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PERIODIC AND WEAKLY PERIODIC GROUND STATES CORRESPONDING TO SUBGROUPS OF INDEX THREE FOR THE ISING MODEL ON CAYLEY TREE

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For the Ising model on a Cayley tree of order two, we describe a set of periodic and weakly periodic ground states that correspond to subgroups of index 3 of the group representation of the Cayley tree.

Keywords: Cayley tree, configuration, Ising model, ground state.

1. Introduction

The Ising model, with two values of spin ± 1 was considered in [10, 13] and became actively researched in the 1990's and afterwards (see for example [1–7, 9, 11]).

Each Gibbs measure is associated with a single phase of a physical system. Existence of two or more Gibbs measures means that phase transitions exist. One of fundamental problems is to describe the extreme Gibbs measures corresponding to a given Hamiltonian. As is known, the phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure, see [12, 16]. The problem naturally arises on description of periodic and weakly periodic ground states. For the Ising model with competing interactions on the Cayley tree, translation-invariant and periodic ground states corresponding to normal subgroups of even indices are studied in [1, 15]. As usual, more simple and interesting ground states are the periodic ones. On the other hand, it is necessary to find weakly periodic ground states for some parameters for which a periodic ground state does not exist.

The main concepts and notations for weakly periodic ground states are introduced in [17]. For the Ising model with competing interactions, weakly periodic ground states corresponding to normal subgroups of indices two and four are described in [17, 18]. For the Potts model, such states for normal subgroups of index 2 are studied in [19, 20], also in [21]. For the Potts model, periodic and weakly periodic ground states for normal subgroups of index 4 are studied.

A full description of normal subgroups of indices $2i$, $i = \overline{1,5}$, for the group representation of the Cayley tree is given e.g. in [8]. Also, the existence of all subgroups of finite index for the group is proved and a full description of (not normal) subgroups of index 3 is given (see [14]). Note that there are some papers devoted to periodic and weakly periodic ground states for normal groups of finite index. In this paper, for the first time we study periodic and weakly periodic ground states for (not normal) subgroups of index 3. This paper is organized as follows. In Section 2, we recall the main definitions and known facts. In Section 3, we describe periodic and weakly periodic ground states.

2. Main definitions and known facts

The Cayley tree. The Cayley tree Γ^k (see [2]) of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighbouring vertices*, and we write $l = \langle x, y\rangle$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree is the shortest path from x to y.

For the fixed $x^0 \in V$ (as usual, x^0 is called a root of the tree) we set

$$W_n = \{x \in V \quad | \quad d(x, x^0) = n\},$$

$$V_n = \{x \in V \quad | \quad d(x, x^0) \leq n\}, \qquad L_n = \{l = \langle x, y\rangle \in L \quad | \quad x, y \in V_n\}. \tag{1}$$

We write $x < y$ if the path from x^0 to y goes through x and $|x| = d(x, x^0)$, $x \in V$.

It is known (see [6]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, \ldots, k+1$, of the second-order (i.e. $a_i^2 = e$, $a_i \neq e$) with generators $a_1, a_2, \ldots, a_{k+1}$.

Let $S(x)$ be the set of "direct successors" of $x \in G_k$, i.e.

$$S(x) = \{y \in W_{n+1} | \ d(y, x) = 1\}, \qquad x \in W_n.$$

Also, $S_1(x)$ is the set of all nearest neighbouring vertices of $x \in G_k$, i.e. $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$ and $x_{\downarrow} = S_1(x) \setminus S(x)$.

The Ising model. At first, we give main definitions and facts about the Ising model (for more details see [1]). We consider models where the spin takes values in the set $\Phi = \{-1, 1\}$. For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations is denoted by $\Omega_A = \Phi^A$. Put $\Omega = \Omega_V$, $\sigma = \sigma_V$ and $-\sigma_A = \{-\sigma_A(x), x \in A\}$. Define a *periodic configuration* as a configuration $\sigma \in \Omega$ which is invariant under cosets of a subgroup $G_k^* \subset G_k$ of finite index. More precisely, a configuration $\sigma \in \Omega$ is called G_k^* -periodic if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$.

The index of a subgroup is called the *period of the corresponding periodic configuration*. A configuration that is invariant with respect to all cosets is called *translation-invariant*.

Let $G_k/G_k^* = \{H_1, \dots, H_r\}$ be a family of cosets, where G_k^* is a subgroup of index $r \geq 1$. Configuration $\sigma(x)$, $x \in V$, is called G_k^* *weakly periodic*, if $\sigma(x) = \sigma_{ij}$ for $x \in H_i$, $x_\downarrow \in H_j, \forall x \in G_k$.

The Ising model with two competing interactions has the form

$$H(\sigma) = J_1 \sum_{\langle x,y \rangle} \sigma(x)\sigma(y) + J_2 \sum_{\substack{x,y \in V: \\ d(x,y)=2}} \sigma(x)\sigma(y), \tag{2}$$

where $J = (J_1, J_2) \in \mathbb{R}^2$ are coupling constants and $\sigma \in \Omega$.

For a pair of configurations σ and φ that coincide almost everywhere, i.e. everywhere except for a finite number of positions, we consider a relative Hamiltonian $H(\sigma, \varphi)$, the difference between the energies of the configurations σ and φ has the form

$$H(\sigma, \varphi) = J_1 \sum_{\langle x,y \rangle} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) + J_2 \sum_{\substack{x,y \in V: \\ d(x,y)=2}} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)), \tag{3}$$

where $J = (J_1, J_2) \in \mathbb{R}^2$ is an arbitrary fixed parameter.

Let M be the set of unit balls with vertices in V. We call the restriction of a configuration σ to the ball $b \in M$ a *bounded configuration* σ_b.

Define the energy of a ball b for configuration σ by

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x,y \rangle} \sigma(x)\sigma(y) + J_2 \sum_{d(x,y)=2} \sigma(x)\sigma(y), \quad x, y \in b, \tag{4}$$

where $J = (J_1, J_2) \in R^2$.

We shall say that two bounded configurations σ_b and $\sigma'_{b'}$ belong to the same class if $U(\sigma_b) = U(\sigma'_{b'})$ and we write $\sigma'_{b'} \sim \sigma_b$.

Let A be a set, then $|A|$ is the cardinality of A.

Lemma 1 [1]. 1) *For any configuration* σ_b *we have*

$$U(\sigma_b) \in \{U_0, U_1, \dots, U_{k+1}\},$$

where

$$U_i = \left(\frac{k+1}{2} - i\right) J_1 + \left(\frac{k(k+1)}{2} + 2i(i - k - 1)\right) J_2, \qquad i = 0, 1, \dots, k+1. \tag{5}$$

2) *Let* $C_i = \Omega_i \cup \Omega_i^-$, $i = 0, \dots, k+1$, *where*

$$\Omega_i = \big\{\sigma_b : \sigma_b(c_b) = +1, \quad |\{x \in b \setminus \{c_b\} : \sigma_b(x) = -1\}| = i\big\},$$

$$\Omega_i^- = \big\{ -\sigma_b = \{-\sigma_b(x), x \in b\} : \sigma_b \in \Omega_i\big\},$$

and c_b *is the center of the ball* b. *Then for* $\sigma_b \in C_i$ *we have* $U(\sigma_b) = U_i$.

3) *The class* C_i *contains* $\frac{2(k+1)!}{i!(k-i+1)!}$ *configurations.*

Definition 1. A configuration φ is called a *ground state* for the relative Hamiltonian H if it satisfies the following condition

$$U(\varphi_b) = \min\{U_0, U_1, \dots, U_{k+1}\}, \qquad \text{for any} \quad b \in M. \tag{6}$$

Denote

$$U_i(J) = U(\sigma_b, J), \quad \text{if} \quad \sigma_b \in C_i, \quad i = 0, 1, \dots, k+1.$$

The quantity $U_i(J)$ is a linear function of the parameter $J \in R^2$. For every fixed $m = 0, 1, \dots, k+1$ we denote

$$A_m = \{J \in R^2 : U_m(J) = \min\{U_0(J), U_1(J), \dots, U_{k+1}(J)\}\}. \tag{7}$$

It is easy to check that

$$A_0 = \{J \in R^2 : J_1 \le 0; \quad J_1 + 2kJ_2 \le 0\},$$

$$A_m = \{J \in R^2 : J_2 \ge 0; \quad 2(2m-k-2)J_2 \le J_1 \le 2(2m-k)J_2\}, \quad m = 1, 2, \dots, k,$$

$$A_{k+1} = \{J \in R^2 : J_1 \ge 0; \quad J_1 - 2kJ_2 \ge 0\} \text{ and } R^2 = \cup_{i=0}^{k+1} A_i.$$

3. Periodic and weakly periodic ground states

In this section we study periodic and weakly periodic ground states corresponding to subgroups of the group representation of the Cayley tree of index 3.

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$, respectively. Then from Theorem 1 in [15], it is known that:

- The group G_k does not have normal subgroups of odd index ($\neq 1$);
- The group G_k has normal subgroups of arbitrary even index.

Now, we give a construction of subgroups of index 3 of the group G_k (for more details see [14]).

Let $N_k = \{1, 2, \dots, k+1\}$ and $B_0 \subset N_k$, $0 \le |B_0| \le k-1$. (B_1, B_2) be a partition of the set $N_k \setminus B_0$. Put m_j as a minimal element of B_j, $j \in \{1, 2\}$. Then we consider the function $u_{B_1B_2} : \{a_1, a_2, \dots, a_{k+1}\} \to \{e, a_1, \dots, a_{k+1}\}$ (where e is identity element) given by

$$u_{B_1B_2}(x) = \begin{cases} e, & \text{if } x = a_i,\ i \in N_k \backslash (B_1 \cup B_2), \\ a_{m_j}, & \text{if } x = a_i,\ i \in B_j,\ j = 1, 2. \end{cases} \tag{8}$$

Let $l(x)$ be the length of x. For $1 \leq q \leq s$, we define $\gamma_s : \langle e, a_{m_1}, a_{m_2}\rangle \to \langle e, a_{m_1}, a_{m_2}\rangle$ by the formula

$$\gamma_s(x) = \begin{cases} e, & \text{if } x = e, \\ a_{m_1}a_{m_2}a_{m_1}\dots a_{m_j}, & \text{if } x \in \{\underbrace{a_{m_1}a_{m_2}a_{m_1}\dots a_{m_j}}_{q}, \underbrace{a_{m_2}a_{m_1}a_{m_2}\dots a_{m_{3-j}}}_{2s+1-q}\}, \\ a_{m_2}a_{m_1}a_{m_2}\dots a_{m_j}, & \text{if } x \in \{\underbrace{a_{m_2}a_{m_1}a_{m_2}\dots a_{m_j}}_{q}, \underbrace{a_{m_1}a_{m_2}a_{m_1}\dots a_{m_{3-j}}}_{2s+1-q}\}, \\ \gamma_s(a_{m_j}\dots\gamma_s(\underbrace{a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}}_{2s})), & \text{if } x = a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}},\ l(x) > 2s, \\ \gamma_s(a_{m_j}\dots\gamma_s(\underbrace{a_{m_{3-j}}a_{m_j}\dots a_{m_j}}_{2s})), & \text{if } x = a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}},\ l(x) > 2s. \end{cases} \tag{9}$$

Denote
$$\Im^s_{B_1B_2}(G_k) = \left\{x \in G_k |\ \gamma_s(u_{B_1B_2}(x)) = e\right\}.$$

LEMMA 2 [14]. *Let (B_1, B_2) be a partition of the set $N_k \setminus B_0$, $0 \leq |B_0| \leq k-1$. Then $x \in \Im^s_{B_1B_2}(G_k)$ if and only if the number $l(u_{B_1B_2}(x))$ is divisible by $2s+1$.*

PROPOSITION 1 [14]. *For the group G_k the following equality holds:*

$$\{K|\ K \text{ is a subgroup of } G_k \text{ of index 3}\} = \left\{\Im^1_{B_1B_2}|\ B_1, B_2 \text{ is a partition of } N_k \setminus B_0\right\}.$$

We consider periodic and weakly periodic ground states on the Cayley tree of order two, i.e. $k = 2$. Let $B_s = \{s\}, s \in \{1, 2, 3\}$, i.e. $m_i = i$, $i \in \{2, 3\}$. Now, we consider functions $u_{B_2B_3} : \{a_1, a_2, a_3\} \to \{e, a_2, a_3\}$ (defined in (8)) and $\gamma :< e, a_2, a_3 >\to \{e, a_2, a_3\}$ (defined in (9))

$$u_{\{2\},\{3\}}(x) = \begin{cases} e, & \text{if } x = a_1, \\ a_i, & \text{if } x = a_i,\ i = \overline{2,3}, \end{cases}$$

$$\gamma(x) = \begin{cases} e & \text{if } x = e, \\ a_2 & \text{if } x \in \{a_2, a_3a_2\}, \\ a_3 & \text{if } x \in \{a_3, a_2a_3\}, \\ \gamma(a_{i+1}a_{4-i}\dots\gamma(a_{i+1}a_{4-i})) & \text{if } x = a_{i+1}a_{4-i}\dots a_{4-i},\ l(x) \geq 3,\ i = \overline{1,2}, \\ \gamma(a_{i+1}a_{4-i}\dots\gamma(a_{4-i}a_{i+1})) & \text{if } x = a_{i+1}a_{4-i}\dots a_{i+1},\ l(x) \geq 3,\ i = \overline{1,2}. \end{cases}$$

Let $H_1 := \Im^1_{\{2\}\{3\}}(G_2)$. Then

$$H_1 = \{x \in G_2|\ \gamma(u_{\{2\}\{3\}}(x)) = e\}.$$

By using H_1 as a subgroup of index 3 of the group G_2 we define a family of cosets,

$$G_2/H_1 = \{H_1, H_2, H_3\},$$

where

$$H_2 = \{x \in G_2 \ |\gamma(u_{\{2\}\{3\}}(x)) = a_2\}, \qquad H_3 = \{x \in G_2 \ |\gamma(u_{\{2\}\{3\}}(x)) = a_3\}.$$

H_1-periodic configurations have the following forms

$$\sigma(x) = \begin{cases} \sigma_1 & \text{if } x \in H_1, \\ \sigma_2 & \text{if } x \in H_2, \\ \sigma_3 & \text{if } x \in H_3, \end{cases}$$

where $\sigma_i \in \Phi$, $i \in \{1, 2, 3\}$.

Note that if $\sigma_1 = \sigma_2 = \sigma_3$ then this configuration is *translation-invariant* and the full details of such configuration are given in [15].

THEOREM 1. *Let* $k = 2$.

1) *If* $(J_1, J_2) \in A_2 \cap A_4$, *then there are six* H_1*-periodic ground states corresponding to the following configurations*

$$\sigma(x) = \pm \begin{cases} \sigma_1 & \text{if } x \in H_1, \\ \sigma_2 & \text{if } x \in H_2, \\ \sigma_3 & \text{if } x \in H_3, \end{cases}$$

where $(\sigma_1, \sigma_2, \sigma_3) \in \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$.

2) *If* $(J_1, J_2) \in R^2 \backslash (A_2 \cap A_4)$, *there are not* H_1*-periodic (except for translation-invariant) ground states.*

Proof: Let $(\sigma_1, \sigma_2, \sigma_3) = (-1, 1, 1)$. Consider the following configuration

$$\varphi_1(x) = \begin{cases} -1 & \text{if } x \in H_1, \\ 1 & \text{if } x \in H_2, \\ 1 & \text{if } x \in H_3. \end{cases}$$

Denote $A_- = \{x \in S_1(x) : \varphi_b(x) = -1\}$, $A_+ = \{x \in S_1(x) : \varphi_b(x) = +1\}$ and $\varphi_{i,b} = (\varphi_i)_b$, for any i. If $c_b \in H_1$, then $\varphi_1(c_b) = -1$ $|A_-| = 1, |A_+| = 2$, $\varphi_{1,b} \in C_4$. For the case $c_b \in H_2$ one gets $\varphi_1(c_b) = 1$ $|A_-| = 1, |A_+| = 2$, $\varphi_{1,b} \in C_2$. Finally, if $c_b \in H_3$, then $\varphi_1(c_b) = 1$ $|A_-| = 1, |A_+| = 2$, $\varphi_{1,b} \in C_2$. Hence, for any $b \in M$ one gets $\varphi_{1,b} \in C_2 \cup C_4$.

From (7) we obtain that $A_2 \cap A_4 = \{(J_1, J_2) : J_1 = 0, J_2 \geq 0\}$. By Lemma 1 one gets that the periodic configuration φ_1 is H_1 periodic ground state on the set $A_2 \cap A_4$. Note that for any $b \in M$ we have $\varphi_{1,b} \sim -\varphi_{1,b}$, i.e. $-\varphi_{1,b} \in C_2 \cup C_4$ for all $b \in M$. Consequently, the periodic configuration $-\varphi_1$ is H_1 periodic ground state on the set $A_2 \cap A_4$.

The cases of periodic configurations $\pm\varphi_2$ and $\pm\varphi_3$, corresponding to $(\sigma_1, \sigma_2, \sigma_3) \in \{(1, -1, 1), (1, 1, -1)\}$ are similar.

Note that there are not periodic (not translation invariant) configuration not mentioned in assertion 1. Above we proved that those configurations are ground states on the set $A_2 \cap A_4$. Hence, if $(J_1, J_2) \in R^2 \setminus (A_2 \cap A_4)$ there are not H_1 periodic (not translation invariant) ground states. This completes the proof. □

REMARK 1. H_1-periodic ground states mentioned in Theorem 1 differ from periodic ground states described in [1]. In addition, in [1] has been proved that for a fixed $J = (J_1, J_2)$ the maximum number of periodic ground states equals four. In our case, it is equal to six.

In [18] and [19] for the normal subgroups of indices two and four, weakly periodic ground states are studied. Now, we study H_1-weakly periodic ground states, i.e. weakly periodic ground states corresponding to subgroups of index 3 of the group representation of the Cayley tree.

For any element x of G_k, we recall that $x_\downarrow$ is an element which satisfies the following condition: $x^{-1} \cdot x_\downarrow \in \{a_i \mid i \in N_k\}$.

Invariance property: If

$$\gamma(u_{\{2\}\{3\}}(x)) = \gamma(u_{\{2\}\{3\}}(y)), \qquad \gamma(u_{\{2\}\{3\}}(x_\downarrow)) = \gamma(u_{\{2\}\{3\}}(y_\downarrow))$$

then

$$\langle \gamma(u_{\{2\}\{3\}}(xa_i)) \mid xa_i \in S(x) \rangle = \langle \gamma(u_{\{2\}\{3\}}(ya_i)) \mid ya_i \in S(y) \rangle,$$

where $\langle \dots \rangle$ stands for ordered k-tuples (for more details see [14]).

In [14] is given a certain condition on subgroups of the group representation of the Cayley tree such that an invariance property holds. Generally speaking, except for the given condition, the invariance property does not hold. H_1 subgroup satisfies the invariance property, that is why H_1-weakly periodic configurations have the following forms:

$$\varphi(x) = \begin{cases} a_{11}, x_\downarrow \in H_1 \text{ and } x \in H_1, \\ a_{12}, x_\downarrow \in H_1 \text{ and } x \in H_2, \\ a_{13}, x_\downarrow \in H_1 \text{ and } x \in H_3, \\ a_{21}, x_\downarrow \in H_2 \text{ and } x \in H_1, \\ a_{22}, x_\downarrow \in H_2 \text{ and } x \in H_2, \\ a_{23}, x_\downarrow \in H_2 \text{ and } x \in H_3, \\ a_{31}, x_\downarrow \in H_3 \text{ and } x \in H_1, \\ a_{32}, x_\downarrow \in H_3 \text{ and } x \in H_2, \\ a_{33}, x_\downarrow \in H_3 \text{ and } x \in H_3, \end{cases} \tag{10}$$

where $\varphi_{ij} \in \Phi$, $i, j \in \{1, 2, 3\}$.

In the sequel, we write $\varphi(x) = (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33})$ for such a weakly periodic configuration φ.

Theorem 2. *Let* $k = 2$. *Then the following assertions hold.*

1. *There are exactly thirty-six* H_1*-weakly periodic ground states on* $\{J_1 = 0; J_2 \geq 0\}$, *which are not periodic, have the form* $\varphi_1 = (i,i,j,i,j,j,i,j,j)$, $\varphi_2 = (i,j,i,i,j,j,i,j,j)$, $\varphi_3 = (i,j,i,j,j,i,j,j,i)$, $\varphi_4 = (i,j,j,i,j,j,i,i,j)$, $\varphi_5 = (i,j,j,i,j,j,i,j,i)$, $\varphi_6 = (j,i,i,j,j,i,j,j,i)$, $\varphi_7 = (j,i,j,i,i,j,j,i,j)$, $\varphi_8 = (j,i,j,j,i,i,j,i,j)$, $\varphi_9 = (j,j,i,i,j,i,j,j,i)$, $\varphi_{10} = (j,j,i,j,i,i,j,j,i)$, $\varphi_{11} = (j,j,i,j,j,i,i,j,i)$, $\varphi_{12} = (j,j,i,j,j,i,j,i,i)$, $\varphi_{13} = (i,i,j,j,i,j,j,i,j)$, $\varphi_{14} = (i,j,j,i,i,j,i,j,j)$, $\varphi_{15} = (i,j,j,i,j,i,i,j,j)$, $\varphi_{16} = (j,i,i,j,i,j,j,i,j)$, $\varphi_{17} = (j,i,j,j,i,j,i,i,j)$, $\varphi_{18} = (j,i,j,j,i,j,j,i,i)$ *and* $\varphi_{25+v} = -\varphi_v$, *where* $v = 1,2,..18$ *and* $i \neq j; i,j \in \Phi$.

2. *All* H_1*-weakly periodic ground states that are not mentioned in assertion* 1 *are periodic.*

Proof: 1. At first, we prove the first part of Theorem 2. Let us consider φ_1.

1.1) Assume that $c_b \in H_1$. Then all possible cases are:
a) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1, |A_-| = 2, |A_+| = 1,\ \varphi_{1,b} \in C_2$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, this case is impossible,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1, |A_-| = 2, |A_+| = 1,\ \varphi_{1,b} \in C_2$,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = -1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_4$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = -1, |A_-| = 2, |A_+| = 1,\ \varphi_{1,b} \in C_2$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, this case is impossible.

1.2) Let $c_b \in H_2$, then all possible cases are:
a) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_4$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$ which is impossible,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 2, |A_+| = 1,\ \varphi_{1,b} \in C_4$,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_2$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, it is impossible.

1.3) If $c_b \in H_3$ then all possible cases are:
a) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_2$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, which is impossible,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 2, |A_+| = 1,\ \varphi_{1,b} \in C_4$;
d) $c_{b\downarrow} \in H_2$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1, |A_-| = 1, |A_+| = 2,\ \varphi_{1,b} \in C_2$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, this case is impossible.

Consequently, we prove that $\varphi_{1,b} \in C_2 \cup C_4$ for all $b \in M$.

From (7) we find that $A_2 \cap A_4 = \{(J_1, J_2) : J_1 = 0, J_2 \geq 0\}$. By Lemma 1 we have that the periodic configuration φ_1 is H_1-weakly periodic ground state on the set $A_2 \cap A_4$. Consequently, we obtain $\varphi_{1,b} \sim (\varphi_{26,b}), b \in M$, i.e. $\varphi_{26,b} \in C_2 \cup C_4$ for all $b \in M$. Then, the weakly periodic configuration φ_{26} is H_1-weakly periodic ground state on the set $A_2 \cap A_4$.

In the same way, we continue the process for φ_2.

1.4) Assume that $c_b \in H_1$. Then all possible cases are as follows:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 2, |A_+| = 1,\ \varphi_{2,b} \in C_2$,

b) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, which is impossible,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, which is impossible,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, then $\varphi_{2,b}(c_b) = -1, |A_-| = 2, |A_+| = 1$, $\varphi_{2,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, then $\varphi_{2,b}(c_b) = -1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_4$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = -1, |A_-| = 2, |A_+| = 1$, $\varphi_{2,b} \in C_2$.

1.5) Let $c_b \in H_2$ then all possible cases are:
a) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_2$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, it is impossible,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, it is impossible,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_2$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 2, |A_+| = 1$, $\varphi_{2,b} \in C_4$.

1.6) Assume that $c_b \in H_3$. Then all possible cases are as follows:
a) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = -1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_4$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, that is impossible case,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, this case is impossible,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{2,b}(c_{b\downarrow}) = 1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{2,b} \in C_2$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = 1, |A_-| = 2, |A_+| = 1$, $\varphi_{2,b} \in C_4$.

Hence, we prove that $\varphi_{2,b} \in C_2 \cup C_4$ for all $b \in M$.

From (7) we find that $A_2 \cap A_4 = \{(J_1, J_2) : J_1 = 0, J_2 \geq 0\}$. By Lemma 1, we obtain that the periodic configuration φ_2 is H_1-weakly periodic ground state on the set $A_2 \cap A_4$. Thus, one gets $\varphi_{2,b} \sim (\varphi_{27,b})$ for all $b \in M$, i.e. $\varphi_{27,b} \in C_2 \cup C_4$, $b \in M$. As a result, the weakly periodic configuration φ_{27} is also H_1-weakly periodic ground state on the set $A_2 \cap A_4$.

Similarly, for $\varphi_j(x)$, $j \in \{3, \ldots, 18; 28, \ldots, 44\}$, we can prove that all of them are H_1-weakly periodic ground states.

2. Now, we prove the second part of Theorem 2. Namely, we show that the configuration $\varphi_{25} = (i, i, i, i, j, j, j, j, j)$ is not weakly periodic ground state.

2.1) If $c_b \in H_1$, then we have to consider the following cases:
a) $c_{b\downarrow} \in H_1$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 3, |A_+| = 0$, $\varphi_{25,b} \in C_1$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 2, |A_+| = 1$, $\varphi_{25,b} \in C_2$,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 3, |A_+| = 0$, $\varphi_{25,b} \in C_1$,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 2, |A_+| = 1$, $\varphi_{25,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 3, |A_+| = 0$, $\varphi_{25,b} \in C_3$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 2, |A_+| = 1$, $\varphi_{25,b} \in C_4$.

2.2) Let $c_b \in H_2$. Then the following cases arise.
a) $c_{b\downarrow} \in H_1$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 1, |A_+| = 2$, $\varphi_{25,b} \in C_4$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 0, |A_+| = 3$, $\varphi_{25,b} \in C_3$,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 2, |A_+| = 1$, $\varphi_{25,b} \in C_2$,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{25,b} \in C_2$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 2, |A_+| = 1$, $\varphi_{25,b} \in C_4$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{25,b} \in C_2$.

2.3) If $c_b \in H_3$ then the following cases arise.
a) $c_{b\downarrow} \in H_1$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 1, |A_+| = 2$, $\varphi_{25,b} \in C_4$,
b) $c_{b\downarrow} \in H_1$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = -1, |A_-| = 0, |A_+| = 3$, $\varphi_{25,b} \in C_3$,
c) $c_{b\downarrow} \in H_2$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{25,b} \in C_2$,
d) $c_{b\downarrow} \in H_2$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 0, |A_+| = 3$, $\varphi_{25,b} \in C_1$,
e) $c_{b\downarrow} \in H_3$ and $\varphi_{25,b}(c_{b\downarrow}) = -1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 1, |A_+| = 2$, $\varphi_{25,b} \in C_2$,
f) $c_{b\downarrow} \in H_3$ and $\varphi_{25,b}(c_{b\downarrow}) = 1$, then $\varphi_{25,b}(c_b) = 1, |A_-| = 0, |A_+| = 3$, $\varphi_{25,b} \in C_1$.

Hence, we prove that for any $b \in M$ we have $\varphi_{25,b} \in C_1 \cup C_2 \cup C_3 \cup C_4$.

From the above, it is known that $A_1 \cap A_2 \cap A_3 \cap A_4 = \{(J_1, J_2) : J_1 = J_2 = 0\}$. Therefore, φ_{25} is not a weakly periodic ground state. In the same way we can prove that H_1-weakly periodic ground states contain only periodic (also, translation-invariant) ground states and 36 weakly periodic ground states mentioned in assertion 1 of Theorem 2. The theorem is proved. □

Remark 2. Note that, on the Cayley tree of order two corresponding to the normal subgroups of indices 2 and 4 do not exist weakly periodic ground states (see [18], [19]), but in our case there exist weakly periodic ground states on the Cayley tree of order two corresponding to the subgroups of index 3.

For a fixed $l \in N_k$, we consider the following functions

$$u_{\{i\},\{j\}}(x) = \begin{cases} e, & \text{if } x = a_l, l \in N_k \setminus \{i, j\}, \\ a_p, & \text{if } x = a_p,\ p \in \{i, j\}. \end{cases}$$

$$\gamma(x) = \begin{cases} e, & \text{if } x = e, \\ a_i, & \text{if } x \in \{a_i, a_j a_i\}, \\ a_j, & \text{if } x \in \{a_j, a_i a_j\}, \\ \gamma(a_i a_j \dots \gamma(a_i a_j)), & \text{if } x = a_i a_j \dots a_j,\ l(x) \geq 3,\ i, j \in N_k \setminus \{l\}, i \neq j, \\ \gamma(a_i a_j \dots \gamma(a_j a_i)), & \text{if } x = a_i a_j \dots a_i,\ l(x) \geq 3,\ i, j \in N_k \setminus \{l\}, i \neq j. \end{cases}$$

By using the above functions we define the following set

$$H_1^{(l)} = \{x \in G_2 \ \ |\gamma(u_{\{i\}\{j\}}(x) = e\}.$$

Then, from Proposition 1, we can conclude that $H_1^{(l)}$ is a subgroup of index 3 of the group G_2. It is easy to see that the analogues of Theorem 1 and Theorem 2 hold for the $H_1^{(l)}$-periodic (weakly periodic) configurations.

REFERENCES

[1] U. A. Rozikov:, A constructive description of ground states and gibbs measures for ising model with two-step interactions on Cayley tree, *J. Statist. Phys.* **122**, 217–235 (2006).
[2] R. J. Baxter: *Exactly Solved Models in Statistical Mechanics.* Academic Press, London/New York 1982.

[3] P. M. Bleher, J. Ruiz and V. A. Zagrebnov: On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice, *J. Stat. Phys.* **79**, 473–482 (1995).

[4] P. M. Bleher, N. N. Ganikhodjaev: On pure phases of the Ising model on the Bethe lattice *Theor. Probab. Appl.* **35**, 216–227 (1990).

[5] P. M. Bleher, J. Ruiz, R. H. Schonmann, S. Shlosman and V. A. Zagrebnov: Rigidity of the critical phases on a Cayley tree, *Moscow Math. J.* **3**, 345–363 (2001).

[6] N. N. Ganikhodjaev and U. A. Rozikov: A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree, *Theor. Math. Phys.* **111**, 480–486 (1997).

[7] F. M. Mukhamedov and U. A. Rozikov: On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras, *J. Stat. Phys.* **114**, 825–848 (2004).

[8] F. H. Haydarov: New normal subgroups for the group representation of the Cayley tree, *Lobach. J. Math.* **39(2)** (2018).

[9] Kh. A. Nazarov and U. A. Rozikov: Periodic Gibbs measures for the Ising model with competing interactions, *Theor. Math. Phys.* **135**, 881–888 (2003).

[10] C. Preston: *Gibbs States on Countable Sets*, Cambridge University Press, London 1974.

[11] U. A. Rozikov: Partition structures of the group representation of the Cayley tree into cosets by finite-index normal subgroups and their applications to the description of periodic Gibbs distributions, *Theor. Math. Phys.* **112**, 929–933 (1997).

[12] Ya. G. Sinai: *Theory of Phase Transitions: Rigorous Results*, Pergamon, Oxford 1982.

[13] S. Zachary: Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* **11**, 894–903 (1983).

[14] U. A. Rozikov and F. H. Haydarov: Invariance property on group representations of the Cayley tree and its applications, *arXiv:1910.13733*.

[15] U. A. Rozikov: *Gibbs Measures on a Cayley Tree*, World Scientific Publishing, Singapore 2013.

[16] R. A. Minlos: *Introduction to Mathematical Statistical Physics*American Mathematical Society, Providence 2000.

[17] U. A. Rozikov and M. M. Rakhmatullaev: Weakly periodic ground states and Gibbs measures for the Ising model with competing interactions on the Cayley tree, *Theor. Math. Phys.* **160**, 1292–1300 (2009).

[18] M. M. Rakhmatullaev: Description of weak periodic ground states of Ising model with competing interactions on Cayley tree, *Appl. Math. Inf. Sci.* **4**, 237–251 (2010).

[19] M. M. Rakhmatullaev:, Weakly periodic Gibbs measures and ground states for the Potts model with competing interactions on the Cayley tree, *Theor. Math. Phys.* **176**, 1236–1251 (2013).

[20] M. M. Rakhmatullaev and M. A. Rasulova: Existence of weakly periodic ground states for the Potts model with competing interactions on the Cayley tree, *Dokl. Akad. nauk Resp. Uzbekistan* **3(10)**, (2013).

[21] M. M. Rahmatullaev and M. A. Rasulova: Periodic and weakly periodic ground states for the Potts model with competing interactions on the Cayley tree, *Sib. Adv. Math.* **26(3)**, 215–229 (2016).