Chapter 14

Differential Games

In previous chapters, we were mainly concerned with the optimal control problems formulated by a single objective function (or a single decision maker). However, there are situations when there may be more than one decision maker, each having one's own objective function that each is trying to maximize, subject to a set of differential equations. This extension of the optimal control theory is referred to as the *theory of differential games*.

The study of differential games was initiated by Isaacs (1965). After the development of Pontryagin's maximum principle, it became clear that there was a connection between differential games and optimal control theory. In fact, differential game problems represent a generalization of optimal control problems in cases where there are more than one controller or player. However, differential games are conceptually far more complex than optimal control problems in the sense that it is no longer obvious what constitutes a solution; see Starr and Ho (1969), Ho (1970), Varaiya (1970), Friedman (1971), Leitmann (1974), Case (1979), Selten (1975), Mehlmann (1988), Berkovitz (1994), Basar and Olsder (1999), Dockner, Jørgensen, Long, and Sorger (2000), and Basar, Bensoussan, and Sethi (2010). Indeed, there are a number of different types of solutions such as minimax, Nash, Stackelberg, along with possibilities of cooperation and bargaining; see, e.g., Tolwinski (1982) and Haurie, Tolwinski, and Leitmann (1983). We will discuss minimax solutions for zero-sum differential games in Section 14.1, Nash solutions for nonzero-sum games in Section 14.2, and Stackelberg differential games in Section 14.3.

14.1 Two-Person Zero-Sum Differential Games

Consider the state equation

$$\dot{x} = f(x, u, v, t),\ x(0) = x_0, \tag{14.1}$$

where we may assume all variables to be scalar for the time being. Extension to the vector case simply requires appropriate reinterpretations of each of the variables and the equations. In this equation, we let u and v denote the controls applied by players 1 and 2, respectively. We assume that

$$u(t) \in U,\ v(t) \in V,\ t \in [0, T],$$

where U and V are convex sets in E^1. Consider further the objective function

$$J(u, v) = S[x(T)] + \int_0^T F(x, u, v, t)dt, \tag{14.2}$$

which player 1 wants to maximize and player 2 wants to minimize. Since the gain of player 1 represents a loss to player 2, such games are appropriately termed *zero-sum games*. Clearly, we are looking for admissible control trajectories u^* and v^* such that

$$J(u^*, v) \geq J(u^*, v^*) \geq J(u, v^*). \tag{14.3}$$

The solution (u^*, v^*) is known as the *minimax* solution. Here u^* and v^* stand for $u^*(t),\ t \in [0, T]$, and $v^*(t),\ t \in [0, T]$, respectively.

The necessary conditions for u^* and v^* to satisfy (14.3) are given by an extension of the maximum principle. To obtain these conditions, we form the Hamiltonian

$$H = F + \lambda f \tag{14.4}$$

with the adjoint variable λ satisfying the equation

$$\dot{\lambda} = -H_x,\ \lambda(T) = S_x[x(T)]. \tag{14.5}$$

The necessary condition for trajectories u^* and v^* to be a minimax solution is that for $t \in [0, T]$,

$$H(x^*(t), u^*(t), v^*(t), \lambda(t), t) = \min_{v \in V} \max_{u \in U} H(x^*(t), u, v, \lambda(t), t), \tag{14.6}$$

which can also be stated, with suppression of (t), as

$$H(x^*, u^*, v, \lambda, t) \geq H(x^*, u^*, v^*, \lambda, t) \geq H(x^*, u, v^*, \lambda, t) \tag{14.7}$$

for $u \in U$ and $v \in V$. Note that (u^*, v^*) is a saddle point of the Hamiltonian function H.

Note that if u and v are unconstrained, i.e., when, $U = V = E^1$, condition (14.6) reduces to the first-order necessary conditions

$$H_u = 0 \text{ and } H_v = 0, \tag{14.8}$$

and the second-order conditions are

$$H_{uu} \leq 0 \text{ and } H_{vv} \geq 0. \tag{14.9}$$

We now turn to the treatment of nonzero-sum differential games.

14.2 Nash Differential Games

In this section, let us assume that we have N players where $N \geq 2$. Let $u^i \in U^i$, $i = 1, 2, \ldots, N$, represent the control variable for the ith player, where U^i is the set of controls from which the ith player can choose. Let the state equation be defined as

$$\dot{x} = f(x, u^1, u^2, \ldots, u^N, t). \tag{14.10}$$

Let J^i, defined by

$$J^i = S^i[x(T)] + \int_0^T F^i(x, u^1, u^2, \ldots, u^N, t)dt, \tag{14.11}$$

denote the objective function which the ith player wants to maximize. In this case, a *Nash solution* is defined by a set of N admissible trajectories

$$\{u^{1*}, u^{2*}, \ldots, u^{N*}\}, \tag{14.12}$$

which have the property that

$$J^i(u^{1*}, u^{2*}, \ldots, u^{N*}) = \max_{u^i \in U^i} J^i(u^{1*}, u^{2*}, \ldots, u^{(i-1)*}, u^i, \ldots, u^{(i+1)*}, \ldots, u^{N*}) \tag{14.13}$$

for $i = 1, 2, \ldots, N$.

To obtain the necessary conditions for a Nash solution for nonzero-sum differential games, we must make a distinction between open-loop and closed-loop controls.

14.2.1 Open-Loop Nash Solution

The open-loop Nash solution is defined when (14.12) is given as functions of time satisfying (14.13). To obtain the maximum principle type conditions for such solutions to be a Nash solution, let us define the Hamiltonian functions

$$H^i = F^i + \lambda^i f \tag{14.14}$$

for $i = 1, 2, \ldots, N$, with λ^i satisfying

$$\dot{\lambda}^i = -H^i_x,\ \lambda^i(T) = S^i_x[x(T)]. \tag{14.15}$$

The Nash control u^{i*} for the ith player is obtained by maximizing the ith Hamiltonian H^i with respect to u^i, i.e., u^{i*} must satisfy

$$\begin{aligned}&H^i(x^*, u^{1*}, \ldots, u^{(i-1)*}, u^{i*}, u^{(i+1)*}, \ldots, u^{N*}, \lambda, t) \geq \\ &\qquad H^i(x^*, u^{1*}, \ldots, u^{(i-1)*}, u^{i}, u^{(i+1)*}, \ldots, u^{N*}, \lambda, t),\ t \in [0, T],\end{aligned} \tag{14.16}$$

for all $u^i \in U^i,\ i = 1, 2, \ldots, N.$

Deal, Sethi, and Thompson (1979) formulated and solved an advertising game with two players and obtained the open-loop Nash solution by solving a two-point boundary value problem. In Exercise 14.1, you are asked to obtain their boundary value problem. See also Deal (1979).

14.2.2 Feedback Nash Solution

A feedback Nash solution is obtained when (14.12) is defined in terms of the current state of the system. To avoid confusion, we let

$$u^{i*}(x, t) = \phi^i(x, t),\ i = 1, 2, \ldots, N. \tag{14.17}$$

For these controls to represent a Nash strategy, we must recognize the dependence of the other players' actions on the state variable x. Therefore, we need to replace the adjoint equation (14.15) by

$$\dot{\lambda}^i = -H^i_x - \sum_{j=1}^{N} H^i_{u^j}\phi^j_x = -H^i_x - \sum_{j=1, j\neq i}^{N} H^i_{u^j}\phi^j_x. \tag{14.18}$$

The presence of the summation term in (14.18) makes the necessary condition for the feedback solution virtually useless for deriving computational algorithms; see Starr and Ho (1969). It is, however, possible to use a dynamic programming approach for solving extremely simple nonzero-sum games, which require the solution of a partial differential equation. We will use this approach in Section 14.3.

The troublesome summation term in (14.18) is absent in three important cases: (a) in optimal control problems ($N = 1$) since $H_u u_x = 0$, (b) in two-person zero-sum games because $H^1 = -H^2$ so that $H^1_{u^2} u^2_x = -H^2_{u^2} u^2_x = 0$ and $H^2_{u^1} u^1_x = -H^1_{u^1} u^1_x = 0$, and (c) in open-loop nonzero-sum games because $u^j_x = 0$. It certainly is to be expected, therefore, that the feedback and open-loop Nash solutions are going to be different, in general. This can be shown explicitly for the linear-quadratic case.

We conclude this section by providing an interpretation to the adjoint variable λ^i. It is the sensitivity of the ith player's profit to a perturbation in the state vector. If the other players are using closed-loop strategies, any perturbation δx in the state vector causes them to revise their controls by the amount $\phi^j_x \delta x$. If the ith Hamiltonian H^i were extremized with respect to u^j, $j \neq i$, this would not affect the ith player's profit; but since $\partial H^i / \partial u^j \neq 0$ for $i \neq j$, the reactions of the other players to the perturbation influence the ith player's profit, and the ith player must account for this effect in considering variations of the trajectory.

14.2.3 An Application to the Common-Property Fishery Resources

Consider extending the fishery model of Section 10.1 by assuming that there are two producers having unrestricted rights to exploit the fish stock in competition with each other. This gives rise to a nonzero-sum differential game analyzed by Clark (1976).

Equation (10.2) is modified by

$$\dot{x} = g(x) - q^1 u^1 x - q^2 u^2 x, \ x(0) = x_0, \tag{14.19}$$

where $u^i(t)$ represents the rate of fishing effort and $q^i u^i x$ is the rate of catch for the ith producer, $i = 1, 2$. The control constraints are

$$0 \leq u^i(t) \leq U^i, \ i = 1, 2, \tag{14.20}$$

the state constraints are

$$x(t) \geq 0, \tag{14.21}$$

and the objective function for the ith producer is the total present value of his profits, namely,

$$J^i = \int_0^\infty (p^i q^i x - c^i) u^i e^{-\rho t} dt, \ i = 1, 2. \tag{14.22}$$

To find the Nash solution for this model, we let $\bar{x}^i$ denote the turnpike (or optimal biomass) level given by (10.12) on the assumption that the ith producer is the sole-owner of the fishery. Let the bionomic equilibrium x_b^i and the corresponding control u_b^i associated with producer i be defined by (10.4), i.e.,

$$x_b^i = \frac{c^i}{p^i q^i} \text{ and } u_b^i = \frac{g(x_b^i)p^i}{c^i}. \tag{14.23}$$

As shown in Exercise 10.2, $x_b^i < \bar{x}^i$, and we assume U^i to be sufficiently large so that $u_b^i \le U^i$. We also assume that

$$x_b^1 < x_b^2, \tag{14.24}$$

which means that producer 1 is more efficient than producer 2, i.e., producer 1 can make a positive profit at any level in the interval $(x_b^1, x_b^2]$, while producer 2 loses money in the same interval, except at x_b^2, where he breaks even. For $x > x_b^2$, both producers make positive profits.

Since $U^1 \ge u_b^1$ by assumption, producer 1 has the capability of driving the fish stock to a level down to at least x_b^1 which, by (14.24), is less than x_b^2. This implies that producer 2 cannot operate at a sustained level above x_b^2; and at a sustained level below x_b^2, he cannot make a profit. Hence, his optimal feedback policy is bang-bang:

$$u^{2*}(x) = \begin{cases} U^2 & \text{if } x > x_b^2, \\ 0 & \text{if } x \le x_b^2. \end{cases} \tag{14.25}$$

As far as producer 1 is concerned, he wants to attain his turnpike level $\bar{x}^1$ if $\bar{x}^1 \le x_b^2$. If $\bar{x}^1 > x_b^2$ and $x_0 \ge \bar{x}^1$, then from (14.25) producer 2 will fish at his maximum rate until the fish stock is driven to x_b^2. At this level, it is optimal for producer 1 to fish at a rate which maintains the fish stock at level x_b^2 in order to keep producer 2 from fishing. Thus, the optimal feedback policy for producer 1 can be stated as

$$u^{1*}(x) = \left\{ \begin{array}{ll} U^1 & \text{if } x > \bar{x}^1 \\ \bar{u}^1 = \frac{g(\bar{x}^1)}{q^1 \bar{x}^1} & \text{if } x = \bar{x}^1 \\ 0 & \text{if } x < \bar{x}^1 \end{array} \right\}, \text{ if } \bar{x}^1 < x_b^2, \tag{14.26}$$

$$u^{1*}(x) = \left\{ \begin{array}{ll} U^1 & \text{if } x > x_b^2 \\ \frac{g(x_b^2)}{q^1 x_b^2} & \text{if } x = x_b^2 \\ 0 & \text{if } x < x_b^2 \end{array} \right\}, \ \text{if } \bar{x}^1 \geq x_b^2. \tag{14.27}$$

The formal proof that policies (14.25)-(14.27) give a Nash solution requires direct verification using the result of Section 10.1.2. The Nash solution for this case means that for all feasible paths u^1 and u^2,

$$J^1(u^{1*}, u^{2*}) \geq J^1(u^1, u^{2*}), \tag{14.28}$$

and

$$J^2(u^{1*}, u^{2*}) \geq J^2(u^{1*}, u^2). \tag{14.29}$$

The direct verification involves defining a modified growth function

$$g^1(x) = \begin{cases} g(x) - q^2 U^2 x & \text{if } x > x_b^2, \\ g(x) & \text{if } x \leq x_b^2, \end{cases}$$

and using the Green's theorem results of Section 10.1.2. Since $U^2 \geq u_b^2$ by assumption, we have $g^1(x) \leq 0$ for $x \geq x_b^2$. From (10.12) with g replaced by g^1, it can be shown that the new turnpike level for producer 1 is $\min(\bar{x}^1, x_b^2)$, which defines the optimal policy (14.26)-(14.27) for producer 1. The optimality of (14.25) for producer 2 follows easily.

To interpret the results of the model, suppose that producer 1 originally has sole possession of the fishery, but anticipates a rival entry. Producer 1 will switch from his own optimal sustained yield $\bar{u}_1$ to a more intensive exploitation policy *prior* to the anticipated entry.

We can now guess the results in situations involving N producers. The fishery will see the progressive elimination of inefficient producers as the stock of fish decreases. Only the most efficient producers will survive. If, ultimately, two or more maximally efficient producers exist, the fishery will converge to a classical bionomic equilibrium, with zero sustained economic rent.

We have now seen that a Nash competitive solution involving $N \geq 2$ producers results in the long-run dissipation of economic rents. This conclusion depends on the assumption that producers face an infinitely elastic supply of all factors of production going into the fishing effort, but typically the methods of licensing entrants to regulated fisheries make some attempt also to control the factors of production such as permitting the licensee to operate only a single vessel of specific size.

In order to develop a model for licensing of fishermen, we let the control variable v^i denote the capital stock of the ith producer and let the concave function $f(v^i)$, with $f(0) = 0$, denote the *fishing mortality function* for $i = 1, 2, \ldots, N$. This requires the replacement of $q^i u^i$ in the previous model by $f(v^i)$. The extended model becomes nonlinear in control variables. You are asked in Exercise 14.3 to formulate this new model and develop necessary conditions for a feedback Nash solution for this model with N producers. The reader is referred to Clark (1976) for further details.

For other papers on applications of differential games to fishery management, see Hämäläinen, Haurie, and Kaitala (1984, 1985) and Hämäläinen, Ruusunen, and Kaitala (1986, 1990). For applications to problems in environmental management, see the edited volume by Carraro and Filar (1995) on the topic.

14.3 A Stochastic Nash Differential Game in Advertising

In this section, we will study a competitive extension of the Sethi advertising model discussed in Section 13.3. This will give us a stochastic differential game, for which we aim to obtain a feedback Nash equilibrium by using a dynamic programming approach developed in Section 13.1. We should note that this approach can also be used to obtain feedback Nash equilibria in deterministic differential games as an alternative to the maximum principle approach developed in Section 14.2.2.

Specifically, we consider a duopoly market in a mature product category where total sales are distributed between two firms, labeled as Firm 1 and Firm 2, which compete for market share through advertising expenditures. We let X_t denote the market share of Firm 1 at time t, so that the market share of Firm 2 is $(1 - X_t)$. Let U_{1t} and U_{2t} denote the advertising effort rates of Firms 1 and 2, respectively, at time t. Using the subscript $i \in \{1, 2\}$ to reference the two firms, let $r_i > 0$ denote the advertising effectiveness parameter, $\pi_i > 0$ denote the sales margin, $\rho_i > 0$ denote the discount rate, and $c_i > 0$ denote the cost parameter so that the cost of advertising effort u by Firm i is $c_i u^2$. Further, let $\delta > 0$ be the churn parameter, z_t be the standard one-dimensional Wiener process, and $\sigma(x)$ be the diffusion coefficient function as defined in Section 13.3. Then, in view of the competition between the firms, Prasad and Sethi (2004) extend the dynamics in (13.33) as the Itô stochastic differential equation

$$dX_t = [r_1 U_{1t}\sqrt{1-X_t} - \delta X_t - r_2 U_{2t}\sqrt{X_t} + \delta(1-X_t) + \sigma(X_t)dz_t], \quad X(0) = x_0 \in [0,1]. \tag{14.30}$$

We formulate the optimal control problem faced by the two firms as

$$\max_{u_1 \geq 0}\left\{V_1(x_0) = E\int_0^\infty e^{-\rho_1 t}\left[\pi_1 X_t - c_1 U_{1t}^2\right]dt\right\}, \tag{14.31}$$

$$\max_{u_2 \geq 0}\left\{V_2(x_0) = E\int_0^\infty e^{-\rho_2 t}\left[\pi_2(1-X_t) - c_2 U_{2t}^2\right]dt\right\}, \tag{14.32}$$

subject to (14.30). Thus, each firm seeks to maximize its expected, discounted profit stream subject to the market share dynamics.

To find the feedback Nash equilibrium solution, we form the Hamilton-Jacobi-Bellman (HJB) equations for the value function $V_1(x)$ and $V_2(x)$:

$$\rho_1 V_1 = \max_{u_1 \geq 0}\{\pi_1 x - c_1 u_1^2 + V_{1x}[r_1 u_1\sqrt{1-x} - r_2 u_2\sqrt{x} - \delta(2x-1)] + (\sigma(x))^2 V_{1xx}/2\}, \tag{14.33}$$

$$\rho_2 V_2 = \max_{u_2 \geq 0}\{\pi_2(1-x) - c_2 u_2^2 + V_{2x}[r_1 u_1\sqrt{1-x} - r_2 u_2\sqrt{x} - \delta(2x-1)] + (\sigma(x))^2 V_{2xx}/2\}. \tag{14.34}$$

We use the first-order conditions to obtain the optimal feedback advertising decisions

$$u_1^*(x) = V_{1x}(x) r_1\sqrt{1-x}/2c_1 \text{ and } u_2^*(x) = -V_{2x}(x) r_2\sqrt{x}/2c_2. \tag{14.35}$$

Since it is reasonable to expect that $V_{1x} \geq 0$ and $V_{2x} \leq 0$, these controls will turn out to be nonnegative as we will see later.

Substituting (14.35) in (14.33) and (14.34), we obtain the Hamilton-Jacobi equations

$$\rho_1 V_1 = \pi_1 x + V_{1x}^2 r_1^2(1-x)/4c_1 + V_{1x}V_{2x}r_2^2 x/2c_2 - V_{1x}\delta(2x-1) + (\sigma(x))^2 V_{1xx}/2, \tag{14.36}$$

$$\rho_2 V_2 = \pi_2(1-x) + V_{2x}^2 r_2^2 x/4c_2 + V_{1x}V_{2x}r_1^2(1-x)/2c_1 - V_{2x}\delta(2x-1) + (\sigma(x))^2 V_{2xx}/2. \quad (14.37)$$

As in Section 13.3, we look for the following forms for the value functions

$$V_1 = \alpha_1 + \beta_1 x \text{ and } V_2 = \alpha_2 + \beta_2(1-x). \quad (14.38)$$

These are inserted into (14.36) and (14.37) to determine the unknown coefficients $\alpha_1, \beta_1, \alpha_2$, and β_2. Equating the coefficients of x and the constants on both sides of (14.36) and the coefficients of $(1-x)$ and the constants on both sides of (14.37), the following four equations emerge, which can be solved for the unknowns $\alpha_1, \beta_1, \alpha_2$, and β_2:

$$\rho_1\alpha_1 = \beta_1^2 r_1^2/4c_1 + \beta_1\delta, \quad (14.39)$$
$$\rho_1\beta_1 = \pi_1 - \beta_1^2 r_1^2/4c_1 - \beta_1\beta_2 r_2^2/2c_2 - 2\beta_1\delta, \quad (14.40)$$
$$\rho_2\alpha_2 = \beta_2^2 r_2^2/4c_2 + \beta_2\delta, \quad (14.41)$$
$$\rho_2\beta_2 = \pi_2 - \beta_2^2 r_2^2/4c_2 - \beta_1\beta_2 r_1^2/2c_1 - 2\beta_2\delta. \quad (14.42)$$

Let us first consider the special case of symmetric firms, i.e., when $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$, $\pi = \pi_1 = \pi_2$, $c = c_1 = c_2$, $r = r_1 = r_2$, and $\rho = \rho_1 = \rho_2$. The four equations in (14.39-14.42) reduce to the following two:

$$\rho\alpha = \beta^2 r^2/4c + \beta\delta \text{ and } \rho\beta = \pi - 3\beta^2 r^2/4c - 2\beta\delta. \quad (14.43)$$

There are two solution for β. One is negative, which clearly makes no sense. Thus, the remaining positive solution is the correct one. This also allows us to obtain the corresponding α. The solution is

$$\alpha = [(\rho-\delta)(W - \sqrt{W^2 + 12R\pi}) + 6R\pi]/18R\rho, \quad (14.44)$$
$$\beta = (\sqrt{W^2 + 12R\pi}) - W/6R, \quad (14.45)$$

where $R = r^2/4c$ and $W = \rho + 2\delta$. With this the value functions in (14.38) are defined, and the controls in (14.35) can be written as

$$u_1^*(x) = \frac{\beta_1 r_1 \sqrt{1-x}}{2c_1} \text{ and } u_2^*(x) = \frac{\beta_2 r_2 \sqrt{x}}{2c_2},$$

which are clearly nonnegative as required.

We return now to the general case of asymmetric firms. For this, we re-express equations (14.39-14.42) in terms of a single variable β_1, which is determined by solving the quartic equation

$$3R_1^2\beta_1^4 + 2R_1(W_1+W_2)\beta_1^3 + (4R_2\pi_2 - 2R_1\pi_1 - W_1^2 + 2W_1W_2)\beta_1^2 + 2\pi_1(W_1 - W_2)\beta_1 - \pi_1^2 = 0. \quad (14.46)$$

This equation can be solved explicitly to give four roots. We will find that only one of these is positive. We select that as our value of β_1. With that, other coefficients can be obtained by solving for α_1 and β_2 and then, in turn, α_2, as follows:

$$\alpha_1 = \beta_1(\beta_1 R_1 + \delta)/\rho_1, \tag{14.47}$$

$$\beta_2 = (\pi_1 - \beta_1^2 R_1 - \beta_1 W_1)/2\beta_1 R_2, \tag{14.48}$$

$$\alpha_2 = \beta_2(\beta_2 R_2 + \delta)/\rho_2, \tag{14.49}$$

where $R_1 = r_1^2/4c_1,\ R_2 = r_2^2/4c_2,\ W_1 = \rho_1 + 2\delta,$ and $W_2 = \rho_2 + 2\delta.$

In Exercise 14.4 you are asked to use *Mathematica* or another suitable software program to solve (14.46) to obtain β_1 and then obtain the coefficients $\alpha_1, \alpha_2,$ and β_2 by using (14.47)-(14.49), when $\rho_1 = \rho_2 = 0.05,\ \pi_1 = \pi_2 = 1,\ \delta = 0.01,\ R_1 = 1,\ R_2 = 4,\ x_0 = 0.5,$ and $\sigma(x) = \sqrt{0.5x(1-x)}$. Figure 14.1 represents a sample path of the market share of the two firms with this data.

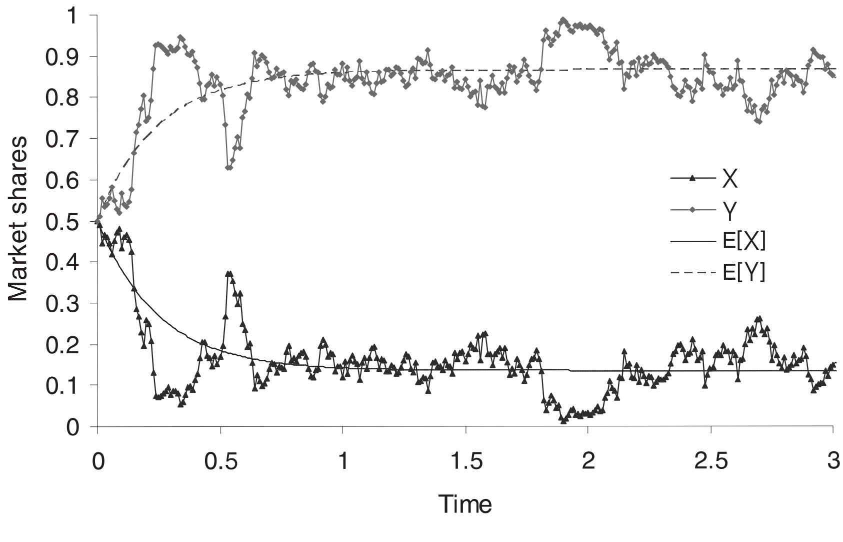

Figure 14.1: A Sample Path of Optimal Market Share Trajectories

14.4 A Stackelberg Differential Game of Cooperative Advertising

The preceding sections in this chapter dealt with differential games in which all players make their decisions simultaneously. We now discuss a differential game in which two players make their decisions in a hierarchical manner. The player having the right to move first is called the leader and the other player is called the follower. In case there are two or more leaders, they play Nash, and the same goes for the followers.

In terms of solutions of Stackelberg differential games, we have open-loop and feedback solutions. An open-loop Stackelberg equilibrium specifies, at the initial time (say, $t = 0$), the decisions over the entire horizon. As in Section 14.1, there is a maximum principle for open-loop solutions. Typically, open-loop solutions are not time consistent in the sense that at any time $t > 0$, the remaining decision may no longer be optimal. A *feedback* or *Markovian Stackelberg equilibrium*, on the other hand, consists of decisions expressed as functions of the current state and time. Such a solution is time consistent.

In this section, we will not develop the general theory, for which we refer the reader to Basar and Olsder (1999) and Dockner et al. (2000). Instead, we will formulate a Stackelberg differential game of cooperative advertising between a manufacturer as the leader and a retailer as the follower, and obtain a feedback Stackelberg solution.

The manufacturer sells a product to end users through the retailer. The product is in a mature category where sales, expressed as a fraction of the potential market, is influenced through advertising expenditures. The manufacturer decides on an advertising support scheme via a *subsidy rate*, i.e., he will contribute a certain percentage of the advertising expenditure by the retailer. Specifically, the manufacturer decides on a subsidy rate W_t and the retailer decides on the advertising effort level $U_t,\ t \geq 0$.

As in Section 13.3, the cost of advertising is quadratic in the advertising effort U_t. Then, with the advertising effort U_t and the subsidy rate W_t, the manufacturer's and the retailer's advertising expenditures are $W_t U_t^2$ and $(1 - W_t)U_t^2$, respectively.

The sequence of events is as follows. First, the manufacturer announces the feedback subsidy rate $w(x) \in [0, 1]$ as a function of the prevailing market share $x \in [0, 1]$. This means that at any time $t \geq 0$, the subsidy rate W_t will be given as $w(X_t)$, where X_t represents the market share at that time. Second, the retailer sets the advertising effort rate U_t as its optimal response to the manufacturer's announced decisions. The retailer accomplishes this by solving an optimization problem to maximize the present value of its profit stream over the infinite horizon. Given the manufacturer's announced policy $w(x)$, the retailer's optimal control problem, using the notation of Section 13.3, is given by the following obvious modification of the problem (13.33):

$$\max_{U_t \geq 0} E\left[\int_0^\infty e^{-\rho t}(\pi X_t - (1 - w(X_t))U_t^2)dt\right] \tag{14.50}$$

subject to

$$dX_t = (rU_t\sqrt{1 - X_t} - \delta X_t)dt + \sigma(X_t)dz_t,\ X_0 = x_0. \tag{14.51}$$

The retailer's problem can be solved as in Section 13.3. With $V^R(x)$ denoting the retailer's value function, the retailer's HJB equation is

$$\rho V^R = \max_{u \geq 0}\left\{\pi x - (1 - w(x))u^2 + V_x^R\left(ru\sqrt{1-x} - \delta x\right) + \frac{(\sigma(x))^2 V_{xx}^R}{2}\right\}. \tag{14.52}$$

The first-order condition for a maximum gives the form of the retailer's response to the manufacturer's announced policy $w(x)$ as

$$u^*(x|w(x)) = \frac{V_x^R r\sqrt{1-x}}{2(1 - w(x))}. \tag{14.53}$$

As in Section 13.3, we will see that this control will turn out to be nonnegative. Substituting $u^*(x|w)$ for u in (14.52), we get the retailer's Hamilton-Jacobi equation as

$$\rho V^R = \pi x + \frac{(V_x^R)^2 r^2(1-x)}{4(1 - w(x))} - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2}. \tag{14.54}$$

Next we consider the formulation of the manufacturer's problem. The manufacturer anticipates the retailer's reaction function (14.53) and incorporates this into his optimal control problem, which can be stated as follows:

$$\max_{0 \leq W_t \leq 1} E\left[\int_0^\infty e^{-\rho t}\left(\pi_M X_t - W_t[u^*(X_t|W_t)]^2\right)dt\right] \tag{14.55}$$

subject to

$$dX_t = (ru^*(X_t|W_t) - \delta X_t)dt + \sigma(X_t)dz_t,\ X_0 = x_0, \tag{14.56}$$

where π_M denotes the manufacturer's margin on sales and where we have, from (14.53),

$$u^*(X_t|W_t) = \frac{V_x^R(X_t)r\sqrt{1-X_t}}{2(1-W_t)}.$$

Letting $V^M(x)$ denote the manufacturer's value function, we can write the HJB equation it should satisfy:

$$\begin{aligned}\rho V^M &= \max_{0\le w\le 1}\left\{\pi_M x - \frac{(V_x^R)^2 r^2(1-x)w}{4(1-w)^2}\right.\\ &\quad \left. + V_x^M\left(\frac{V_x^R r^2(1-x)}{2(1-w)} - \delta x\right) + \frac{(\sigma(x))^2 V_{xx}^M}{2}\right\}. \qquad (14.57)\end{aligned}$$

The first-order condition for maximizing the RHS of (14.57) gives

$$\frac{1+w}{1-w} = \frac{2V_x^M}{V_x^R}. \qquad (14.58)$$

Since V_x^M and V_x^R are expected to be positive, we see that (14.58) will always have a solution $w < 1$. This makes intuitive sense because otherwise, $w \ge 1$ and the retailer would indulge in an infinite amount of advertising, which would not be optimal for the leader. Since, we also restrict the subsidy rate to be nonnegative, we can conclude from (14.58) that

$$w^*(x) = \max\left\{0, \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}\right\}. \qquad (14.59)$$

Next, we investigate the two cases where the subsidy rate is (a) zero or (b) positive, and determine the condition required for no subsidy to be optimal.

Case (a): No Co-op Advertising ($w^* = 0$). Inserting $w(x) = 0$ into (14.53) gives

$$u^*(x) = \frac{rV_x^R\sqrt{1-x}}{2}. \qquad (14.60)$$

Inserting $w(x) = 0$ into (14.57) and (14.54), we have

$$\rho V^M = \pi_M x + \frac{V_x^M V_x^R r^2(1-x)}{2} - V_x^M \delta x + \frac{(\sigma(x))^2 V_{xx}^M}{2}, \qquad (14.61)$$

$$\rho V^R = \pi x + \frac{(V_x^R)^2 r^2(1-x)}{4} - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2}. \qquad (14.62)$$

Let $V^M(x) = \alpha_M + \beta_M x$ and $V^R(x) = \alpha + \beta x$. Then, $V_x^M = \beta_M$ and $V_x^R = \beta$. Substituting these into (14.61) and (14.62) and equating like powers of x, we can express all the unknowns in terms of β, which itself can be explicitly solved. That is, we obtain

$$\begin{aligned}\beta &= \frac{2\pi}{\sqrt{(\rho+\delta)^2 + r^2\pi} + (\rho+\delta)},\ \beta_M = \frac{2\pi_M}{2(\rho+\delta) + \beta r^2}, \\ \alpha &= \frac{\beta^2 r^2}{4\rho},\ \alpha_M = \frac{\beta\beta_M r^2}{2\rho}.\end{aligned} \tag{14.63}$$

Using (14.63) in (14.60), we can write $u^*(x) = \sqrt{\rho\alpha(1-x)}$. Finally, we can derive the required condition from (14.59), which is $2V_x^M \leq V_x^R$, for no co-op advertising ($w^* = 0$) in the equilibrium. This is given by $2\beta^M \leq \beta$, or

$$\frac{4\pi_M}{2(\rho+\delta) + \frac{2\pi r^2}{\sqrt{(\rho+\delta)^2+r^2\pi}+(\rho+\delta)}} \leq \frac{2\pi}{\sqrt{(\rho+\delta)^2 + r^2\pi} + (\rho+\delta)}. \tag{14.64}$$

After a few steps of algebra, this yields the required condition

$$\theta = \frac{\pi_M}{\sqrt{(\rho+\delta)^2 + r^2\pi}} - \frac{\pi}{\sqrt{(\rho+\delta)^2 + r^2\pi} + (\rho+\delta)} \leq 0. \tag{14.65}$$

Next, we obtain the solution when $\theta > 0$.

Case (b): Co-op Advertising ($w^* > 0$). Then, (14.59) reduces to

$$w^*(x) = \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}. \tag{14.66}$$

Inserting this for $w(x)$ into (14.57) and (14.54), we have

$$\begin{aligned}\rho V^M &= \pi_M x - \frac{r^2(1-x)[4(V_x^M)^2 - (V_x^R)^2]}{16} \\ &\quad + \frac{V_x^M r^2(1-x)[2V_x^M + V_x^R]}{4} \\ &\quad - V_x^M \delta x + \frac{(\sigma(x))^2 V_{xx}^M}{2},\end{aligned} \tag{14.67}$$

$$\rho V^R = \pi x + \left[\frac{(V_x^R)^2 r^2 (1-x)}{4}\right]\left[\frac{2V_x^M + V_x^R}{2V_x^R}\right] - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2}. \tag{14.68}$$

Once again, $V^M(x) = \alpha_M + \beta_M x$, $V^R = \alpha + \beta x$, $V_x^M = \beta_M$, $V_x^R = \beta$. Substituting these into (14.67) and (14.68) and equating like powers of x, we have

$$\alpha = \frac{\beta(\beta + 2\beta_M)r^2}{8\rho}, \tag{14.69}$$

$$(\rho + \delta)\beta = \pi - \frac{\beta(\beta + 2\beta_M)r^2}{8}, \tag{14.70}$$

$$\alpha_M = \frac{(\beta + 2\beta_M)^2 r^2}{16\rho}, \tag{14.71}$$

$$(\rho + \delta)\beta_M = \pi_M - \frac{(\beta + 2\beta_M)^2 r^2}{16}. \tag{14.72}$$

From (14.53), (14.66), and (14.71), we can write

$$u^*(x) = \frac{r(V_x^R + 2V_x^M)\sqrt{1-x}}{4} = \sqrt{\rho\alpha_M(1-x)}. \tag{14.73}$$

The four equations (14.69)–(14.72) determine the solutions for the four unknowns, α, β, α_M, and β_M. From (14.70) and (14.72), we can obtain

$$\beta^3 + \frac{2\pi_M}{\rho + \delta}\beta^2 + \frac{8\pi}{r^2}\beta - \frac{8\pi^2}{(\rho + \delta)r^2} = 0. \tag{14.74}$$

If we denote

$$a_1 = \frac{2\pi_M}{\rho + \delta},\ a_2 = \frac{8\pi}{r^2},\ \text{and } a_3 = \frac{8\pi^2}{(\rho + \delta)r^2},$$

then $a_1 > 0,\ a_2 > 0$, and $a_3 < 0$. From Descarte's Rule of Signs, there exists a unique, positive real root. The two remaining roots may be both imaginary or both real and negative. Since this is a cubic equation, a complete solution can be obtained. Using *Mathematica* or following Spiegel, Lipschutz, and Liu (2008), we can write down the three roots as

$$\begin{aligned}
\beta(1) &= S + T - \frac{1}{3}a_1, \\
\beta(2) &= -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{\sqrt{3}}{2}i(S-T), \\
\beta(3) &= -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{\sqrt{3}}{2}i(S-T),
\end{aligned}$$

with

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}},\ T = \sqrt[3]{R - \sqrt{Q^3 + R^2}},\ i = \sqrt{-1},$$

where

$$Q = \frac{3a_2 - a_1^2}{9}, \ R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}.$$

Next, we identify the positive root in each of the following three cases:

Case 1: $(Q > 0)$. We have $S > 0 > T$ and $Q^3 + R^2 > 0$. There is one positive root and two imaginary roots. The positive root is $\beta = S + T - (1/3)a_1$.

Case 2: $(Q < 0$ and $Q^3 + R^2 > 0.)$ There are three real roots with one positive root, which is $\beta = S + T - (1/3)a_1$.

Case 3: $(Q < 0$ and $Q^3 + R^2 < 0)$. S and T are both imaginary. We have three real roots with one positive root. While subcases can be given to identify the positive root, for our purposes, it is enough to identify it numerically.

Finally, we can conclude that $2\beta_M - \beta > 0$ so that $w^* > 0$, since if this were not the case, then w^* would be zero, and we would once again be in Case (a).

We can now summarize the optimal feedback Stackelberg equilibrium in Table 14.1. In Figure 14.2 we also graph the optimal subsidy rate w^* against the retailer's margin π, when the parameters $\pi_M = 0.5, r = .05, \rho = 2, \delta = 1$.

Figure 14.2: Optimal Subsidy Rate vs. the Retailer's Margin

There have been many applications of differential games in marketing in general and optimal advertising in particular. Some references are Bensoussan, Bultez, and Naert (1978), Deal, Sethi, and Thompson (1979), Deal (1979), Jørgensen (1982a), Rao (1984, 1990), Dockner and Jørgensen (1986, 1992), Chintagunta and Vilcassim (1992), Chintagunta and Jain (1994, 1995), Fruchter (1999), Jarrar, Martín-Herrán and Zaccour (2004), Martín-Herrán, Taboubi and Zaccour (2005), Breton, Jarrar and Zaccour (2006), Jørgensen and Zaccour (2007), He and Sethi (2008),

	(a) if $\theta \le 0$ No Co-op Equilibrium	(b) if $\theta > 0$ Co-op Equilibrium
Retailer's profit V^R	$V^R(x) = \alpha + \beta x$	$V^R(x) = \alpha + \beta x$
Manufacturer's profit V^M	$V^M(x) = \alpha_M + \beta_M x$	$V^M(x) = \alpha_M + \beta_M x$
Coefficients of profit functions, $\alpha, \beta, \alpha_M, \beta_M$ obtained from:	$\beta = \frac{2\pi}{\sqrt{(\rho+\delta)^2+r\pi}+(\rho+\delta)}$ $\beta_M = \frac{2\pi_M}{2(\rho+\delta)+\beta r^2}$ $\alpha = \frac{\beta^2 r^2}{4\rho}$ $\alpha_M = \frac{\beta\beta_M r^2}{2\rho}$	$\beta = \frac{\pi}{\rho+\delta} - \frac{\beta(\beta+2\beta_M)r^2}{8(\rho+\delta)}$ $\beta_M = \frac{\pi_M}{\rho+\delta} - \frac{(\beta+2\beta_M)^2 r^2}{16(\rho+\delta)}$ $\alpha = \frac{\beta(\beta+2\beta_M)r^2}{8\rho}$ $\alpha_M = \frac{(\beta+2\beta_M)^2 r^2}{16\rho}$
Subsidy rate $w^*(x) =$	0	$\frac{2\beta_M-\beta}{2\beta_M+\beta} = 1 - \frac{\alpha}{\alpha_M}$
Advertising effort $u^*(x) =$	$\frac{r\beta\sqrt{1-x}}{2} = \sqrt{\rho\alpha(1-x)}$	$\frac{r(\beta+2\beta_M)\sqrt{1-x}}{4} = \sqrt{\rho\alpha_M(1-x)}$

Table 14.1: Optimal Feedback Stackelberg Solution

Naik, Prasad and Sethi (2008), Zaccour (2008a), Jørgensen, Kort and Zaccour (2009), Prasad and Sethi (2009). The literature on advertising differential games is surveyed by Jørgensen (1982a) and the literature on management applications of Stackelberg differential games is reviewed by He et al. (2007). Monographs are written by Erickson (1991) and Jørgensen and Zaccour (2004).

For applications of differential games to economics and management science in general, see the book by Dockner, Jørgensen, Long, and Sorger (2000).

EXERCISES FOR CHAPTER 14

E 14.1 *A Bilinear Quadratic Advertising Model* (Deal, Sethi, and Thompson, 1979). Let x_i be the market share of firm i and u_i be its advertising rate, $i = 1, 2$. The state equations are

$$\begin{aligned} \dot{x}_1 &= b_1 u_1 (1 - x_1 - x_2) + e_1 (u_1 - u_2)(x_1 + x_2) - a_1 x_1 \\ & \quad x_1(0) = x_{10}, \\ \dot{x}_2 &= b_2 u_2 (1 - x_1 - x_2) + e_2 (u_2 - u_1)(x_1 + x_2) - a_2 x_2 \\ & \quad x_2(0) = x_{20}, \end{aligned}$$

where b_i, e_i, and a_i are given positive constants. Firm i wants to maximize

$$J_i = w_i e^{-\rho T} x_i(T) + \int_0^T (c_i x_i - u_i^2) e^{-\rho t} dt,$$

where w_i, c_i, and ρ are positive constants. Derive the necessary conditions for the open-loop Nash solution, and formulate the resulting boundary value problem. In a related paper, Deal (1979) provides a numerical solution to this problem with $e_1 = e_2 = 0$.

E 14.2 Let $x(t)$ denote the stock of pollution at time $t \in [0, T]$ that affects the welfare of two countries. The state dynamics is

$$\dot{x} = u + v, \quad x(0) = x_0,$$

where u and v are emission rates of the leader and the follower, respectively. Let their instantaneous utility functions be

$$u - (u^2 + x^2)/2 \text{ and } v - (v^2 + x^2)/2,$$

respectively. Obtain the open-loop Nash solution. By re-solving this problem at time τ, $0 < \tau < T$, show that the first solution obtained is time inconsistent.

E 14.3 Develop the nonlinear model for licensing of fisherman described toward the end of Section 14.2.3 by rewriting (14.19) and (14.22) for the model. Derive the adjoint equation for λ^i for the ith producer, and show that the feedback Nash policy for producer i is given by

$$f'(v^i) = \frac{c^i}{(p^i - \lambda^i)x}.$$

E 14.4 Use *Mathematica* or another suitable software program to solve the quartic equation (14.46). Show that for $\rho_1 = \rho_2 = 0.05$, $\pi_1 = \pi_2 = 1$, $\delta = 0.01$, $R_1 = 1$, $R_2 = 4$, the only positive solution for β_1 is 0.264545. Figure 14.1 gives a sample path of the optimal market shares of the two firms for this problem. Draw another sample path.

E 14.5 Solve the Stackelberg differential game of Section 14.4 for $\pi = 0.25$, $\pi_M = 0.5$, $r = 2$, $\rho = 0.05$, $\delta = 1$, and $\sigma(x) = \sqrt{0.25x(1-x)}$, obtain the coefficients $\alpha, \beta, \alpha_M, \beta_M$, and show that $v^* = 0.58$.

E 14.6 Assume no cooperative advertising in Exercise 14.5 and obtain the value functions of the manufacturer and the retailer. Compare the manufacturer's profit in this case with that in Exercise 14.5 when $x_0 = 0.5$, and obtain the manufacturer's loss.

E 14.7 Suppose that the manufacturer and the retailer in the problem of Section 14.4 are integrated into a single firm. Then, formulate the stochastic optimal control problem of the integrated firm. Also, using the data in Exercise 14.5, obtain the value function $V^I(x) = \alpha_I + \beta_I x$ of the integrated firm. Finally, compare the profit of the integrated channel to the sum of the profits of the manufacturer and the retailer when $x_0 = 0.5$.

E 14.8 Consider an N-firm oligopoly. Let $S_i(t)$ denote the cumulative sales by time t of firm $i \in \{1, 2, ..., N\}$ and define $S(t) = \sum_{i=1}^{N} S_i(t)$. Let $A_i(t)$ denote firm i's advertising rate. With positive constants a, b, and d, assume that the differential game has the diffusion dynamics

$$\dot{S}_i(t) = [a + b \log A_i(t) + dS(t)][M - S(t)], \quad S_i(0) = S_{i0} \geq 0,$$

which means that a firm can stimulate its sales through advertising (but subject to decreasing returns) and that demand learning effects (imitation) are industry-wide. (If these effects were firm-specific we would have S_i instead of S in the brackets on the right-hand side of the dynamics.) Payoffs are given by

$$J^i = \int_0^T [(p_i - c_i)\dot{S}_i(t) - A_i(t)]dt,$$

in which prices and unit costs are constant. Since $\dot{S}_i(t)$ in the expression for J^i is stated in terms of the state variable $S(t)$ and the control variables $A_i(t),\ i \in \{1, 2, \ldots, N\}$, formulate the differential game problem with $S(t)$ as the state variable. In the open-loop Nash equilibrium, show that the advertising rates are monotonically decreasing over time.

HINT: Assume $\partial^2 H^i/\partial S^2 \leq 0$ so that H^i is concave in S. Use this condition to prove the monotone property.