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Differential Games with the Langenhop Type Constrains on Controls

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Abstract—In this article, the concept of a Langenhop type constraint on the controls of players is introduced. The relationship between geometric, Langenhop and integral constraints is investigated. For these three classes, the attainability domains of the players are found by admissible controls. The parallel pursuit strategy (Π-strategy) is constructed for differential game of pursuit with Langenhop type constraints and the sufficient solvability conditions of the problem are obtained. To solve the evasion problem, the lower bounds of the convergence, depending on the given parameters, is found. This work develops the works of Isaacs, Petrosjan, Pshenichnyi, Azamov and other researchers, including the authors.

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1. INTRODUCTION

Differential games was initiated by Isaacs [1]. Fundamental results in this field were obtained by Pontryagin [2], Krasovskii [3], Bercovitz [4], Elliot and Kalton [5], Fleming [6], Friedman [7], Hajek [8], Ho, Bryson and Baron [9], Subbotin [10], Petrosjan [11], Pshenichnyi [12] and others.

Finding the initial positions, from which the pursuit can be ended in a finite time, is one of the interesting problems in the theory of differential games. To study the pursuit problems devoted many research in the world. First of all, noteworthy are the studies [2, 3, 10, 12], whose methods and results are generalized and developed in [13–15] and etc.

The construction of optimal strategies of players and finding the value of the game are important problems of differential games. In works [16–20] were studied differential games of optimal approach of many pursuers and one evader.

Problems, involving linear differential games with integral constraints, were studied by many authors (see [21–27]) and their works served as a basis for studying pursuit-evasion problems in linear differential games with integral constraints. In the works [28–30], sufficient conditions for the solvability of linear pursuit problems with integral constraints on the control of the players were obtained, when there is delay to information.

From the standpoint of the application, of a substantial interest is the study of differential games, in which constraints of different types are imposed on the players' controls. Differential games with different types constraints studied in [31–41]. But, it should be noted that the differential games with different types of constraints have generally been studied relatively little.

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In the theory of differential games, control functions are mainly subjected to geometric, integral or mixed constraints. However, different types of constraints on controls are also arisen in solving some applied problems in ecology, engineering sciences and etc. [42, 43]. The book [1] of Isaacs contains many specific game problems that were discussed in details and proposed for further study. One of them is Life-line problem, which rather was comprehensively studied by Petrosyan in [11] by the aid of approximating measurable controls with most efficient piecewise constant controls that presents the strategy of parallel approach. Later this strategy was called Π-strategy. The strategies proposed in [11, 12, 31] for a simple motion pursuit game with geometric constraints became the starting point for the development of the pursuit methods in games with multiple pursuers (see [13–15, 26, 27, 32]).

In the work [44], the pursuit-evasion differential games with the Grönwall type constraints (see also [45, 46]) on controls are studied and the concept Gr-constraint on the controls of the players, which in a certain sense generalizes a geometric constraints, is introduced. In [44] a Π-strategy for the Grönwall type constraints is constructed and proved that it is an optimal strategy of pursuer. The present work is also devoted to simple motion pursuit-evasion problems. We propose Langenhop [46, 47] type constraints on controls of the players for differential games of pursuit-evasion and construct Π-strategy of pursuer. The new sufficient solvability conditions are obtained for problems of the pursuit-evasion.

2. STATEMENT OF PROBLEMS

Suppose that in $\mathbb{R}^n$ a controlled object P called the Pursuer, chases another object E called the Evader. Denote by x the position of the Pursuer and denote by y the position of the Evader in $\mathbb{R}^n$. In the present work, we consider the pursuit-evasion problems, when the objects move in accordance with the equations

$$\dot{x} = u, \quad x(0) = x_0, \tag{1}$$

$$\dot{y} = v, \quad y(0) = y_0, \tag{2}$$

where $x, y, u, v \in \mathbb{R}^n$, $n \geq 1$; x_0 and y_0 are initial positions of the objects. It is assumed that $x_0 \neq y_0$; u and v are the velocity vectors, which serve as parameters of the equations. Here u and v must be a measurable functions $u(\cdot) : [0, +\infty) \to \mathbb{R}^n$ and $v(\cdot) : [0, +\infty) \to \mathbb{R}^n$, respectively.

There is number of works, where simple motion differential games with integral constraints on controls of the form

$$\|u(\cdot)\|_p = \left[\int_0^{t^*} |u(t)|^p dt\right]^{1/p} \leq \varrho, \quad \varrho > 0, \quad p \geq 1, \tag{3}$$

were studied.

Problems with integral constraints (3) are more complicated, than problems with geometrical constraints (see [21–27]). But, both types of constraints are important in practice: the former type expresses the constrained nature of the dynamical possibilities of the object (for example, a constraint on the thrust), constraints of the latter type express the finite nature of resources (for example, fuel).

In the present paper, we propose new classes of controls for pursuer and evader. Introduce classes admissible controls for pursuer:

P_1) Langenhop constraint (briefly, La-constraint)

$$|u(t)|^2 \leq \rho^2 - 2k \int_0^t |u(s)|^2 ds, \quad \text{for almost every} \quad 0 \leq t < \bar{t}, \tag{4}$$

and in the sequel, we denote such class by U_{La}, where

$$\bar{t} = \sup\left\{t : \rho^2 - 2k \int_0^t |u(s)|^2 ds \geq 0\right\};$$

P_2) Geometric constraint (briefly, G-constraint) of the form

$$|u(t)| \leq \rho e^{-kt} \quad \text{for almost every} \quad t \geq 0 \tag{5}$$

and we denote this class by U_G;

P_3) Integral constraint (briefly, I-constraint) in the form

$$\int_0^t |u(s)|^2 ds \leq \frac{\rho^2}{2k}\left(1 - e^{-2kt}\right), \quad t \geq 0 \tag{6}$$

and we denote such class of admissible controls by U_I, where in (4)–(6) ρ and k are positive numbers.

Above, for the pursuer we introduced the three classes of admissible controls. Similarly, to (4)–(6) for evader we introduce the following classes of admissible controls:

E_1) Langenhop constraint (briefly, La-constraint)

$$|v(t)|^2 \leq \sigma^2 - 2k \int_0^t |v(s)|^2 ds, \text{ for almost every } 0 \leq t < \tilde{t}, \tag{7}$$

and in the sequel, we denote the class of all admissible controls of the form (7) by V_{La}, where

$$\tilde{t} = \sup\left\{t : \sigma^2 - 2k \int_0^t |v(s)|^2 ds \geq 0\right\};$$

E_2) Geometric constraint (briefly, G-constraint)

$$|v(t)| \leq \sigma e^{-kt}, \quad \text{for almost every} \quad t \geq 0 \tag{8}$$

and we denote by V_G the class of admissible controls satisfying G-constraint (8);

E_3) Integral constraint (briefly, I-constraint)

$$\int_0^t |v(s)|^2 ds \leq \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right), \quad t \geq 0 \tag{9}$$

and we denote by V_I the class of admissible controls satisfying I-constraint (9), where in (7)–(9) σ and k are positive numbers, too.

If U (correspondingly, V) is one of the introduced classes U_G, U_I, U_{La} (correspondingly, V_G, V_I, V_{La}), then the pairs $\big(x_0, u(\cdot) \in U\big)$ and $\big(y_0, v(\cdot) \in V\big)$ generate the following trajectories

$$x(t) = x_0 + \int_0^t u(s)ds, \quad y(t) = y_0 + \int_0^t v(s)ds$$

of the pursuer and evader, respectively. The goal of the pursuer P is capture, i.e. achievement of the equality $x(t) = y(t)$ (*Pursuit problem*) and the evader E strives to avoid an encounter (*Evasion problem*), i.e., to achieve the inequality $x(t) \neq y(t)$ for all $t \geq 0$, and in the opposite case, to postpone the instant of encounter as long as possible. This is a preliminary formulation of the Pursuit-Evasion problems under consideration.

This paper is devoted to solve the following problems:

1. Pursuit problem in the game (1)–(2) with the Langenhop constraints (4) and (7) (briefly, La-Game of Pursuit).

2. Evasion problem in the game (1)–(2) with the Langenhop constraints (4) and (7) (briefly, La-Game of Evasion).

For solution La-Game of Pursuit assume that at each current time t, the pursuer is allowed to know the initial states x_0, y_0, the constants ρ, σ, k, the current time t, and the value of evader's control $v(t)$.

Definition 1. *We say that a function* $\mathbf{u}(t,v):\mathbb{R}_+\times S_\sigma\to S_\rho$ *is the strategy of pursuer, if* $\mathbf{u}$ *is Lebesgue measurable on* t *for each fixed* v*, Borel measurable on* v *for each fixed* t*, where* $S_\alpha=\{w\in\mathbb{R}^n:|w|\le\alpha\}$ *i.e. the ball of the radius* α *in* $\mathbb{R}^n$*.*

Definition 2. *We say that a strategy* $\mathbf{u}=\mathbf{u}\,(t,v)$ *guarantees completion of pursuit for the time* $T(\mathbf{u})$*, if for any control of the evader* $v(\cdot)\in V$ *we have* $x(t^*)=y(t^*)$ *at some time* $t^*\in[0,T(\mathbf{u})]$*, where the pair of* $x(t)$ *and* $y(t)$ *is the solution of the initial value problem*

$$\dot{x}=\mathbf{u}(t,v(t)),\quad x(0)=x_0,$$
$$\dot{y}=v(t),\quad y(0)=y_0,\quad t\ge 0.$$

For the solution of La-Game of Evasion assume that at each current time t, the evader is allowed to know the initial states x_0, y_0, the constants ρ, σ, k, the current time t, and the value of pursuer's control $u(t-\varepsilon)$, where ε is certain positive number (delay information).

Definition 3. *Suppose*

$$\mathbf{v}_\varepsilon(t,u)=\begin{cases}0, & \text{if}\quad 0\le t<\varepsilon,\\ \mathbf{v}^*(t-\varepsilon,u), & \text{if}\quad t\ge\varepsilon,\end{cases}\tag{10}$$

and let a function $\mathbf{v}^*(t,u):\mathbb{R}_+\times S_\rho\to S_\sigma$ *be Lebesgue measurable on* t *for each fixed* u *and Borel measurable on* u *for each fixed* t*. Then* (10) *is called a strategy of evader.*

Definition 4. *We say that a strategy* $\boldsymbol{v}_\varepsilon=\boldsymbol{v}_\varepsilon(t,\,u)$ *guarantees evasion on* $[0,+\infty)$*, if for any control of pursuer* $u\,(\cdot)\in U$ *the condition* $x\,(t)\ne y\,(t)$ *holds for all* $t\in[0,+\infty)$*, where the pair of* $x(t)$ *and* $y(t)$ *is the solution of the initial value problem*

$$\dot{x}=u(t),\quad x(0)=x_0,$$
$$\dot{y}=\mathbf{v}_\varepsilon(t,\,u(t-\varepsilon))),\quad y(0)=y_0,\quad t\ge 0.$$

3. THE CLASSES OF CONTROLS

Lemma 1. *$U_G\subset U_{La}\subset U_I$ for almost every $t\ge 0$.*

Proof. a) From (5) we have

$$|u(t)|^2\le\rho^2e^{-2kt}=\rho^2-2k\int\limits_0^t\rho^2e^{-2ks}ds\le\rho^2-2k\int\limits_0^t|u(s)|^2ds.$$

Hence, we obtain $U_G\subset U_{La}$.

b) Let $w(t)=-2k\int\limits_0^t|u(s)|^2ds$. Then $\dot{w}(t)=-2k|u(t)|^2$ and $w(0)=0$. From (4) we have $\dot{w}(t)\ge -2kw(t)-2k\rho^2$ for almost every $0\le t\le\bar{t}$. Now multiplying both sides to e^{2kt} of last inequality, we obtain $d\Big(w(t)e^{2kt}\Big)\ge-2k\rho^2e^{2kt}dt$. Integrating both sides of this inequality, we obtain

$$w(t)e^{2kt}\ge\rho^2\Big(1-e^{2kt}\Big)\Rightarrow-2k\int\limits_0^t|u(s)|^2ds\ge\rho^2\Big(e^{-2kt}-1\Big).$$

Hence, follows that $U_{La}\subset U_I$ for almost every $t\ge 0$.

Lemma 1 is proved. □

Lemma 2. *$V_G\subset V_{La}\subset V_I$ for almost every $t\ge 0$.*

The proof of the Lemma 2 is analogous to proof of the Lemma 1.

Lemma 3. *If $u(\cdot)\in U_G$ and $k>0$, then the trajectory of pursuer is $x(t)\in S_\mu(x_0)$ for all $t\ge 0$, where $S_\mu(x_0)$ is the ball of the radius $\mu=\rho/k$ and centered at the point x_0.*

Proof. Let $u(\cdot) \in U_G$. Then from (1) and (5) we have

$$|x(t) - x_0| \leq \int_0^t |u\,(s)|ds \leq \rho \int_0^t e^{-ks}ds = \rho\left(1 - e^{-kt}\right)/k < \rho/k$$

for all $t \geq 0$. □

Lemma 4. *Let for any $p \in \mathbb{R}^n$, $p \neq x_0$, there hold*

$$u^* = (p - x_0)\left[\sqrt{k^2 + (\rho/a)^2} - k\right] \tag{11}$$

and

$$t^* = 1/\left[\sqrt{k^2 + (\rho/a)^2} - k\right], \tag{12}$$

where $a = |p - x_0|$. Then: a) $x(t^*) = p$; and b) $u^* \in U_{La}$ for $t \in [0,\, t^*]$.

Proof. a) From (1) and (11)–(12) we have $x(t^*) = x_0 + t^*u^* = p$; b) from (4) and (11)–(12) follows

$$|u^*|^2 = a^2\left[\sqrt{k^2 + (\rho/a)^2} - k\right]^2 = \rho^2 - 2ka^2\left[\sqrt{k^2 + (\rho/a)^2} - k\right] = \rho^2 - 2kt^*\left|u^*\right|^2 \leq \rho^2 - 2kt|u^*|^2,$$

i.e. $u^* \in U_{La}$ for $t \in [0,\, t^*]$. □

Lemma 5. *If $u(\cdot) \in U_I$ and $k > 0$, then for the trajectory of the pursuer holds $x\,(t) \in S_{\nu\,(t)}(x_0)$ for all $t \geq 0$, where $\nu\,(t) = \rho\sqrt{\frac{t}{2k}}$.*

Proof. Let $u\,(\cdot) \in U_G$. Then from (1), (6) and by Cauchy–Bunyakovskii inequality we obtain

$$|x(t) - x_0| \leq \int_0^t |u(s)|ds \leq \sqrt{t}\sqrt{\int_0^t |u(s)|^2 ds} \leq \rho\sqrt{\frac{t}{2k}\left(1 - e^{-2kt}\right)} < \rho\sqrt{\frac{t}{2k}}$$

for all $t \geq 0$. □

Corollary 1. *If $u\,(\cdot) \in U_G$ (respectively, $v\,(\cdot) \in V_G$) and $k > 0$, then the pursuer (evader) contains in $S_\mu(x_0)$ ($S_\nu(y_0), \nu = \sigma/k$) for all $t \geq 0$. If $u\,(\cdot) \in U_{La} \subset U_I$ (respectively, $v\,(\cdot) \in V_{La} \subset V_I$), then the pursuer (evader) can be reach any point $\mathbb{R}^n$.*

4. THE La-GAME OF PURSUIT

Definition 5. *If $\rho \geq \sigma$, then the function*

$$\mathbf{u}_{La}(t,\, v) = v - \lambda_{La}(t,\, v)\xi_0, \quad \lambda_{La}(t,\, v) = \langle v,\, \xi_0\rangle + \sqrt{\langle v,\, \xi_0\rangle^2 + \delta e^{-2kt}} \tag{13}$$

is called a Π_{La}-strategy of pursuer in the La-Game, where $\delta = \rho^2 - \sigma^2$, $\xi_0 = z_0/|z_0|$ and by $\langle v,\, \xi\rangle$ denotes the inner product of vectors v and ξ in $\mathbb{R}^n$.

Note that

$$|\mathbf{u}_{La}(t,\, v)|^2 = |v|^2 + \delta e^{-2kt}, \quad t \geq 0. \tag{14}$$

Definition 6. *If there exists positive root of the equation*

$$\sqrt{\Phi_P(t) + \Psi_P(t)} - \sqrt{\Phi_P(t)} = |z_0| \tag{15}$$

with respect to t, where $\Phi_P(t) = t\sigma^2(1 - e^{-2kt})/2k$, $\Psi_P(t) = (\rho^2 - \sigma^2)(1 - e^{-kt})^2/k^2$, then the smallest positive root of the equation (15) we call a guaranteed pursuit time and denote it by T_{La}.

We prove the following statement.

Theorem 1. *If $\rho > \sigma$ and there exists the smallest positive root of the equation (15), then Π_{La} -strategy guarantees the completion of pursuit in the La-Game on the time interval $[0,\, T_{La}]$.*

Proof. Suppose the evader chooses an arbitrary control $v(\cdot) \in V_{La}$. If the pursuer P implements the Π_{La} -strategy, then using equations (1)–(2) and (13), we have the following initial value problem

$$\dot{z} = \mathbf{u}_{La}(t,\, v(t)) - v(t) = -\lambda_{La}(t,\, v(t))\xi_0, \quad z(0) = z_0,$$

where $z = x - y$. Hence, we obtain

$$z(t) = \Lambda_{La}(t, v(\cdot))z_0, \tag{16}$$

where

$$\Lambda_{La}(t,\, v(\cdot)) = 1 - \frac{1}{|z_0|}\int\limits_0^t \lambda_{La}(s,\, v(s))ds.$$

We will study the nature of the decay of the approach function $\Lambda_{La}(t,\ v(\cdot))$ with respect to t. The function $\Lambda_{La}(t,\ v(\cdot))$ is continuous and decreasing with respect to $t \geq 0$ and the following estimate holds

$$\Lambda_{La}(t,\, v(\cdot)) \leq 1 - \frac{1}{|z_0|}\int\limits_0^t \left[\sqrt{|v\,(s)|^2 + \delta e^{-2ks}} - |v\,(s)|\right]ds$$

or

$$\Lambda_{La}(t,\, v(\cdot)) \leq 1 - \frac{1}{|z_0|}\int\limits_0^t g\,(s)\, f\,(w\,(s))ds, \tag{17}$$

where $g(s) = e^{-ks},\ w(s) = e^{ks}|v\,(s)|$ and $f\,(w) = \sqrt{w^2 + \delta} - w$. We will apply Jensen's inequality for integrals. Since $f : [0, +\infty) \to (0, \gamma]$ is a convex function ($\ddot{f}(w) > 0$) and $g : [0, +\infty) \to (0,\, 1]$ is integrable, we obtain

$$\int\limits_0^t g(s)f(w(s))ds \geq \int\limits_0^t g(s)ds f\left(\frac{\int_0^t g(s)w(s)ds}{\int_0^t g(s)ds}\right). \tag{18}$$

Using (18) for (17), we have

$$\Lambda_{La}(t,\, v\,(\cdot)) \leq 1 - \frac{1}{|z_0|}\left[\sqrt{\left(\int\limits_0^t |v\,(s)|ds\right)^2 + \Psi_P(t)} - \int\limits_0^t |v(s)|ds\right].$$

From the Cauchy–Bunyakovskii inequality

$$\int\limits_0^t |v\,(s)|ds \leq \sqrt{t}\left(\int\limits_0^t |v\,(s)|^2 ds\right)^{1/2}$$

and from the decreasing function $f(t, r) = \sqrt{r^2 + \Psi_P(t)} - r$ with respect to r, where $r = \int_0^t |v(s)|ds$, we obtain

$$\Lambda_{La}(t,\, v(\cdot)) \leq 1 - \frac{1}{|z_0|}\left[\sqrt{t\int\limits_0^t |v(s)|^2 ds + \Psi_P(t)} - \sqrt{t\int\limits_0^t |v(s)|^2 ds}\right]. \tag{19}$$

Hence, from (9) and (19) we find that $\Lambda_{La}(t,\, v(\cdot)) \le \Lambda_{La}(t)$, where

$$\Lambda_{La}(t) = 1 - \frac{1}{|z_0|}\left[\sqrt{\Phi_P(t) + \Psi_P(t)} - \sqrt{\Phi_P(t)}\right]. \tag{20}$$

Consequently, from the conditions of the Theorem 1 follow that $\Lambda_{La}(T_{La}) = 0$ and there exists time t^* such that $\Lambda_{La}(t^*,\, v(\cdot)) = 0$. So, by the aid of (16) we have that $z(t^*) = 0$ or $x\,(t^*) = y\,(t^*)$, where $t^* \le T_{La}$.

Now, prove the admissibility of the realization (13) for all $t \in [0,\, t^*]$. Let the evader chooses an arbitrary control $v\,(\cdot) \in V_{La}$. Using (7) for (14), we have

$$|\mathbf{u}_{La}(t,\, v(t))|^2 = |v(t)|^2 + \delta e^{-2kt} \le \sigma^2 - 2k\int_0^t |v(s)|^2 ds + \delta e^{-2kt}$$

$$= \rho^2 - 2k\int_0^t (|v(s)|^2 + \delta e^{-2ks}) ds = \rho^2 - 2k\int_0^t (|\mathbf{u}_{La}(s, v(s))|^2 ds,$$

which completes the proof of the Theorem 1. □

5. THE La-GAME OF EVASION

In the present section the Evasion problem is consider as a control problem from the point of view of the Evader. For solving this problem we assume, that at each current instant $t \ge \varepsilon$ the evader E has information about the value of function $u(t-\varepsilon) \in \mathbb{R}^n$ chosen by pursuer P, where ε is certain positive number (delay information). In such situation, the pursuer P is said to be discriminated and the game itself is called a game with discrimination for the pursuer P.

Definition 7. *Let $\sigma \ge \rho$ in the La-game. Then by the E_{La}-strategy of the Evader we mean the function*

$$\mathbf{v}_{La}(t, u_\varepsilon(t)) = \begin{cases} 0, & \text{if} \quad 0 \le t < \varepsilon, \\ -\sqrt{|u(t-\varepsilon)|^2 + \theta e^{-2k(t-\varepsilon)}}\,\xi_0, & \text{if} \quad t \ge \varepsilon, \end{cases} \tag{21}$$

where $u_\varepsilon(t) = \begin{cases} 0, & \text{if } 0 \le t < \varepsilon, \\ u(t-\varepsilon), & \text{if } t \ge \varepsilon, \end{cases}$ $\quad u_\varepsilon(t) \in \mathbb{R}^n,\ \theta = \sigma^2 - \rho^2,\ \xi_0 = z_0/|z_0|.$

Definition 8. *The E_{La}-strategy (21) is called winning for the evader, if for every $u\,(\cdot) \in U_{La}$:*

a) the solution $z(t)$ to the Cauchy problem

$$\dot{z} = u(t) - \mathbf{v}_{La}(t,\, u_\varepsilon(t)), \quad z(0) = z_0 \tag{22}$$

is nonzero for all $t \ge 0$, i.e. $z(t) \ne 0$ for all $t \ge 0$;

b) there holds $\mathbf{v}_{La}(t,\, u_\varepsilon(\cdot)) \in V_{La}$ in the time interval $[0,\, t]$ for all $t \ge 0$.

Theorem 2. *If in the La-game $\rho \le \sigma$, $k > 0$ and $0 < \varepsilon \le \frac{2k|z_0|^2}{\rho^2}$, then the E_{La}-strategy (21) is winning for E and the following estimate for the distance between the players holds for all $t \ge 0$:*

$$|z(t)| > \begin{cases} 0, & \text{if} \quad 0 \le t < \varepsilon, \\ \sqrt{\Phi_E(t-\varepsilon) + \Psi_E(t-\varepsilon)} - \sqrt{\Phi_E(t-\varepsilon)}, & \text{if} \quad t \ge \varepsilon, \end{cases} \tag{23}$$

where $\Phi_E(t) = t\rho^2\left(1 - e^{-2kt}\right)/(2k)$, $\Psi_E(t) = \left(\sigma^2 - \rho^2\right)\left(1 - e^{-kt}\right)^2/k^2$.

Proof. Suppose that the pursuer P moves under the arbitrary control $u\,(\cdot) \in U_{La}$, and the evader E implements the E_{La}-strategy (21). If $0 \le t < \varepsilon$, then from (21) and (22) we have

$$|z(t)| \ge |z_0| - \int_0^t |u(s)| ds.$$

From Lemma 5 and condition $0 < \varepsilon \le \frac{2k|z_0|^2}{\rho^2}$ we derive that

$$\int_0^\varepsilon |u(s)|ds < \rho\sqrt{\frac{\varepsilon}{2k}} \le |z_0|.$$

Hence, $|z(t)| > 0$ for $0 \le t < \varepsilon$.

Let $t \ge \varepsilon$. Then from (21) and (22) we obtain

$$|z(t)| \ge |z_0 - \int_\varepsilon^t \mathbf{v}_{La}(s, u_\varepsilon(s))ds| - \int_0^t |u(s)|ds$$

$$= |z_0| + \int_\varepsilon^t \sqrt{|u(s-\varepsilon)|^2 + \theta e^{-2k(s-\varepsilon)}}ds - \int_0^t |u(s)|ds$$

$$= |z_0| + \int_0^{t-\varepsilon} \left[\sqrt{|u(s)|^2 + \theta e^{-2ks}} - |u(s)|\right] ds - \int_{t-\varepsilon}^t |u(s)|ds.$$

Since

$$\int_{t-\varepsilon}^t |u(s)|ds < \rho\sqrt{\frac{\varepsilon}{2k}} \le |z_0|,$$

for $t \ge \varepsilon$ we find

$$|z(t)| > \int_0^{t-\varepsilon} \left[\sqrt{|u(s)|^2 + \theta e^{-2ks}} - |u(s)|\right] ds = \int_0^{t-\varepsilon} e^{-ks}\left[\sqrt{e^{2ks}|u(s)|^2 + \theta} - e^{ks}|u(s)|\right] ds.$$

By analogy of proof of the Theorem 1, using Jensen's inequality, obtain the following inequality

$$|z(t)| > \left[\sqrt{\left(\int_0^{t-\varepsilon} |u(s)|ds\right)^2 + \Psi_E(t-\varepsilon)} - \int_0^{t-\varepsilon} |u(s)|ds\right].$$

Using the Cauchy–Bunyakovskii inequality

$$\int_0^t |u(s)|ds \le \sqrt{t}\left(\int_0^t |u(s)|^2 ds\right)^{1/2},$$

we obtain

$$|z(t)| > \sqrt{(t-\varepsilon)\int_0^{t-\varepsilon} |u(s)|^2 ds + \Psi_E(t-\varepsilon)} - \sqrt{(t-\varepsilon)\int_0^{t-\varepsilon} |u(s)|^2 ds}.$$

Hence, from (6) we find the estimates (23) for the distance between the players.

It remains to show the admissibility of the implementation of the E_{La}-strategy for all $t \ge \varepsilon$ (for $0 \le t < \varepsilon$ obviously). Let the purser chooses an arbitrary control $u(\cdot) \in U_{La}$. Then from (4), (7) and (21) we have

$$|\mathbf{v}_{La}(t, u_\varepsilon(t))|^2 = |u(t-\varepsilon)|^2 + (\sigma^2 - \rho^2)e^{-2k(t-\varepsilon)} \le \rho^2 - 2k\int_0^{t-\varepsilon} |u(s)|^2 ds + (\sigma^2 - \rho^2)e^{-2k(t-\varepsilon)}$$

$$= \sigma^2 - 2k \int\limits_{\varepsilon}^{t} (|u(s-\varepsilon)|^2 + \theta e^{-2k(s-\varepsilon)}) ds = \sigma^2 - 2k \int\limits_{0}^{t} |\mathbf{v}_{La}(s, u_\varepsilon(s))|^2 ds,$$

which completes the proof of the Theorem 2. □

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