbf{w} = \mathbf{u}*(\mathbf{v}*\mathrm{tom}[\mathbf{v};\mathbf{u}]\mathbf{w})$	Left weak associative law
$\mathrm{tom}[\mathbf{u};\mathbf{v}] = \mathrm{tom}[\mathbf{u}*\mathbf{v};\mathbf{v}]$	Loop property

The weakly associative-commutative group $\mathbb{R}_c^3 = (\mathbb{R}_c^3, *, \mathrm{tom})$ is the Lorentz counterpart of the ordinary group $\mathbb{R}^3 = (\mathbb{R}^3, +)$, which we interpret as the group of Galilean velocities. Exploiting this analogy between the Galilean and the Lorentz transformation groups, the aim of this article is to present an abstract product group called the *quasidirect product group*.

Taking the Galilean transformation group as a model for a *semidirect* product group, the Lorentz transformation group may be considered as a model for an extended product group that we call the *quasidirect* product group. The Lorentz group is a natural generalization of the Galilean group to which it specializes in the limit of large speed of light and with which it shares many analogous properties. Since the Galilean group is a semidirect product group and since the Lorentz group is not a semidirect product group, one may hope that the Lorentz group gives rise to some generalized product group in terms of which the analogy between the Galilean and the Lorentz groups is retained. Furthermore, one may hope that the usefulness of the resulting generalized product will be similar to that of the semidirect product and, hence, will have impact in the study of abstract groups rather than merely being restricted to the study of the Lorentz group. Accordingly, in generalizing the concept of the semidirect product group into that of the quasidirect product group we are guided in this article by a hint hidden in the structure of the Galilean group and having its echo in the structure of the Lorentz group.

The (homogeneous, proper) Galilean group has a well-known semidirect product structure: It is isomorphic to the semidirect product of the normal subgroup of *boosts* and the group $SO(3)$ of 3×3 (proper) space rotations. Boosts are rotation-free Galilean transformations, that is, Galilean acceleration transformations. The structure of the (homogeneous, proper, orthochronous) Lorentz group is more complicated than that of the Galilean group. The Lorentz group contains $SO(3)$ as a subgroup, and it also contains boosts, which in this context are rotation-free Lorentz transformations, that is, Lorentz acceleration transformations. Like Galilean boosts, Lorentz boosts form a subset which is normal with respect to $SO(3)$. Unlike Galilean boosts, however, Lorentz boosts do not form a subgroup due to the presence of the *Thomas rotation*[14)]. Since the Lorentz group is analogous to the Galilean group and since the Lorentz group does not have a semidirect product structure, its structure may lead us to a new product structure which is analogous to the semidirect product structure. This is indeed the case; the structure of the Lorentz group gives a clue as to how to define the new concept of the *quasidirect product* in such a way that the Lorentz group appears as the quasidirect product of boosts and rotations in analogy with the Galilean group which appears as the semidirect product of boosts and rotations.

The structure of the Lorentz group, viewed as naturally analogous to the structure of the Galilean group, thus, suggests a group theoretic extension of the notion of the semidirect product group into that of the quasidirect product group. The suggested extension turns out to be a natural one along the line of an existing extension of the concept of the direct product group into the concept of the semidirect product group. A group possessing the extended structure, that is, the quasidirect product structure, is called a quasidirect product group. The concept of the quasidirect product group generalizes the concept of the semidirect product group in a way similar to the way in which the latter generalizes the concept of the direct product group.

While the semidirect product is a product between two groups, the quasidirect product is a product between a *weakly associative group* and a group. The weakly associative group turns out to possess interesting properties some of which have been discovered by Karzel in a totally different context, and studied by Kerby, Wefelscheid and others since the 1960's[10,11,22,23)].

The definitions of the direct and the semidirect product groups have several equivalent forms in the literature. The form which suits the aim of this article is presented in Section 2 and, as a relevant example, the semidirect product structure of the Galilei group is illustrated in Section 3. The extension in Section 2 of the notion of the direct product group into the notion of the semidirect product group is further extended in Section 4 into the notion of the quasidirect product group. In Section 5 the Lorentz group is shown to possess a quasidirect product structure, resulting in the newly discovered composition law for Lorentz transformations in terms of parameter composition[14)]. This novel composition law of Lorentz transformations is the natural extension to higher dimensions of the well-known composition law of (1+1)-Lorentz transformations in terms of Einstein's addition law of parallel velocities, and is identified in Section 5 as the quasidirect product between elements of a quasidirect product group. Finally, we present in Section 6 a nonstandard relativistic velocity composition law, as an example of an elegant weakly associative-commutative group which, in turn, gives rise to a group by means of the quasidirect product.

2. *DIRECT PRODUCT GROUPS, SEMIDIRECT PRODUCT GROUPS, AND QUASIDIRECT PRODUCT GROUPS*

The definition of the direct product group has several equivalent forms in the literature. Our purpose will be best served by the following definition.

DEFINITION 1 *(Direct product group)* A group F is a *direct product group* if it possesses two subsets G and H such that

(a1) G and H are normal subgroups of F;

(b1) G and H have only the identity element in common; and

(c1) every element of F can be written as a product of an element of G with an element of H.

F is said to be isomorphic to $G \otimes H$.

Commonly, condition (a1) of Definition 1 is replaced by the simpler, but equivalent condition (a1′) which reads:

(a1′) The elements of G commute with the elements of H.

Proof of the equivalence between conditions (a1) and (a1′) may be found in Cornwell[4].

Elements f of the direct product group $F = G \otimes H$ can be written uniquely as $f = gh = (g, h)$ where $g \in G$ and $h \in H$. The multiplication law for (g, h) is then

$$f = f_1 f_2 = (g_1, h_1)(g_2, h_2) = g_1 h_1 g_2 h_2 = g_1 g_2 h_1 h_2 = (g_1 g_2, h_1 h_2) \tag{D}$$

If condition (a1) of Definition 1 is weakened to the requirement that only the subgroup G must be normal we obtain the more general notion of the semidirect product group:

DEFINITION 2 *(Semidirect product group)* A group F is a *semidirect product group* if it possesses two subsets G and H such that

(a2) G is a normal subgroup of F, and H is a subgroup of F;

(b1) G and H have only the identity element in common; and

(c1) every element of F can be written as a product of an element of G with an element of H.

F is said to be isomorphic to G Ⓢ H.

In both Definitions 1 and 2 the requirement (b1) implies that the decomposition (c1) is unique. Elements f of the semidirect product group $F = G$ Ⓢ H can uniquely be written as $f = gh = (g, h)$ where $g \in G$ and $h \in H$. The multiplication law for (g, h) is then

$$\begin{aligned} f &= f_1 f_2 = (g_1, h_1)(g_2, h_2) \\ &= g_1 h_1 g_2 h_2 = g_1 h_1 g_2 h_1^{-1} h_1 h_2 \\ &= (g_1 h_1 g_2 h_1^{-1}, h_1 h_2) \end{aligned} \tag{S}$$

If condition (a2) of Definition 2 is weakened to the point where the *normal* subset G need not be a subgroup we obtain the more general notion of the quasidirect product group:

DEFINITION 3 *(Quasidirect product group)* A group F is a *quasidirect product group* if it possesses two subsets G and H such that

(a3) G is a normal subset of F with respect to H (that is, $h^{-1} g h \in G$ for all $g \in G$ and all $h \in H$), and H is a subgroup of F;

(b3) $\mathrm{Ext}(G) = \{g_1 g_2^{-1} : g_1, g_2 \in G\}$ and H have only the identity element in common; and

(c1) every element of F can be written as a product of an element of G with an element of H.

F is said to be isomorphic to G Ⓠ H.

The set $\mathrm{Ext}(G) = \{g_1 g_2^{-1} : g_1, g_2 \in G\}$ in Definition 3 is called the *extension* of the subset G in F. Clearly, if G contains the identity element of F then $G \subset \mathrm{Ext}(G)$; and $G = \mathrm{Ext}(G)$ if and only if G is a subgroup of F. Hence, Definition 3 reduces to Definition 2 in the special case when the subset G of F is a subgroup of F.

As in Definitions 1 and 2, the requirement ($b3$) implies that the decomposition ($c1$) in Definition 3 is unique. To establish this uniqueness let us assume that $g_1'h_1=g_2'h_2$ where $g_1', g_2' \in G$ and $h_1, h_2 \in H$. Then $h_1g_1=h_2g_2$ where $g_1=h_1^{-1}g_1'h_1 \in G$ and $g_2=h_2^{-1}g_2'h_2 \in G$, implying $g_1g_2^{-1}=h_1^{-1}h_2$. But $g_1g_2^{-1} \in Ext(G)$ and $h_1^{-1}h_2 \in H$. Hence, by ($b3$), $g_1g_2^{-1}=h_1^{-1}h_2=1$ so that $g_1=g_2$, $h_1=h_2$ and $g_1'=g_2'$.

Elements f of the quasidirect product group $F=G \circledR H$ can be written uniquely as $f=gh=(g,h)$ where $g \in G$ and $h \in H$. The multiplication law for (g,h) is then

$$\begin{aligned} f=f_1f_2&=(g_1,h_1)(g_2,h_2)=g_1h_1g_2h_2 \\ &=g_1h_1g_2h_1^{-1}h_1h_2=g_1g_3h_1h_2 \\ &=g_{13}h_{13}h_1h_2=(g_{13},h_{13}h_1h_2) \end{aligned} \tag{Q}$$

In eq. (Q) $g_3=h_1g_2h_1^{-1} \in G$; and $g_{13} \in G$ and $h_{13} \in H$ are determined by the equation

$$g_1g_3 = g_{13}h_{13} \tag{1}$$

which gives the unique decomposition of the element $g_1g_3 \in F$ as a product of an element of G with an element of H. The semidirect product and the quasidirect product in eqs. (S) and (Q) form successive generalizations of the direct product in eq. (D).

Definition 3 provides a natural extension along the line of the extension of Definition 1 into Definition 2. Definition 3 is suggested by the structure of the Lorentz group, which provides a natural extension of the semidirect product structure of the Galilean group. The Lorentz group is, accordingly, a *quasidirect product group*. In Theorem 2 we will see that a product of Galilean transformations is a semidirect product, having the form in eq. (S), and in Theorem 4 we will see that a product of Lorentz transformations is a quasidirect product, having the form in eq. (Q). As we will see in the sequel, the Lorentz group L, its subset of boosts B and its subgroup of space rotations $SO(3)$ form a realization of the group F, its subset G and its subgroup H in Definition 3. Since the extension of the Galilean group into the Lorentz group serves as a model for the extension of Definition 2 into Definition 3, it would be instructive to illustrate the semidirect product structure of the Galilean group before studying the quasidirect product structure of the Lorentz group.

3. THE SEMIDIRECT PRODUCT STRUCTURE OF THE GALILEAN GROUP

The elements, Galilean transformations, of the Galilean group are transformations between time-space coordinates which will be specified below. We identify the Galilean group with its generic element $G(\mathbf{v};V)$ which is a (homogeneous, proper) Galilean transformation parametrized by a (3-dimensional) velocity parameter $\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^3$, and an orientation parameter V, $V \in SO(3)$. The Galilean transformation $G(\mathbf{v};V)$ relates the time-space coordinates of an event resolved in two inertial frames with relative velocity $\mathbf{v}$ and relative orientation V, as shown in Fig. 1 and in eq. (2). The velocity parameter space, $\mathbb{R}^3$, is the Euclidean 3-space, and the orientation parameter group, $SO(3)$, is the group of all 3×3 real orthogonal, unit determinant matrices.

Let $(t',x',y',z')^t$ and $(t,x,y,z)^t$ (the exponent t indicates transposition) be the respective time-space coordinates of an event resolved in two inertial frames Σ' and Σ, the origins of which coincided at time $t=0$. These coordinates are related by the equation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = G(\mathbf{v};V)\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \tag{2}$$

where $\mathbf{v}$ is the velocity of the *rocket frame* Σ' relative to the *lab frame* Σ and where V is the orientation of the rocket frame Σ' relative to the lab frame Σ; see Fig. 1.

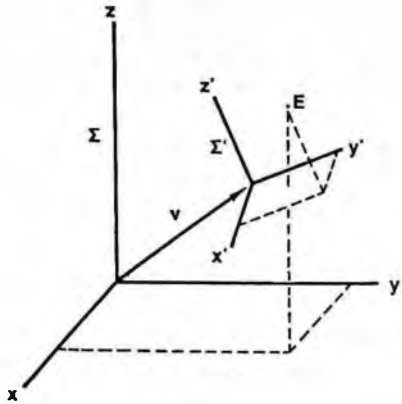

Fig. 1 Σ′ and Σ are inertial Galilean (Lorentz) frames of reference moving apart with relative velocity v, $\mathbf{v} \in \mathbb{R}^3$ ($\mathbf{v} \in \mathbb{R}^3_c$), and relative orientation V, $V \in SO(3)$, the origins of which coincided at time $t = 0$. For clarity, time dimension is suppressed. The time-space coordinates of an event E measured in the *rocket frame* Σ′ and in the *lab frame* Σ are respectively (t', x', y', z') and (t, x, y, z). These are linked by the Galilean transformation $G\{\mathbf{v}; V\}$ of eq. (2) (by the Lorentz transformation $L\{\mathbf{v}; V\}$ of eq. (18)).

The Galilean transformation $G\{\mathbf{v}; V\}$ in eq. (2) is a linear transformation which, in terms of its effects on time-space coordinates, has the matrix representation

$$G\{\mathbf{v};V\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & V & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & & & \\ v_2 & & V & \\ v_3 & & & \end{pmatrix}$$

where $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is a representation of v by its coordinates relative to Σ, and where $V \in SO(3)$ is a 3×3 unimodular orthogonal matrix representing the orientation of Σ′ relative to Σ.

Clearly, the Galilean transformation $G\{\mathbf{v}; V\}$ can be written as a boost $B_\infty(\mathbf{v})$ preceded by a space rotation $\rho(V)$,

$$G\{\mathbf{v};V\} = B_\infty(\mathbf{v})\rho(V), \qquad \mathbf{v} \in \mathbb{R}^3, \quad V \in SO(3) \tag{3}$$

where, anticipating the limit in eq. (11), we use the notation

$$B_\infty(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

and where

$$\rho(V) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & V & \\ 0 & & & \end{pmatrix}$$

ρ being a homomorphism, $\rho: SO(3) \to SO(4)$. Galilean boosts, $B_\infty(\mathbf{v})$, are thus rotation-free Galilean transformations.

The matrix representation in eq. (4) of the boost $B_\infty(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3$, forms a one-parameter matrix group where matrix multiplication corresponds to parameter addition,

$$B_\infty(\mathbf{u})B_\infty(\mathbf{v}) = B_\infty(\mathbf{u}+\mathbf{v}), \qquad \mathbf{u},\mathbf{v} \in \mathbb{R}^3 \tag{5}$$

The identity element of this boost matrix group is $B_\infty(0)$ and the inverse of $B_\infty(\mathbf{v})$ is $B_\infty(-\mathbf{v})$. Normally, one-parameter matrix groups involve a single *scalar* parameter[9]. To avoid confusion we should therefore emphasize that the single parameter involved in the Galilean boost matrix group, $B_\infty(\mathbf{v})$, is equivalent to three scalar parameters.

THEOREM 1 The Galilean group $G\{\mathbf{v};V\}$ is isomorphic to the semidirect product of the normal subgroup $B_\infty(\mathbf{v})$ of Galilean boosts and the subgroup $\rho(V)$ of space rotations,

$$G\{\mathbf{v};V\} \cong B_\infty(\mathbf{v}) \circledS \rho(V) \tag{6}$$

Proof of Theorem 1 Both $B_\infty(\mathbf{v})$ and $\rho(V)$ are subgroups of the Galilean group having only the identity element in common. The subgroup $B_\infty(\mathbf{v})$ is normal, as we see from eq. (5) and from the equation

$$\rho(V)B_\infty(\mathbf{v})\rho(V^{-1}) = B_\infty(V\mathbf{v}), \qquad \mathbf{v} \in \mathbb{R}^3, \quad V \in SO(3) \tag{7}$$

Finally, by eq. (3), every element of the Galilean group is the product of an element of $B_\infty(\mathbf{v})$ with an element of $\rho(V)$. The result of the Theorem, thus, follows from Definition 2. •

THEOREM 2 Two successive Galilean transformations are equivalent to a Galilean transformation,

$$G\{\mathbf{u};U\}\,G\{\mathbf{v};V\} = G\{\mathbf{u}+U\mathbf{v};\,UV\} \tag{8}$$

Proof of Theorem 2 Forming a semidirect product group, the composition law of Galilean transformations is the multiplication law of eq. (S): If we use the notation

$$G\{\mathbf{v};V\} = B_\infty(\mathbf{v})\rho(V) = (B_\infty(\mathbf{v}),\rho(V))$$

then for all $\mathbf{u},\mathbf{v} \in \mathbb{R}^3$ and $U,V \in SO(3)$, as in eq. (S),

$$\begin{aligned} G\{\mathbf{u};U\}\,G\{\mathbf{v};V\} &= (B_\infty(\mathbf{u}),\rho(U))(B_\infty(\mathbf{v}),\rho(V)) \\ &= B_\infty(\mathbf{u})\rho(U)B_\infty(\mathbf{v})\rho(V) \\ &= B_\infty(\mathbf{u})\rho(U)B_\infty(\mathbf{v})\rho(U^{-1})\rho(U)\rho(V) \\ &= B_\infty(\mathbf{u})B_\infty(U\mathbf{v})\rho(UV) \\ &= B_\infty(\mathbf{u}+U\mathbf{v})\rho(UV) \\ &= (B_\infty(\mathbf{u}+U\mathbf{v}),\rho(UV)) \\ &= G\{\mathbf{u}+U\mathbf{v};\,UV\} \end{aligned}$$

where eqs. (7) and (5) have been employed. •

Eq. (8) demonstrates that the well-known composition law of Galilean transformations[7,8,12,21] is the semidirect product between elements of a semidirect product group.

4. THE QUASIDIRECT PRODUCT STRUCTURE OF THE LORENTZ GROUP

Let

$$\mathbb{R}_c^3 = \left\{ \mathbf{v} \in \mathbb{R}^3 : \ |\mathbf{v}| < c \right\}$$

be the set of all 3-vectors with magnitude smaller than some positive constant c. In special relativity the constant c represents the speed of light in empty space; and $\mathbb{R}_c^3$ is the *weakly associative-commutative group* of relativistically admissible velocities with the group operation given by relativistic velocity composition[14–17], as explained in Section 5. The relativistic velocity composition $\mathbf{u}*\mathbf{v}$ of $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$ is given by the equation

$$\mathbf{u}*\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} + \frac{1}{c^2}\frac{\gamma_u}{\gamma_u+1}\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \tag{9}$$

where γ_u is the *Lorentz factor,*

$$\gamma_u = \frac{1}{\sqrt{1-(\frac{\mathbf{u}}{c})^2}} = \frac{1}{\sqrt{1-(\frac{u}{c})^2}}$$

associated with the velocity $\mathbf{u}$ whose magnitude is u, $u = |\mathbf{u}|$, and where $\cdot$ and $\times$ signify the usual dot (scalar) and cross (vector) product between two vectors. Clearly, when $c \to \infty$ the weakly associative-commutative group $(\mathbb{R}_c^3, *)$, which is neither commutative nor associative, reduces to the Euclidean 3-group $(\mathbb{R}^3, +)$, which is both commutative and associative.

The Lorentz boost $B(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}_c^3$, is a rotation-free Lorentz transformation which, in terms of its effects on time-space coordinates, is represented by the matrix[13]

$$B(\mathbf{v}) = B_c(\mathbf{v}) = \begin{bmatrix} \gamma_v & c^{-2}\gamma_v v_1 & c^{-2}\gamma_v v_2 & c^{-2}\gamma_v v_3 \\ \gamma_v v_1 & 1+c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1^2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1 v_2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1 v_3 \\ \gamma_v v_2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1 v_2 & 1+c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_2^2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_2 v_3 \\ \gamma_v v_3 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1 v_3 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_2 v_3 & 1+c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_3^2 \end{bmatrix} \tag{10}$$

where (v_1, v_2, v_3) are the components of $\mathbf{v}$ in a frame relative to which $B(\mathbf{v})$ is represented; see Fig. 1. The identity Lorentz boost is $B(0)$ and the inverse Lorentz boost of $B(\mathbf{v})$ is $B(-\mathbf{v})$. An important relationship between the Lorentz boost $B_c(\mathbf{v}) = B(\mathbf{v})$ of eq. (10) and the Galilean boost $B_\infty(\mathbf{v})$ of eq. (4) is clear:

$$B_\infty(\mathbf{v}) = \lim_{c\to\infty} B_c(\mathbf{v}) \tag{11}$$

Galilean boosts form a normal subgroup of the Galilean group. In particular, since the Galilean group contains the group of space rotations, $SO(3)$, Galilean boosts form a subgroup which is normal with respect to $SO(3)$ (here, for simplicity, we identify $SO(3)$ with its image $\rho(SO(3)) \subset SO(4)$). Unlike Galilean boosts, and as a peculiarity of special relativity, Lorentz boosts do not form a group. Like Galilean boosts, however, Lorentz boosts form a subset of the Lorentz group which is *normal* with respect to $SO(3)$. This is due to eq. (7), which remains valid for Lorentz boosts,

$$\rho(V)B(\mathbf{v})\rho(V^{-1}) = B(V\mathbf{v}), \qquad \mathbf{v} \in \mathbb{R}_c^3, \quad V \in SO(3) \tag{12}$$

In order to expose the quasidirect product structure of the Lorentz group it is necessary to resolve the composition of two boosts as a boost preceded by a space rotation, as we see from eq. (1). This resolution is known[14–18],

$$B(\mathbf{u})B(\mathbf{v}) = B(\mathbf{u}*\mathbf{v})\,\mathrm{Tom}[\mathbf{u};\,\mathbf{v}], \qquad \mathbf{u},\mathbf{v}\in \mathbb{R}^3_c \tag{13}$$

where

$$\mathrm{Tom}[\mathbf{u};\,\mathbf{v}] = \rho(\mathrm{tom}[\mathbf{u};\,\mathbf{v}]) \in SO(4) \tag{14}$$

and where $\mathrm{tom}[\mathbf{u};\,\mathbf{v}] \in SO(3)$ is the 3×3 Thomas rotation of space coordinates generated by two successive boosts with velocity parameters $\mathbf{v}$ and $\mathbf{u}$. Eq. (13) presents two successive (Lorentz) boosts as a boost preceded by a (Thomas) rotation. The Thomas rotation $\mathrm{tom}[\mathbf{u};\,\mathbf{v}]$, generated by two successive boosts with velocity parameters $\mathbf{v}$ and $\mathbf{u}$, is given by the equation[14]

$$\mathrm{tom}[\mathbf{u};\,\mathbf{v}] = I + c_1\Omega + c_2\Omega^2, \qquad \mathbf{u},\mathbf{v}\in \mathbb{R}^3_c, \quad \mathrm{tom}[\mathbf{u};\,\mathbf{v}]\in SO(3) \tag{15a}$$

where I is the 3×3 identity matrix, and where the matrix $\Omega=\Omega(\mathbf{u},\mathbf{v})$ and the coefficients $c_1=c_1(\mathbf{u},\mathbf{v})$ and $c_2=c_2(\mathbf{u},\mathbf{v})$ are functions of $\mathbf{u}$ and $\mathbf{v}$, given in eqs. (15b-d) below.

The matrix $\Omega=\Omega(\mathbf{u},\mathbf{v})$ in eq. (15a) is skew symmetric,

$$\Omega(\mathbf{u},\mathbf{v}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{15b}$$

representing the linear transformation of cross product with $\boldsymbol{\omega}$, that is, $\Omega\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}$ for a 3-vector $\mathbf{r}$. The entries ω_k, $1\le k\le 3$, of the matrix Ω are the components of the vector product $\boldsymbol{\omega}=\mathbf{u}\times\mathbf{v}$ measured in the frame Σ of Fig. 1,

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \mathbf{u}\times\mathbf{v} = (u_2v_3-u_3v_2,\, u_3v_1-u_1v_3,\, u_1v_2-u_2v_1) \tag{15c}$$

The coefficients $c_1=c_1(\mathbf{u},\mathbf{v})$ and $c_2=c_2(\mathbf{u},\mathbf{v})$ in eq. (15a) are given by the equations

$$\begin{aligned} c_1(\mathbf{u},\mathbf{v}) &= -\frac{1}{c^2}\,\frac{\gamma_u\gamma_v(\gamma_u+\gamma_v+\gamma_{u*v}+1)}{(\gamma_u+1)(\gamma_v+1)(\gamma_{u*v}+1)} \\ c_2(\mathbf{u},\mathbf{v}) &= \frac{1}{c^4}\,\frac{\gamma_u^2\gamma_v^2}{(\gamma_u+1)(\gamma_v+1)(\gamma_{u*v}+1)} \end{aligned} \tag{15d}$$

The Thomas rotation is a rotation about a screw axis parallel to the vector $\mathbf{u}\times\mathbf{v}$ through an angle ε which is related to $\mathbf{u}$ and $\mathbf{v}$ and to the rotation angle θ from $\mathbf{u}$ to $\mathbf{v}$ by the equations

$$\begin{aligned} \cos\varepsilon &= \frac{(k+\cos\theta)^2-\sin^2\theta}{(k+\cos\theta)^2+\sin^2\theta} \\ \sin\varepsilon &= \frac{-2(k+\cos\theta)\sin\theta}{(k+\cos\theta)^2+\sin^2\theta} \end{aligned} \tag{16a}$$

where

$$k^2 = \frac{\gamma_u+1}{\gamma_u-1}\,\frac{\gamma_v+1}{\gamma_v-1}, \qquad k>1 \tag{16b}$$

or, equivalently,

$$k = \frac{\gamma_u+1}{\gamma_u}\,\frac{\gamma_v+1}{\gamma_v}\,\frac{c^2}{uv} \tag{16c}$$

where $u=|\mathbf{u}|$ and $v=|\mathbf{v}|$. Eqs. (16) can readily be derived from eqs. (26) of ref. 14.

Since

$$\lim_{|\mathbf{u}|,|\mathbf{v}|\to c} k = 1$$

we see from eqs. (16) that the Thomas rotation angle, ε, is not defined for $\theta=\pi$ when

$|u| = |v| = c$. This singularity is associated with the singularity in the velocity composition $u*v$ when $|u| = |v| = c$ and $\theta = \pi$ which, in turn, asserts that there is no transformation from the rest frame of a photon into a lab frame, or, in Wigner's words: moving particles "either can, or cannot, be transformed to rest".[24] Graphs of $\cos\varepsilon$ and $-\sin\varepsilon$ as functions of their generating angle θ, for several values of k, are shown in Fig. 2 of ref. 14 where the singularity at $\theta = \pi$ when $|u| = |v| = c$ is clearly observed. An interesting property of the Thomas angle ε is discussed by Shahar Ben-Menahem[2]. The group-theoretic importance of the Thomas rotation rests on the weakly associative-commutative structure for $\mathbb{R}_c^3$ to which it gives rise:

(*i*)	$u*v = \text{tom}[u; v](v*u)$	Weak commutative law of velocity composition
(*iia*)	$u*(v*w) = (u*v)*\text{tom}[u; v]w$	Right weak associative law of velocity composition
(*iib*)	$(u*v)*w = u*(v*\text{tom}[v; u]w)$	Left weak associative law of velocity composition

In analogy with eq. (3), the general (homogeneous, proper, orthochronous) Lorentz transformation $L\{v; V\}$ has the form

$$L\{v;V\} = B(v)\rho(V), \qquad v \in \mathbb{R}_c^3, \quad V \in SO(3) \tag{17}$$

where it is parametrized by velocity and orientation parameters. Lorentz boosts, $B(v)$, are thus rotation-free Lorentz transformations and the general Lorentz transformation is a boost preceded by a space rotation. The Lorentz transformation $L\{v; V\}$ links the time-space coordinates of an event resolved in two inertial frames with relative velocity v and relative orientation V, the origins of which coincided at time $t = 0$, as depicted in Fig. 1:

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = L\{v;V\} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \tag{18}$$

Eq. (18) for the Lorentz transformation is analogous to eq. (2) for the Galilean transformation.

THEOREM 3. The Lorentz group $L\{v; V\}$ is isomorphic to the quasidirect product group of the subset $B(v)$ of boosts and the subgroup $\rho(V)$ of space rotations,

$$L\{v;V\} \cong B(v) \circledQ \rho(V) \tag{19a}$$

Proof of Theorem 3. The inverse of the boost $B(v)$ is the boost $B(-v)$. The subset of composite boosts $B(u)B(-v)$, and the subgroup $\rho(V)$ of the Lorentz group, $L\{v; V\}$, have only the identity element in common. The subset of boosts, $B(v)$, of the Lorentz group is normal with respect to $\rho(V)$ as we see from eq. (12). Finally, by eq. (17), every element of the Lorentz group is the product of an element of $B(v)$ with an element of $\rho(V)$. The result of the Theorem, thus, follows from Definition 3 of the quasidirect product group. •

Eq. (19a) may be written as

$$L\{v;V\} \cong B_c(v) \circledQ \rho(V) \tag{19b}$$

in order to display the dependence of Lorentz boosts on the speed of light, c, emphasizing the limit in eq. (11) and the analogy between the product structure of the Galilean and the Lorentz groups in eqs. (6) and (19). By letting the speed of light approach infinity, the operators $L\{v; V\}$, $B_c(v)$ and Ⓠ in eq. (19b) are respectively deformed into $G\{v; V\}$, $B_\infty(v)$ and Ⓢ of eq. (6).

THEOREM 4. Two successive Lorentz transformations are equivalent to a Lorentz transformation,

$$L\{u;U\}L\{v;V\} = L\{u*Uv; \text{tom}[u;Uv]UV\}$$

Proof of Theorem 4 Forming a quasidirect product group, the composition law of Lorentz transformations is the multiplication law of eq. (Q): If we use the notation

$$L\{\mathbf{v};V\} = B(\mathbf{v})\rho(V) = (B(\mathbf{v}),\rho(V))$$

then for all $\mathbf{u},\mathbf{v}\in \mathbb{R}_c^3$ and $U,V\in SO(3)$ we have, as in eq. (Q),

$$\begin{aligned} L\{\mathbf{u};U\}L\{\mathbf{v};V\} &= (B(\mathbf{u}),\rho(U))(B(\mathbf{v}),\rho(V)) \\ &= B(\mathbf{u})\rho(U)B(\mathbf{v})\rho(V) \\ &= B(\mathbf{u})\rho(U)B(\mathbf{v})\rho(U^{-1})\rho(U)\rho(V) \\ &= B(\mathbf{u})B(U\mathbf{v})\rho(UV) \\ &= B(\mathbf{u}{*}U\mathbf{v})\rho(\mathrm{tom}[\mathbf{u};U\mathbf{v}])\rho(UV) \\ &= (B(\mathbf{u}{*}U\mathbf{v}),\rho(\mathrm{tom}[\mathbf{u};U\mathbf{v}]\,U\,V)) \\ &= L\{\mathbf{u}{*}U\mathbf{v};\mathrm{tom}[\mathbf{u};U\mathbf{v}]\,U\,V\} \end{aligned} \tag{20}$$

In the chain of equations (20) we have employed eq. (17), eq. (12), and eqs. (13) and (14). •

In Theorem 4 we have recovered the Lorentz transformation composition law,

$$L\{\mathbf{u};U\}L\{\mathbf{v};V\} = L\{\mathbf{u}{*}U\mathbf{v};\mathrm{tom}[\mathbf{u};U\mathbf{v}]\,U\,V\} \tag{21L}$$

which reduces to the well-known Galilean transformation composition law, eq.(8),

$$G\{\mathbf{u};U\}G\{\mathbf{v};V\} = G\{\mathbf{u}+U\mathbf{v};U\,V\} \tag{21G}$$

when $c\to\infty$. We have, furthermore, identified the Lorentz transformation composition law as the quasidirect product between elements of a quasidirect product group, in a way analogous to the one in which the Galilean transformation composition law is identified as the semidirect product between elements of a semidirect product group.

The special case of Composite Lorentz transformations associated with collinear velocities, $\mathbf{u}$ and $\mathbf{v}$, is popular in the literature of special relativity since it does not involve orientations and Thomas rotations and is, therefore, simple: for collinear relativistically admissible velocities $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$, $\mathbf{u}\parallel\mathbf{v}$, we have

$$L\{\mathbf{u}\}L\{\mathbf{v}\} = L\{\mathbf{u}{*}\mathbf{v}\} \tag{22}$$

where

$$\mathbf{u}{*}\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \tag{23}$$

is Einstein's velocity addition law for parallel velocities which, being both commutative and associative, is well-behaved.

Notations similar to the one in eq. (21G) for the composition of Galilean transformations are common in the literature. The analogous notation in eq. (21L) for the composition of Lorentz transformations is however novel[14]. The naturality and the usefulness of the notation in eq. (21L) rest on the obvious way in which it extends the standard notations in eq. (21G) and in eq. (22). This clearly indicates the naturality and the expected usefulness of the generalization of the notions of the direct product group and the semidirect product group between two groups into that of the quasidirect product group between a weakly associative group and a group. In this connection we may note that the operation (23) is frequently being cited in the literature as an elegant example of a group operation. In contrast, the common generalization, (9), of (23) to noncollinear velocities is not a group operation! It is, however, an elegant example of a weakly associative-commutative group operation.

5. THE WEAKLY ASSOCIATIVE-COMMUTATIVE GROUP OF RELATIVISTICALLY ADMISSIBLE VELOCITIES

The Galilean transformation group is commonly parametrized, Fig. 1, by a velocity parameter v, $\mathbf{v}\in I\!R^3$, and an orientation parameter V, $V\in SO(3)$ in such a way that the Galilean transformation composition is given by parameter composition, eq. (21G). Since the correspondence between a Galilean transformation $G\{\mathbf{v};V\}$ and its parameters $(\mathbf{v},V)$ is one-to-one, the Galilean group is isomorphic to the group of pairs $(\mathbf{v},V)$ with composition given by the equation

$$(\mathbf{u},U)(\mathbf{v},V) = (\mathbf{u}+U\mathbf{v},UV) \tag{24G}$$

We recognize this product of pairs as a semidirect product. The group of pairs $(\mathbf{u},U)$, where $\mathbf{u}\in I\!R^3$ and $U\in SO(3)$, is thus the semidirect product group

$$I\!R^3 ⓢ SO(3) \tag{25G}$$

of the group $(I\!R^3,+)$ and the group $SO(3)$. The group

$$I\!R^3 = (I\!R^3,+) \tag{26G}$$

is the common Euclidean 3-space possessing the binary operation +, the common vector addition, which is both associative and commutative.

Following the introduction of the notion of the quasidirect product in Section 4, eqs. (24G), (25G) and (26G) can be generalized to accommodate the Lorentz group. The Lorentz transformation group is parametrized, Fig. 1, by two parameters, $(\mathbf{v},V)$, as it is the case with the Galilean transformation group. The first parameter is a velocity parameter v, $\mathbf{v}\in I\!R_c^3$, and the second one is an orientation parameter V, $V\in SO(3)$. Since the correspondence between a Lorentz transformation $L\{\mathbf{v};V\}$ and its parameters $(\mathbf{v},V)$ is one-to-one, and since the Lorentz transformation composition is given by parameter composition, eq. (21L), the Lorentz group is isomorphic to the group of pairs $(\mathbf{v},V)$ with composition given by the equation

$$(\mathbf{u},U)(\mathbf{v},V) = (\mathbf{u}*U\mathbf{v},\mathrm{tom}[\mathbf{u};U\mathbf{v}]UV) \tag{24L}$$

We recognize this product as a quasidirect product. The group of pairs $(\mathbf{u},U)$, where $\mathbf{u}\in I\!R_c^3$ and $U\in SO(3)$, is thus the quasidirect product group

$$I\!R_c^3 ⓠ SO(3) \tag{25L}$$

of the *weakly associative group*

$$I\!R_c^3 = (I\!R_c^3,*,\mathrm{tom}) \tag{26L}$$

and the group $SO(3)$. The weakly associative group $(I\!R_c^3,*,\mathrm{tom})$ possesses (i) a binary operation $*$, the common relativistic velocity addition law, and a precession-mapping *tom*,

$$\mathrm{tom}: I\!R_c^3\times I\!R_c^3 \rightarrow SO(3)$$

the Thomas precession of special relativity. While the Galilean counterpart of the binary operation $*$ is the binary operation +, there is no Galilean counterpart to the Lorentz precession-mapping *tom*.

Employing the associativity of the composition (24L) in the group $I\!R_c^3 ⓠ SO(3)$ of pairs $(\mathbf{u},U)$ one may readily find that for any three elements $\mathbf{u},\mathbf{v},\mathbf{w}\in I\!R_c^3$ we have

$$\mathbf{u}*(\mathbf{v}*\mathbf{w}) = (\mathbf{u}*\mathbf{v})*\mathrm{tom}[\mathbf{u};\mathbf{v}]\mathbf{w} \tag{27}$$

Eq. (27) expresses a weak form of an associative law for the triple $(I\!R_c^3,*,\mathrm{tom})$, discovered in ref. 14. It is known in the literature that the Thomas rotation gives rise to a weak commutative law for the triple $(I\!R_c^3,*,\mathrm{tom})$,

$$\mathbf{u}*\mathbf{v} = \mathrm{tom}[\mathbf{u};\mathbf{v}](\mathbf{v}*\mathbf{u}) \tag{28}$$

for any two elements $\mathbf{u},\mathbf{v}\in I\!R_c^3$; see for instance refs. 1,5,20.

The weakly associative-commutative group $I\!R_c^3 = (I\!R_c^3, *, \text{tom})$, thus, possesses properties similar to those of a group:

(*i*)	$\mathbf{u}*\mathbf{v} \in I\!R_c^3$	Closure
(*ii*)	$\mathbf{u}*\mathbf{v} = \text{tom}[\mathbf{u};\mathbf{v}](\mathbf{v}*\mathbf{u})$	Weak commutative law
(*iiia*)	$\mathbf{u}*(\mathbf{v}*\mathbf{w}) = (\mathbf{u}*\mathbf{v})*\text{tom}[\mathbf{u};\mathbf{v}]\mathbf{w}$	Right weak associative law
(*iiib*)	$(\mathbf{u}*\mathbf{v})*\mathbf{w} = \mathbf{u}*(\mathbf{v}*\text{tom}[\mathbf{v};\mathbf{u}]\mathbf{w})$	Left weak associative law
(*iv*)	$\mathbf{0}*\mathbf{u} = \mathbf{u}*\mathbf{0} = \mathbf{u}$	Existence of identity
(*v*)	$(-\mathbf{u})*\mathbf{u} = \mathbf{u}*(-\mathbf{u}) = \mathbf{0}$	Existence of inverse.
(*vi*)	$\text{tom}[\mathbf{u};\mathbf{v}] = \text{tom}[\mathbf{u}*\mathbf{v};\mathbf{v}]$	Loop property

Due to the loop property the weakly associative-commutative group $(I\!R_c^3, *, \text{tom})$ forms a *loop*[3], that is, a binary system in which each of the two equations $\mathbf{u}*\mathbf{x}=\mathbf{v}$ and $\mathbf{y}*\mathbf{u}=\mathbf{v}$ can be solved for $\mathbf{x}$ and $\mathbf{y}$. These properties have been discovered in 1965 by Karzel in a totally different context, and studied by Kerby, Wefelscheid and others[10,11,22,23]. It is thus interesting to realize that the algebraic structure underlying the harmonious interplay of the Thomas precession of special relativity and relativistically admissible velocities has already been discovered elsewhere. In order to demonstrate the importance of the loop property, let us consider the role it plays, together with the right weak associative law, in solving the equation $\mathbf{x}*\mathbf{u} = \mathbf{v}$ for the unknown $\mathbf{x} \in I\!R_c^3$, where $\mathbf{u}$ and $\mathbf{v}$ are any two given elements of $I\!R_c^3 = (I\!R_c^3, *, \text{tom})$.

$$\mathbf{x} = \mathbf{x}*\mathbf{0} = \mathbf{x}*(\mathbf{u}*(-\mathbf{u})) = (\mathbf{x}*\mathbf{u})*\text{tom}[\mathbf{x};\mathbf{u}](-\mathbf{u})$$

$$= (\mathbf{x}*\mathbf{u})*\text{tom}[\mathbf{x}*\mathbf{u};\mathbf{u}](-\mathbf{u}) = \mathbf{v}*\text{tom}[\mathbf{v};\mathbf{u}](-\mathbf{u})$$

Since our study of the quasidirect product group as a natural generalization of the semi-direct product group is guided by the generalization of the Galilean group provided by the Lorentz group, it would be instructive to consider a basic distinction between the analogous parametrized generic elements, $G\{\mathbf{v};V\}$ and $L\{\mathbf{v};V\}$, of the Galilean and the Lorentz groups. The second parameter, V, in the parametrized Galilean transformation $G\{\mathbf{v};V\}$ is *ignorable* in the sense that one may require, by convention, all inertial frames to be constructed parallel to one another, so that the only relative orientation between inertial frames is given by the identity matrix I, $I \in SO(3)$. The general Galilean transformation, $G\{\mathbf{v};I\}$, can then be parametrized by a single parameter, $G\{\mathbf{v}\}$, with composition given by the equation

$$G\{\mathbf{u}\}G\{\mathbf{v}\} = G\{\mathbf{u}+\mathbf{v}\} \tag{29}$$

as we see from eq. (21G) which, for the special case when $U = V = I$, takes the form

$$G\{\mathbf{u};I\}G\{\mathbf{v};I\} = G\{\mathbf{u}+\mathbf{v};I\} \tag{30G}$$

In contrast to the Galilean transformation, the second parameter, V, in the parametrized Lorentz transformation $L\{\mathbf{v};V\}$ is not ignorable since, as we see from eq. (21L), the Lorentz counterpart of eq. (30G) is

$$L\{\mathbf{u};I\}L\{\mathbf{v};I\} = L\{\mathbf{u}*\mathbf{v};\text{tom}[\mathbf{u};\mathbf{v}]\} \tag{30L}$$

where, in general, $\text{tom}[\mathbf{u};\mathbf{v}] \neq I$. A Lorentz counterpart of eq. (29), thus, does not exist.

Finally, we may remark that groups possessing a quasidirect product structure, like that of the Lorentz group, are not rare. Thus, for instance, it follows from well-known properties of matrices[6] that the group P_n of all $n \times n$ real matrices with positive determinant possesses a quasidirect product structure,

$$P_n \equiv S_n \circledQ SO(n)$$

where $S_n \subset P_n$ is the subset of all $n \times n$ real symmetric matrices with positive determinant, and $SO(n)$ is the group of all $n \times n$ real orthogonal matrices with determinant 1. Indeed, the matrices in S_n do not form a group, but they do form a weakly associative-commutative group. By introducing the concept of the quasidirect product group into abstract group theory as a natural generalization of the well-known concepts of the semidirect and the direct product group, and by demonstrating its relevance we have completed the task we faced in this article.

6. AN ELEGANT EXAMPLE OF A WEAKLY ASSOCIATIVE-COMMUTATIVE GROUP

The semidirect product is a product between two *groups*, plenty concrete examples of which are available in the literature. In contrast, the quasidirect product is a product between a *weakly associative group* and a group; and the first published concrete example of a weakly associative group appeared only in 1988[14,15]. It is therefore useful to indicate that following wide interest in the weakly associative group, many new concrete examples are likely to be discovered, either by the technique developed in ref. 19 or by other methods. Such an indication is provided in this section by presenting a *nonstandard* relativistic velocity composition law, in addition to the standard one in (9), which gives rise to an elegant, interesting weakly associative-commutative group.

Let $\oplus$ be a binary operation on $\mathbb{R}_c^3$ given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}-\frac{1}{c^2}(\mathbf{u}\times\mathbf{v})\times}{(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})^2+\frac{1}{c^4}(\mathbf{u}\times\mathbf{v})^2}(\mathbf{u}+\mathbf{v}), \qquad \mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$$

giving rise to a nonstandard relativistic velocity composition law which reduces to the standard, Einstein's velocity addition law (23) for parallel velocities when $\mathbf{u}$ and $\mathbf{v}$ are parallel.

The composite velocity $\mathbf{u}\oplus\mathbf{v}$ is the sum in $\mathbb{R}_c^3$ of two vectors, $(\mathbf{u}\oplus\mathbf{v})_\parallel$ and $(\mathbf{u}\oplus\mathbf{v})_\perp$, which are respectively parallel and perpendicular to $\mathbf{u}+\mathbf{v}$ in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$,

$$(\mathbf{u}\oplus\mathbf{v})_\parallel = \frac{1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}}{(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})^2+\frac{1}{c^4}(\mathbf{u}\times\mathbf{v})^2}(\mathbf{u}+\mathbf{v})$$

$$(\mathbf{u}\oplus\mathbf{v})_\perp = -\frac{1}{c^2}\frac{1}{(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})^2+\frac{1}{c^4}(\mathbf{u}\times\mathbf{v})^2}(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}+\mathbf{v})$$

The *parallel sum* $(\mathbf{u}\oplus\mathbf{v})_\parallel$ is symmetric in $\mathbf{u}$ and $\mathbf{v}$ while The *perpendicular sum* $(\mathbf{u}\oplus\mathbf{v})_\perp$ is antisymmetric in $\mathbf{u}$ and $\mathbf{v}$. The square magnitude of $\mathbf{u}\oplus\mathbf{v}$ is symmetric in $\mathbf{u}$ and $\mathbf{v}$,

$$(\mathbf{u}\oplus\mathbf{v})^2 = \frac{(\mathbf{u}+\mathbf{v})^2}{(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})^2+\frac{1}{c^4}(\mathbf{u}\times\mathbf{v})^2}$$

satisfying $(\mathbf{u}\oplus\mathbf{v})^2<c^2$; and $\lim_{\mathbf{v}^2\to c^2}(\mathbf{u}\oplus\mathbf{v})^2=c^2$ for any $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$.

As in the standard case, the nonstandard velocity composition operation $\oplus$ is, in general, neither commutative nor associative; and this "deficiency" is rectified by means of a *nonstandard* Thomas rotation (or precession) in the same way that it is rectified in the standard case by means of the standard Thomas precession.

The nonstandard Thomas rotation is the mapping

$$\tau:\mathbb{R}_c^3\times\mathbb{R}_c^3\to \mathrm{Aut}(\mathbb{R}_c^3)$$

defined as

$$\tau[\mathbf{u};\mathbf{v}] = I+\frac{2}{c^2}\frac{-(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})+\frac{1}{c^2}(\mathbf{u}\times\mathbf{v})\times}{(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})^2+\frac{1}{c^4}(\mathbf{u}\times\mathbf{v})^2}(\mathbf{u}\times\mathbf{v})\times$$

where I is the identity automorphism. In calculating the effect of $\tau[\mathbf{u};\mathbf{v}]$ on, say, $\mathbf{w}\in\mathbb{R}_c^3$ one should note that the cross product is not associative and that the effect involves the product $(\mathbf{u}\times\mathbf{v})\times((\mathbf{u}\times\mathbf{v})\times\mathbf{w})$ rather than the trivial product $((\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{v}))\times\mathbf{w}\equiv 0$.

Similarly to the triple $(\mathbb{R}_c^3, *, \mathrm{tom})$, the triple $(\mathbb{R}_c^3, \oplus, \tau)$ turns out to be a weakly associative-commutative group, that is, it possesses the following properties for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}_c^3$.

(*i*)	$\mathbf{u}\oplus\mathbf{v}\in\mathbb{R}_c^3$	Closure
(*iia*)	$\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})=(\mathbf{u}\oplus\mathbf{v})\oplus\tau[\mathbf{u};\mathbf{v}]\mathbf{w}$	Right weak associative law
(*iib*)	$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w}=\mathbf{u}\oplus(\mathbf{v}\oplus\tau[\mathbf{v};\mathbf{u}]\mathbf{w})$	Left weak associative law
(*iii*)	$\mathbf{u}\oplus\mathbf{v}=\tau[\mathbf{u};\mathbf{v}](\mathbf{v}\oplus\mathbf{u})$	Weak commutative law
(*iv*)	$\mathbf{0}\oplus\mathbf{v}=\mathbf{v}$	Existence of identity
(*v*)	$-\mathbf{v}\oplus\mathbf{v}=\mathbf{0}$	Existence of inverse
(*vi*)	$\tau[\mathbf{u};\mathbf{v}]=\tau[\mathbf{u}\oplus\mathbf{v};\mathbf{v}]$	Loop property

The weakly associative-commutative group $(\mathbb{R}_c^3, \oplus, \tau)$ gives rise to an obvious group, that is, to the quasidirect product group $(\mathbb{R}_c^3, \oplus, \tau)\circledS SO(3)$ where the composition law is given by the equation

$$(\mathbf{u},U)(\mathbf{v},V)=(\mathbf{u}\oplus U\mathbf{v}, \tau[\mathbf{u};U\mathbf{v}]UV)$$

for all $\mathbf{u},\mathbf{v}\in(\mathbb{R}_c^3, \oplus, \tau)$ and $U, V\in SO(3)$.

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Abraham A. Ungar

Department of Mathematics
North Dakota State University
Fargo, North Dakota 58105

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ON FAMILIES OF HOLOMORPHIC FUNCTIONS WITH RESTRICTED BOUNDARY VALUES

Elias Wegert

Constantin Carathéodory[1)2)4)] and Leopold Fejér[4)] initiated with their work on bounded harmonic functions a research project that is up to now an area of intensive study and presently includes more and more qualitative, quantitative and algorithmic aspects.

From a first, mainly function theoretically oriented period especially the papers of G. Pick[11)], I.Schur[12)] and R. Nevanlinna[10)] have enduring influence to forthcoming developments.

The irresistible progress of what is now frequently called Schur analysis started at the moment when methods of functional analysis and operator theory entered the scene, which placed the problem in a new perspective. Due to a variety of interrelations to other themes, such as invariant subspaces, reproducing kernel Hilbert spaces, Krein space geometry, commutant lifting theorems, extensions of positive/contractive operators, et al., and in view of an ample list of applications (pediction theory, scattering theory, control theory, signal processing, digital filter design) the subject has attracted considerable interest (cf. I. Gohberg[6)]).

Replacing, in the classical versions of the problem alluded to above, the unit ball in H^∞ by a set of bounded holomorphic functions whose boundary values are subject to a more universal sort of restrictions originates a new program of investigation. The kind of generalization we have in mind effects that methods of nonlinear functional analysis become relevant for tackling these problems. In particular, it turned out that there is a close interrelation with an old nonlinear boundary value problem posed by B. Riemann, which was satisfactorily solved for the first time by A.I. Shnirel'man[13].

In the paper at hand we present a couple of results which have proven to be useful in various applications, for instance in approximation theory (Wegert[15]), control theory (Wegert[15]; cf. Helton and Howe[7], Helton, Schwarz, and Warschawski[8], Hui[9] for different approaches to the problem), and nonlinear singular integral equations (Wegert[17]). The basis of our approach is a variational principle which may, in a more general context, substitute the maximum modulus principle or the Schwarz lemma (cf. Theorem 3). We wish to take the opportunity to demonstrate the handling of this technique proving a generalization of the following result, commonly attributed to Carathéodory:

Let B denote the closed unit ball in H^∞, the Hardy space of functions holomorphic and bounded in the open complex unit disk D. Carathéodory's result says that B is the closure of the set of all finite Blaschke products

$$c \prod_{k=1}^{n} \frac{z-a_k}{1-\bar{a}_k z}, \quad a_k \in D, \quad c \in T := \partial D,$$

with respect to the topology induced by uniform convergence on compact subsets of D.

The subject of our investigation is the set of bounded holomorphic functions

$$A := \{w \in H^{\infty}: w(t) \in \text{clos int } M_t \quad \text{a.e. on} \quad T\}$$

where $\{M_t\}_{t \in T}$ is a prescribed family of simple closed curves in the complex plane. Here $\text{int } M_t$ denotes the bounded component of $\mathbb{C} \setminus M_t$ and clos refers to the closure of a set.

We suppose that the so-called restriction manifold

$$M := \bigcup_{t \in T} \{t\} \times M_t$$

is a C^1-submanifold of $T \times \mathbb{C}$ which is transverse to each plane $\{t\} \times \mathbb{C}$, $(t \in T)$. The special choice $M_t = T$ $(t \in T)$ yields that $A = B$.

To classify the restriction manifolds according to the structure of the corresponding sets A we give the following definitions.

A restriction manifold M is called **regularly (holomorphically) traceable** if there exists a function $w_M \in H^{\infty} \cap C$ ($H^{\infty} \cap C$ denoting the space of functions holomorphic in D which are continuously extendible onto clos D) with $w_M(t) \in \text{int } M_t$ $(t \in T)$.

A restriction manifold M is said to be **singularly (holomorphically) traceable** if it is not regularly traceable and there exists a function $w_M \in H^{\infty} \cap C$ such that $w_M(t) \in \text{clos int } M_t$ $(t \in T)$.

If M is neither regularly nor singularly traceable we speak of a **nontraceable** M.

In dependence on whether M is regularly traceable, singularly traceable, or nontraceable, we write $M \in \mathcal{R}$, $M \in \mathcal{S}$, or $M \in \mathcal{N}$.

This classification of the set of admissible restriction manifolds is closely related to the number $\#A$ of elements in A.

<u>Theorem 1</u>.

(i) $M \in \mathcal{N}$ if and only if $\#A=0$.

(ii) $M \in \mathcal{S}$ if and only if $\#A=1$.

(iii) $M \in \mathcal{R}$ if and only if $\#A>1$.

A proof can be found in Wegert[14].

The next question is to ask for the natural generalization of the finite Blaschke products. For instance, putting $M_t:=T$, the finite Blaschke products can be characterized as those functions in $H^\infty \cap C$ for which $|w(t)| = 1$ on T, i.e.

$$w(t) \in M_t \quad \text{on} \quad T. \tag{1}$$

Therefore we are looking for the solutions of the nonlinear boundary value problem (1). Problems of this kind are frequently called Riemann-Hilbert problems (RHPs).

The solutions of the RHP (1) can be classified by their winding number about M, defined as

$$\text{wind}_M w := \text{wind}(w-m),$$

where $m \in C(T)$ satisfies $m(t) \in \text{int}\, M_t$ and the wind on the right stands for the usual winding number about zero. In the standard case $M_t=T$ the winding number $\text{wind}_M w$ is the order of the Blaschke product but in general it may also be a negative number.

We denote the set of all solutions to (1) (endowed with the topology of $H^\infty \cap C$) by $\mathbb{W}$ and the subset of solutions with $\text{wind}_M w=n$ by $\mathbb{W}_n$. The main result about the solvability of (1) shows that $\mathbb{W}$ has the same structure as the set of finite Blaschke products, provided that $\#A>1$.

Theorem 2.

(i) Suppose that $M \in \mathcal{S}$ and let $w_M \in H^\infty \cap C$ be a function with $w_M(t) \in \text{clos int } M_t$ $(t \in T)$. Then $A = W = \{w_M\}$ and $w_M \in W_n$ for some $n<0$.

(ii) Suppose that $M \in \mathcal{R}$ and let $w_M \in H^\infty \cap C$ be a function with $w_M(t) \in \text{int } M_t$ $(t \in T)$. Choose a point t_o on T, a point w_o on M_t, m different points $z_1, z_2, \ldots, z_m$ in D $(m \geq 0)$, and m natural numbers $n_1, \ldots, n_m$; put $n := n_1 + \ldots + n_m$. Then there exists exactly one solution in $H^\infty \cap C$ of (1) which belongs to W_n and satisfies the additional constraints

$$w(t_o) = w_o, \tag{2}$$

$$\frac{d^j}{dz^j}(w - w_M)(z_k) = 0 \quad (k=1,\ldots,m,\ j=0,\ldots,n_k-1). \tag{3}$$

Further $W_n = 0$ if $n<0$.

(iii) Fix $z_1, \ldots, z_m \in D$, $n_1, \ldots, n_m \in \mathbb{Z}$ $(n_j \geq 1)$, and let W^* denote the set of solutions to (1), (3) which belong to W_n $(n := n_1 + \ldots + n_m)$. Then the map

$$W^* \longrightarrow M_t, \quad w \longmapsto w(t)$$

is a homeomorphism (if W^* and M_t are endowed with the topology of $H^\infty \cap C$ and $\mathbb{C}$, respectively).

For a proof we refer the reader to Wegert[14] and the references therein. Some of the assertions in (ii) (under slightly stronger assumptions) are already contained in a paper by A.I. Shnirel'man[13].

Another common property of the solutions to the RHP (1) and of Blaschke products is their occurence as solutions of certain extremal problems. For instance, an immediate consequence of the maximum principle says that if b is a finite Blaschke product of order n with the zeros $z_1,\ldots,z_m$ and $w \in B$ satisfies $w(z_k)=0$ $(k=1,\ldots,m)$ then

$$|w(z)| \leqq |b(z)| \qquad z \in D. \tag{4}$$

Equality in (4) for one $z \in D^* := D \setminus \{z_1,\ldots,z_m\}$ implies that $w = e^{i\alpha} b$.

Now replace the unit ball B by the set A, fix m different points $z_1,\ldots,z_m \in D$ and $n := n_1+\ldots+n_m$ complex numbers w_k^j $(k=1,\ldots,m,\ j=0,\ldots,n_k-1)$. We study the following interpolation problem of Pick-Nevanlinna type:

Find all functions in A which meet the interpolation conditions

$$\frac{d^j w}{dz^j}(z_k) = w_k^j \quad (k=1,\ldots,m, \quad j=0,\ldots,n_k-1). \tag{5}$$

The set of solutions to this problem is denoted by A^*. Further we pick a point $z \in D^*$ and ask for the variability region

$$E^*(z) := \{w(z): \ w \in A^*\}.$$

If $z=z_k$ $(k\in\{1,\ldots,m\})$ we put

$$E^*(z_k) := \{ \frac{d^{n_k}}{dz^{n_k}} w(z_k): \ w \in A^*\}.$$

Of course, $E^*(z)$ degenerates to a point if $\#A^*=1$. The converse can be proved using Theorems 1 and 2. Notice, however, that the case where (5) has at least two solutions is more interesting.

Theorem 3. Let $z \in D$ and suppose that $\# A^* > 1$.

(i) The set $E^*(z)$ is a closed Jordan domain.

(ii) If $W \in E^*(z) \setminus \partial E^*(z)$ then there exists an infinitely differentiable function $w \in A^*$ with $w(t) \in \text{int}\, M_t$ $(t \in T)$ and $w(z)=W$.
Further, $\{w \in A \cap \mathbb{W}_k : w(z)=W\} \neq \emptyset$, for each $k \in \mathbb{Z}$ with $k \geq n$.

(iii) If $W \in \partial E^*(z)$ then there exists exactly one function $w \in A$ with $w(z)=W$. This function belongs to $\mathbb{W}_n$.

(iv) The map $A \to E^*(z)$, $w \mapsto w(z)$ induces a homeomorphism between the set $\mathbb{W}^* := A^* \cap \mathbb{W}_n$ and $E^*(z)$ (endowed with the topologies of $H^\infty \cap C$ and $\mathbb{C}$ respectively).

(v) If z approaches $t \in T$ then $E^*(z)$ tends to clos int M_t, in the sense that for each compact subset F of int M_t and each open set G containing clos int M_t there exists a $\delta > 0$ such that $F \subset E^*(z) \subset G$, provided that $|z-t| < \delta$.

Proof. The assertions (i)-(iv) easily follow from the results in Wegert[14]. It remains to verify (v). By (i) and (iv), $\mathbb{W}^*$ is homeomorphic to T and thus it is a compact subset of $H^\infty \cap C$. Consequently, the family $\mathbb{W}^*$ is equicontinuous on $D \cup T$ and hence there exists a $\delta > 0$ with the property that

$$w \in \mathbb{W}^* \quad |z-t| < \delta \quad \Longrightarrow \quad |w(z)-w(t)| < \varepsilon. \tag{6}$$

Recall that according to Theorem 2(iii) the curve M_t can be described as

$$M_t = \{w(t): \; w \in \mathbb{W}^*\}.$$

If z_o denotes any point of F, the index (winding number) of the oriented curve M_t about z_o is one. Since the index is stable with respect to small perturbations, the Jordan curve

$$E^*(z) = \{w(z): \ w \in W^*\}$$

has also index one about z_o if $|z-t| < \delta$ (cf.(6)). Here δ can be chosen independently of $z_o \in F$. So we have $F \subset$ clos int $\partial E^*(z) = E^*(z)$. An analogous reasoning proves that $E^*(z) \subset G$. ∎

Theorem 3 can be used to examine the solvability of the Pick-Nevanlinna interpolation problem introduced above (see Wegert[14)17)]). The crucial point is to find the solutions of the corresponding nonlinear Riemann-Hilbert problems, which can be done in practice using numerical methods (see Wegert[16)]).

Here we shall apply Theorem 3 with $m=1$, $z_1=0$, $n_1=n$ to generalize Carathéodory's result mentioned above.

<u>Theorem 4</u>. The set A is the closure of W (the set of solutions to $w(t) \in M_t$ $(t \in T)$) with respect to the uniform convergence on compact subsets of D:

$$A = \text{clos } W.$$

<u>Proof</u>. We take an arbitrary function $w_o \in A$ and let

$$w_o(z) = c_o + c_1 z + \ldots + c_k z^k + \ldots$$

be its Taylor expansion. Our claim is the existence of a sequence of functions $w_n \in W_n$ such that

$$w_n(z) - w_o(z) = O(z^n).$$

This would imply that w_n converges to w_o uniformly on compact subsets of D since all functions w_n are uniformly bounded in H^∞ and hence have uniformly bounded

Taylor coefficients, which yields the estimate

$$|w_n(z)-w_o(z)| \leq C|z|^n/(1-|z|).$$

So the inclusion $A \subset \text{clos}\, \mathbb{W}$ is proven once we have shown the existence of the sequence $\{w_n\}$. To construct $\{w_n\}$ we adapt Carathéodory's[3] (p.13) approach and define

$$A[c_o,\ldots,c_{n-1}] := \{w \in A:\ w(z)-c_o-c_1 z-\ldots-c_{n-1}z^{n-1} = O(z^n)\},$$

$$E := \{w(0):\ \ w \in A\},$$

$$E[c_o,\ldots,c_{n-1}] := \{\ \frac{1}{n!}\frac{d^n w}{dz^n}(0):\ \ w \in A[c_o,\ldots,c_{n-1}]\}.$$

Theorem 3 tells us that $E[c_o,\ldots,c_{n-1}]$ is a closed Jordan domain if and only if $\#A[c_o,\ldots,c_{n-1}] > 1$. Since $w_o \in A[c_o,\ldots,c_{n-1}]$ we have $c_o \in E$ and $c_n \in E[c_o,\ldots,c_{n-1}]$ for all $n \geq 1$.

If $\#E=1$ then $A=\{w_o\}$ and $w_o \in \mathbb{W}_n = \mathbb{W}$ for some $n<0$, and thus there is nothing to prove.

So let us assume that $\#E[c_o,\ldots,c_{k-1}]>1$ for some $k \geq 0$ (we put $E[c_o,\ldots,c_{k-1}]:=E$ if $k=0$). Further we suppose that the sequence $w_1,\ldots,w_k$ is already constructed.

If $c_k \in \partial E[c_o,\ldots,c_{k-1}]$ then, by Theorem 3(iii), the set $A[c_o,\ldots,c_k]$ contains only one function (namely w_o) and this function must belong to $\mathbb{W}_k$. In this case Theorem 4 is trivial.

If $c_k \in E[c_o,\ldots,c_{k-1}] \setminus \partial E[c_o,\ldots,c_{k-1}]$, Theorem 3(ii) shows the existence of a function $w_{k+1} \in \mathbb{W}_{k+1}$ with

$$w_{k+1}(z) = c_o + c_1 z + \dots + c_k z^k + O(z^{k+1}).$$

Furthermore, applying Theorem 3(ii) we get a function w^*_{k+1} in $H^\infty \cap C$ with

$$w^*_{k+1}(z) = c_o + c_1 z + \dots + c_k z^k + O(z^{k+1})$$

and $w^*_{k+1}(t) \in \text{int } M_t$ on T. Then for arbitrarily chosen $c \in \mathbb{C}$ with sufficiently small absolute value the function $w = w^*_{k+1} + cz^{k+1}$ belongs to the set $A[c_o,\dots,c_k]$. Hence $\#E[c_o,\dots,c_k] > 1$ and repeating the above construction yields inductively the sequence $\{w_n\}$.

It still remains to show that $\text{clos } \mathbb{W} \subset A$. Let $\{w_n\} \subset \mathbb{W}$ be a sequence which converges to w_o uniformly on each compact subset of D. Since $\{w_n\}$ is uniformly bounded, w_o belongs to H^∞. Let us assume that $w_o \notin A$. Then there is a point $t \in T$ where the nontangential limit $W_o := \lim_{z \to t} w_o(z)$ exists but does not belong to $\text{clos int } M_t$. In this case, $d := \text{dist}(W_o, \text{clos int } M_t) > 0$, and hence one can find a sequence of points $z^*_k \in D$ converging to t such that

$$\text{dist}(w_o(z^*_k), \text{clos int } M_t) > d/2.$$

This implies the existence of a sequence of numbers n_k with the property that

$$\text{dist}(w_{n_k}(z^*_k), \text{clos int } M_t) > d/4. \tag{7}$$

Obviously, $w_{n_k}(z_k^*) \in E(z_k^*) := \{w(z_k^*) : w \in A\}$, and, by Theorem 3(v), $E(z_k^*) \to \text{clos int } M_t$ as $k \to \infty$, which contradicts (7). Consequently $w_o \in A$. ∎

Remark. M.A. Efendiev[5)] studied the case where the origin belongs to int M_t for each $t \in T$. His result is that the zero-function can be uniformly approximated by a sequence of functions $\{w_n\} \subset W$ with a certain prescribed asymptotic behavior of arg w on the boundary.

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Elias Wegert
Bergakademie Freiberg
Sektion Mathematik
Akademiestrasse 6
Freiberg
DDR - 9200

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ON EQUATIONS IN BANACH SPACES INVOLVING COMPOSITION PRODUCTS OF SET-VALUED MAPPINGS

F. Williamson

In this paper we derive existence results for equations in Banach spaces involving composition products of a finite number of set-valued mappings with convex compact images.

Introduction

The aim of this paper is to derive existence results for nonlinear set-value problems in Banach spaces, of the form:

(P) $$x \in \Phi_r \circ \Phi_{r-1} \circ \dots \Phi_1(x)\,,\ x \in \overline{U}$$

where U denotes a bounded subset of a Banach space and $\Phi_r \circ \Phi_{r-1} \circ \dots \Phi_1$ a finite composition product of uppersemicontinuous set-valued mappings with convex compact images.

Such problems are met in the study of processes having several stages and provided with boundary conditions.

1. Notations and Definitions

We first recall some definitions about composition products of set-valued mappings with convex compact images. We denote by X and Y two topological spaces and by $\Phi : X \to 2^Y$ a given set-valued mapping. This mapping is called *uppersemicontinuous* at the point $x_0 \in X$ if for each open neighborhood V of the image $\Phi(x_0)$ in Y, there exists a neighborhood

$U_V(x_0)$ of x_0 in X, depending on V, such that $\Phi(U_V(x_0)) \subset V$.

Definition 1.1. Let X_1 be a Banach space and U a given subset of X_1. A set-valued mapping Φ from $\overline{U}$ to $2^{X}1$ will be called a ***convex compact product*** of set-valued mappings if and only if there exists a finite number of Banach spaces $X_2, X_3, \ldots X_r$ together with uppersemicontinuous set-valued mappings having convex compact images: $\Phi_1 : \overline{U} \to \Gamma_K(X_2)$ with $\Phi_1(\overline{U})$ relatively compact (where we denote by $\Gamma_K(E)$ the family of all convex compact subsets of the vector space E), $\Phi_i : X_i \to \Gamma_K(X_{i+1}), i = 2, \ldots r$ satisfying: $\Phi = \Phi_r \circ \ldots \circ \Phi_1$.

Definition 1.2. Let U denote an open bounded set of the Banach space X. Let for $i = 0, 1, \Phi^{(i)} = \Phi_r^{(i)} \circ \ldots \Phi_1^{(i)}$ from $\overline{U}$ to 2^X denote two convex compact products of set-valued mappings. The mappings $f^{(0)} = \mathrm{Id}_X - \Phi^{(0)}$ and $f^{(1)} = \mathrm{Id}_X - \Phi^{(1)}$ are called ***homotopically equivalent*** with respect to U and 0 if there exists a family of convex compact products of set-valued mappings defined on $\overline{U} \times I\, \Phi(\cdot, t) = \Phi_r(\cdot, t) \circ \ldots \circ \Phi_1(\cdot, t)$ such that: $\Phi_j(\cdot, 0) = \Phi_j^{(0)}, \Phi_j(\cdot, 1) = \Phi_j^{(1)}, j = 1, \ldots r$ and if in addition the following condition is satisfied: $0 \notin f(\partial U \times I)$ with $f(\cdot, t) = \mathrm{Id}_X - \Phi(\cdot, t)$ and $I = [0, 1]$.

2. A Degree Notion for Convex Compact Products of Set-Valued Mappings

For convex compact products of set-valued mappings it is possible to establish approximation and homotopy properties that may be found in [6]. If $\mathrm{Id}_X - \Phi$ with $\Phi = \Phi_r \circ \ldots \circ \Phi_1$ denotes a convex compact product of set-valued mappings such that: $0 \notin [\mathrm{Id}_X - \Phi](\partial U)$ and if $\mathrm{Id}_X - \bar{\Phi}_{r,\eta} \circ \ldots \bar{\Phi}_{1,\eta}$ and $\mathrm{Id}_X - \bar{\bar{\Phi}}_{r,\eta} \circ \ldots \circ \bar{\bar{\Phi}}_{1,\eta}$ denote two single-valued η-approximations of the same composed mapping in the closure $\overline{U}$ of the set U of the Banach space X where $\bar{\Phi}_{k,\eta}$ resp. $\bar{\bar{\Phi}}_{k,\eta}$ denotes a single-valued continuous approximation of Φ_k in the sense of [2] Thm. 7.3.3, then these approximations are by Proposition 2.2 of [6] homotopically equivalent with respect to U and to the zero vector and therefore they have the same Leray-Schauder degree (for this degree, or shortly L. S. degree, of a single-valued mapping $\mathrm{Id}_X - \varphi$, where $\varphi : \overline{U} \to X$ is a continuous compact mapping, the notation $d(\mathrm{Id}_X -$

$\varphi, U, 0)$ will be used).

On account of the above mentioned fact the following extension of this degree notion to the case of convex compact products of set-valued mappings seems to have some interest.

Definition 2.1. Let U denote an open bounded subset of the Banach space X and $\Phi = \Phi_r \circ \ldots \circ \Phi_1 : \overline{U} \to 2^X$ a convex compact product of set-valued mappings for which the following holds: $0 \notin [\mathrm{Id}_X - \Phi](\partial U)$. Let for $k = 1, 2, \ldots r$, $\bar{\Phi}_{k,\eta}$ denote a single-valued η-approximation of Φ_k in $M_k = \overline{\mathrm{Co}}\, \Phi_{k-1}(M_{k-1})$ with $M_1 = \overline{U}$. Then we define the *degree* of $\mathrm{Id}_X - \Phi$ with respect to the set U and the null vector by:

$$d(\mathrm{Id}_X - \Phi, U, 0) = \lim_{\eta \to 0+} d(\mathrm{Id}_X - \bar{\Phi}_{r,\eta} \circ \ldots \circ \bar{\Phi}_{1,\eta}, U, 0)\,.$$

For $p \notin [\mathrm{Id}_X - \Phi](\partial U)$ we set similarly:

$$d(\mathrm{Id}_X - \Phi, U, p) = \lim_{\eta \to 0+} d(\mathrm{Id}_X - \bar{\Phi}_{r,\eta} \circ \ldots \bar{\Phi}_{1,\eta}, U, p)\,. \qquad (2.1)$$

It next follows from the translation invariance of the L. S. degree for single-valued continuous mappings that:

$$d(\mathrm{Id}_X - \Phi, U, p) = d(\mathrm{Id}_X - \Phi - p, U, 0)\,. \qquad (2.1)'$$

We now prove a few properties of the same degree.

Proposition 2.1. Under the assumptions of Definition 2.1 the following properties of $\mathrm{Id}_X - \Phi$ with $\Phi = \Phi_r \circ \ldots \circ \Phi_1$ hold:

1) If $d(\mathrm{Id}_X - \Phi, U, 0) \neq 0$, then there exists $x^+ \in X$ for which:

$$x^+ \in \Phi(x^+) = \Phi_r \circ \ldots \circ \Phi_1(x^+), x^+ \in U\,.$$

2) Let $(U_\lambda)_{\lambda \in \Lambda}$ denote an arbitrary family of disjoint open subsets of U and let the following condition be satisfied:

$$0 \notin [\mathrm{Id}_X - \Phi]\,(\overline{U} \backslash (\bigcup_{\lambda \in \Lambda} U_\lambda))\,.$$

Then the following relation holds in which on the right hand side an at most finite number of terms vanishes:

$$d(\mathrm{Id}_X - \Phi, U, 0) = \sum_{\lambda \in \Lambda} d([\mathrm{Id}_X - \Phi]|_{\overline{U}_\lambda}, U_\lambda, 0)\,.$$

3) Let $\Phi^{(i)} = \Phi_r^{(i)} \circ \ldots \circ \Phi_1^{(i)}, i = 0, 1,$ denote two convex compact products of set-valued mappings in the same set $\overline{U}$. If $\mathrm{Id}_X - \Phi^{(0)}$ and $\mathrm{Id}_X - \Phi^{(1)}$ are homotopic in the sense of Definition 1.2, then the following equality holds:

$$d(\mathrm{Id}_X - \Phi^{(0)}, U, 0) = d(\mathrm{Id}_X - \Phi^{(1)}, U, 0)\,.$$

4) Let Y denote a closed vector subspace of the Banach space X such that:

$$Y \supset \overline{\mathrm{Co}}\, \Phi_r(\overline{\mathrm{Co}}\Phi_{r-1}(\ldots(\overline{\mathrm{Co}}\Phi_1(\overline{U}))) \cup \{p\} \text{with}\, U \cap Y \neq \emptyset \tag{2.2}$$

then:

$$d(\mathrm{Id}_X - \Phi, U, p) = d(\mathrm{Id}_X - \Phi|_{\overline{U \cap Y}}, U \cap Y, p)\,. \tag{2.3}$$

5) Let p_0 and p_1 be two points from the same connected component of $X \backslash [\mathrm{Id}_X - \Phi]\,(\partial U)$, then:

$$d(\mathrm{Id}_X - \Phi, U, p_0) = d(\mathrm{Id}_X - \Phi, U, p_1)\,. \tag{2.4}$$

6a) Let $\Phi_1, \ldots, \Phi_r$ and $\Psi_1, \ldots, \Psi_r$ denote uppersemicontinuous mappings such that:

$$\Phi_1, \Psi_1\ : \overline{U} \to \Gamma_K(X_2) \text{ with } \Phi_1(\overline{U})\,, \Psi_1(\overline{U}) \text{ relatively compact}\,,$$
$$\Phi_i, \Psi_i\ : X_i \to \Gamma_K(X_{i+1})\,, i = 2, 3, \ldots r$$

and let us assume that:

$$0 \notin [\mathrm{Id}_X - \mathrm{Co}(\Phi_r(\cdot) \cup \Psi_r(\cdot)) \circ \ldots \circ \mathrm{Co}(\Phi_1(\cdot) \cup \Psi_1(\cdot))]\,(\partial U) \tag{2.5}$$

then:

$$d(\mathrm{Id}_X - \Phi_r \circ \ldots \circ \Phi_r, U, 0) = d(\mathrm{Id}_X - \Psi_r \circ \ldots \Psi_1, U, 0)\,. \tag{2.6}$$

b) Let the following boundary condition be satisfied in which $V_i \subset X_{i+1}$, $i = 1, 2, \ldots r$, denote convex neighborhoods of the null vector:

$$0 \notin \{\mathrm{Id}_X - [\Phi_r(\cdot) + V_r] \circ \ldots \circ [\Phi_1(\cdot) + V_1]\}\,(\partial U) \tag{2.7}$$

and let $\Psi_1 : \overline{U} \to \Gamma_K(X_2)$ with $\Psi_1(\overline{U})$ relatively compact, $\Psi_i : X_i \to \Gamma_K(X_{i+1}), i = 2, 3, \ldots r$ denote r uppersemicontinuous set-valued mappings for which the following holds:

$$\begin{cases} \Psi_1(x_1) \subset \Phi_1(x_1) + V_1 \ \forall x_1 \in \partial U \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \Psi_r(x_r) \subset \Phi_r(x_r) + V_r \ \forall x_r \in [\Phi_{r-1}(\cdot) + V_{r-1}] \circ \ldots \circ [\Phi_1(\cdot) + V_1](\partial U) . \end{cases} \tag{2.8}$$

Then relation (2.6) still holds.

Proof.

1) Let $\alpha_n = \frac{1}{n}$ with $n > 0$ integer. By definition (2.1) there exist for n large enough and $i = 1, \ldots, r$ single-valued continuous α_n-approximations $\tilde{\Phi}_{i,\alpha_n}$ of the set-valued mappings Φ_i in the sets $M_i = \overline{\text{Co}}\Phi_{i-1}(M_{i-1})$ with $M_1 = \overline{U}$ such that:

$$d(\text{Id}_X - \tilde{\Phi}_{r,\alpha_n} \circ \ldots \circ \tilde{\Phi}_{1,\alpha_n}, U, 0) = d(\text{Id}_X - \Phi_r \circ \ldots \circ \Phi_1, U, 0) \neq 0 .$$

Therefore by the corresponding property of the L. S. degree there exists $x_n \in \overline{U}$ such that:

$$x_n \in \tilde{\Phi}_{r,\alpha_n} \circ \ldots \circ \tilde{\Phi}_{1,\alpha_n}(x_n) \tag{2.9}$$

and as the range of $\tilde{\Phi}_{r,\alpha}$ is contained in $M_{r+1} = \overline{\text{Co}}\Phi_r(M_r^n)$, it follows that $x_n \in M_{r+1}$. By the uppersemicontinuity of the Φ_i and the compactness of their images as well as of $\Phi_1(\overline{U})$, the set M_{r+1} is compact and hence (x_n) contains a subsequence converging to a limit $x^+ \in \overline{U}$. Moreover it follows from the definition of the $\tilde{\Phi}_{i,\alpha_n}$ that:

$$\tilde{\Phi}_{1,\alpha_n}(x_1) \in \Phi_1((x_1 + \alpha_n B_{X_1}) \cap \overline{U}) + \alpha_n B_{X_2} \quad \forall x_1 \in \overline{U}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\tilde{\Phi}_{r,\alpha_n}(x_r) \in \Phi_r((x_r + \alpha_n B_{X_r}) \cap M_r) + \alpha_n B_{X_1} \quad \forall x_r \in M_r$$

where we denote by B_{X_i} the unit ball of the space X_i. Therefore it follows from (2.8) that there exist vectors $w_n^{X_1}, (u_n^{X_k})_{k=1,2,\ldots,r}, (\zeta_{k,n})_{k=1,\ldots,r-1}$ such that:

$$x_n - \alpha_n w_n^{X_1} \in \Phi_r(\zeta_{r-1,n} + 2\alpha_n u_n^{X_r}) \quad \text{with} : x_n + \alpha_n u_n^{X_1} \in \overline{U}$$

$$\zeta_{k-1,n} \in \Phi_{k-1}(\zeta_{k-2,n} + 2\alpha_n u_n^{X_{k-1}}) \quad k = 3, \ldots, r$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\zeta_{1,n} \in \Phi_1(x_n + \alpha_n u_n^{X_1}), \ w_n^{X_1} \in B_{X_1}, u_n^{X_k} \in B_{X_k}, \ k = 1, \ldots r$$

$$\zeta_{k-1,n} + 2\alpha_n u_n^{X_k} \in M_k , \ k = 2, \ldots, r .$$

By the uppersemicontinuity of the Φ_i and the compactness of their images there exists a further subsequence (x_μ) of (x_ν) such that $\zeta_{k,\mu} \to \zeta_k^+, k = 1,\dots,r-1$ with:

$$\zeta_1^+ \in \Phi_1(x^+), \zeta_2^+ \in \Phi_2(\zeta_1^+), \dots, \zeta_{r-1}^+ \in \Phi_{r-1}(\zeta_{r-2}^+), x^+ \in \Phi_r(\zeta_{r-1}^+).$$

These relations yield together:

$$x^+ \in \overline{U}, x^+ \in \Phi(x^+).$$

As $d(\mathrm{Id}_X - \Phi, U, 0)$ is well defined, we have: $0 \notin [\mathrm{Id}_X - \Phi](\partial U)$ and this implies $x^+ \in U$.

2) As the U_λ are open subsets of U we have:

$$\partial U, \partial U_\gamma \subset \overline{U} \setminus (\bigcup_{\lambda \in \Lambda} U_\lambda) \quad \forall \gamma \in \Lambda. \tag{2.10}$$

We next consider α_n-approximations $\tilde{\Phi}_{k,\alpha_n}$ with $\alpha_n = \frac{1}{n}$ and $k = r,\dots,1$ in the subsets $M_k = \overline{\mathrm{Co}}\Phi_{k-1}(M_{k-1})$ with $M_1 = \overline{U}$ and set $\tilde{\Phi}_{\alpha_n} = \tilde{\Phi}_{r,\alpha_n} \circ \dots \circ \tilde{\Phi}_{1,\alpha_n}$. By application of Proposition 2.1 of [6] with $M = U, Q = \overline{U} \setminus (\bigcup_{\lambda \subset \Lambda} U_\lambda)$ the condition $0 \notin [\mathrm{Id}_X - \Phi](Q)$ implies the existence of n_1 large enough such that for $n \geq n_1$:

$$\begin{cases} 0 \notin x - ((1-t)\tilde{\Phi}_{r,\alpha_n} + t\Phi_r) \circ \dots \circ ((1-t)\tilde{\Phi}_{1,\alpha_n} + t\Phi_1)(x) \\ \text{for all} (x,t) \in Q \times I \end{cases} \tag{2.11}$$

and in particular by (2.10) for all $(x,t) \in \partial U \times I$ resp. $\partial U_\lambda \times I, \lambda \in \Lambda$. Therefore we have for the same values of n:

$$d(\mathrm{Id}_X - \Phi, U, 0) = d(\mathrm{Id}_X - \tilde{\Phi}_{\alpha_n}, U, 0) \tag{2.12}$$

$$d(\mathrm{Id}_X - \Phi, U_\lambda, 0) = d(\mathrm{Id}_X - \tilde{\Phi}_{\alpha_n}, U_\lambda, 0) \quad \forall \lambda \in \Lambda. \tag{2.13}$$

It follows from (2.10) that:

$$0 \notin [\mathrm{Id}_X - \tilde{\Phi}_{\alpha_n}](\overline{U} \setminus (\bigcup_{\lambda \in \Lambda} U_\lambda))$$

and therefore the generalized excision property for the L. S. degree yields:

$$d(\mathrm{Id}_X - \tilde{\Phi}_{\alpha_n}, U, 0) = \sum_{\lambda \in \Lambda} d(\mathrm{Id}_X - \tilde{\Phi}_{\alpha_n}|_{\overline{U}_\lambda}, U_\lambda, 0) \tag{2.14}$$

where at most finitely many terms on the right-hand side are non zero. The announced excision property 2) follows from (2.12), (2.13), (2.14).

3) We denote by $\Phi^{(i)} = \Phi_r^{(i)} \circ \ldots \circ \Phi_1^{(i)}, i = 0, 1$, two convex compact products of set-valued mappings and assume that $\mathrm{Id}_X - \Phi^{(0)}$ and $\mathrm{Id}_X - \Phi^{(1)}$ are homotopically equivalent with respect to the set U and the zero vector. Then by Proposition 2.2 of [6] for $\alpha > 0$ their approximations $\tilde{\Phi}_\alpha^{(i)} = \tilde{\Phi}_{r,\alpha}^{(i)} \circ \ldots \circ \tilde{\Phi}_{1,\alpha}^{(i)}$ are also homotopically equivalent as single-valued continuous approximations and therefore we have:

$$d(\mathrm{Id}_X - \tilde{\Phi}_\alpha^{(0)}, U, 0) = d(\mathrm{Id}_X - \tilde{\Phi}_\alpha^{(1)}, U, 0). \tag{2.15}$$

It further follows from Definition 2.1 that:

$$d(\mathrm{Id}_X - \Phi^{(i)}, U, 0) = \lim_{\alpha \to 0+} d(\mathrm{Id}_X - \tilde{\Phi}_\alpha^{(i)}, U, 0), i = 0, 1. \tag{2.16}$$

The desired homotopy invariance property results from (2.15) and (2.16).

4) For simplicity we assume $p = 0$. It follows from Definition 2.1 of the degree that for $\eta > 0$ small enough:

$$d(\mathrm{Id}_X - \Phi, U, 0) = d(\mathrm{Id}_X - \tilde{\Phi}_\eta, U, 0) \tag{2.17}$$

with $\tilde{\Phi}_\eta = \tilde{\Phi}_{r,\eta} \circ \ldots \circ \tilde{\Phi}_{1,\eta}$. It follows from (2.2) together with: $\tilde{\Phi}_\eta(\overline{U}) \subset \overline{\mathrm{Co}}\Phi_r(\ldots(\overline{\mathrm{Co}}\Phi_1(\overline{U}))\ldots)$, where the set on the right-hand side is relatively compact, that $\tilde{\Phi}_\eta(\overline{U}) \subset Y$. Therefore the reduction property for the L. S. degree yields:

$$d(\mathrm{Id}_X - \tilde{\Phi}_\eta, U, 0) = d(\mathrm{Id}_X - \tilde{\Phi}_\eta|_{\overline{U \cap Y}}, U \cap Y, 0). \tag{2.18}$$

As $\partial_Y(U \cap Y) \subset \partial U \cap Y$, it follows from the boundary condition satisfied by Φ together with Proposition 2.1 from [6] and for $\eta > 0$ small enough that:

$$\begin{cases} 0 \notin x - ((1-t_r)\tilde{\Phi}_{r,\eta} + t_r\Phi_r) \circ \ldots \circ ((1-t_1)\tilde{\Phi}_{1,\eta} + t_1\Phi_1)(x) \\ \text{for all } x \in \partial_Y(U \cap Y) \quad \text{and} \quad t_1, \ldots t_r \in I \end{cases}$$

and hence the mappings $\mathrm{Id}_X - \Phi$ and $\mathrm{Id}_X - \tilde{\Phi}_\eta$ are homotopically equivalent with respect to $U \cap Y$, from where it follows that

$$d(\mathrm{Id}_X - \Phi|_{\overline{U \cap Y}}, U \cap Y, 0) = d(\mathrm{Id}_X - \tilde{\Phi}_\eta|_{\overline{U \cap Y}}, U \cap Y, 0). \tag{2.19}$$

The relations (2.17), (2.18) and (2.19) yield together:

$$d(\mathrm{Id}_X - \Phi, U, 0) = d(\mathrm{Id}_X - \Phi|_{\overline{U \cap Y}}, U \cap Y, 0).$$

5) As the space X admits a neighborhood basis consisting of convex sets, it is arcwise connected and as p_0 and p_1 belong to the same connected component of $X \backslash [\mathrm{Id}_X - \Phi](\partial U)$ there exists a single-valued continuous mapping $p : [0,1] \to X \backslash [\mathrm{Id}_X - \Phi](\partial U)$ with $p(0) = p_0, p(1) = p_1$ and we have for all $t \in I : p(t) \notin [\mathrm{Id}_X - \Phi](\partial U)$. Thus the mapping γ defined for all $(x,t) \in \overline{U} \times I$ by $\gamma(x,t) = [\mathrm{Id}_X - \Phi](x) - p(t)$ is a homotopy of convex compact products connecting $\mathrm{Id}_X - \Phi - p_0$ and $\mathrm{Id}_X - \Phi - p_1$ and therefore we have:

$$\begin{aligned} d(\mathrm{Id}_X - \Phi, U, p_0) &= d(\mathrm{Id}_X - \Phi - p_0, U, 0) = d(\mathrm{Id}_X - \Phi - p_1, U, 0) \\ &= d(\mathrm{Id}_X - \Phi, U, p_1). \end{aligned}$$

6a) The relations:

$$[t\Phi_k + (1-t)\Psi_k](x_k) \subset \mathrm{Co}(\Phi_k(x_k) \cup \Psi_k(x_k))$$
$$\text{for all}\,(x_k, t) \in X_k \times I \quad \text{and} \quad k = 2, \dots r$$

yield together with (2.5):

$$\left.\begin{array}{l} 0 \notin x - [t\Phi_r + (1-t)\Psi_r] \circ \dots \circ [t\Phi_1 + (1-t)\Psi_1](x) \\ \text{for all} \quad (x,t) \in \partial U \times I \end{array}\right\} \tag{2.20}$$

whence (2.6) results by the homotopy invariance property.

6b) Assuming now that (2.8) is satisfied by $\Psi_1, \dots, \Psi_r$ we have for arbitrary $t \in [0,1]$:

$$t\Phi_1(x_1) + (1-t)\Psi_1(x_1) \subset \Phi_1(x_1) + V_1 \quad \text{for any } x_1 \in \partial U$$
$$t\Phi_2(x_2) + (1-t)\Psi_2(x_2) \subset \Phi_2(x_2) + V_2 \quad \text{for any } x_2 \in \Phi_1(\partial U) + V_1 .$$

Taking into account relation (2.7) we obtain again relation (2.20), so that (2.6) still holds.

Q. E. D.

Our next result is concerned with an existence property for solutions of equations involving convex compact products of set-valued mappings.

Proposition 2.2. Let U and Φ be given as above and let ω be a fixed point of U. if the following boundary condition is satisfied:

(B) $\qquad \Phi(x) \not\ni \omega + \theta(x - \omega) \quad \text{for any} \quad x \in \partial U \text{ and } \theta > 1$

then we have:

1) $d(\mathrm{Id}_X - \Phi, U, 0) = 1$ if $0 \notin [\mathrm{Id}_X - \Phi](\partial U)$
2) there exists $x^+ \in \overline{U}$ such that $x^+ \in \Phi(x^+)$.

Proof. By replacing Φ resp. U by $\Phi_\omega(\cdot) = \Phi(\cdot + \omega) - \omega$ resp. $U_\omega = U - \omega$ condition (B) may be rewritten as:

$$[\mathrm{Id}_X - t\Phi_\omega]\ (y) \not\ni 0 \quad \text{for any} \quad (y,t) \in \partial U_\omega \times]0,1[\,. \tag{2.21}$$

If $0 \notin [\mathrm{Id}_X - \Phi](\partial U)$, then $0 \notin [\mathrm{Id}_X - \Phi_\omega](\partial U_\omega)$ so that relation (2.21) is satisfied for all $(y,t) \in \partial U_\omega \times [0,1]$ and it follows from assertion 3) of Proposition 2.1 that:

$$d(\mathrm{Id}_X - \Phi, U, 0) = d(\mathrm{Id}_X - \Phi_\omega, U_\omega, 0) = d(\mathrm{Id}_X, U_\omega, 0) = 1\,.$$

Let us assume moreover that $0 \notin [\mathrm{Id}_X - \Phi](\overline{U})$, then it follows from assertion 2) of Proposition 2.1 that $d(\mathrm{Id}_X - \Phi, U, 0) = 0$, which implies a contradiction with assertion 1). Thus we obtain $0 \in [\mathrm{Id}_X - \Phi](\overline{U})$.

Q. E. D.

We will finally extend the preceding fixed point result to set-valued mappings which are approximable in the sense of the asymmetric Hausdorff metric:

(Hf)
$$\Delta(M_1, M_2) = \operatorname*{Sup}_{x_1 \in M_1} d(x, M_2)$$

by convex compact products of set-valued mappings.

Proposition 2.3. Let U denote as above a non-empty open and bounded subset of the Banach space X and let $\Phi : \overline{U} \to 2^X$ be a set-valued mapping having closed images and such that moreover $[\mathrm{Id}_X - \Phi](\overline{U})$ is closed. We assume that Φ satisfies the following boundary condition in which ω denotes some given point of U and η a positive eventually small constant:

(B_ω^η)
$$\Phi(x) + \eta B_X \not\ni \omega + \theta(x - \omega) \quad \text{for all } x \in \partial U \text{ and } \theta > 1\,.$$

Under the additional assumption that there exists a family $\Phi^{(k)}$ of convex compact products of set-valued mappings approximating Φ in the following sense:

(AP)
$$\lim_{k \to +\infty} \sup_{u \in \overline{U}} \Delta(\Phi^{(k)}(u), \Phi(u)) = 0$$

with Δ defined as in (Hf), the set-valued mapping Φ admits a fixed point in $\overline{U}$.

Proof. We will consider the following family of approximate fixed point problems:

$$(\mathrm{P_k}) \qquad \text{Find } u \in \overline{U} \quad \text{such that} \quad u \in \Phi^{(k)}(u)\,.$$

As a consequence of the approximation property (AP) there exists for any given $\varepsilon > 0$ less than η some integer k_ε such that $k \geq k_\varepsilon$ implies:

$$\Phi^{(k)}(u) \subset \Phi(u) + \varepsilon B_X \quad \text{for all} \quad u \in \overline{U}\,. \tag{2.22}$$

It thus follows from (B^η_ω) together with (2.22) that:

$$\Phi^{(k)}(x) \not\ni \omega + \theta(x - \omega) \quad \text{for any} \quad x \in \partial U \quad \text{and} \quad \theta > 1\,.$$

As a consequence of Proposition 2.2, $\Phi^{(k)}$ admits a fixed point $u_k \in \overline{U}$ and we have further:

$$0 \in [\mathrm{Id}_X - \Phi^{(k)}](u_k) \subset u_k - \Phi(u_k) + \varepsilon B_X \subset [\mathrm{Id}_X - \Phi](\overline{U}) + \varepsilon B_X$$

and hence $[\mathrm{Id}_X - \Phi](\overline{U}) \cap \varepsilon B_X \neq \emptyset$ for any $\varepsilon \in]0, \eta]$. As $[\mathrm{Id}_X - \Phi](\overline{U})$ is by assumption closed, it follows that $0 \in [\mathrm{Id}_X - \Phi](\overline{U})$ which terminates the proof.

Q. E. D.

The closedness assumption on the set $[\mathrm{Id}_X - \Phi](\overline{U})$ is certainly satisfied if Φ has closed graph and if $\Phi(\overline{U})$ is relatively compact.

Moreover assumption (B^η_ω) may be replaced by simpler ones as this is shown in the following Corollary.

Corollary 2.1. The result of Proposition 2.3 still holds if condition (B^η_ω) is replaced by either of the following ones:

$$(\overline{\mathrm{R}}) \qquad \begin{cases} \text{the set } U \text{ is convex and } \Phi \text{ is such that } \overline{\Phi(\partial U)} \subset U \\ \text{with } \Phi(\partial U) \text{ relatively compact} \end{cases}$$

$$(\overline{\mathrm{S}}) \qquad \begin{cases} X \text{ is a Hilbert space and } \Phi \text{ satisfies:} \\ \displaystyle \operatorname*{Sup}_{\substack{y \in \Phi(x) \\ x \in \partial U}} (y - x, x - \omega) < 0\,. \end{cases}$$

Proof. It suffices to show for each of the conditions $(\overline{\mathrm{R}})$ and $(\overline{\mathrm{S}})$ that they imply condition (B_ω^η) for some $\eta > 0$.

Case of condition $(\overline{\mathrm{R}})$

This condition may be rewritten as $\overline{\Phi(\partial U)} \cap [X \backslash U] = \emptyset$ and as $\overline{\Phi(\partial U)}$ is compact and $X\backslash U$ is closed, there exists by a well-known result some $\eta > 0$ such that:

$$[\Phi(\partial U) + \eta B_X] \cap [(X\backslash U) + \eta B_X] = \emptyset \tag{2.23}$$

and thus $\Phi(\partial U) + \eta B_X \subset U$. Therefore for an arbitrary $x \in \partial U$ and $z \in \Phi(x) + \eta B_X$ we have $z \in U$ and hence by the convexity of this set:

$$x \neq (1-t)\omega + tz \quad \text{for any} \quad t \in]0,1[\,,$$

a relation which is equivalent to (B_ω^η).

Case of condition $(\overline{\mathrm{S}})$

This condition together with the boundedness of U implies the existence of $\eta > 0$ small enough such that:

$$\operatorname*{Sup}_{y \in \Phi(x)} (y - x, x - \omega) + \eta \|x - \omega\| \leq 0 \quad \text{for any} \quad x \in \partial U\,. \tag{2.24}$$

By performing on U and Φ the same translation as in the proof of Proposition 2.2 we may assume that $\omega = 0$. As in the proof of Proposition 2.3 for given $\varepsilon > 0$ at most equal to η there exists an integer k_ε such that (2.22) holds for $k \geq k_\varepsilon$. We will show that the following boundary condition is satisfied by $\Phi^{(k)}$ for $k \geq k_\varepsilon$:

$$\Phi^{(k)}(x) \not\ni \theta x \quad \text{for any} \quad x \in \partial U \quad \text{and} \quad \theta > 1\,. \tag{2.25}$$

Indeed if there would exist $x_0 \in \partial U, z_0 \in \Phi^{(k)}(x_0)$ and $\theta_0 > 1$ such that $z_0 = \theta_0 x_0$, then there would exist by (2.22) $y_0 \in \Phi(x_0)$ and $u_0 \in B_X$ such that $z_0 = y_0 + \varepsilon u_0$ and we would have:

$$(y_0 + \varepsilon u_0, x_0) = (y_0, x_0) + \varepsilon(u_0, x_0) = \theta_0 \|x_0\|^2 \tag{2.26}$$

with $\theta_0 > 1$.

Further it follows from (2.24) that:

$$(y_0 - x_0, x_0) + \eta \|x_0\| \leq 0 . \qquad (2.27)$$

As $\varepsilon \in]0, \eta]$ the relations (2.26) and (2.27) imply together that:

$$\begin{aligned} \theta_0 \|x_0\|^2 &= (y_0, x_0) + \varepsilon(u_0, x_0) \\ &\leq (y_0, x_0) + \varepsilon \|x_0\| \\ &\leq (y_0, x_0) + \eta \|x_0\| \leq \|x_0\|^2 \end{aligned}$$

which cannot hold since $\theta_0 > 1$ and $0 \notin \partial U$. Therefore relation (2.25) is satisfied and hence for all $k \geq k_\varepsilon$, $\Phi^{(k)}$ admits a fixed point $u^{(k)}$ in $\overline{U}$. The proof may be completed as with Proposition 2.3. Q.E.D.

To conclude this paper we point out that it is possible to show by an argument similar to that of Proposition 2.3 that under the conditions:

$$\begin{cases} \text{Graph}\, \Phi + \eta(B_X \times B_X) \not\ni (x, \omega + \theta(x - \omega)) \,\text{for any}\, x \in \partial U \,\text{and}\, \theta > 1 , \\ \lim\limits_{k \to \infty} \Delta(\text{Graph}\, \Phi^{(k)}, \text{Graph}\, \Phi) = 0 \end{cases}$$

the convex compact product $\Phi^{(k)}$ approximating Φ in the sense of Graphs still admits for k large enough, a fixed point $x^{(k)}$ in $\overline{U}$ and that any cluster point x^* of the sequence $x^{(k)}$ is a fixed point of Φ in $\overline{U}$.

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F. Williamson
16 Avenue de la Commune de Paris
94400 Vitry sur Seine
France

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MAPPINGS CONNECTED WITH HARMONIC FUNCTIONS OF SEVERAL VARIABLES

Algimantas Yanushauskas

A conformal mapping T realized by the analytical function $w = f(z) = u(x,y) + iv(x,y)$ of a complex variable may be considered as the mapping, realized by the gradient of some harmonic function $\varphi(x,y)$. Imaginary and real parts of the analytic function $f(z)$ satisfy the Cauchy-Riemann system [1]

$$u_x - v_y = 0\,,\; u_y + v_x = 0 \tag{1}$$

from which it follows that there exists a harmonic function such that $u = \varphi_y, v = \varphi_x$. The Jacobian of the conformal mapping T is of the form

$$J(T) = u_x v_y - u_y v_x = u_x^2 + u_y^2 = v_y^2 + v_x^2\,.$$

Consequently, this Jacobian may vanish only in such a set of points which is the set of zeroes of a gradient of some harmonic function, where the zero set of gradients of two conjugated harmonic functions coincide. Conformal mappings may be also approached from other positions [2,3]. Let the harmonic function $\varphi(x,y)$ be regular in some domain D. Let us try to construct the mapping of the domain D which transfers the level lines of the function φ into the straight ones $u =$ const whereas orthogonal trajectories of level lines of the function φ transfer into the family of orthogonal straights $v =$ const. We shall require additionally the tension coefficients along the level lines of the function φ and their orthogonal trajectories to coincide.

Let the mapping, which interests us, be given by the correlation

$$u = \varphi(x,y), v = \psi(x,y)$$

then from the above formulated properties of the mapping it follows that functions φ and ψ satisfy the correlation

$$\varphi_x\psi_x + \varphi_y\psi_y = 0\,, \; \varphi_x^2 + \varphi_y^2 = \psi_x^2 + \psi_y^2\,, \tag{2}$$

whence it follows

$$\varphi_x = \lambda\psi_y, \varphi_y = -\lambda\psi_x\,, \; \lambda^2(\psi_x^2 + \psi_y^2) = \psi_x^2 + \psi_y^2\,.$$

Thus, $\lambda = \pm 1$ and functions φ and ψ satisfy one of the systems

$$\varphi_x - \psi_y = 0\,, \; \varphi_y + \psi_x = 0\,; \varphi_x + \psi_y = 0\,, \varphi_y - \psi_x = 0\,,$$

i.e., the mapping which we are interested is either conformal or anti-conformal.

Let us find the substitutions of the variables

$$\xi = \varphi(x, y)\,, \; \eta = \psi(x, y)\,,$$

which transform the Laplace operator into the operator of the type

$$L = A\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \alpha\frac{\partial}{\partial\xi} + \beta\frac{\partial}{\partial\eta}\,.$$

By a direct calculation we obtain that the functions φ and ψ must satisfy Eqs. (2). Thus, once again we approach the mappings realized by holomorphic or antiholomorphic functions. Changes of the variables are realized by the conformal mappings of first or second kind.

Each of these three approaches to the concept of conformal mappings of flat domains allows us to generalize for a multivarite case. We shall begin with the mappings realized by the gradients of harmonic functions. Let $u(X), X = (x_1, \dots, x_n)$ be a regular harmonic function of the domain D in the space R^n. The mapping $E(u)$, realized by a gradient of this function, is described by the equations

$$v_j = \frac{\partial u}{\partial x_j}\,, \; j = 1, \dots, n\,.$$

Let $H(u)$ denote a matrix, composed from the second derivatives of the function u, i.e.,

$$H(u) = \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|\,.$$

Local homeomorphism of the mapping $E(u)$ may be disturbed only at such points at which $\det H(u) = 0$. This set contains all zeroes of the gradient of each derivative $\partial u/\partial x_j$. However, the set of zeroes of the Hessian $\det H(u)$ of the function u is not exhausted by the set of zeroes of the gradients of the derivatives of the function u. It is clearly illustrated by the example $g(x, y, z) = z(x^2 + y^2) - \frac{2}{3}z^3$. In this case we have

$$\det H(g) = 8 \begin{vmatrix} z & 0 & x \\ 0 & z & y \\ x & y & -2z \end{vmatrix} = -8z(x^2 + y^2 + 2z^2).$$

Here zeroes of the gradients of the derivatives of the function complete two straight lines $x = z = 0$ and $y = z = 0$, and the Hessian vanishes on the plane $z = 0$, containing these straight lines.

The mapping, realized by the gradient of the function g is given by the correlations

$$E(g) : u = 2zx\,, v = 2zy\,, w = x^2 + y^2 - 2z^2\,. \tag{3}$$

The entire plane $z = 0$ is transferred into a ray of the straight line $u = v = 0, 0 < w < \infty$. Everywhere outside the plane $z = 0$ the mapping $E(g)$ is a local homeomorphism. An inverse mapping maps the entire space R^3 of the variables u, v, w with the rejected beam $u = v = 0, w > 0$ onto the upper half-space $E^+ : \{z > 0\}$ and is given by the formulas

$$\begin{aligned} z &= \frac{1}{2}\{[w^2 + 2(u^2 + v^2)]^{1/2} - w\}^{1/2}\,, \\ x &= u\{[w^2 + 2(u^2 + v^2)]^{1/2} - w\}^{-1/2}\,, \\ y &= v\{[w^2 + 2(u^2 + v^2)]^{1/2} - w\}^{-1/2}\,. \end{aligned} \tag{4}$$

Though the mapping $E(g)$, given by formulas (3), is a local homeomorphism outside the plane $z = 0$, it is not homeomorphic on the whole. Its inverse mapping, the space of the variables u, v, w with the rejected beam $u = v = 0, w > 0$ also maps homeomorphically on the lower half-space $E^- : \{z < 0\}$. Moreover, it is necessary to take the right parts of formulas (4) with the opposite signs in order to get another branch of the inverse mapping. The points of the beam $u = v = 0, w > 0$ and the plane $z = 0$ are nouniformizable singular points of the corresponding mappings.

Let us consider the mapping realized by the gradient of the fundamental solution of Laplace equation which is in the form of $cr^{2-n}, r^2 = x_1^2 + \ldots + x_n^2$, c is a fixed constant. This mapping is given by the formulas

$$u_j = (2-n)cx_j[x_1^2 + \ldots + x_n^2]^{-n/2}, \quad j = 1, \ldots, n. \tag{5}$$

From these correlations we obtain

$$(x_1^2 + \ldots + x_n^2)^{-1} = [(u-2)^2 c^2]^{-\frac{1}{n-1}} [u_1^2 + \ldots + u_n^2]^{\frac{1}{n-1}}$$

hence it easily follows that the inversion of formulas (5) is expressed by

$$x_j = -[(n-2)c]^{\frac{1}{n-1}} u_j (u_1^2 + \ldots + u_n^2)^{-\frac{n}{2(n-1)}}, \quad j = 1, \ldots, n. \tag{6}$$

It is possible to consider this mapping an analogue of the inversion

$$\begin{aligned} u = -x(x^2+y^2)^{-1}, v = -y(x^2+y^2)^{-1}, \\ x = -u(u^2+v^2)^{-1}, y = -v(u^2+v^2)^{-1} \end{aligned} \tag{7}$$

which is connected with the fundamental solution of the two-dimensional Laplace equation $-\frac{1}{2}\ln(x^2+y^2)$. The inversion, expressed by formulas (7), mapps one-to-one one-point compactification of the plane on itself, besides the interior of the circle $K : \{x^2+y^2 < 1\}$ transfers to the exterior and vice versa. It also transfers circumferences and straights into circumferences and straights [1]. If in (5) we assume $c = (2-n)^{-1}$, then mapping (5) will map one-to-one one-point compactification of the space R^n on itself, besides, the interior of the ball $\sum : \{x_1^2 + \ldots + x_n^2 < 1\}$ transfers to the exterior and vice versa.

Mapping (5) for $c = (2-n)^{-1}$ the surface

$$Ar^2 + 2\lambda r\cos\theta = \gamma$$

$r^2 = x_1^2 + \ldots + x_n^2, r\cos\theta = x_1\alpha_1 + \ldots + x_n\alpha_n$, where A, γ are constants and $B = (\alpha_1, \ldots, \alpha_n)$ is a fixed point, transfers to the surfaces $\gamma\rho^{\frac{2}{n-2}} + 2\lambda\rho^{\frac{1}{n-1}}\cos\theta = A$. Consequently this mapping transfers into spheres only the spheres with the centre at the beginning of coordinates. As for the surfaces given by the equality

$$Ar^n + 2\lambda r\cos\theta = \gamma$$

it transfers to the surfaces

$$\gamma\rho^{-\frac{n}{n-1}} + 2\lambda\rho\cos\theta = A\,.$$

If $\gamma = 0$, i.e., the surface passes through the beginning of coordinates, then it turns into the plane.

The fundamental solution of the Laplace equation is determined in all the space and vanishes at infinity. In the case of the bounded domain D a similar role is played by the Green function with a fixed pole $A \in D$. Let us suppose that the domain D has so smooth a boundary Γ that the second derivatives of the Green function $G(X, A)$ are continuous up to the boundary Γ. Sometimes it will be sufficient to require such a smoothness of Γ only in the neighbourhood of some fixed point we are interested in.

Theorem 1. If at the boundary Γ of the domain D there is a tangent plane Γ, the tangency point set of which fills out the isolated variety without an edge of dimensionality above one, then on the boundary Γ of the domain D the mapping homeomorphism realized by the gradient of Green function of the domain D with a pole at any inner point is necessarily violated.

Proof. Since the variety N of tangency points of the plane T and the boundary Γ of the domain D has the dimensionality not below one and no edge, then it cannot be homeomorphic to an intercept of the straight line if it is connected. If N still splits into several coherent components, then none of them being a coherent variety without edge cannot be homeomorphic to an intercept of the straight, presenting one-dimensional variety with the edge. On the other hand, because of the fact that the Green function gradient on the boundary of the domain is orthogonal to the domain's boundary, the image of the variety N is on the straight collinear with the normal of the plane T, moreover the image of each connected component N is an intercept of this straight line.

Hence, there follows the validity of the proof of the theorem.

In particular, if the boundary Γ of the domain D contains a piece of the plane, then the mapping realized by the Green function gradient of this domain transfers this flat piece into an intercept of the straight. Consequently, in this case the homeomorphism of the mapping is obviously violated. However, the mapping may preserve the homeomorphism within the domain.

If the domain D is an exterior of the domain B homeomorphic to the ball, then the mapping, realized by the Green function gradient of the domain D with a pole at any finite point, cannot be a homeomorphism. At the infinity the Green function itself and its gradient tend to zero. Besides the Green function of the domain D with any finite pole always has at least one finite critical point C [2]. Consequently the mapping realized by the Green function gradient of the domain D transfers into one point at the infinity and the point C.

The given examples show that properties of the mapping realized by the gradients of harmonic functions of many variables differ from the properties of conformal mappings of flat domains. The properties of null sets of the Hessian harmonic function of many variables also differ from the case of two variables. In the case of two variables this set is a gradient null set of some harmonic function, and at such points where Hessian vanishes and the rank of the matrix $H(u)$, composed from the second derivatives of the given function is also equal to zero. The situation is more complicated for the harmonic functions of many variables. For example, it is known [4] that the Hessian of the harmonic function of three variables changes the sign if it vanishes but not vanishing identically. In a general case the following statement is valid.

Theorem 2. Let the Hessian $\det H(u)$ of the harmonic function $u(X)$ of the variables $X = (x_1, \dots, x_n), n \geq 2$, vanishes at the point O, where the rank of the matrix $H(u)$ at this point equals to $n-1$. Then the Hessian $\det H(u)$ changes the sign at the point O.

Proof. The fact that $\det H(u) = 0$ at the point O which without a limitation of generality may be considered the beginning of coordinates denotes that among the lines of the matrix $H(u)$ at this point there is a linear dependence. By a linear orthogonal substitution of independent variables it is always possible to achieve that at the point O correlations

$$\begin{aligned} &\operatorname{grad} \frac{\partial u}{\partial x_1} = 0\,; \\ &\det \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\| \neq 0\,, i, j = 2, \dots, n\,, \end{aligned} \tag{8}$$

would be realized under the conditions of the theorem. Consequently, we can assume at once that conditions of (8) are realized. The harmonic function u in the neighbourhood of the point O may be presented in a series according to homogeneous harmonic polynomials [5] which by virtue of conditions (8) is of the shape

$$u(X)=\sum_{j=2}^{k}P_j(X')+\sum_{j=k+1}^{\infty}P_j(X),$$
$$X'=(x_2,\ldots,x_n).$$

By virtue of the second condition (8) the system of equations

$$\xi_l=\sum_{j=2}^{k}\frac{\partial}{\partial x_l}P_j(X'),l=2,\ldots,n,$$

can be solved with respect to X' in the neighbourhood of the point O and it has a unique solution

$$x_l=x_l(\xi_2,\ldots,\xi_n),l=2,\ldots,n. \tag{9}$$

Let us make a substitution of variables (9) in the Laplace equation. It will take the form

$$\frac{\partial^2 u}{\partial x_1^2}+L(u)=0,$$
$$L=\sum_{i,j=2}^{n}A_{ij}(\Xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j}+\sum_{i=1}^{n}B_i(\Xi)\frac{\partial}{\partial\xi_i} \tag{10}$$

whereas the correlation which defines the mapping realized by the gradient u will take the form

$$v_1=\frac{\partial\psi}{\partial x_1}v_l=\xi_l+\chi_l(\Xi,x_1),$$
$$l=2,\ldots,n, \tag{11}$$

where ψ is the solution of equation (10) which in the neighbourhood of the point O acts as $0(r^{k+1})$ of the function χ_l and also as $0(r^k)$, and the derivatives of $\frac{\partial\psi}{\partial x_1}$ and all χ_l act as $0(r^{k-1})$. The Jacobian of the mapping defined by correlation (11) has the shape

$$J=\begin{vmatrix}\frac{\partial^2\psi}{\partial x_1^2} & \frac{\partial^2\psi}{\partial x_1\partial\xi_2} & \cdots & \frac{\partial^2\psi}{\partial x_1\partial\xi_n}\\ \frac{\partial\chi_2}{\partial x_1} & 1+\frac{\partial\chi_2}{\partial\xi_2} & \cdots & \frac{\partial\chi_2}{\partial\xi_n}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\chi_n}{\partial x_1} & \frac{\partial\chi_n}{\partial\xi_2} & \cdots & 1+\frac{\partial\chi_n}{\partial\xi_n}\end{vmatrix}=\frac{\partial^2\psi}{\partial x_1^2}+0(r^{2k-2}).$$

Since the operator L does not depend on x_1, then along with ψ this equation is also satisfied by $\partial^2\psi/\partial x_1^2$. From the maximum principle for the elliptical equations it follows that $\partial^2\psi/\partial x_1^2$ changes the sign at the point O. By virtue of the fact that in the neighbourhood of the point O we have

$$\frac{\partial^2\psi}{\partial x_1^2} = 0(r^{k-1}), J - \frac{\partial^2\psi}{\partial x_1^2} = 0(r^{2k-2})$$

and $k \geq 2$, a determinant J also changes the sign at the point O. Hence, it follows that det $H(u)$ also changes the sign at the point O, since the substitution of variables $x_1, x_2, \dots, x_n$ for $x_1, \xi_2, \dots, \xi_n$ has a different from zero Jacobian.

The theorem is proved.

It is obvious that by virtue of the harmonic function u the rank of the matrix $H(u)$ can never be equal to one. All the points of zero sets of the Hessian det $H(u)$ of the function u in which the rank $H(u)$ is smaller than $n-1$ are the critical points of the function $G(X) = \det H(u)$. By virtue of the fact that the rank $H(u)$ is smaller than $n-1$ between the lines of this matrix there are at least two linear dependencies, i.e., by linear substitution of independent variables it is possible to achieve that two lines of the matrix $H(u)$ vanish. Since a derivative of the determinant equals to the sum of determinants obtained from the original, substituting one of the lines by a line composed of the derivative elements in the line to be substituted, therefore the derivative $\frac{\partial}{\partial x_j}G(X), j = 1, \dots, n$, is the sum of determinants in each of which at least one line vanishes. Consequently at the points under consideration $\operatorname{grad} G(X) = 0$. At such points $G(X)$ may not change the sign. If at some point

$$G(X) = 0, \operatorname{grad} G(X) = 0,$$

but the function $G(X)$ does not change the sign at this point, then the given point is the extremum point of the function $G(X)$. Only at such points a problem on the homeomorphic mapping, realized by the gradient of the harmonic function requires an additional investigation.

A generalization of the interpretation of conformal mappings as the mappings transferring level lines and their orthogonal trajectories of one harmonic function into the level lines and their orthogonal trajectories of another harmonic function for a multivariate case leads to another class of mappings. At first let us consider a particular case of such mappings which are connected with the Green function [2]. Let a point A and an outgoing straight beam l_0 from it be given, let, further, a linear mapping T, transferring a unit sphere with centre at the point A into the unit sphere with centre at the point B be given, in this connection A passes to B and the beam l_0 passes to the preassigned beam $\lambda_0 = T(l_0)$, outgoing from the point B. The mapping T is the composition of parallel translation and rotation. By $G_0(X, A)$ let us denote the Green function of the domain $D \subset R^n$ with a pole at the point$A \in D$, and by $G(X, B)$ the Green function of the ball $\sum(R) : \{x_1^2 + \ldots + x_n^2 < R^2\}$ with a pole at the point B of this ball. We shall define the mapping $X : D \to \sum(R)$ in the following way. Let the point $y \in D$ be a point of surface intersection of the level $G_0(X, A) = c, 0 < c < \infty$ with the trajectory of the field grad G_0, outgoing to the point with a tangent l, then as the image $Z = \chi(y)$ of the point y in the mapping χ, consider the point of surface intersection of the level $G(X, B) = c$ with the trajectory of the field grad G entering the point B with the tangent $T(l)$. It is obvious that the mapping χ determined in such a way is determined uniquely by the corresponding Green functions and the linear orthogonal mapping T. It is also obvious that the mapping χ reflects homeomorphically some neighbourhood of the point onto the neighbourhood of the point B and if $G_0(X, A)$ has no critical points in D, the χ is the homeomorphism of D on $\sum(R)$. The linear mapping T sets the correspondence n of orthogonal directions at the point A, n orthogonal directions at the point B, i.e., sets the correspondence of frames. From the neighbourhood of the point A the mapping χ is uniquely continuing along the trajectories of the fields grad G_0 and grad G, in addition the uniqueness may be violated only at such points, at which two or more trajectories of the field grad G_0 intersect. The Green function $G(X, \gamma), \gamma = \nu^{-1}R, \nu > 1$, of the ball $\sum(R) : \{x_1^2 + \ldots + x_n^2 < R^2\}$ with the pole $B = (0, \ldots, 0, \gamma)$ writes out clearly

$$G(X, \gamma) = \left[\sum_{i=1}^{n} x_i^2 + (x_n - \nu^{-1}R)^2\right]^{\frac{2-n}{2}}$$

$$- \nu^{n-2}\left[\sum_{i=1}^{n-1} x_i^2 + (x_n - \nu R)^2\right]^{\frac{2-n}{2}}$$

moreover, without limitation of generality we assume that the pole B is on the axis Ox_n. If the pole is in the centre of the ball $\sum(R)$, then we have

$$G(X) = \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{2-n}{2}} - R^{2-n}.$$

It is obvious that the Green function gradient of the ball does not vanish anywhere in the ball. The Green function of the exterior $\Omega(R)$: $\{x_1^2 + \ldots + x_n^2 > R^2\}$ of the ball $\sum(R)$ with the pole $B = (0, \ldots, 0, \delta), \delta = \nu R$, has the shape

$$G_1(X, \delta) = \left[\sum_{i=1}^{n-1} x_i^2 + (x_n - \nu R)^2\right]^{\frac{2-n}{2}} - \nu^{2-n}\left[\sum_{i=1}^{n-1} x_i^2 + (x_n - \nu^{-1} R)^2\right]^{\frac{2-n}{2}}$$

and with the pole at infinity

$$G_1(X) = R^{2-n} - \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{2-n}{2}}.$$

By a direct calculation it can be easily checked that the Green function of the domain $\Omega(R)$ with any finite pole B has a critical point C lying on a negative semi-axis Ox_n.

The class of mappings described above, connected with Green functions, may naturally be called the class of Green mappings. It is also obvious that any flat conformal mapping of simply connected domains may be interpreted as some Green mapping, i.e., as the mapping, connected with Green functions of these flat domains [2]. For $n \geq 3$ the Green mapping of the exterior of the ball onto its interior, as a rule, is not a homeomorphism. It is homeomorphic only in such a case when it is determined by the Green function of the ball and the exterior of the ball $\Omega(R)$ with the pole to the infinity, i.e., the functions $G(X, \gamma)$ and $G_1(X)$.

By virtue of the Zaremba-Girord principle [5] on the boundary $S(R)$ of the ball $\sum(R)$ we have $\frac{\partial}{\partial n} G(X, \gamma) > 0$, where n is an inner with respect to $\sum(R)$ normal $S(R)$, consequently the trajectories of the field grad G at all the points $S(R)$ are directed inside $\sum(R)$. All these trajectories enter the pole B of the function $G(X,\gamma)$. Now let an arbitrary domain D, the boundary Γ of which has a continuous changing normal, be given; let, further, $G_0(X,\gamma)$ be the Green function of this domain with the pole A. On the boundary Γ all the trajectories of the field $\operatorname{grad} G_0$ are directed inside the domain D. While continuing inside D these trajectories either reach the pole A or part of them enters the critical point G_0 which may be contained in D. Let $y_0, \dots, y_k$ be the critical points of the function $G_0(X,A)$, numerated so that $G_0(y_0,A) \geq \dots \geq G_0(y_k,A) > 0$. Thus, $G_0(y_0,A)$ is the largest critical value of the function $G_0(X,A)$ and $G_0(y_k,A)$ is the smallest one, besides it is possible to restrict ourselves with the case when all these critical points are not degenerated. By N_j^+ let us denote a point set of the domain D, lying on the trajectories of the field $\operatorname{grad} G_0$, entering the point y_j, N_j^- is a point set of D, lying on the trajectories outgoing from the point y_j. Further let us suppose $N^+ = U_j N_j^+$, $N^- = U_j N_j^-$, $N_0 = N^+ \cap N^-$. Denote the M^--set which is obtained from $N^- \cup N_0$. The trajectories of the field $\operatorname{grad} G_0$, filling the set M^-, outgoing from the critical points $y_0, \dots, y_k$ and enter the pole A of the Green function $G_0(X,A)$. Denote by M_j^- a point set of the domain D, lying on the trajectories of the field $\operatorname{grad} G_0$, connecting the critical point y_j with the pole A, i.e., outgoing from y_j and entering A.

The Green mapping χ, determined by the Green functions $G_0(X,A)$ of the domain D and $G_1(X,\gamma)$ of the ball $\sum(R)$ maps homeomorphically the part of the domain D satisfying the inequality $G_0(X,A) > G_0(y_0,A)$ to the ball $\sum(R)$. Denote by L_j^- a point set of the ball $\sum(R)$, lying on the trajectories of the field lying in M_j^-, further denote by k_j^- a subset L_j^-, satisfying the inequalities $0 \leq G_1(X,\gamma) \leq G_0(y_j,A)$ and suppose $k^- = \cup_j k_j$. It is obvious that under the Green mapping $\chi : D \to \sum(R)$ the points of the mapping k^- have no prototypes in D while the points of the set N^+ from D have no prototypes in $\sum(R)$. If we eliminate the set N^+ from the domain D and the remaining part of D we denote by $D^+(A)$, whereas from the ball $\sum(R)$ we eliminate the set k^- and the remaining part of the ball we denote by $\sum^-(R,A)$, then the mapping χ will be a homeomorphism $D^+(A)$ on $\sum^-(R,A)$.

The sets $D^+(A)$ and $\sum^-(R,A)$ are the domains, i.e., connected by open sets. Their openness is obvious while their connectedness can be easily proved. Let us suppose that $D^+(A)$ splits into more than one connected component, then there will be at least one connected component R, to which the pole A of the Green function G_0 of the domain D does not belong. The harmonic function $G_0(X, A)$ is regular in the domain R. The boundary R may consist of pieces of the boundary Γ of the domain D and of the pieces of the variety N^+. We have

$$\frac{\partial}{\partial n} G_0(X, A) = 0, X \in N^+; G_0(X, A) = 0, X \in \Gamma.$$

Hence by virtue of the Zaremba-Giraud principle and the maximum principle [5] it follows that $G_0(X, A) = 0$ on the entire set R, therefore R cannot be an open connected component $D^+(A)$. Consequently, $D^+(A)$ consists of one connected component. Similarly, $\sum^-(R, A)$ is considered, too.

Thus, contractible by itself to the point, domain D may be mapped homeomorphically by the Green mapping on the ball only when there exists at least one point $A \in D$ such that the Green function $G_0(X, A)$ of the domain D with a pole at the point A has no critical points in D. Such domains are called harmonically weak simply connected [2]. If the Green function of the domain D with any pole $A \in D$ has no critical points, then we shall call such a domain harmonically simply connected. Some simple criteria of the weak harmonic simple connection of the domains can be easily given. For example, if the domain D is star relatively to its point A, then the Green function of this domain with the pole A has no critical points, since having taken A as a pole of the spherical system of coordinates from the Zaremba-Giraud principle, we obtain [5]

$$r \frac{\partial}{\partial r} G(X, A) < 0, X \in D.$$

Hence it follows that a convex domain in R^n is harmonically simply connected because it is star with respect to its every point. In a general case the problem of the harmonic simple connection of the domain or its weak harmonic simple connection is sufficiently complicated and little investigated until now. Generally speaking, not every homeomorphic to the ball domain is harmonically simply connected [2], while the problem on the weak harmonic simple connection remains open.

In the above description of the Green mappings one of the domains was fixed by the ball $\sum(R)$. From a more general view-point it is possible to take two domains D_1 and D_2 and their Green functions $G_1(X, A)$ and $G_2(X, B)$ with the arbitrary fixed poles $A \in D_1$ and $B \in D_2$. Now we shall give the following.

Definition 1. Let there be given two domains D_1 and D_2, the points

$A \in D_1$ and $B \in D_2$ and a linear orthogonal mapping T of the space R^n on itself, transferring the point A to B. Further let $G_1(X,A)$ and $G_2(X,B)$ be the Green functions of the domains D_1 and D_2 with the poles $A \in D_1$ and $B \in D_2$, respectively. The mapping χ, which at the point y lying on the intersection of the surface $G_1(X,A) = e, 0 < e < \infty$, with an orthogonal trajectory of the G_1 level surfaces entering the pole A with a tangent l compares the point $Z \in D_2$ of surface intersection of the level $G_2(X,B) = e$ with an orthogonal trajectory of surfaces of the level G_2, entering the pole B with a tangent $T(l)$ we shall call the Green mapping of the domain D_1 in D_2.

Now the Green function $G_2(X,B)$ of the domain D_2 may also have critical points. In order that the mapping χ should homeomorphically continue onto the entire domain D_1, a mutual coincidence of critical values of the functions G_1 and G_2 is necessary, i.e., the functions must have identical critical values. The correspondence of the trajectories of the fields $\operatorname{grad} G_j, j = 1,2$, at the points A and B must be given so that the trajectories outgoing from a critical point y_j of the function G_1 and entering the pole A one-to-one correspond to the trajectory of the field $\operatorname{grad} G_2$, entering the B and outgoing from the critical point Z_j of the function G_2, where $G_1(y_j, A) = G_2(Z_j, B)$. Here the critical points are also assumed degenerated.

In Definition 1 the domains D_1 and D_2 may coincide, i.e., G_1 and G_2 are the Green functions of one and the same domain but with different poles. In this case we shall call the Green mapping, defined by the functions G_1 and G_2, the Green automorphism of the domain D, while the poles of Green functions A and B shall be called the poles defining the automorphism. If the domain D is harmonically simply connected, then any pair of its points may serve as the poles, determining the Green automorphism. In a harmonically weak simply connected domain already not every pair of points may serve as the poles of the Green automorphism. It would be interesting to investigate the structural properties of the set, whose each pair of points may serve as the poles, determining the Green automorphism of the given domain.

As an example, we shall consider a family of domains $D(\alpha, R)$, which are obtained a result of rotating around the axis Ox_n of the ball segment

$$Q(\alpha, R) : \{(x_1 - \alpha)^2 + x_2^2 + \ldots + x_{n-1}^2 < R^2, x_{n-1} \geq 0\}, \alpha \leq 0 .$$

It is obvious that for $\alpha = 0$ the domain $D(\alpha, R)$ turns into a ball. For

$0 < \alpha < R$ the domain $D(\alpha, R)$ is homeomorphic to the ball looks as a ball impressed at both poles. When $\alpha > R$, the domain $D(\alpha, R)$ looks as a bagel. When $\alpha > R$, the Green function of the domain $D(\alpha, R)$ with a pole at any interior point has at least one critical point, i.e., this domain is not harmonically simply connected. When $\alpha = 0$, the domain $D(0, R)$ is harmonically simply connected, therefore even for sufficiently small $\alpha > 0$, the domain $D(\alpha, R)$ is harmonically simply connected, too. With an increase of α the harmonic simple connection of the domain $D(\alpha, R)$ may be violated [6], though the domain remains harmonically simply connected because the Green function with the pole at any intersection point of the axis Ox_n, contained in the domain $D(\alpha, R)$ is axially symmetric and has no critical points. This is also a property of all the points of the domain, located sufficiently close to this segment. Due to the symmetry of the domain any two points $D(\alpha, R), 0 < \alpha < R$, lying on the circumference, described by a fixed point of the segment $Q(\alpha, R)$ during its rotation, may serve as the poles, defining the automorphism of the domain $D(\alpha, R)$. It is possible to show that this is also valid for $\alpha > R$.

The Green mappings were considered as far as in the thirties of this century [3], they are a special case of a more wide class of mappings, to which we shall come generalizing the third interpretation of the conformal mappings. In the Laplace operator Δ we shall substitute the independent variables

$$\xi_j = \xi_j(X), j = 1, \ldots, n, X = (x_1, \ldots, x_n). \tag{12}$$

In the new variables the Laplace operator will take the form

$$\sum_{i,j=1}^{n} a_{ij}(\Xi) \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{n} B_i(\Xi) \frac{\partial}{\partial \xi_i},$$

$$a_{ij} = a_{ji} = \sum_{k=1}^{n} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_k}, B_i = \Delta \xi_i,$$

$$i, j = 1, \ldots, n.$$

Let us demand that the correlations $a_{1s} = 0, s = 2, \ldots, n$, be fulfilled, which have the form

$$\sum_{k=1}^{n} \frac{\partial \xi_1}{\partial x_k} \frac{\partial \xi_j}{\partial x_k} = 0, j = 2, \ldots, n. \tag{13}$$

We shall consider correlations (13) as a system $n-1$, of linear algebraic equations with respect to unknown $\partial\xi_1/\partial x_k$. From (13) we find

$$\frac{\partial\xi_1}{\partial x_k} = \lambda A_l(\xi_2,\dots,\xi_n)\,, l = 1,\dots,n\,, \tag{14}$$

where λ is an arbitrary function, and A_l is a cofactor of the element $\partial\xi_1/\partial x_l$ of the matrix

$$M = \left\| \frac{\partial\xi_i}{\partial x_j} \right\|, i,j = 1,\dots,n\,.$$

Taking into account the expression a_{11} and correlations (14), we find

$$a_{11} = \sum_{k=1}^{n} \left(\frac{\partial\xi_1}{\partial x_k}\right)^2 = \lambda \sum_{k=1}^{n} \frac{\partial\xi_1}{\partial x_k} A_k = \lambda \det M\,.$$

By a direct calculation the correlation

$$\sum_{l=1}^{n} \frac{\partial A_l}{\partial x_l} = 0$$

is checked. If in (14) we suppose $\lambda = 1$, then it follows from this correlation that $\xi_1(X)$ is harmonic, and for a_{11} we have

$$a_{11} = J(X) = \det M\,.$$

Considering $\lambda = 1$, in (14) we shall make the variables $\Xi = (\xi_1,\dots,x_n)$ independent, and $X = (x_1,\dots,x_n)$ dependent. The obvious correlations

$$\frac{\partial x_1}{\partial x_j} = \sum_{l=1}^{n} \frac{\partial x_i}{\partial \xi_l}\frac{\partial \xi_i}{\partial x_j} = \delta_{ij}\,, \qquad \delta_{ii} = 1\,, \delta_{ij} = 0\,, i \neq j\,, \tag{15}$$

take place, from which it follows that the matrix

$$N = \left\| \frac{\partial x_i}{\partial \xi_j} \right\|, i,j = 1,\dots,n\,.$$

is reciprocal for the matrix M, i.e., $MN = NM = E$, where E is a unit matrix. From equalities (15) it follows that

$$\frac{\partial x_j}{\partial \xi_1} = J^{-1} A_j(\xi_2,\dots,x_n)\,, \frac{\partial \xi_1}{\partial x_j} = \Delta^{-1} B_j$$

where $J = \det M, \Delta = \det N$, and $B_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ is a cofactor of the element of the j-th line and the first column of the matrix N. Taking into account (14) and $\lambda = 1$, we find

$$B_j \Delta^{-1} = A_j = J \frac{\partial x_j}{\partial \xi_1}$$

and by virtue of the fact that $MN = E$ we have $J\Delta = 1$, consequently

$$\frac{\partial x_l}{\partial \xi_1} = B_l(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n), l = 1, \dots, n. \tag{16}$$

System (16) is a system of the Cauchy-Kovalevskaya type only by variable ξ_1 and is not such by any other variable. The system, obtained from (14) for $\lambda = 1$, has the form

$$\frac{\partial \xi_1}{\partial x_l} = A_l(\xi_2, \dots, \xi_n), l = 1, \dots, n. \tag{17}$$

Both systems (16) and (17) substitute the Cauchy-Riemann system.

Consider the matrix

$$A = \|a_{ij}\|, i, j = 1, \dots, n.$$

It is obvious that $A = MM^*$, where M^* is a conjugations matrix for M, hence $\det A = (\det M)^2 = J^2$. If the functions $\xi_j(X)$ satisfy system (17), then

$$a_{1j} = a_{j1} = 0, j = 2, \dots, n \quad \text{and} \quad \det A = a_{11} \det B, \text{where}$$
$$B = \|a_{ij}\|, i, j = 2, \dots, n,$$

i.e., B is a $(n-1) \times (n-1)$ matrix. Taking into account that $a_{11} = J$, we find $\det B = J(X)$. Thus the substitution of the variables, satisfying system (17), transforms the Laplace operator Δ into the operator

$$L = a_{11} \frac{\partial^2}{\partial \xi_1^2} + \sum_{i,j=2}^{n} a_{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{j=2}^{n} B_j \frac{\partial}{\partial \xi_j},$$

in this connection the correlation

$$a_{11} = \det B, B = \|a_{ij}\|, i, j = 2, \dots, n, \tag{18}$$

is valid.

If, however, the functions $x_l(\Xi)$ satisfy system (16), then this substitution of the variables transforms the operator L into Δ. By virtue of conditions (13), det B is the Jacobian of the trace of the mapping, realized by the functions $\xi_j(X)$, on the surface of the level of the function $\xi_1(X)$. However, an analogous feature is typical for the Green mappings [2], too.

Now it is possible to give the definition of a most wide class of mappings, connected with harmonic functions and generalizing flat conformal mappings.

Definition 2. Let two harmonic functions $u_1(X)$ and $u_2(X)$ be given, regular in the domains D_1 and D_2, respectively; let, further, a mapping χ of the domains D_1 in D_2 be given. If this mapping has the following features:

1) the mapping transfers the surfaces of the level $M_\nu \; : \; \{u_1(X) = \nu\}$ into the surfaces of the level $N_\nu \; : \; \{u_2(X) = \nu\}$, and the orthogonal trajectories of the surfaces of the level $u_1(X)$ are transferred by it into the orthogonal trajectories of the surfaces $u_2(X)$;

2) the coefficient of tension along the orthogonal surfaces trajectories of the level of the functions $u_i(X), i = 1, 2$, is equal to the Jacobian of mapping ψ_ν, which is a trace χ on the surface M_ν; that mapping will be called harmonic in the M. A. Lavrentyev's sense.

The harmonic in the M. A. Lavrentyev's sense mappings of multivariate domains generalize a hydrodynamic interpretation of the conformal mappings of flat domains.