Uzbek Mathematical Journal
2021, Volume 65, Issue 1, pp.110-117
DOI: 10.29229/uzmj.2021-1-11

Periodic Gibbs measures for the antiferromagnetic Potts model on a Cayley tree of order k

Makhammadaliev M.T.

Abstract. This paper is the continuation of [7]. We study the antiferromagnetic q-state Potts model on the Cayley tree. Under certain conditions, for this model on a Cayley tree of order $k \geq 3$ the exact number of the periodic (non translation-invariant) Gibbs measures is found.

Keywords: Cayley tree, configuration, Potts model, Gibbs measure, periodic Gibbs measures, translation-invariant measure.

MSC (2010): 53C12, 57R25, 57R35

1 Introduction

It is known that each limit Gibbs measure is associated with one phase of a physical system. Therefore, a challenging problem in the theory of Gibbs measures is the problem of the existence of a phase transition, i.e., a change in the state of the physical system as the temperature varies. This occurs when the Gibbs measure is not unique. In this case, the temperature at which the state of the physical system changes is usually called the critical temperature (see f.e. [1]-[3]).

In [4] the uniqueness of the translation-invariant Gibbs measure of the antiferromagnetic Potts model with an external field was proved. Periodic Gibbs measures were studied in [5], where it was proved that under certain conditions, all periodic Gibbs measures are translation-invariant. In particular, under certain conditions, for the ferromagnetic Potts model with three states on a Cayley tree of an arbitrary order and for the antiferromagnetic Potts model with three states on a second-order Cayley tree, all periodic Gibbs measures are translation-invariant. Moreover, the conditions were found under which the Potts model with a nonzero external field has periodic Gibbs measures. The results in [5] were followed up in [6], where the existence of at least three periodic Gibbs measures with the period two on a third- or fourth-order Cayley tree for the antiferromagnetic Potts model with three states and a zero external field was proved.

In [7], the antiferromagnetic Potts model with q states on a Cayley tree of order $k \geq 3$ was studied, and on certain invariant sets, the existence of periodic (not translation-invariant) Gibbs measures was shown under certain conditions on the model parameters. Moreover, the lowest bound of the number of the existing periodic Gibbs measures was shown. In [8] previously obtained results that the exact number of periodic Gibbs measures with the period two on a Cayley tree of order $k \geq 3$ that are defined on invariant set I_m from [7] are improved. In [9] for the antiferromagnetic

Potts model with q states on a Cayley tree order of two and for the ferromagnetic Potts model with q states on a Cayley tree order of k, was shown that all periodic Gibbs measures are translation-invariant for all parameter values. Other properties of the Potts model on a Cayley tree were studied in [3] (p.105-121).

In this paper, we improved one of the results in [7]. For the antiferromagnetic q-state Potts model on the Cayley tree of order $k \geq 3$ the exact number of periodic Gibbs measures with the period two that are defined on invariant set $I_m^{'}$ in [7] is found.

2 Definitions and known facts

The Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Im^k = (V, L, i)$, where V is the set of vertices $\Im^k$, L the set of edges and i is the incidence function setting each edge $l \in L$ into correspondence with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then the vertices x and y are called the *nearest neighbors*, denoted by $l = \langle x, y\rangle$. The distance $d(x, y), x, y \in V$ on the Cayley tree is the number of edges of the minimum path connecting the vertices x and y.

For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

We assume that $\Phi = \{1, 2, \dots, q\}$, $q \geq 2$, and $\sigma \in \Omega = \Phi^V$ is a configuration, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. For the subset $A \subset V$, we define Ω_A as the set of all configurations defined on A and taking values in Φ; the set of all configurations coincides with $\Omega = \Phi^V$.

The Hamiltonian of the Potts model is defined as

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2.1}$$

where $J \in R$, $\langle x, y\rangle$ are nearest neighbors and δ_{ij} is the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 0, & \text{if } \ i \neq j \\ 1, & \text{if } \ i = j. \end{cases}$$

Define a finite-dimensional distribution of a probability measure μ in the volume V_n as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right\}, \tag{2.2}$$

where $\beta = 1/T$, $T > 0$-temperature, Z_n^{-1} is the normalizing factor, $\{h_x = (h_{1,x}, \dots, h_{q,x}) \in R^q, x \in V\}$ is a collection of vectors and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma(x)\sigma(y)}$$

is the restriction of Hamiltonian on V_n.

We say that the probability distributions (2.2) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}).$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. In this case, there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such a measure is called a *splitting Gibbs measure* corresponding to the Hamiltonian (2.1) and vector-valued function $h_x, x \in V$.

The following statement describes conditions on h_x, guaranteeing compatibility of $\mu_n(\sigma_n)$.

Theorem 2.1. *[4] The probability distributions $\mu_n(\sigma_n)$, $n = 1, 2, \dots$ in (2.2) are compatible for Potts model iff, for any $x \in V$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \tag{2.3}$$

where $F : h = (h_1, \dots, h_{q-1}) \in R^{q-1} \to F(h, \theta) = (F_1, \dots, F_{q-1}) \in R^{q-1}$ is defined as

$$F_i = \ln\left(\frac{(\theta-1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}}\right),$$

and $\theta = \exp(J\beta)$, $S(x)$ is the set of direct successors of x and $h_x = (h_{1,x}, \dots, h_{q-1,x})$ with

$$h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}, \quad i = 1, \dots, q-1.$$

It is known that there exists a one-to-one correspondence between the set of vertices V of the Cayley tree $\Im^k$ and the group G_k that is the free product of $k+1$ cyclic groups of second order with the generators $a_1, a_2, \dots, a_{k+1}$.

Let $\widehat{G}_k$ be a subgroup of the group G_k.

Definition 2.2. The set of vectors $h = \{h_x,\, x \in G_k\}$ is said to be $\widehat{G}_k$-periodic if $h_{yx} = h_x$ for all $x \in G_k, y \in \widehat{G}_k$.

The G_k-periodic sets are said to be translation-invariant.

Definition 2.3. The measure μ is said to be $\widehat{G}_k$-periodic if it corresponds to the $\widehat{G}_k$-periodic set of vectors h.

The following theorem characterizes periodic Gibbs measures.

Theorem 2.4. *[5] Let K be a normal divisor of finite index of the group G_k. Then for the Potts model, all K-periodic Gibbs measures are either $G_k^{(2)}$-periodic or translation-invariant, where $G_k^{(2)} = \{x \in G_k :$ the length of x is even$\}$.*

3 Periodic Gibbs measures

We consider case $q \geq 3$, i.e. $\sigma : V \to \Phi = \{1, 2, 3, ..., q\}$. By Theorem 2, we have only $G_k^{(2)}$-periodic Gibbs measures corresponding to the set of vectors $h = \{h_x \in R^{q-1} : x \in G_k\}$ of the form

$$h_x = \begin{cases} h, & \text{if } |x| - \text{is even}, \\ l, & \text{if } |x| - \text{is odd}, \end{cases}$$

where $h = (h_1, h_2, ..., h_{q-1})$, $l = (l_1, l_2, ..., l_{q-1})$. From equality (2.3), we then obtain

$$\begin{cases} h_i = k \ln \frac{(\theta-1)\exp(l_i)+\sum_{j=1}^{q-1} exp(l_j)+1}{\sum_{j=1}^{q-1} exp(l_j)+\theta}, \\ l_i = k \ln \frac{(\theta-1)\exp(h_i)+\sum_{j=1}^{q-1} exp(h_j)+1}{\sum_{j=1}^{q-1} exp(h_j)+\theta}, \end{cases} \quad i = \overline{1, q-1}.$$

We introduce the notations $\exp(h_i) = x_i,\ \exp(l_i) = y_i$. We can then rewrite the last system of equations for $i = \overline{1, q-1}$ as

$$\begin{cases} x_i = \left(\frac{(\theta-1)y_i+\sum_{j=1}^{q-1} y_j+1}{\sum_{j=1}^{q-1} y_j+\theta} \right)^k, \\ y_i = \left(\frac{(\theta-1)x_i+\sum_{j=1}^{q-1} x_j+1}{\sum_{j=1}^{q-1} x_j+\theta} \right)^k. \end{cases} \tag{3.1}$$

Remark 3.1. 1) In the case $q = 2$, the Potts model coincides with the Ising model which was studied in [4].

2) In the case $k = 2,\ q = 3$ and $J < 0$, it was proved that all $G_k^{(2)}$-periodic Gibbs measures on invariant set $I = \{(x_1, x_2, y_1, y_2) \in R^4 : x_1 = x_2,\ y_1 = y_2\}$ are translation-invariant (see [5]).

3) In the case $k \geq 1,\ q = 3$ and $J > 0$, it was proved that all $G_k^{(2)}$-periodic Gibbs measures are translation-invariant (see [5]).

4) In [7] two invariant sets $I_m,\ I_m^{'}$ have been introduced and the lowest bounder of the number of periodic Gibbs measures on I_m and $I_m^{'}$ are found.

5) In [8] an exact number of periodic Gibbs measures on invariant set I_m is found.

The case $I_m^{'}$. We consider the invariant set $I_m^{'}$ defined in [7]:

$$I_m^{'} = \{z = (u, v) \in R^{q-1} \times R^{q-1} : x_i = x,\ i = \overline{1, m},\ x_i = 1,\ i = \overline{m+1, q-1-m},$$
$$x_i = y, i = \overline{q-m, q-1}, y_i = y,\ i = \overline{1, m},\ y_i = 1,\ i = \overline{m+1, q-1-m},$$
$$y_i = x, i = \overline{q-m, q-1}\},$$

i.e.,

$$u = (\underbrace{x, x, ..., x}_{m}, 1, 1, ..., 1, \underbrace{y, y, ..., y}_{m}),\ v = (\underbrace{y, y, ..., y}_{m}, 1, 1, ..., 1, \underbrace{x, x, ..., x}_{m}),$$

where $2m \leq q - 1$.

We rewrite the system of equations (3.1) on this set as

$$\begin{cases} x = \left(\frac{(\theta-1)y+my+(q-2m-1)+mx+1}{\theta+mx+my+(q-2m-1)}\right)^k \\ y = \left(\frac{(\theta-1)x+mx+(q-2m-1)+my+1}{\theta+mx+my+(q-2m-1)}\right)^k. \end{cases} \tag{3.2}$$

Proposition 3.2. *[7] Let $k \geq 3$, $3 \leq q < k + 1$, $\theta_{cr} = \frac{k-q+1}{k+1} < 1$. For the system of equations (3.1) on $I_m^{'}$*

(i) For $0 < \theta < \theta_{cr}$ has at least three solutions;

(ii) For $\theta = \theta_{cr}$ has at least one solution;

(iii) For $\theta > \theta_{cr}$ has only one solution.

Remark 3.3. 1) For $m = 0$ we obtain $u = (1, 1, ..., 1)$, $v = (1, 1, ..., 1)$, which corresponds to the translation-invariant Gibbs measure. Thus we consider the case $m \geq 1$.

2) In the case $k = 2$, $q = 3$, $m = 1$ on $I_m^{'}$ it was proved that all $G_k^{(2)}$-periodic Gibbs measures are translation-invariant (see [5]).

In the last system substituting $\sqrt[k]{x} = z$, $\sqrt[k]{y} = t$ we obtain

$$\begin{cases} z = \frac{(\theta+m-1)t^k+mz^k+q-2m}{\theta+mz^k+mt^k+q-2m-1} \\ t = \frac{(\theta+m-1)z^k+mt^k+q-2m}{\theta+mz^k+mt^k+q-2m-1}. \end{cases} \tag{3.3}$$

We have the following

Proposition 3.4. *Let $k \geq 3$, $3 \leq q < k + 1$, $\theta_{cr} = \frac{k-q+1}{k+1} < 1$. For the system of equations (3.1) on $I_m^{'}$*

1. If $0 < \theta < \theta_{cr}$, then there exist exactly three solutions;

2. If $\theta \geq \theta_{cr}$, then there is only one solution.

Proof. Subtracting the second equation in (3.3) from the first, we obtain :

$$z - t = \frac{(1-\theta)(z^k - t^k)}{mt^k + mz^k + q - 2m - 1 + \theta}.$$

After simplifications we get

$$(z-t)\Big[mt^k + mz^k + q - 2m - 1 + \theta - (1-\theta)\Big(z^{k-1} + z^{k-2}t + \ldots + t^{k-1}\Big)\Big] = 0 \quad (3.4)$$

If $z = t$, then it corresponds to a translation-invariant Gibbs measure. More precisely, taking into account [9] we have $z = t = 1$. For the second multiplication of (3.4) we have

$$F(z,t) = mt^k + mz^k + q - 2m - 1 + \theta - (1-\theta)(z^{k-1} + z^{k-2}t + z^{k-3}t^2 ... + t^{k-1}) = 0.$$

We rewrite above equation in the following form:

$$\begin{cases} F(z) = h(z) + d(t) \\ F(t) = h(t) + d(z) \end{cases} \quad (3.5)$$

where

$$h(z) = mz^k - (1-\theta)(z^{k-1} + z^{k-2}t + z^{k-3}t^2 ... + zt^{k-1}),$$
$$d(t) = mt^k - (1-\theta)t^{k-1} + q - 2m - 1 + \theta.$$

Rewriting (3.3) as

$$\begin{cases} z - 1 = \frac{(\theta-1)(t^k-1)}{\theta + mz^k + mt^k + q - 2m - 1} \\ t - 1 = \frac{(\theta-1)(z^k-1)}{\theta + mz^k + mt^k + q - 2m - 1} \end{cases} \quad (3.6)$$

Recall $q \geq 2m+1$ and $0 < \theta < \frac{k-q+1}{k+1} < 1$. It follows from (3.6) that if $t < 1$ then $z > 1$ and if $t > 1$ then $z < 1$. Also, $d(t) > 0$ for $t > 1$. The existence of roots satisfying the condition $z \neq t$ in (3.5) ensures the existence of roots satisfying the condition $z \neq t$ in (3.3). If $z < 1$ and $t > 1$, the number of sign changes of the first equation of the system (3.5) is two. Similarly, if $z > 1$ and $t < 1$, the number of sign changes of the second equation of the system (3.5) is two. Due to the Descartes theorem, $F(t, z) = 0$ has at most two solutions. Consequently, the system of equations (3.3) has up to three solutions. Moreover, it was proved that under the condition $\theta < \frac{k-q+1}{k+1}$, the system of equations (3.3) has at least three solutions. Therefore, the system of equations (3.3) has exactly three positive solutions.

In [7], the system of equations (3.3) are rewritten after some algebra in the following form

$$f(z) = [(\theta + 2m - 1)z^k - mz^{k+1} + mz + q - 2m]^k(\theta + m - 1 - mz) -$$

$$-(mz^k + q - m - 1 + \theta)^k[mz^{k+1} - mz^k + (\theta + q - 2m - 1)z - q + 2m] = 0. \quad (3.7)$$

We are aiming to prove that the system of equations (3.3) has a unique solution when $\theta = \theta_{cr} = \frac{k-q+1}{k+1}$. For this, it is sufficient to show $z = 1$ is a solution having multiplicity of three in (3.7). For the sake of simplicity, we denote

$$A(z) = (\theta + 2m - 1)z^k - mz^{k+1} + mz + q - 2m, \quad B(z) = \theta + m - 1 - mz,$$

$$C(z) = mz^k + q - m - 1 + \theta \quad \text{and} \quad D(z) = mz^{k+1} - mz^k + (\theta + q - 2m - 1)z - q + 2m.$$

Then due to notations we have

$$f(z) = A^k B - C^k D. \tag{3.8}$$

According to the theorem about zeroes of holomorphic functions, if $z = 1$ is a three-fold solution of $f(z)$, then $f'(1) = 0$ and $f''(1) = 0$. Let's compute derivatives of (3.8):

$$f'(z) = kA^{k-1}BA' + A^k B' - kC^{k-1}DC' - C^k D'$$

$$f''(z) = A^{k-2}(k(k-1)BA'^2 + 2kAA'B' + kAA''B + A^2B'')-$$

$$-C^{k-2}(k(k-1)DC'^2 + 2kCC'D' + kCC''D + C^2D'')$$

where

$$A(1) = \theta + q - 1, B(1) = \theta - 1, C(1) = \theta + q - 1, D(1) = \theta - 1,$$

$$A'(1) = k(\theta + m - 1), B'(1) = -m, C'(1) = mk, D'(1) = \theta + q - m - 1,$$

$$A''(1) = k(\theta(k-1) + (m-1)k - 3m + 1), B''(1) = 0, C''(1) = mk(k-1), D''(1) = 2mk$$

$$f'(1) = k^2(\theta + q - 1)^{k-1}(\theta - 1)^2 - (\theta + q - 1)^{k+1}.$$

Taking into account that $\theta = \theta_{cr} = \frac{k-q+1}{k+1}$ and $A(1) = C(1)$, we have

$$f'(1) = (\theta + q - 1)^{k-1}\left(\frac{k^2q^2}{(k+1)^2} - \frac{k^2q^2}{(k+1)^2}\right) = 0.$$

$$f''(1) = A^{k-2}(k^3(k-1)(\theta - 1)[(\theta + m - 1)^2 - m^2] - 2mk^2(\theta + q - 1)(2(\theta - 1) + q)+$$
$$+k^2(\theta - 1)(\theta + q - 1)(\theta(k-1) + k(m-1) - 3m + 1 - m(k-1)) - 2mk(\theta + q - 1)^2).$$

Putting $\theta = \theta_{cr} = \frac{k-q+1}{k+1}$ into the last equation and after dividing to $A^{k-2}\frac{k^3q^2}{(k+1)^2}$ we obtain

$$f''(1) = -(k-1)(\frac{q}{k+1} - 2m) - 2m(k-1) + \frac{k-1}{k+1}q + 2m - 2m = 0,$$

i.e., $z = 1$ is the three-fold solution.

The following theorem follows from Theorem 3 in [7].

Theorem 3.5. *For the Potts model for* $k \geq 3$, $3 \leq q < k+1$ *and* $0 < \theta < \theta_{cr}$ *on the sets* $\bigcup_{m=1}^{q} I_m$ *and* $\bigcup_{m=1}^{q} I_m'$ *there are exactly*

$$2 \cdot \left(2^q - 1 + \sum_{m=1}^{[q/2]} \binom{q}{m} \cdot \binom{q-m}{m}\right)$$

$G_k^{(2)}$*-periodic (non translation-invariant) Gibbs measures.*

References

1. Georgii H.-O. , Gibbs Measures and Phase Transitions (de Gruyter Stud. Math., Vol.9), Walter de Gruyter, Berlin (1988).
2. Sinai Ya.G., Theory of Phase Transitions: Rigorous Results [in Russian], Nauka, Moscow, (1980); English transl. (Intl.Ser. Nat. Philos., Vol. 108, Pergamon, Oxford (1982)).
3. Rozikov U.A., Gibbs measures on Cayley trees, World Scientific, Singapore, 2013.
4. Ganikhodzhaev N.N., Rozikov U.A., Description of periodic extreme Gibbs measures of some lattice models on a Cayley tree, Theor. Math. Phys., 111(1), (1997), 480-486.
5. Rozikov U.A. , Khakimov R.M., Periodic Gibbs measures for Potts model on the Cayley tree, Theor. Math. Phys., 175(2), (2013), p.699-709.
6. Khakimov R.M. , The existence periodic Gibbs measures for the Potts model on the Cayley tree., To appear in Uzbek Math.Jour.
7. Khakimov R.M. , New Periodic Gibbs Measures for q-state Potts Model on a Cayley Tree, Journal of Siberian Federal Uni., Math. and Phy., 2014, 7(3), 297-304
8. Khakimov R.M., Khaydarov F.Kh., Translation-invariant and periodic Gibbs measures for the Potts model on a Cayley tree, Theor. Math. Phy.,189(2),(2016),1651-1659
9. Khakimov R.M., Makhammadaliev M.T., Translation invariance of the periodic Gibbs measures for the Potts model on the Cayley tree, Theor. Math. Phy., 199(2),(2019), 726-735

Makhammadaliev M.T.,
Namangan State University, 316, Uychi str., Namangan, 160136,
Uzbekistan.
e-mail: mmtmuxtor93@mail.ru