Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Classical statistical mechanics, equilibrium and non-equilibrium

Extremality of translation-invariant Gibbs measures for the Potts–SOS model on the Cayley tree

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Received 2 April 2021
Accepted for publication 11 May 2021
Published 16 July 2021

Online at stacks.iop.org/JSTAT/2021/073201
https://doi.org/10.1088/1742-5468/ac08ff

Abstract. In this paper, we consider the Potts–SOS model where the spin takes values in the set $\{0, 1, 2\}$ on the Cayley tree of order two. We describe all the translation-invariant splitting Gibbs measures (GMs) for this model in some conditions. Moreover, we investigate whether these GMs are extremal or non-extremal in the set of all GMs.

Keywords: classical phase transitions, extreme value, phase diagrams, nonlinear dynamics

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 1742-5468/21/073201+18$33.00

Contents

1. Introduction **2**
2. Main definitions and known facts **3**
3. The model and a system vector-valued functional equations **3**
4. Translation-invariant GMs **4**
5. Tree-indexed Markov chains of TISGMs **11**
5.1. Conditions of non-extremality 12
5.2. Conditions for extremality 13
Acknowledgments 17
References **18**

1. Introduction

One of the central problems in the theory of Gibbs measures (GMs) is to describe infinite-volume (or limiting) GMs corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the ground-breaking work of Dobrushin (see, e.g. [2]). However, a complete analysis of the set of limiting GMs for a specific Hamiltonian is often a difficult problem.

In this paper, we consider the Potts–SOS model, with spin values $0, 1, 2$ on the Cayley tree (CT). Models on a CT were discussed in [3, 4–7]. A classical example of such a model is the Ising model, with two values of spin -1 and 1. It was considered in [1, 3, 7, 16, 17] and became a focus of active research in the first half of the 90s and afterwards; see [1, 8–14].

In [18] all translation-invariant splitting Gibbs measures (TISGMs) for the Potts model on the CT are described. In [19, 20] periodic GMs are studied, and in [21–23] weakly periodic GMs for the Potts model are studied.

In [25, 26] translation-invariant and periodic Gibbs measures for the SOS model on the CT are studied.

The model considered in this paper (Potts–SOS model) is a generalization of the Potts and SOS (solid-on-solid) models. In [15] some translation-invariant GMs for the Potts–SOS model on the CT are studied. Periodic GMs are studied for the Potts–SOS model on the CT in [24]. In this paper we will study all the TISGMs for this model under some conditions. Next we investigate whether these GMs are extremal or non-extremal in the set of all GMs.

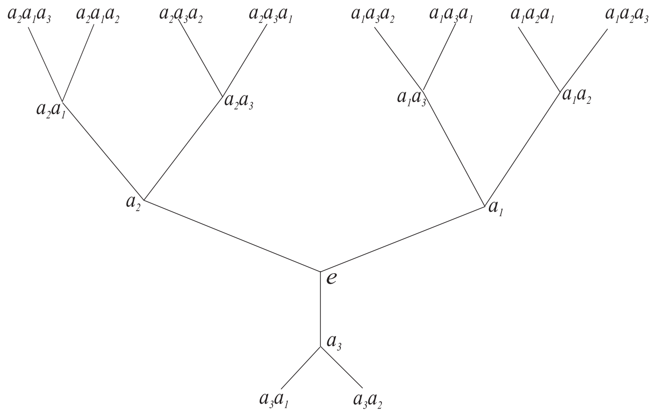

Figure 1. The CT τ^2 and elements of the group G_2 representation of vertices.

2. Main definitions and known facts

The CT Γ^k (see [1]) of order $k \geqslant 1$ is an infinite tree, i.e. a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y\rangle$.

The distance $d(x, y), x, y \in V$ on the CT is defined by the formula

$$d(x,y) = \min\{d|\exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \quad \text{such that} \\ \langle x_0, x_1\rangle \cdots \langle x_{d-1}, x_d\rangle\}.$$

For the fixed $x^0 \in V$, we set $W_n = \{x \in V | d(x, x^0) = n\}$,

$$V_n = \{x \in V | d(x, x^0) \leqslant n\}, \qquad L_n = \{l = \langle x, y\rangle \in L | x, y \in V_n\}. \tag{1}$$

Denote $|x| = d(x, x^0)$, $x \in V$.

A collection of the pairs $\langle x, x_1\rangle \cdots \langle x_{d-1}, y\rangle$ is called a path from x to y and we write $\pi(x, y)$. We write $x < y$ if the path from x^0 to y goes through x.

It is known (see [1]) that there exists a one-to-one correspondence between the set V of vertices of the CT of order $k \geqslant 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, \ldots, k+1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$, see figure 1.

Denote the set of 'direct successors' of $x \in G_k$ by $S(x)$. Let $S_1(x)$ be the set of all nearest neighboring vertices of $x \in G_k$, i.e. $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$ and $\{x_\downarrow\} = S_1(x) \backslash S(x)$.

J. Stat. Mech. (2021) 073201

3. The model and a system vector-valued functional equations

Here, we shall give main definitions and facts about the model. Consider a model where the spin takes values in the set $\Phi = \{0, 1, 2, \dots, m\}, m \geqslant 1$. For $A \subseteq V$, a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

A configuration that is invariant with respect to all shifts is called *translational-invariant*.

The Hamiltonian of the Potts–SOS model with nearest-neighbor interaction has the form

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} |\sigma(x) - \sigma(y)| - J_p \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2}$$

where $J, J_p \in R$ are nonzero coupling constants.

It is known [15] that any SGM of the model (2) corresponds to a solution of the following equation:

$$h_x^* = \sum_{y \in S(x)} F\left(h_y^*, m, \theta, r\right), \tag{3}$$

where $x \in V\backslash\{x^0\}$,

$$\theta = \exp(J\beta), \qquad r = \exp(J_p\beta) \tag{4}$$

and also $\beta = 1/T$ is the inverse temperature. Here, h_x^* represents the vector $(h_{0,x} - h_{m,x}, h_{1,x} - h_{m,x}, \dots, h_{m-1,x} - h_{m,x})$ and the vector function $F(., m, \theta, r) : R^m \to R^m$ is defined as follows

$$F(h, m, \theta, r) = (F_0(h, m, \theta, r), F_1(h, m, \theta, r), \dots, F_{m-1}(h, m, \theta, r)),$$

where

$$F_i(h, m, \theta, r) = \ln \frac{\sum_{j=0}^{m-1} \theta^{|i-j|} r^{\delta_{ij}} e^{h_j} + \theta^{m-i} r^{\delta_{mi}}}{\sum_{j=0}^{m-1} \theta^{m-j} r^{\delta_{mj}} e^{h_j} + r}, \tag{5}$$

$h = (h_0, h_1, \dots, h_{m-1}), i = 0, 1, 2, \dots, m-1$.

Namely, for any collection of functions satisfying the functional equation (3) there exists a unique splitting GM, the correspondence being one-to-one.

4. Translation-invariant GMs

Definition 1. For an SGM μ, if $h_{j,x}$ is independent from $\{x : h_{j,x} \equiv h_j, x \in V, j \in \Phi\}$, μ is called translation-invariant(**TI**).

Let $m = 2$, that is $\Phi = \{0, 1, 2\}$. In this case, for the TISGMs (3) has the form

$$h = kF(h, \theta, r),$$

J. Stat. Mech. (2021) 073201

where $h = (h_0, h_1)$. Introducing the notation $l_0 = e^{h_0}, l_1 = e^{h_1}$, we obtain the following the system of equations

$$\begin{cases} l_0 = \left(\dfrac{rl_0 + \theta l_1 + \theta^2}{\theta^2 l_0 + \theta l_1 + r}\right)^k, \\ l_1 = \left(\dfrac{\theta l_0 + rl_1 + \theta}{\theta^2 l_0 + \theta l_1 + r}\right)^k. \end{cases} \tag{6}$$

Let $k = 2$. Denote $\sqrt{l_0} = x, \sqrt{l_1} = y$. Then from (6) we get

$$\begin{cases} x = \dfrac{rx^2 + \theta y^2 + \theta^2}{\theta^2 x^2 + \theta y^2 + r}, \\ y = \dfrac{\theta x^2 + ry^2 + \theta}{\theta^2 x^2 + \theta y^2 + r}. \end{cases} \tag{7}$$

After simplifying the system of equation (7) above, we have

$$\begin{cases} \theta^2 x^3 - rx^2 + (\theta y^2 + r)x - \theta y^2 - \theta^2 = 0, \\ \theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0. \end{cases} \tag{8}$$

The system of equation (8) can be rewritten as

$$\begin{cases} (x-1)(\theta^2 x^2 + \theta^2 x + \theta^2 - rx + \theta y^2) = 0, \\ \theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0. \end{cases} \tag{9}$$

Obviously, the solutions of (9) are the solutions of the following system of equations

$$\begin{cases} x - 1 = 0, \\ \theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0, \end{cases} \tag{10}$$

or the solutions of the following system of equations

$$\begin{cases} \theta^2 x^2 + \theta^2 x + \theta^2 - rx + \theta y^2 = 0, \\ \theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0. \end{cases} \tag{11}$$

Let us consider (10). Substituting $x = 1$ into the second equation of (10), we get

$$\theta y^3 - ry^2 + (\theta^2 + r)y - 2\theta = 0. \tag{12}$$

For

$$y = z + \frac{r}{3\theta}, \tag{13}$$

we reduce (12) to the equation

$$z^3 + \left(\frac{r}{\theta} + \theta - \frac{r^2}{3\theta^2}\right) z + \left(\frac{r}{3} + \frac{r^2}{3\theta^2} - \frac{2r^3}{27\theta^3} - 2\right) = 0. \tag{14}$$

Denote

$$p=\frac{r}{\theta}+\theta-\frac{r^2}{3\theta^2},\qquad q=\frac{r}{3}+\frac{r^2}{3\theta^2}-\frac{2r^3}{27\theta^3}-2. \tag{15}$$

After solving the equation $p=0$ in terms of r, we have the solutions $r_{1,2}=\frac{3\pm\sqrt{9+12\theta}}{2}\theta$. Since $r>0, \theta>0$, we get $r_1=\frac{3+\sqrt{9+12\theta}}{2}\theta$. Putting r_1 into q in (15) and solving the equation $q=0$ in terms of θ, we have the solution $\theta_1=3\sqrt[3]{2}(\sqrt[3]{2}-1)$.

Substituting r_1,θ_1 into equation (14), we get the equation $z^3=0$. It follows that equation (12) has one positive root $y=\frac{r_1}{3\theta_1}$.

From (15), we obtain

$$\begin{aligned}Q(r,\theta)&=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2=\frac{1}{27}\left(-\frac{1}{3}\frac{r^2}{\theta^2}+\frac{r}{\theta}+\theta\right)^3+\frac{1}{4}\left(-\frac{2}{27}\frac{r^3}{\theta^3}+\frac{1}{3}\frac{r^2}{\theta^2}+\frac{1}{3}r-2\right)^2\\&=-\frac{1}{108\theta^4}\left(r^4+2r^3\theta^2+r^2\theta^4-12r^3\theta-12r^2\theta^3-12\theta^5r-4\theta^7+36\theta^2r^2\right.\\&\quad\left.+36\theta^4r-108\theta^4\right).\end{aligned} \tag{16}$$

For $\theta=\theta_1=3\sqrt[3]{2}(\sqrt[3]{2}-1)$, we have

$$Q(r,\theta_1)=\frac{116+73\sqrt[3]{4}+92\sqrt[3]{2}}{34\,992}\left(-r^2+36(1-2\sqrt[3]{2}+\sqrt[3]{4})r+324(13-4\sqrt[3]{2}-5\sqrt[3]{4})\right)\cdot\\ \cdot\left(r-18+9\sqrt[3]{4}\right)^2$$

Using Cardano's formula, one can prove the following

Lemma 1. *Let* $\theta=3\sqrt[3]{2}(\sqrt[3]{2}-1)$. *There exists* $r_c(\approx 4.221\,293\,186)$ *such that*

- *If* $r\in(0,r_c)$ *then the equation* (12) *has one positive solution.*
- *If* $r=r_c$ *then the equation* (12) *has two positive solutions.*
- *If* $r\in(r_c,\infty)$ *then the equation* (12) *has three positive solutions.*

Now we consider (11). From (11), we get

$$x=\frac{\theta y(\theta^2-y+ry-r)}{-\theta^3y+\theta^2+\theta ry-r}. \tag{17}$$

Substituting (17) into the first equation of (11), we obtain

$$\begin{aligned}f(y,r,\theta)&=\theta^2(\theta+1)(r^2-2\theta r+\theta^3-\theta^2+\theta)y^4-\theta(r-\theta^2)(r^2+(\theta^2+1)r-3\theta^2)y^3\\&\quad+((\theta+1)r+\theta^3)(r-\theta^2)^2y^2-(r+\theta^2)(r-\theta^2)^2y+\theta(r-\theta^2)^2=0.\end{aligned} \tag{18}$$

Equation (18) can be rewritten as

$$f(y,r,\theta)=(ay^2+by+c)(\mathrm{d}y^2+ey+f),$$

where

$$ad = \theta^2(\theta+1)(r^2 - 2\theta r + \theta^3 - \theta^2 + \theta),$$

$$ae + bd = -\theta(r-\theta^2)(r^2 + (\theta^2+1)r - 3\theta^2),$$

$$af + be + cd = ((\theta+1)r + \theta^3)(r-\theta^2)^2,$$

$$bf + ce = -(r+\theta^2)(r-\theta^2)^2,$$

$$cf = \theta(r-\theta^2)^2.$$

Let $D_1(r,\theta) = b^2 - 4ac$ and $D_2(r,\theta) = e^2 - 4df$.
We denote the following sets

$$B_1 = \{(r,\theta) \in \mathbb{R}_+^2 : D_1(r,\theta) > 0, D_2(r,\theta) > 0\},$$

$$B_2 = \{(r,\theta) \in \mathbb{R}_+^2 : D_1(r,\theta) > 0, D_2(r,\theta) = 0 \vee D_1(r,\theta) = 0, D_2(r,\theta) > 0\},$$

$$B_3 = \{(r,\theta) \in \mathbb{R}_+^2 : D_1(r,\theta) = 0, D_2(r,\theta) = 0 \vee D_1(r,\theta) > 0, D_2(r,\theta) < 0 \vee \vee D_1(r,\theta) < 0, D_2(r,\theta) > 0\},$$

$$B_4 = \{(r,\theta) \in \mathbb{R}_+^2 : D_1(r,\theta) = 0, D_2(r,\theta) < 0 \vee D_1(r,\theta) < 0, D_2(r,\theta) = 0\},$$

$$B_5 = \{(r,\theta) \in \mathbb{R}_+^2 : D_1(r,\theta) < 0, D_2(r,\theta) < 0\}.$$

Thus, we can prove the following

Lemma 2. *Let $\theta = 3\sqrt[3]{2}(\sqrt[3]{2}-1)$, then the following assertions hold*

- *If $r \in B_1(r)$ then the equation (18) has four solutions which are positive.*
- *If $r \in B_2(r)$ then the equation (18) has three positive solutions.*
- *If $r \in B_3(r)$ then the equation (18) has two positive solutions.*
- *If $r \in B_4(r)$ then the equation (18) has one positive solution.*
- *If $r \in B_5(r)$ then the equation (18) has no solution.*

With respect to (15) and (16) we denote the following sets

$$A_1 = \{(r,\theta) \in \mathbb{R}_+^2 : r \leqslant 3\theta^2, Q > 0\} \cup \{(r,\theta) \in \mathbb{R}_+^2 : r \leqslant 3\theta^2, p = 0, q = 0\},$$

$$A_2 = \{(r,\theta) \in \mathbb{R}_+^2 : r \leqslant 3\theta^2, Q = 0\} \cap \{(r,\theta) \in \mathbb{R}_+^2 : p \neq 0 \vee q \neq 0\},$$

$$A_3 = \{(r,\theta) \in \mathbb{R}_+^2 : r \leqslant 3\theta^2, Q < 0\},\ A_4 = \{(r,\theta) \in \mathbb{R}_+^2 : r > 3\theta^2, Q > 0\},$$

$$A_5 = \{(r,\theta) \in \mathbb{R}_+^2 : r > 3\theta^2, Q = 0\} \cap \{(r,\theta) \in \mathbb{R}_+^2 : p \neq 0 \vee q \neq 0\},$$

$$A_6 = \{(r,\theta) \in \mathbb{R}_+^2 : r > 3\theta^2, Q < 0\}.$$

Let N be the number of TISGMs for the Potts–SOS model.

Theorem 1. *Let $k=2, m=2$. The following statements hold for the N*

$$N=\begin{cases}1, & if\ (r,\theta)\in A_1,\\ 2, & if\ (r,\theta)\in A_2\cup(A_4\cap B_4)\cup(A_5\cap B_5),\\ 3, & if\ (r,\theta)\in A_3\cup(A_4\cap B_3)\cup(A_5\cap B_4),\\ 4, & if\ (r,\theta)\in(A_4\cap B_2)\cup(A_5\cap B_3)\cup(A_6\cap B_4),\\ 5, & if\ (r,\theta)\in(A_4\cap B_1)\cup(A_5\cap B_2)\cup(A_6\cap B_3),\\ 6, & if\ (r,\theta)\in(A_5\cap B_1)\cup(A_6\cap B_2),\\ 7, & if\ (r,\theta)\in A_6\cap B_1.\end{cases} \tag{19}$$

Proof. We consider the first equation of (11). We write this in the following form

$$\theta^2x^2+(\theta^2-r)x+\theta^2=-\theta y^2. \tag{20}$$

The rhs of (20) is negative, thus

$$\theta^2x^2+(\theta^2-r)x+\theta^2<0. \tag{21}$$

For the lhs of (21), we calculate its discriminant $D=(\theta^2-r)^2-4\theta^4$. If the discriminant is positive, then the inequality (21) has real solutions. Therefore, we should solve

$$(-r-\theta^2)(3\theta^2-r)>0.$$

Since $-r-\theta^2<0$, it follows that $r>3\theta^2$.

Inequality (21) has a positive solution as soon as $\theta^2-r<0$ or $r>\theta^2$. If $r>3\theta^2$, then $r>\theta^2$ also holds. If $r>3\theta^2$, the solutions of the inequality (21) belong to

$$\left(\frac{r-\theta^2-\sqrt{D}}{2\theta^2},\frac{r-\theta^2+\sqrt{D}}{2\theta^2}\right).$$

Moreover, (20) holds in this interval.

Consequently, if $r>3\theta^2$ then the first equation of (11) has a positive real solution, and if $r\leqslant 3\theta^2$ then the first equation of (11) cannot have a positive solution, i.e. any positive real pair (x,y), which is the solution of the first equation of (11), does not satisfy $r\leqslant 3\theta^2$. Then the TISGM's corresponding roots of (11) do not exist under condition $r\leqslant 3\theta^2$.

According to the Descartes theorem, the number of positive roots of equation (12) is at least 1 and at most 3.

If $Q>0$, then equation (14) has one positive real root and two conjugate complex roots. If $Q=0$, then all roots of equation (14) are positive real and two of them are equal or if $p=q=0$, then (14) has one positive real root (one real zero of multiplicity three). If $Q<0$, then equation (14) has three distinct positive real roots. Hence, we can say this about the number of TISGM's corresponding positive roots from equation (12).

From lemmas 1 and 2, we can see that

$$\left\{(r,\theta)\in R^2 : \theta = 3\sqrt[3]{2}(\sqrt[3]{2}-1), r\in(r_c,\infty)\cap B_1(r)\right\}\subset A_6\cap B_1.$$

Thus, the set $A_6\bigcap B_1$ is not empty, i.e. the number of TISGMs corresponding positive solutions of (8) for the Potts–SOS model is up to seven. □

Remark 1. Note that theorem 1 (for $k=m=2$) generalizes results of [18, 26].

If $J=0$, then the Potts–SOS model changes to the Potts model. In this case, theorem 1 can be restated as follows

Theorem 2. *Let $k=2, m=2$. The following statements hold for the number n of the TISGMs for the Potts model*

$$n=\begin{cases}1, & if\ r\in(0,1+2\sqrt{2}),\\ 4, & if\ r=1+2\sqrt{2}\ or\ r=4,\\ 7, & if\ r\in(1+2\sqrt{2},4)\cup(4,\infty)\end{cases} \tag{22}$$

(see [18] for more details).

If $J_p=0$, then the Hamiltonian (2) of the Potts–SOS model changes to the Hamiltonian of the SOS model. In this case, theorem 1 can be restated as follows

Theorem 3. *Let $k=2, m=2$. The following statements are appropriate for the number n of the TISGMs for the SOS model*

$$n=\begin{cases}1, & if\ \theta\in(\theta_2,\infty),\\ 3, & if\ \theta=\theta_2,\\ 5, & if\ \theta\in(\theta_1,\theta_2),\\ 6, & if\ \theta=\theta_1,\\ 7, & if\ \theta\in(0,\theta_1),\end{cases} \tag{23}$$

where $\theta_1\approx 0.1414$ and $\theta_2\approx 0.2956$

(see [26] for more details).

Now we study the extremality of the TISGMs for the Potts–SOS model. In general, a complete analysis of extremality or non-extremality of the TISGMs is a difficult problem. Therefore, we assume $r=\theta^2$.

Lemma 3. *Let $r=\theta^2$. There exists a unique $\theta_c(\approx 7.729814)$ such that*

- *If $\theta\in(0,\theta_c)$ then system (7) has one positive root.*
- *If $\theta=\theta_c$ then system (7) has two positive roots.*
- *If $\theta\in(\theta_c,\infty)$ then system (7) has three positive roots.*

J. Stat. Mech. (2021) 073201

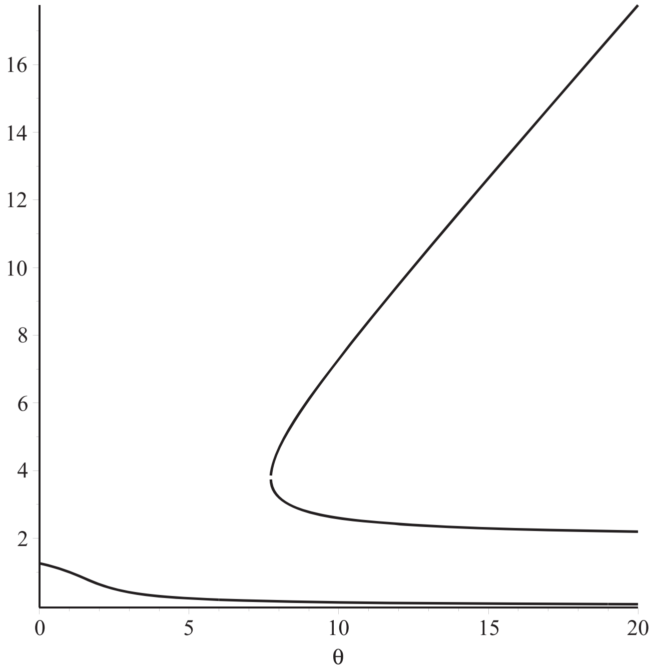

Figure 2. The graphs of functions $y_i = y_i(\theta), i = 1, 2, 3$.

Proof. Substituting $r = \theta^2$ into (7) we have

$$\begin{cases} x = 1, \\ y = \dfrac{2 + \theta y^2}{2\theta + y^2}. \end{cases} \tag{24}$$

Simplifying the second equation of (24), we obtain the cubic equation

$$y^3 - \theta y^2 + 2\theta y - 2 = 0. \tag{25}$$

We calculate its discriminant

$$D = 4(\theta^4 - 10\theta^3 + 18\theta^2 - 27). \tag{26}$$

Denote $\theta_c \approx 7.729814$. If $D < 0$ $(\theta < \theta_c)$ equation (25) has one real and two conjugate complex roots. If $D = 0$ $(\theta = \theta_c)$ then all roots of equation (25) are real, in which two of them are equal. If $D > 0$ $(\theta > \theta_c)$ then equation (25) has three distinct real roots (see figure 2). The obtained real roots are positive due to the Descartes theorem (see [5]). □

The lower curve is y_1, the middle curve is y_2, and the upper curve is y_3.

Using lemma 3, we have the following

Theorem 4. *Let $k = m = 2$. If $r = \theta^2$ then the following statements hold for the N*

$$N = \begin{cases} 1, & if\ \theta \in (0, \theta_c), \\ 2, & if\ \theta = \theta_c, \\ 3, & if\ \theta \in (\theta_c, \infty), \end{cases} \tag{27}$$

where $\theta_c \approx 7.729\,814$.

Remark 2. Note that theorem 4 is a particular case of theorem 1.

We denote the obtained TISGMs corresponding to y_i in theorem 4 by $\mu_i, i = 1, 2, 3$, respectively.

5. Tree-indexed Markov chains of TISGMs

A tree-indexed Markov chain is defined as follows. Suppose we are given with a vertices set V both a probability measure ν and a transition matrix $P = (p_{i,j})_{i,j\in\Phi}$ on the single-site space, which is the finite set here $\Phi = \{0, 1, \ldots, m\}$. We can obtain a tree-indexed Markov chain $X : V \to \Phi$ by choosing $X(x_0)$ according to ν and choosing $X(v)$, for each vertex $v \neq x^0$, using the transition probabilities given the value of its parent, independently of everything else. See definition 12.2 in [4] for a detailed definition.

We note that a TISGM corresponding to a vector $v = (x, y) \in R^2$ (which is the solution to system (7)) is a tree-indexed Markov chain with states $\{0, 1, 2\}$ and transition probabilities matrix:

$$P = \begin{pmatrix} \dfrac{rx^2}{rx^2+\theta y^2+\theta^2} & \dfrac{\theta y^2}{rx^2+\theta y^2+\theta^2} & \dfrac{\theta^2}{rx^2+\theta y^2+\theta^2} \\ \dfrac{\theta x^2}{\theta x^2+ry^2+\theta} & \dfrac{ry^2}{\theta x^2+ry^2+\theta} & \dfrac{\theta}{\theta x^2+ry^2+\theta} \\ \dfrac{\theta^2 x^2}{\theta^2x^2+\theta y^2+r} & \dfrac{\theta y^2}{\theta^2x^2+\theta y^2+r} & \dfrac{r}{\theta^2x^2+\theta y^2+r} \end{pmatrix}. \tag{28}$$

Since (x, y) is a solution to the system (7), this matrix can be written in the following form

$$P = \frac{1}{Z}\begin{pmatrix} rx & \dfrac{\theta y^2}{x} & \dfrac{\theta^2}{x} \\ \dfrac{\theta x^2}{y} & ry & \dfrac{\theta}{y} \\ \theta^2 x^2 & \theta y^2 & r \end{pmatrix}, \tag{29}$$

where $Z = \theta^2x^2 + \theta y^2 + r$.

Simple calculations show that the matrix (29) has three eigenvalues: 1 and

$$\lambda_1(x, y, \theta, r) = \frac{(x+y+1)r - Z + \sqrt{D^*}}{2Z}, \qquad \lambda_2(x, y, \theta, r) = \frac{(x+y+1)r - Z - \sqrt{D^*}}{2Z}, \tag{30}$$

J. Stat. Mech. (2021) 073201

where λ_1 and λ_2 are solutions to

$$Z^3\lambda^2 + (Z - (1 + x + y)r)Z^2\lambda + (2\theta^4 - \theta^4 r - 2\theta^2 r + r^3)xy = 0 \tag{31}$$

and $D^* = ((1 + x + y)r - Z)^2 - 4xyZ^{-1}(2\theta^4 - \theta^4 r - 2\theta^2 r + r^3)$.

5.1. Conditions of non-extremality

In this subsection we are going to find the regions of the parameter θ where the TISGMs $\mu_i, i = 1, 2, 3$ are not extreme in the set of all GMs (including the non-translation invariant ones).

It is known that a sufficient condition (Kesten–Stigum condition) for non-extremality of a GM μ corresponding to the matrix P on a CT of order $k \geqslant 1$ is that $k\lambda^2_{\max} > 1$, where $\lambda_{\max}$ is the second largest (in absolute value) eigenvalue of P [27]. We are going to use this condition for TISGMs $\mu_i, i = 1, 2, 3$ in theorem 4. We have all solutions of the system (7) in condition $r = \theta^2$ (see theorem 4) and the eigenvalues of the matrix P in the explicit form.

Let us denote

$$\lambda_{\max,i}(\theta, r) = \max\{|\lambda_1(x_i, y_i, \theta, r)|, |\lambda_2(x_i, y_i, \theta, r)|\}, \quad i = 1, 2, 3.$$

Using a computer, we have

$$\lambda_{\max,i}(\theta) = \begin{cases} |\lambda_2(1, y_1, \theta)|, & \text{if } i = 1, \theta < 1, \\ |\lambda_1(1, y_1, \theta)|, & \text{if } i = 1, \theta > 1, \\ |\lambda_1(1, y_i, \theta)|, & \text{if } i = 2, 3. \end{cases}$$

Denote

$$\eta_i(\theta) = 2\lambda^2_{\max,i}(\theta) - 1, \quad i = 1, 2, 3.$$

Let $\theta < \theta_c$. Using the Cardano formula, we solve equation (25). It has one real solution

$$y_1 = \frac{1}{3}\left(\theta + \sqrt[3]{\theta^3 - 9\theta^2 + 27 + 1.5\sqrt{-3D}} + \frac{\theta^2 - 6\theta}{\sqrt[3]{\theta^3 - 9\theta^2 + 27 + 1.5\sqrt{-3D}}}\right), \tag{32}$$

where D is defined in (26). In this case, we are aiming to check the Kesten–Stigum condition of the non-extremality of the measure μ_1. To determine the non-extremality interval of TISGM μ_1, we should check the condition

$$2\lambda^2_{\max,1} - 1 > 0.$$

Using a *Maple* program, one can see that the last inequality holds for $\theta \in (0, \theta_1)(\theta_1 \approx 0.166\,699\,3311)$, which implies that the TISGM μ_1 is not-extreme in this interval (see figure 3).

To check that the TISGMs $\mu_i, i = 2, 3$ are non-extreme, we should solve the following inequality: $\eta_i(\theta) > 0, i = 2, 3$ (see figure 4).

Proposition 1. *Let $r = \theta^2$. Then the following statements hold*

J. Stat. Mech. (2021) 073201

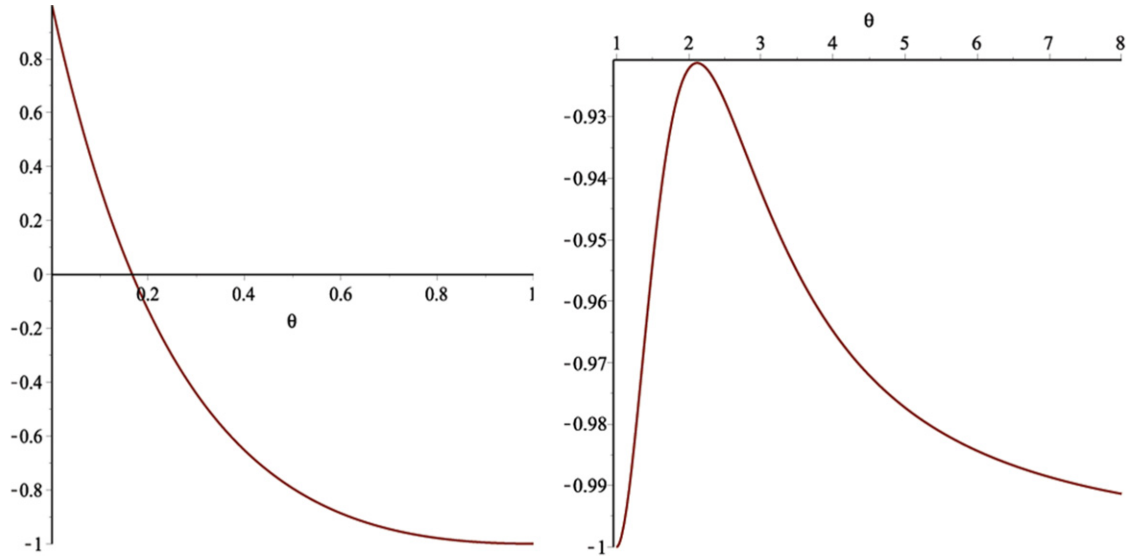

Figure 3. The graphs of functions $\eta_1(\theta)$ for $\theta \in (0, 1)$ (left) and for $\theta \in (1, \infty)$ (right).

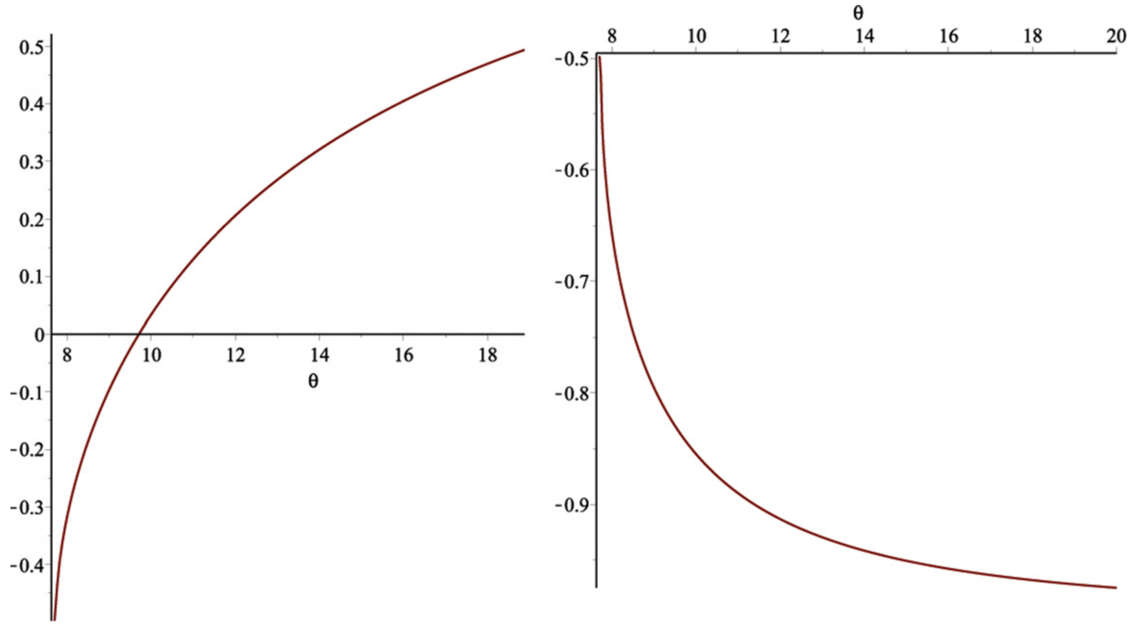

Figure 4. The graphs of functions $\eta_2(\theta)$ (left) and $\eta_3(\theta)$ (right).

(a) *There exists* $\theta_1(\approx 0.166\,699\,3311)$ *such that the measure* μ_1 *is non-extreme if* $\theta \in (0, \theta_1)$;

(b) *There exists* $\theta_2(\approx 9.706\,301\,628)$ *such that the measure* μ_2 *is non-extreme if* $\theta \in (\theta_2, \infty)$.

J. Stat. Mech. (2021) 073201

5.2. Conditions for extremality

In [26, 28] the key ingredients are two quantities, κ and γ, which bound the rates of percolation of disagreement down and up the tree, respectively.

For two measures μ_1 and μ_2 on Ω, $\|\mu_1 - \mu_2\|_x$ denotes the variation distance between the projections of μ_1 and μ_2 onto the spin at x, i.e.

$$\|\mu_1 - \mu_2\|_x = \frac{1}{2}\sum_{i=0}^{2} |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.$$

Let $\eta^{x,s}$ be the configuration η with the spin at x set to s. Following [26, 28] define

$$\kappa \equiv \kappa(\mu) = \sup_{x\in\Gamma^k} \max_{x,s,s'} \|\mu^s_{\tau_x} - \mu^{s'}_{\tau_x}\|_x\,;$$

$$\gamma \equiv \gamma(\mu) = \sup_{A\subset\Gamma^k} \max \|\mu^{\eta^{y,s}}_A - \mu^{\eta^{y,s'}}_A\|_x,$$

where the maximum is taken over all boundary conditions η, all sites $y \in \partial A$, all neighbors $x \in A$ of y, and all spins $s, s' \in \{0, 1, 2\}$.

The criterion of extremality of a TISGM is $k\kappa\gamma < 1$ [26, 28]. Note that κ has the particularly simple form $\kappa = \frac{1}{2}\max_{i,j}\sum_l|P_{i,l} - P_{j,l}|$ and γ is a constant which does not have a clear general formula.

Let $r = \theta^2$. For the solution $(1, y)$, we shall compute κ

$$\kappa = \frac{2\cdot|1 - \theta y| + y^2\cdot|\theta - y|}{2y(2\theta + y^2)}. \tag{33}$$

For $\theta < 1$ from the system (7), we get the following inequalities

$$1 - \theta y = \frac{\theta(1 - \theta^2)y^2}{Z} > 0, \qquad y - \theta = \frac{2\theta(1 - \theta^2)}{Z} > 0.$$

Using these inequalities, we obtain

$$\kappa = \begin{cases} \dfrac{y^3 - \theta y^2 - 2\theta y + 2}{2y(2\theta + y^2)}, & \text{if } 0 < \theta < 1,\\[2mm] \dfrac{-y^3 + \theta y^2 + 2\theta y - 2}{2y(2\theta + y^2)}, & \text{if } \theta \geqslant 1. \end{cases}$$

For the solution $(1, y)$, we shall calculate γ.

$$\gamma = \max\left\{\|\mu^{\eta^{y,0}}_A - \mu^{\eta^{y,1}}_A\|_x, \|\mu^{\eta^{y,0}}_A - \mu^{\eta^{y,2}}_A\|_x, \|\mu^{\eta^{y,1}}_A - \mu^{\eta^{y,2}}_A\|_x\right\},$$

where

$$\|\mu^{\eta^{y,0}}_A - \mu^{\eta^{y,1}}_A\|_x = \frac{1}{2}\sum_{s\in\{0,1,2\}} |\mu^{\eta^{y,0}}_A(\sigma(x) = s) - \mu^{\eta^{y,1}}_A(\sigma(x) = s)|$$

$$= \frac{1}{2}\left(|P_{0,0} - P_{1,0}| + |P_{0,1} - P_{1,1}| + |P_{0,2} - P_{1,2}|\right)$$

$$= \begin{cases} \dfrac{y^3 - \theta y^2 - 2\theta y + 2}{2y(2\theta + y^2)}, & \text{if } 0 < \theta < 1, \\ \dfrac{-y^3 + \theta y^2 + 2\theta y - 2}{2y(2\theta + y^2)}, & \text{if } \theta \geqslant 1, \end{cases}$$

$$\|\mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,2}}\|_x = \frac{1}{2} \sum_{l \in \{0,1,2\}} |P_{0,l} - P_{2,l}| = 0,$$

$$\|\mu_A^{\eta^{y,1}} - \mu_A^{\eta^{y,2}}\|_x = \frac{1}{2} \sum_{l \in \{0,1,2\}} |P_{1,l} - P_{2,l}|$$

$$= \begin{cases} \dfrac{y^3 - \theta y^2 - 2\theta y + 2}{2y(2\theta + y^2)}, & \text{if } 0 < \theta < 1, \\ \dfrac{-y^3 + \theta y^2 + 2\theta y - 2}{2y(2\theta + y^2)}, & \text{if } \theta \geqslant 1. \end{cases}$$

Hence, when $0 < \theta < 1$

$$\gamma = \max\left\{0, \frac{y^3 - \theta y^2 - 2\theta y + 2}{2y(2\theta + y^2)}\right\} = \frac{y^3 - \theta y^2 - 2\theta y + 2}{2y(2\theta + y^2)},$$

and when $\theta \geqslant 1$

$$\gamma = \max\left\{0, \frac{-y^3 + \theta y^2 + 2\theta y - 2}{2y(2\theta + y^2)}\right\} = \frac{-y^3 + \theta y^2 + 2\theta y - 2}{2y(2\theta + y^2)}.$$

Now for TISGMs $\mu_i, i = 1, 2, 3$ we want to check the extremality condition $2\kappa\gamma < 1$. When $\theta > 0$, this condition has the form

$$2\kappa\gamma - 1 = 2\left(\frac{y_i^3 - \theta y_i^2 - 2\theta y_i + 2}{2y_i(2\theta + y_i^2)}\right)^2 - 1 < 0.$$

We check this condition for the TISGM μ_2. Denote

$$U_2(\theta) = \frac{(y_2^3 - \theta y_2^2 - 2\theta y_2 + 2)^2}{2y_2^2(2\theta + y_2^2)^2} - 1.$$

The function $U_2(\theta)$ only depends on θ and has no additional parameters. From its graph, one can see the region of θ where the function is negative. Thus, looking at the graph of $U_2(\theta)$ (see figure 5) completes the arguments.

We check extremality of TISGMs $\mu_1,\ \mu_3$. Thus, consider the following functions

$$U_1(\theta) = \frac{(y_1^3 - \theta y_1^2 - 2\theta y_1 + 2)^2}{2y_1^2(2\theta + y_1^2)^2} - 1,$$

$$U_3(\theta) = \frac{(y_3^3 - \theta y_3^2 - 2\theta y_3 + 2)^2}{2y_3^2(2\theta + y_3^2)^2} - 1.$$

The extremality interval of TISGMs $\mu_1,\ \mu_3$ are seen from figure 6.

J. Stat. Mech. (2021) 073201

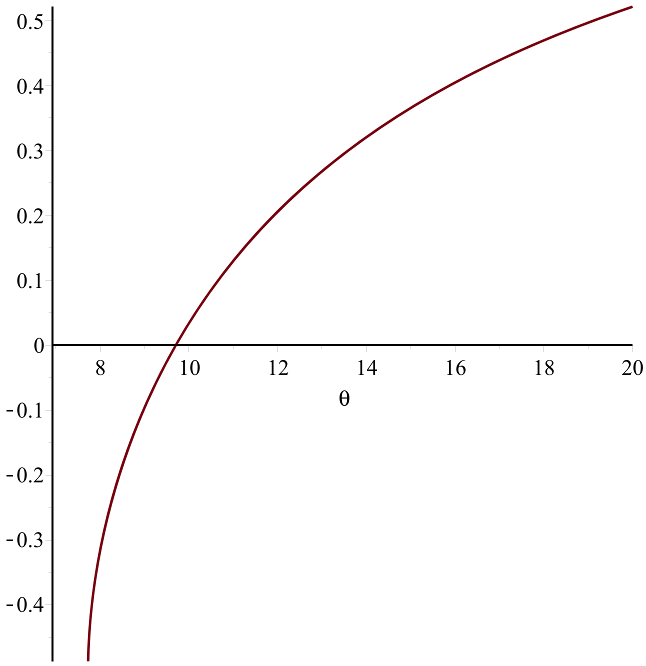

Figure 5. The graph of function $U_2(\theta)$.

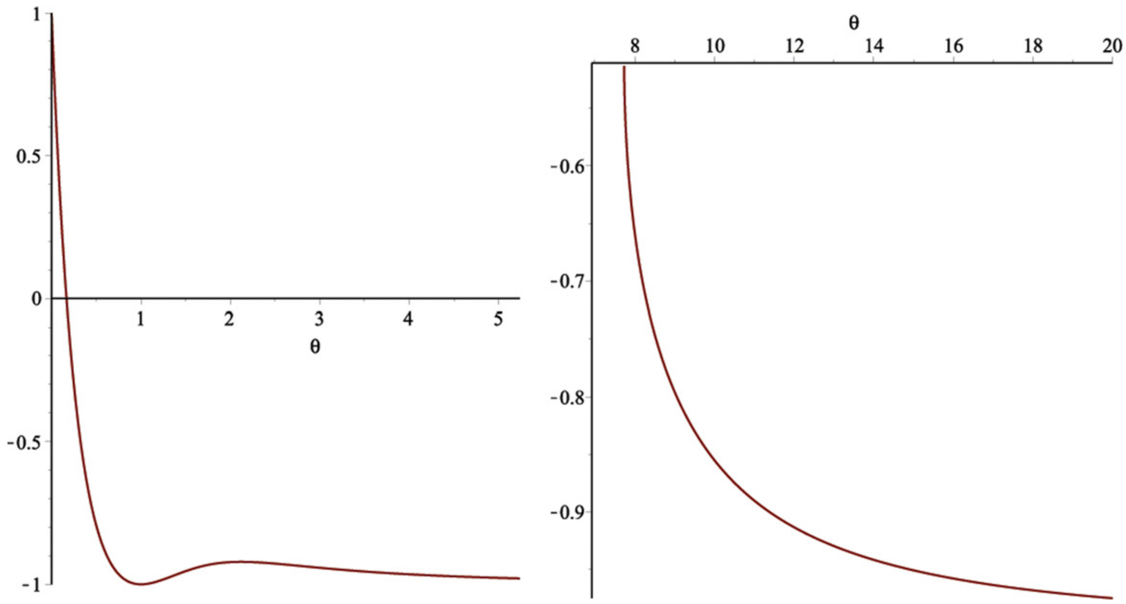

Figure 6. The graphs of functions $U_1(\theta)$ (left) and $U_3(\theta)$ (right).

Proposition 2. *Let* $r = \theta^2$. *Then the following statements hold*

(a) There exists $\theta_1 (\approx 0.166\,699\,3311)$ *such that the measure* μ_1 *is extreme if* $\theta \in (\theta_1, \infty)$;

J. Stat. Mech. (2021) 073201

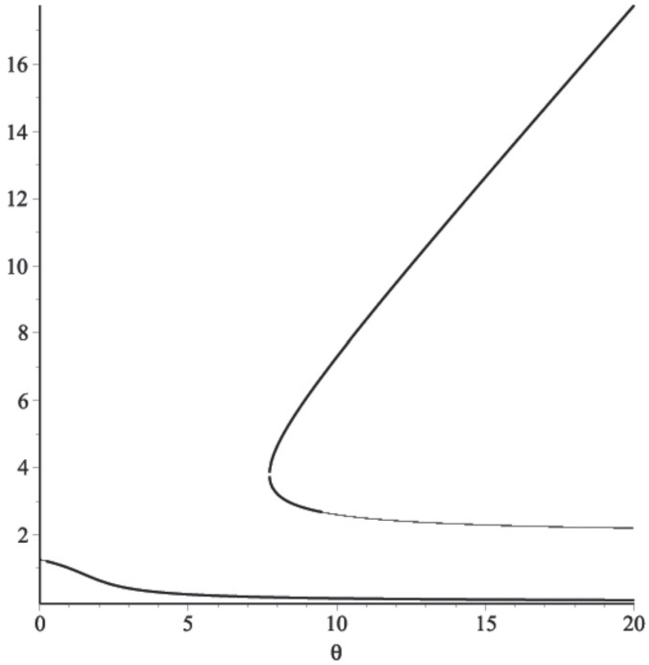

Figure 7. The graphs of functions $y_i(\theta), i = 1, 2, 3$. The *bold curves* correspond to regions of the functions where the corresponding TISGM is extreme. The *thin curves* correspond to regions of the functions where the corresponding TISGM is non-extreme.

(b) *There are values* $\theta^*(\approx 7.729\,813\,675)$ *and* $\theta_2(\approx 9.706\,301\,628)$ *such that the measure* μ_2 *is extreme if* $\theta \in [\theta^*, \theta_2)$*;*

(c) *The measure* μ_3 *is extreme (where it exists, that is* $\theta \in [\theta^*, \infty)$*).*

From propositions 1 and 2, we have the following

Theorem 5. *Let* $r = \theta^2$*. Then the following statements hold*

(a) *There exists* $\theta_1(\approx 0.166\,699\,3311)$ *such that the measure* μ_1 *is non-extreme if* $\theta \in (0, \theta_1)$ *and is extreme if* $\theta \in (\theta_1, \infty)$*;*

(b) *There are values* $\theta^*(\approx 7.729\,813\,675)$ *and* $\theta_2(\approx 9.706\,301\,628)$ *such that the measure* μ_2 *is extreme if* $\theta \in [\theta^*, \theta_2)$ *and is non-extreme if* $\theta \in (\theta_2, \infty)$*;*

(c) *The measure* μ_3 *is extreme (where it exists, that is* $\theta \in [\theta^*, \infty)$*) (see figure 7).*

Acknowledgments

The authors are greatly indebted to Professor U A Rozikov for suggesting the problem and for many stimulating conversations.

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https://doi.org/10.1088/1742-5468/ac08ff