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p-Adic $(1,2)$-rational dynamical systems with two fixed points on $\mathbb{C}_p$

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Abstract. In this work, the general view of the two fixed point $(1,2)$-rational function given on the field $\mathbb{C}_p$ is simplified using a topological conjugation. Then, the discrete-time dynamical systems of the functions in this view were studied, and all invariant spheres, Siegel discs and Attractors were found.

Keywords: p-Adic numbers, (n,m)-rational function, topological conjugate, discrete-time dynamical system, fixed point, invariant set, Siegel disc.

MSC (2010): 46S10, 12J12, 11S99, 30D05, 54H20.

1 Introduction

It is well known that the completion of the field $\mathbb{Q}$ of rational numbers with respect to the p-adic norm defines a field of p-adic numbers, which is denoted by $\mathbb{Q}_p$ (see [8]). The algebraic closure $\mathbb{Q}_p^{ac}$ of $\mathbb{Q}_p$ is an infinite extension, this follows from the fact that there exist irreducible polynomials of any degree over $\mathbb{Q}_p$. Unfortunately, $\mathbb{Q}_p^{ac}$ is not complete with the metric induced by the extended p-adic absolute value. We complete $\mathbb{Q}_p^{ac}$ and obtain a new field $\mathbb{C}_p$ which is algebraically closed. We call $\mathbb{C}_p$ the field of complex p-adic numbers.

We study discrete-time dynamical systems generated by a rational function given on the field of p-adic numbers. For basic definitions and motivations of such investigations see [1]-[19] and references therein.

A function is called a (n,m)-rational function if and only if it can be written in the form $f(x)=\frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ are polynomial functions with degree n and m respectively, $Q_m(x)$ is non zero polynomial.

In [6] the trajectories of $(2,1)$-rational p-adic dynamical system with the form $f(x)=\frac{ax^2}{bx+1}$ in a complex p-adic field $\mathbb{C}_p$ are studied.

In [1] the $(2,1)$-rational dynamical systems on the field of p-adic complex numbers $\mathbb{C}_p$ are studied. In this study, the cases in which a function has a unique fixed point, two fixed points, and no fixed point have been studied in detail separately.

In [19] it is considered $(2,1)$-rational dynamical systems with unique fixed point on $\mathbb{Q}_p$. It is founded all invariant spheres, and investigated ergodicity of such dynamical system on invariant sphere.

In this paper we consider a $(1,2)$-rational function with two distinct fixed points on the field $\mathbb{C}_p$.

2 $(1,2)$-rational function with two distinct fixed points

In this paper we consider the dynamical system associated with the $(1,2)$-rational function $f:\mathbb{C}_p\to\mathbb{C}_p$ defined by

$$f(x)=\frac{ax+b}{x^2+cx+d},\quad a\neq 0,\quad a,b,c,d\in\mathbb{C}_p. \tag{2.1}$$

where $x\neq\hat{x}_{1,2}=\frac{-c\pm\sqrt{c^2-4d}}{2}$.

We study p-adic dynamical systems generated by the rational function (2.1).

The equation $f(x)=x$ for fixed points of function (2.1) is equivalent to the equation

$$x^3+cx^2+(d-a)x-b=0. \tag{2.2}$$

Since $\mathbb{C}_p$ is algebraic closed the equation (2.2) may have three solutions with one of the following relations:

(i). One solution having multiplicity three;

(ii). Two solutions, one of which has multiplicity two;

(iii). Three distinct solutions.

In this paper we assume that equation (2.2) has two distinct solutions x_1 and x_2, such that x_2 has multiplicity two. Then we have $x^3+cx^2+(d-a)x-b=(x-x_1)(x-x_2)^2$ and

$$\begin{cases} x_1+2x_2=-c,\\ x_2^2+2x_1x_2=d-a,\\ x_1x_2^2=b, \end{cases} \tag{2.3}$$

Let homeomorphism $h:\mathbb{C}_p\to\mathbb{C}_p$ is defined by $h(t)=t+x_2$. Note that, the function f is topologically conjugate to function $h^{-1}\circ f\circ h$. We have

$$(h^{-1}\circ f\circ h)(t)=\frac{-x_2t^2+Bt}{t^2+Dt+B}, \tag{2.4}$$

where $B=x_2^2+cx_2+d$ and $D=2x_2+c$.

In (2.4), the case $x_2\neq 0$ is studied in [17].

Thus in this paper we consider the case $x_2=0$ in (2.4). If $x_2=0$, then $B=d=a$ and $D=c$. Thus we have the following proposition

Proposition 2.1. *Any (1,2)-rational function having two distinct fixed points is topologically conjugate to one of the functions in the following forms*

$$f(x)=\frac{ax^2+bx}{x^2+cx+b},\quad ab\neq 0,\quad a\neq c,\quad a,b,c\in\mathbb{C}_p, \tag{2.5}$$

and

$$f(x)=\frac{ax}{x^2+cx+a}, \quad a\neq 0, \quad a,c,\in \mathbb{C}_p. \tag{2.6}$$

where $x\neq \hat{x}_{1,2}=\frac{-c\pm\sqrt{c^2-4a}}{2}$.

Thus we study the dynamical system $(\mathbb{C}_p, f)$ with f given by (2.6).

3 Behavior of dynamical systems

Note that, function (2.6) has two fixed points $x_1=-c$ and $x_2=0$. So we have

$$f'(x_1)=\frac{a-c^2}{a}, \quad f'(x_2)=1.$$

Thus, the point x_2 is an indifferent point for (2.6).

For any $x\in\mathbb{C}_p$, $x\neq \hat{x}_{1,2}$, by simple calculations we get

$$|f(x)|_p=|x|_p\cdot\frac{|a|_p}{|x-\hat{x}_1|_p|x-\hat{x}_2|_p}. \tag{3.1}$$

Denote

$$\mathcal{P}=\{x\in\mathbb{C}_p : \exists n\in\mathbb{N}\cup\{0\}, f^n(x)\in\{\hat{x}_1,\hat{x}_2\}\}, \quad \alpha=|\hat{x}_1|_p \text{ and } \beta=|\hat{x}_2|_p.$$

Since $\hat{x}_1+\hat{x}_2=-c$ and $\hat{x}_1\hat{x}_2=a$, we have $|c|_p\leq\max\{\alpha,\beta\}$ and $|a|_p=\alpha\beta$.

Remark 3.1. It is easy to see that $\hat{x}_1$ and $\hat{x}_2$ are symmetric in (3.1), i.e., if we replace them then RHS of (3.1) does not change. Therefore we consider the dynamical system $(\mathbb{C}_p\setminus\mathcal{P}, f)$ for cases $\alpha=\beta$ and $\alpha<\beta$.

By using (3.1) we define the following functions.

1. For $\alpha=\beta$ define the function $\varphi:[0,+\infty)\to[0,+\infty)$ by

$$\varphi_\alpha(r)=\begin{cases} r, & \text{if } r<\alpha,\\ \alpha^*, & \text{if } r=\alpha,\\ \frac{\alpha^2}{r}, & \text{if } r>\alpha,\end{cases}$$

where α^* is a positive number with $\alpha^*\geq\alpha$.

2. For $\alpha<\beta$ define the function $\varphi_{\alpha,\beta}:[0,+\infty)\to[0,+\infty)$ by

$$\varphi_{\alpha,\beta}(r)=\begin{cases} r, & \text{if } r<\alpha,\\ \hat{\alpha}, & \text{if } r=\alpha,\\ \alpha, & \text{if } \alpha<r<\beta,\\ \hat{\beta}, & \text{if } r=\beta,\\ \frac{\alpha\beta}{r}, & \text{if } r>\beta.\end{cases}$$

where $\hat{\alpha}$, $\hat{\beta}$ are some positive numbers with $\hat{\alpha} \geq \alpha$ and $\hat{\beta} \geq \alpha$.

Using formula (3.1) we easily get the following:

Lemma 3.2. *If $x \in S_r(0)$, $x \neq \hat{x}_{1,2}$ then the following formula holds for function (2.6)*

$$|f^n(x)|_p = \begin{cases} \varphi_\alpha^n(r), & \text{if } \alpha = \beta \\ \varphi_{\alpha,\beta}^n(r), & \text{if } \alpha < \beta. \end{cases}$$

Proof. We will give the proof for the case $\alpha = \beta$. Since $|x|_p = r$, $|a|_p = \alpha\beta$, using formula (3.1) and the strong triangle inequality of the p-adic norm, we get the following

$$|f(x)|_p = |x|_p \cdot \frac{|a|_p}{|x - \hat{x}_1|_p |x - \hat{x}_2|_p} = \varphi_\alpha(r) = \begin{cases} r, & \text{if } r < \alpha, \\ \geq \alpha, & \text{if } r = \alpha, \\ \frac{\alpha^2}{r}, & \text{if } r > \alpha. \end{cases} \tag{3.2}$$

Now consider the case $n = 2$. Since $|f(x)|_p = \varphi_\alpha(r)$ (by (3.2)), we obtain

$$|f^2(x)|_p = |f(x)|_p \cdot \frac{|a|_p}{|f(x) - \hat{x}_1|_p |f(x) - \hat{x}_2|_p} = \varphi_\alpha(\varphi_\alpha(r))$$

$$= \begin{cases} \varphi_\alpha(r), & \text{if } \varphi_\alpha(r) < \alpha, \\ \geq \alpha, & \text{if } \varphi_\alpha(r) = \alpha, \\ \frac{\alpha^2}{\varphi_\alpha(r)}, & \text{if } \varphi_\alpha(r) > \alpha. \end{cases}$$

Iterating this argument for any $n \geq 1$ and any $x \in S_r(0) \setminus \mathcal{P}$, we obtain the following formula

$$|f^n(x)|_p = \varphi_\alpha^n(r), \quad \text{if } \alpha = \beta.$$

The other case can be similarly proved.

Thus the p-adic dynamical system $f^n(x)$, $n \leq 1$, $x \in \mathbb{C}_p \setminus \mathcal{P}$ is related to the real dynamical systems generated by functions (2.5)-(2.6) and we have two cases.

3.1 Case: $\alpha = \beta$.

Lemma 3.3. *If $\alpha = \beta$, then the dynamical system generated by $\varphi_\alpha(r)$ has the following properties:*

1. $\mathrm{Fix}(\varphi_\alpha) = \{r : 0 \leq r < \alpha\} \cup \{\alpha\text{: if } \alpha^* = \alpha\}$.
2. *If $r > \alpha$, then $\varphi_\alpha^n(r) = \frac{\alpha^2}{r}$ for any $n \geq 1$.*
3. *If $r = \alpha$ and $\alpha^* > \alpha$, then $\varphi_\alpha^n(r) = \frac{\alpha^2}{\alpha^*}$ for any $n \geq 2$.*

Proof.

1. This is the result of a simple analysis of the equation $\varphi_\alpha^n(r) = r$.
2. If $r > \alpha$, then $\varphi_\alpha(r) = \frac{\alpha^2}{r}$.
   Consequently,
   $$r > \alpha, \Rightarrow \frac{\alpha^2}{r} < \alpha \Rightarrow \varphi_\alpha(r) < \alpha.$$
   Thus $\varphi_\alpha(\varphi_\alpha(r)) = \varphi_\alpha(r)$, i.e., $\varphi_\alpha(r)$ is a fixed point of φ_α for any $r > \alpha$. Consequently, for each $n \geq 1$ we have
   $$\varphi_\alpha^n(r) = \frac{\alpha^2}{r}.$$
3. Part 3 easily follows from parts 1 and 2.

Now we shall apply these lemmas to study the p-adic dynamical system generated by the function (2.6).

For $\alpha = \beta$ denote the following

$$\alpha^*(x) = |f(x)|_p, \quad \text{if} \quad x \in S_\alpha(0).$$

Then using Lemmas 3.2 and 3.3, we obtain the following

Theorem 3.4. *If $\alpha = \beta$, then the p-adic dynamical system generated by the function (2.6) has the following properties:*

1. *1.1) $SI(x_2) = U_\alpha(0)$.*

   1.2) $\mathcal{P} \subset S_\alpha(0)$.
2. *If $r > \alpha$ and $x \in S_r(0)$, then for any $n \geq 1$, $f^n(x) \in S_{\frac{\alpha^2}{r}}(0)$.*
3. *If $\alpha^*(x) > \alpha$ and $x \in S_\alpha(0)$, then for any $n \geq 2$, $f^n(x) \in S_{\frac{\alpha^2}{\alpha^*(x)}}(0)$.*
4. *If $|c|_p < \alpha$, then $|f'(x_1)|_p = 1$ and*
   $$SI(x_1) = SI(x_2).$$
5. *Let $|c|_p = \alpha$. Then $x_1 \in S_\alpha(0)$ and*

   5.1) if $|a - c^2|_p < \alpha^2$, then x_1 is an attractive fixed point for f and its basin of attraction is
   $$A(x_1) = U_\alpha(x_1) \subset S_\alpha(0).$$

   5.2) if $|a - c^2|_p = \alpha^2$, then x_1 is an indifferent fixed point for f and
   $$SI(x_1) = U_\alpha(x_1) \subset S_\alpha(0).$$

Proof. We will prove part 1, by using parts 2 and 3.

Parts 2 and 3 easily follow from Lemma 3.2 and part 2 of Lemma 3.3.

1. By parts 2 and 3 of the theorem we know that $S_r(0)$ is not invariant for f if $r \geq \alpha$. Consequently, $SI(x_2) \subset U_\alpha(0)$.

By Lemma 3.2 and part 1 of Lemma 3.3 if $r < \alpha$ and $x \in S_r(0)$ then $|f^n(x)|_p = \varphi_\alpha^n(r) = r$, i.e., $f^n(x) \in S_r(0)$. Hence $U_\alpha(0) \subset SI(x_2)$ and thus $SI(x_2) = U_\alpha(0)$.

Since $|\hat{x}_1|_p = |\hat{x}_2|_p = \alpha$, we have $\hat{x}_i \notin U_\alpha(0)$, $i = 1, 2$. From $f(U_\alpha(0)) \subset U_\alpha(0)$ it follows that

$$U_\alpha(0) \cap \mathcal{P} = \{x \in U_\alpha(0) : \exists n \in N \cup \{0\},\ f^n(x) \in \{\hat{x}_1, \hat{x}_2\}\} = \emptyset.$$

By part 2 of the theorem for $r > \alpha$ we have $f(S_r(0)) \subset U_\alpha(0)$. Let $V_\alpha(0)$ be the closed ball with the center 0 and radius α. Then

$$(\mathbb{C}_p \setminus V_\alpha(0)) \cap \mathcal{P} = \emptyset,$$

i.e., $\mathcal{P} \subset S_\alpha(0)$.

4. Note that $|c|_p \leq \alpha$. If $|c|_p < \alpha$, then $|x_1|_p = |-c|_p < \alpha$. So $x_1 \in U_\alpha(0) = SI(x_2)$ and

$$|f'(x_1)|_p = \frac{|a - c^2|_p}{|a|_p} = \frac{\alpha^2}{\alpha^2} = 1.$$

Consequently, x_1 is an indifferent fixed point for f and

$$SI(x_1) \subset SI(x_2). \tag{3.3}$$

By simple calculation we get

$$|f(x) - x_1|_p = |x - x_1|_p \cdot \frac{|c(x - x_1) + cx_1 + a|_p}{|(x - x_1) + (x_1 - \hat{x}_1)|_p |(x - x_1) + (x_1 - \hat{x}_2)|_p}. \tag{3.4}$$

If $x \in S_\rho(x_1) \subset U_\alpha(0)$, for some $\rho < \alpha$, then in (5.2) we have $|c(x - x_1) + cx_1 + a|_p = \alpha^2$. Moreover, $|x_1 - \hat{x}_1|_p = |\hat{x}_2|_p = \alpha$ and $|x_1 - \hat{x}_2|_p = |\hat{x}_1|_p = \alpha$. Therefore, $|f(x) - x_1|_p = |x - x_1|_p$, i.e. $f(x) \in S_\rho(x_1)$ holds for every $x \in S_\rho(x_1) \subset U_\alpha(x_1)$. Then $U_\alpha(x_1) = U_\alpha(0) = SI(x_2) \subset SI(x_1)$ and by (3.3) we have $SI(x_1) = SI(x_2)$.

5. If $|c|_p = \alpha$, then $|x_1|_p = |-c|_p = \alpha$, i.e., $x_1 \in S_\alpha(0)$. Moreover, if $x \in U_\alpha(x_1)$, then $|x|_p = |(x - x_1) + x_1|_p = \alpha$, i.e., $U_\alpha(x_1) \subset S_\alpha(0)$.

Note that

$$|f'(x_1)|_p = \frac{|a - c^2|_p}{|a|_p}.$$

We have $|a - c^2|_p \leq \alpha^2$ and $|a|_p = \alpha^2$.

5.1. If $|a - c^2|_p < \alpha^2$, then $|f'(x_1)|_p < 1$, i.e., x_1 is an attractive fixed point for f. If $x \in U_\alpha(x_1)$, then in (5.2) we have $|c(x - x_1) + cx_1 + a|_p < \alpha^2$. Therefore, $|f(x) - x_1|_p < |x - x_1|_p$ for all $x \in U_\alpha(x_1)$. So

$$U_\alpha(x_1) \subset A(x_1).$$

If $x \in S_\alpha(0) \setminus (U_\alpha(x_1) \cup \mathcal{P})$, then $|x - x_1|_p = \alpha$ and by (5.2) we have $|f(x) - x_1|_p \geq |x - x_1|_p$, i.e., $x \notin A(x_1)$. Consequently,

$$A(x_1) = U_\alpha(x_1).$$

5.2. If $|a - c^2|_p = \alpha^2$, then $|f'(x_1)|_p = 1$, i.e., x_1 is an indifferent fixed point for f. If $x \in S_\rho(x_1) \subset U_\alpha(x_1)$, then in (5.2) we have $|c(x - x_1) + cx_1 + a|_p = \alpha^2$ and $|f(x) - x_1|_p = |x - x_1|_p$. Therefore, $f(x) \in S_\rho(x_1)$ for all $x \in S_\rho(x_1)$. So $U_\alpha(x_1) \subset SI(x_1)$.

If $x \in S_\alpha(0) \setminus (U_\alpha(x_1) \cup \mathcal{P})$, then $|x - x_1|_p = \alpha$ and by (5.2) we have $|f(x) - x_1|_p$ is some given number with $|f(x) - x_1|_p > 0$, i.e., the sphere $S_\alpha(x_1)$ is not invariant for f. Consequently,

$$SI(x_1) = U_\alpha(x_1).$$

Theorem is proved.

3.2 Case: $\alpha < \beta$.

Lemma 3.5. *If $\alpha < \beta$, then the dynamical system generated by $\varphi_{\alpha,\beta}(r)$ has the following properties:*

1. $\mathrm{Fix}(\varphi_{\alpha,\beta}) = \{r : 0 \leq r < \alpha\} \cup \{\alpha\text{: if } \hat{\alpha} = \alpha\} \cup \{\beta\text{: if } \hat{\beta} = \beta\}$.
2. *If $\alpha < r < \beta$, then $\varphi_{\alpha,\beta}(r) = \alpha$.*
3. *If $r > \beta$, then $\varphi^n_{\alpha,\beta}(r) = \frac{\alpha\beta}{r}$ for any $n \geq 1$.*
4. *Let $r = \alpha$.*

   4.1) If $\alpha < \hat{\alpha} < \beta$, then $\varphi^2_{\alpha,\beta}(\alpha) = \alpha$.

   4.2) If $\hat{\alpha} = \beta$, then $\varphi_{\alpha,\beta}(\alpha) = \beta$.

   4.3) If $\hat{\alpha} > \beta$, then $\varphi^n_{\alpha,\beta}(\alpha) = \frac{\alpha\beta}{\hat{\alpha}}$ for any $n \geq 2$.
5. *Let $r = \beta$.*

   5.1) If $\alpha < \hat{\beta} < \beta$, then $\varphi^2_{\alpha,\beta}(\beta) = \alpha$.

   5.2) If $\hat{\beta} > \beta$, then $\varphi^n_{\alpha,\beta}(\beta) = \frac{\alpha\beta}{\hat{\beta}}$ for any $n \geq 2$.

Proof.
1. This is the result of a simple analysis of the equation $\varphi_{\alpha,\beta}(r) = r$.
2. If there is $\alpha < r < \beta$, then function $\varphi_{\alpha,\beta}$ will be $\varphi_{\alpha,\beta}(r) = \alpha$ by definition.
3. If $r > \beta$, then $\varphi_{\alpha,\beta}(r) = \frac{\alpha\beta}{r}$. Consequently,

$$\beta < r \ \Rightarrow\ \frac{\alpha\beta}{r} < \alpha \ \Rightarrow\ \varphi_{\alpha,\beta}(r) < \alpha.$$

Thus $\varphi_{\alpha,\beta}(\varphi_{\alpha,\beta}(r)) = \varphi_{\alpha,\beta}(r)$, i.e., $\varphi_{\alpha,\beta}(r)$ is a fixed point of $\varphi_{\alpha,\beta}$ for any $r > \beta$. Consequently, for each $n \geq 1$ we have $\varphi^n_{\alpha,\beta}(r) = \frac{\alpha\beta}{r}$.

Parts 4 and 5 easily follow from parts 1, 2 and 3.

For $\alpha < \beta$ we denote the following

$$\hat{\alpha}(x) = |f(x)_p| \ \text{ if } \ x \in S_\alpha(0) \setminus \{\hat{x}_1, \hat{x}_2\}; \quad \hat{\beta}(x) = |f(x)|_p \ \text{ if } \ x \in S_\beta(0) \setminus \{\hat{x}_1, \hat{x}_2\}.$$

Then using Lemma 3.2 and Lemma 3.5, we obtain the following

Theorem 3.6. *If $\alpha < \beta$ and $x \in S_r(0) \setminus \mathcal{P}$, then the p-adic dynamical system generated by function (2.6) has the following properties:*

1. $SI(x_2) = U_\alpha(0)$.
2. *If $\alpha < r < \beta$, then $f(x) \in S_\alpha(0)$.*
3. *Let $r > \beta$, then for any $n \geq 1$, $f^n(x) \in S_{\frac{\alpha\beta}{r}}(0)$.*
4. *Let $x \in S_\alpha(0) \setminus \mathcal{P}$.*
   - *4.1) If $\hat{\alpha}(x) = \alpha$, then $f(x) \in S_\alpha(0)$.*
   - *4.2) If $\alpha < \hat{\alpha}(x) < \beta$, then $f^2(x) \in S_\alpha(0)$.*
   - *4.3) If $\hat{\alpha}(x) = \beta$, then $f(x) \in S_\beta(0)$.*
   - *4.4) If $\hat{\alpha}(x) > \beta$, then for any $n \geq 2$, $f^n(x) \in S_{\frac{\alpha\beta}{\hat{\alpha}}}(0)$.*
5. *Let $x \in S_\beta(0) \setminus \mathcal{P}$.*
   - *5.1) If $\hat{\beta}(x) = \alpha$, then $f(x) \in S_\alpha(0)$.*
   - *5.2) If $\alpha < \hat{\beta}(x) < \beta$, then $f^2(x) \in S_\alpha(0)$.*
   - *5.3) If $\hat{\beta}(x) = \beta$, then $f(x) \in S_\beta(0)$.*
   - *5.4) If $\hat{\beta}(x) > \beta$, then for any $n \geq 2$, $f^n(x) \in S_{\frac{\alpha\beta}{\hat{\beta}(x)}}(0)$.*
6. *6.1) $x_1 \in S_\beta(0)$.*
   - *6.2) The fixed point x_1 is repeller and the inequality $|f(x) - x_1|_p > |x - x_1|_p$ holds for $x \in U_\beta(x_1)$, $x \neq x_1$.*

Proof. Parts 1-5 of this theorem can be shown by Lemma 3.2 and Lemma 3.5.

6. In this case we note that $\alpha < \beta = |c|_p$. Then $|x_1|_p = |-c|_p = \beta$, i.e., $x_1 \in S_\beta(0)$, and we have

$$|f'(x_2)|_p = \left|\frac{a - c^2}{a}\right|_p = \frac{\max\{\alpha\beta, \beta^2\}}{\alpha\beta} = \frac{\beta}{\alpha} > 1.$$

So, x_1 is a repelling fixed point for f.

In (5.2) we have $|x_1 - \hat{x}_1|_p = \beta$ and $|x_1 - \hat{x}_2|_p = \alpha$. Indeed, $|x_1 - \hat{x}_1|_p = \max\{\alpha, \beta\} = \beta$ and $|x_1 - \hat{x}_2|_p = |\hat{x}_1|_p = \alpha$. Moreover, by formula (5.2) we have

$$|f(x) - x_1|_p = \begin{cases} \frac{\beta}{\alpha}|x - x_1|_p, & \text{if } |x - x_1|_p < \alpha \\ \geq \beta, & \text{if } |x - x_1|_p = \alpha \\ \beta, & \text{if } \alpha < |x - x_1|_p < \beta \\ \beta_0, & \text{if } |x - x_1|_p = \beta \\ \beta, & \text{if } |x - x_1|_p > \beta, \end{cases}$$

where $\beta_0 > 0$.

That is the inequality $|f(x) - x_1|_p > |x - x_1|_p$ is satisfied for $|x - x_1|_p < \beta$, i.e., for $x \in U_\beta(x_1)$, $x \neq x_2$. Theorem is proved.

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