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RESEARCH ARTICLES

On G_2-Periodic Quasi Gibbs Measures of p-Adic Potts Model on a Cayley Tree

Akbarkhuja Tukhtabaev*

Namangan State University, P.O. Box, 160136, 316 Uychi street, Namangan, Uzbekistan

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Abstract—In the present paper we study G_2-periodic p-adic quasi Gibbs measures for p-adic Potts model on a Cayley tree of order two. In the case $q = 3$, we prove the occurrence of a phase transition and construct ART quasi Gibbs measures for p-adic Potts model on a Cayley tree of order $k \geq 3$.

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Key words: *p-adic numbers, Potts model, p-adic quasi Gibbs measure, periodic, translation-invariant, ART Gibbs measure.*

1. INTRODUCTION

During the last few decades ultrametric analysis has grown from a relatively small and remote area maintained by few enthusiastic pioneers to widely recognized and mature discipline. Consequently, various models in physics described in language of p-adic analysis (see [2, 36, 37]),and numerous applications of such an analysis to mathematical physics have been studied in [3, 7, 8, 12, 13, 35]. These investigations proposed to study new probability models (namely p-adic measure theory in [15, 20]), the theories of p-adic and non-Archimedian stochastic processes have been developed. By means of p-adic stochastic process we have constructed wide classes of process using finite dimensional probability distributions [4, 9, 10, 15, 27]. In [4-6], [16, 42] and [21-34] it has been developed p-adic statistical mechanics within the scheme of theory of p-adic probability and p-adic stochastic processes. Namely, authors have studied p-adic Ising and Potts models on Cayley trees. Note that there are also several p-adic models of complex hierarchic system [14].

In the mentioned investigations, mostly, translation-invariant p-adic Gibbs measures have been described and studied. It is interesting to know how large is the class of p-adic Gibbs measures for the given model. In [25-28] the existence of periodic p-adic Gibbs measures for the p-adic Potts model on the Cayley tree has been carried our by means of chaotic behavior of the function associated with renormalization group (see (3.3)). We point out that such kind of measures (non trivial ones) may not exist in general. Namely, in [32] it was shown that there is no periodic p-adic Gibbs measures except for translation-invariant ones for the Potts model. In [22, 23, 31] studied p-adic quasi Gibbs measures, which contain p-adic Gibbs measure as a particular case. Moreover, it has been described translation-invariant p-adic quasi Gibbs measures for the p-adic Potts model on a Cayley tree. This allows us to establish the existence of a phase transition. It is known from [6] that any periodic p-adic Gibbs measure is either translation-invariant or G_2-periodic. These investigations lead us to the explicit construction of periodic p-adic Gibbs measures which is highly non trivial task. Therefore, one of the main aims of this paper is to construct and investigate G_2-periodic p-adic quasi Gibbs measures for p-adic q-state Potts model of the Cayley tree of order two. In the case $q = 3$, we prove the occurrence of a phase transition and construct ART quasi Gibbs measures for p-adic Potts model of the Cayley tree of order $k \geqslant 3$.

*E-mail: akbarxoja.toxtaboyev@mail.ru

2. PRELIMINARIES

2.1. p-Adic numbers and p-adic measure.

Let $\mathbb{Q}$ be a field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$ where, $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, \ x \neq 0, \\ 0, \quad x = 0. \end{cases}$$

This norm is non-Archimedean, i.e. it satisfies the strong triangle inequality:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \quad \forall x, y \in \mathbb{Q}.$$

From this property immediately get the following facts:
1) if $|x|_p \neq |y|_p$, then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$;
2) if $|x|_p = |y|_p$, then $|x - y|_p \leq |x|_p$.
The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form $x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$, where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0, \ 0 \leq x_j \leq p - 1$. In this case $|x|_p \ = p^{-\gamma(x)}$. We recall that an integer $a \in \mathbb{Z}$ is called *quadratic residue modulo p* if the congruent equation $x^2 \equiv a(\mathrm{mod}\, p)$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. *[35] The equation $x^2 = a, \ 0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + a_2 p^2 + ...), \ 0 \leq a_j \leq p - 1, \ a_0 > 0$ has a solution in $x \in \mathbb{Q}_p$ iff hold true the following:*
i) $\gamma(a)$ is even;
ii) $x^2 \equiv a_0(\mathrm{mod}\, p)$ is solvable for $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

In [29] authors have introduced new symbols "O" and "o" which allowed to simplify certain calculations. Roughly speaking, these symbols replace the notation $\equiv \ (mod\, p^k)$ without noticing about power of k. Let us recall them. A given p-adic number x by $O[x]$ we mean a p-adic number with the norm $p^{-\gamma(x)}$, i.e. $|x|_p = |O(x)|_p$. By $o[x]$, we mean a p-adic number with a norm strictly less than $p^{-\gamma(x)}$, i.e. $|o(x)|_p < |x|_p$. For instance, if $x = 1 - p + p^2$, we can write $O[1] = x$, $o[1] = x - 1$ or $o[p] = x - 1 + p$. Therefore, the symbols $O[\cdot]$ and $o[\cdot]$ make our work easier when we need to calculate the p-adic norm of p-adic numbers. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

We can see below some basic properties of $O[\cdot]$ and $o[\cdot]$, which will be used later on.

Lemma 2.2. *[29] Let $x, y \in \mathbb{Q}_p$. Then the following statements hold:*

1°. $O[x]O[y] = O[xy]$;

2°. $xO[y] = O[y]x = O[xy]$;

3°. $O[x]o[y] = o[xy]$;

4°. $o[x]o[y] = o[xy]$;

5°. $xo[y] = o[y]x = o[xy]$;

6°. $\frac{O[x]}{O[y]} = O\left[\frac{x}{y}\right]$, *if $y \neq 0$;*

7°. $\frac{o[x]}{O[y]} = o\left[\frac{x}{y}\right]$, *if* $y \neq 0$.

For $a \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : \ |x - a|_p < r\},$$

and the set of all p-adic integers $\mathbb{Z}_p := B(0, p)$. The set $\mathbb{Z}_p^* = \mathbb{Z}_p \backslash p\mathbb{Z}_p$ is called a set of p-adic units. p-adic *logarithm* is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n!},$$

which converges for $x \in B(1, 1)$ and p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, \frac{1}{2})$ if $p = 2$ and $x \in B(0, 1)$ if $p \neq 2$.

Put

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)}\right\}.$$

As corollary of Lemma 2.2 we have the following

Lemma 2.3. *Let p be a prime. Then the set $\mathcal{E}_p$ has the following properties:*
(a) $\mathcal{E}_p$ is a group under multiplication;
(b) $|a - b|_p < \begin{cases} \frac{1}{2}, & p = 2; \\ 1, & p \neq 2 \end{cases}$ *for all $a, b \in \mathcal{E}_p$;*
(c) $|a + b|_p = \begin{cases} \frac{1}{2}, & p = 2; \\ 1, & p \neq 2 \end{cases}$ *for all $a, b \in \mathcal{E}_p$;*
(d) If $a \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $a = \exp_p(h)$.

A more detailed description of p-adic calculus and p-adic mathematical physics can be found in [17, 18].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, A_2, ..., A_n \in \mathcal{B}$ such that
$A_i \cap A_j = \emptyset, \ i \neq j$, the following holds:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called *probability* if $\mu(X) = 1$. One of the important condition is boundedness, namely a p-adic measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. For more detail information about p-adic measures we refer to [1, 11, 17].

2.2. Cayley tree.

Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k + 1$ edges, except for the root x^0, which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from the point x to the point y. The distance $d(x, y)$ on the Cayley tree, is the length (number of edges) of the shortest path from x to y.

Let us set

$$W_n = \{x \in V : \ d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m,$$

$$L_n = \{\langle x, y\rangle \in L : \ x, y \in V_n\}.$$

We recall a coordinate structure in Γ_+^k: every vertex x (except for x^0) of Γ_+^k has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}, \ 1 \le m \le n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \dots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice. Let us define on Γ_+^k binary operation $\circ : \Gamma_+^k \times \Gamma_+^k \to \Gamma_+^k$ as follows: for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m) \tag{2.1}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \dots, i_n) \circ (0) = (i_1, \dots, i_n). \tag{2.2}$$

By means of the defined operation Γ_+^k becomes a noncommutative semigroup with a unit. Let us denote this group $(G^k, \circ)$. Using this semigroup structure one defines translations $\tau_g : G^k \to G^k, \ g \in G_k$ by

$$\tau_g(x) = g \circ x.$$

It is clear that $\tau_{(0)} = id$.

Let $G \subset G^k$ be a sub-semigroup of G^k and $h : G^k \to Y$ be a Y-valued function defined on G^k. We say that h is G- *periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in G^k$. Any G^k-periodic function is called *translation invariant*.

Now for each $m \ge 2$ we put

$$G_m = \{x \in G^k : d(x, x^0) \equiv 0 (\operatorname{mod} m)\}. \tag{2.3}$$

One can check that G_m is a sub-semigroup of G^k.

2.3. p-Adic quasi Gibbs measure for the Potts model.

Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\Phi = \{1, 2, ..., q\}$ be a finite set. A configuration σ on V is defined as $x \in V \mapsto \sigma(x) \in \Phi$; in a similar fashion one defines a configuration σ_n and σ^n on V_n and W_n, respectively. The set of all configurations on V (resp. V_n , W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi_n^V, \Omega_{W_n} = \Phi_n^W$). Using this, for given configurations $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), \ if \ \ x \in V_{n-1}, \\ \omega(x), \ \ if \ \ x \in W_n. \end{cases}$$

It is clear that $\sigma \vee \omega \in \Omega_{V_n}$.

We consider *p-adic Potts model* on a Cayley tree, where the spin takes $\Phi := \{1, 2, ..., q\}$.

The (formal) Hamiltonian of p-adic Potts model is

$$H(\sigma) = J \sum_{<x,y> \in l} \delta_{\sigma(x)\sigma(y)} \tag{2.4}$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\langle x, \, y\rangle$ stands for nearest neighbor vertices and δ_{ij} is the Kronecker's symbol, i.e.

$$\delta_{i,j} = \begin{cases} 0, & \text{if } \ i \ne j, \\ 1, & \text{if } \ i = j. \end{cases}$$

Assume that $h : V\backslash\{x^{(0)}\} \to \mathbb{Q}_p^{\Phi}$ is a mapping, i.e $h_x = (h_{1,x}, h_{2,x}, ..., h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ $(i \in \Phi)$ and $x \in V\backslash\{x^{(0)}\}$. Given $n \in \mathbb{N}$, we consider a p-adic probability measure $\mu_{h,\sigma}^{(n)}$ on Ω_{V_n} defined by

$$\mu_h^{(n)}(\sigma) = \frac{1}{Z_n^{(h)}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{2.5}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(h)}$ is the corresponding normalizing factor

$$Z_n^{(h)} = \sum_{\sigma\in\Omega_{V_n}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{2.6}$$

We say that p-adic probability distributions (2.5) are compatible if all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega\in\Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \omega) = \mu_h^{(n-1)}(\sigma_{n-1}). \tag{2.7}$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. We note that a non-Archimedean analogue of the Kolmogorov's extension theorem was proved in [4]. According to this theorem there exists a unique p-adic measure μ_h on $\Omega = \Phi^V$ for all $n \geq 1$ and $\sigma \in \Phi^{V_{n-1}}$, i.e.

$$\mu(\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \text{ for all } \sigma_n \in \Omega_{V_n},\ n \in \mathbb{N}.$$

Such measure is called a *p-adic quasi Gibbs measure* corresponding to the Hamiltonian (2.4) and vector-valued function $\mathbf{h}_x$, $x \in V$. By $QG(H)$ we denote the set of all p-adic quasi Gibbs measure associated with function $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$. If all values of h_x belong to the set $\mathcal{E}_p$ then it is called p-adic Gibbs measure. If there are at least two distinct p-adic quasi Gibbs measures $\mu, \nu \in QG(H)$ such that μ is bounded and ν is unbounded, then we say that a *a phase transition* occurs. The following statement describes conditions $\mathbf{h}_x$ guaranteing compatibility of $\mu_{\mathbf{h}}^{(n)}(\sigma)$.

Theorem 2.4. *[22] The measure $\mu_h^{(n)}(\sigma), n = 1, 2, ...$ (see (2.5)) associated with q-state Potts model (2.4) satisfy the compatibility condition (2.7) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\widehat{\boldsymbol{h}_x} = \prod_{y\in S(x)} F(\widehat{\boldsymbol{h}_y}, \theta). \tag{2.8}$$

Here and below a vector $\widehat{h} = \left(\widehat{h}_1, \widehat{h}_2, ..., \widehat{h}_{q-1}\right) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\mathbf{h} = (h_1, h_1, ..., h_q) \in \mathbb{Q}_p^q$ as follows

$$\widehat{h}_i = \frac{h_i}{h_q},\ i = 1, 2, ..., q-1 \tag{2.9}$$

and mapping

$F :\ \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \to \mathbb{Q}_p^{q-1}$ is defined by $F(x;\theta) = (F_1(x;\theta), ..., F_{q-1}(x;\theta))$ with

$$F_i(x;\theta) = \frac{(\theta-1)x_i + \sum\limits_{j=1}^{q-1} x_j + 1}{\sum\limits_{j=1}^{q-1} x_j + \theta},\ \ x = \{x_i\} \in \mathbb{Q}_p^{q-1},\ i = 1, 2, ..., q-1. \tag{2.10}$$

Remark 2.5. *Without loss of generality, we may assume that $h_q = 1$. Otherwise, in (2.5) we multiply and divide the expression on the right hand side by $\prod\limits_{x\in W_n} h_{q,x}$ and after replacing h_i by h_i/h_q we get the desired equality.*

Remark 2.6. *In [23] the existence of the phase transition for the considered model was established. In [32] it was shown that G_2-periodic solution of (2.8) belonging to $\mathcal{E}_p$ coincides with translation-invariant ones. Therefore, it is natural to find periodic solution of (2.8) in a general setting, which allows to find periodic p-adic Gibbs measures.*

Remark 2.7. *In [26] the existence of m-periodic quasi Gibbs measures was found under certain conditions such that $p \geq 3$, $k = 2$, $q = mp^n$ and $0 < |\theta - 1|_p \leq p^{-2n-1}$ for some $m, n \in \mathbb{N}$ and $(m, p) = 1$. However, the explicit form of that measures is not given. In present paper we find the explicit form of 2-periodic quasi Gibbs measures. Besides, 2-periodic quasi Gibbs measures we found are not required to satisfy the aforementioned conditions in [26]. Using the explicit form of solutions we can construct some non-periodic quasi Gibbs measures (see section 5).*

Let us first observe that the set $(\underbrace{1, ..., 1, h}_{m}, 1, ..., 1), (m = 1, ..., q-1)$ is invariant for the equation (2.8). Therefore, in what follows, we restrict ourselves to one of such lines, let us say $(h, 1, ..., 1)$.

3. NON TRANSLATION-INVARIANT TWO PERIODIC QUASI GIBBS MEASURE FOR p-ADIC POTTS MODEL

In [34] authors studied all translation-invariant p-adic Gibbs measures for the Potts model on the Cayley tree. Recently, in [1, 27, 28] the existence of periodic p-adic Gibbs measures for the p-adic q-state Potts model on the Cayley tree has been carried our by means of chaotic behavior of the function associated with renormalization group (see (3.3)). However, explicit construction of such kind of p-adic Gibbs measures is highly a non-trivial task. Therefore, in this section, we are going to construct G_2-periodic p-adic quasi Gibbs measures for the considered model.

Let G_2 be a sub-semigroup of G_k (see (2.3)). Recall that h_x is called G_2-periodic, if for all $x \in G^k$ and $y \in G_2$ it holds $\sigma(yx) = \sigma(x)$. We denote that

$$h_x = \begin{cases} h_1, \ x \in G^k \\ h_2, \ x \in G^k \backslash G_2. \end{cases}$$

From the equation (2.8) we can get the following system

$$\begin{cases} \widehat{h_1} = (F(\hat{h}_2, \theta))^k \\ \widehat{h_2} = (F(\hat{h}_1, \theta))^k. \end{cases} \tag{3.1}$$

We assume $\widehat{h}_i = \left(h_i^{(1)}, h_i^{(2)}, ..., h_i^{(q-1)}\right)$. Let $h_i^{(j)} = 1, j = \overline{2, q-1}$. Unless otherwise stated, we concentrate on the simplest case where $k = 2$.

In this case we can obtain following system of equations from (3.1)

$$\begin{cases} \widehat{h}_1^{(1)} = \left(\dfrac{\theta \widehat{h}_2^{(1)} + q - 1}{\widehat{h}_2^{(1)} + \theta + q - 2}\right)^2 \\ \widehat{h}_2^{(1)} = \left(\dfrac{\theta \widehat{h}_1^{(1)} + q - 1}{\widehat{h}_1^{(1)} + \theta + q - 2}\right)^2. \end{cases} \tag{3.2}$$

Let

$$f_\theta(h) = \left(\frac{\theta h + q - 1}{h + \theta + q - 2}\right)^2. \tag{3.3}$$

Rewriting (3.3) we have

$$f_\theta(h) = \left(\theta - \frac{(\theta - 1)(\theta + q - 1)}{h + \theta + q - 2}\right)^2.$$

Clearly that if $\theta = 1$ or $\theta = -q + 1$, then $f_\theta(h)$ is a constant function, i.e. $f_\theta(h)$ is a periodic function of any order.

In [25, 28] authors studied dynamics of the function $f_\theta(h)$ and found translation-invariant p-adic quasi Gibbs measures on a Cayley tree of order two. In [30, 33] several methods from Diophantine p-adic equations were used.

There are three fixed points of the function $f_\theta(h)$, i.e. $h_0 = 1$,

$$\begin{aligned} h_1 &= \frac{(\theta - 1)^2 - 2(q-1) + (\theta - 1)\sqrt{(\theta-1)^2 - 4(q-1)}}{2}, \\ h_2 &= \frac{(\theta - 1)^2 - 2(q-1) - (\theta - 1)\sqrt{(\theta-1)^2 - 4(q-1)}}{2}. \end{aligned} \tag{3.4}$$

Denote $\Delta(\theta, q) = -8 + (\theta - 1)^2$.

If $\widehat{h}_1^{(1)} = \widehat{h}_2^{(1)}$ then we get translation-invariant Gibbs measures. Finding G_2-periodic (non translation-invariant) quasi Gibbs measures is equivalent to find $h_1^{(1)} \neq h_2^{(1)}$ solutions of the system (3.2). It is equivalent to solve the following equation

$$\frac{f_\theta(f_\theta(h)) - h}{f_\theta(h) - h} = 0. \tag{3.5}$$

Simplifying the last equation we obtain

$$Ah^2 + Bh + C = 0, \tag{3.6}$$

where

$$\begin{aligned} &A(\theta, q) = (\theta^2 + \theta + q - 2)^2, \\ &B(\theta, q) = \theta^4 + 4(q-1)\theta^3 + (q^2 + 6q - 12)\theta^2 + (10q^2 - 36q + 32)\theta + 2q^3 - 13q^2 + 26q - 17, \\ &C(\theta, q) = (\theta^2 + (3q - 5)\theta + (q-2)^2)^2. \end{aligned}$$

We notice that (3.6) has solutions in $\mathbb{Q}_p$ if and only if $\sqrt{D(\theta, q)} \in \mathbb{Q}_p$ where $D(\theta, q) = B^2(\theta, q) - 4A(\theta, q)C(\theta, q)$. For the sake of convenience we denote

$$D_1(\theta, q) = 4q^2(1-q) + (24q - 24q^2)(\theta - 1) + (36 - 36q - 3q^2)(\theta - 1)^2 - 10q(\theta - 1)^3 - 3(\theta - 1)^4.$$

After some calculations we can rewrite $D(\theta, q)$ as follows

$$D(\theta, q) = (\theta - 1)^2(\theta + q - 1)^2 D_1(\theta, q).$$

Thus, we can conclude that $\sqrt{D(\theta, q)}$ exists in $\mathbb{Q}_p$ if and only if $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$.

Since (3.6) is a quadratic equation we can write formal solutions of (3.6) as follows

$$\begin{aligned} h_{3,4} = &\frac{(2q^2 - 2q^3) + (12q - 12q^2)(\theta - 1) + (18 - 18q - q^2)(\theta - 1)^2 - 4q(\theta - 1)^3 - (\theta - 1)^4}{2(q + 3(\theta - 1) + (\theta - 1)^2)^2} \pm \\ &\pm \frac{(\theta - 1)(\theta + q - 1)\sqrt{D_1(\theta, q)}}{2(q + 3(\theta - 1) + (\theta - 1)^2)^2}. \end{aligned} \tag{3.7}$$

Now we show that for a given $q \geq 2$ and for each $\theta \in \mathcal{E}_p$ one can find prime integer p such that $h_{3,4} \in \mathbb{Q}_p$.

Theorem 3.1. *For every $q \geq 2$ one can find a prime p such that the periodic solutions $h_{3,4}$ exist in $\mathbb{Q}_p$.*

Proof. Since the existence of $h_{3,4}$ is equivalent to the existence of $\sqrt{D_1(\theta,q)}$. Let us, we rewrite $D_1(\theta,q)$ as follows

$$\begin{aligned} D_1(\theta,q) \ = \ & 12^2-288(\theta-1)+117(\theta-1)^2+30(\theta-1)^3-3(\theta-1)^4 \\ & -(132-168(\theta-1)+18(\theta-1)^2+10(\theta-1)^3)(q+3) \\ & +(40-24(\theta-1)-3(\theta-1)^2)(q+3)^2-4(q+3)^3 \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} D_1(\theta,q) \ = \ & 48^2-1728(\theta-1)+132(\theta-1)^2+80(\theta-1)^3-3(\theta-1)^4 \\ & -(832-408(\theta-1)-12(\theta-1)^2+10(\theta-1)^3)(q+8) \\ & +(100-24(\theta-1)-3(\theta-1)^2)(q+8)^2-4(q+8)^3. \end{aligned} \tag{3.9}$$

In order to establish the existence $\sqrt{D_1(\theta,q)}$ we consider several cases.

Case 1. Let p be a prime divisor of $q+3$ such that $p\notin\{2,3\}$. We are going to show that $\sqrt{D_1(\theta,q)}\in\mathbb{Q}_p$. Since $\theta\in\mathcal{E}_p$. By (3.8), we get

$$D_1(\theta,q)=12^2(1+o[1]).$$

Using the last equality and due to Lemma 2.1 we have $\sqrt{D_1(\theta,q)}\in\mathbb{Q}_p$.

Case 2. Let $q+3=2^a3^b$, where a and b are some non negative integers.

First we assume that $b\geq 2$ and $p=3$. Then we get $q+3=o[3]$. Due to $\theta-1=o[1]$, from (3.8), we obtain

$$D_1(\theta,q)=12^2(1+o[1]).$$

Hence, thanks to Lemma 2.1 we have $\sqrt{D_1(\theta,q)}\in\mathbb{Q}_3$.

Let us suppose that $b=1$. Then we have $q+8=3\cdot 2^a+5$. One can see that $q+8$ has a prime divisor $p\notin\{2,3\}$. Then due to (3.9), in $\mathbb{Q}_p$ we have the following representation

$$D_1(\theta,q)=48^2(1+o[1]).$$

Again according to Lemma 2.1 we conclude that $\sqrt{D_1(\theta,q)}\in\mathbb{Q}_p$.

Now we assume that $b=0$. Then keeping in mind $q\geq 2$ one has $a\geq 3$. It is easy to see that $2\nmid(q+8)$. So, we have to consider two cases: either $q+8=3^c$ or $q+8$ has a prime divisor $p>3$.

One can see that if $q+8=3^c$ holds then $c\geq 3$. In this case we consider $D_1(\theta,q)$ in $\mathbb{Q}_3$. We have

$$\begin{aligned} & 1728(\theta-1)=o[3^2], \quad 132(\theta-1)^2=o[3^2], \\ & 80(\theta-1)^3=o[3^2], \quad 3(\theta-1)^4=o[3^4], \\ & (832-408(\theta-1)-12(\theta-1)^2+10(\theta-1)^3)(q+8)=o[3^2], \\ & (100-24(\theta-1)-3(\theta-1)^2)(q+8)^2=o[3^5], \\ & 4(q+8)^3=o[3^8]. \end{aligned}$$

Hence, for $p=3$ from (3.9) we obtain $D_1(\theta,q)=48^2+o[9])$, which yields that $D_1(\theta,q)=48^2(1+o[1])$. Then due to Lemma 2.1 we conclude that $\sqrt{D_1(\theta,q)}\in\mathbb{Q}_3$.

Suppose that $q+8$ has a prime divisor $p>3$. Then for this prime number from (3.9) we immediately get $D_1(\theta,q)=48^2+o[1]$ in $\mathbb{Q}_p$. The last equality together with $48^2=O[1]$ implies that $D_1(\theta,q)=48^2(1+o[1])$. Again by Lemma 2.1 we conclude that $\sqrt{D_1(\theta,q)}\in\mathbb{Q}_p$ where $p>3$ is a prime divisor of $q+8$.

The theorem is proved.

4. BOUNDEDNESS OF TRANSLATION-INVARIANT AND TWO PERIODIC p-ADIC QUASI GIBBS MEASURES AND PHASE TRANSITIONS

Lemma 4.1. *[21] Let $\boldsymbol{h}$ be a solution of* (2.8)*, and $\mu_{\boldsymbol{h}}$ be an associated p-adic quasi Gibbs measure. Then for the corresponding partition function $Z_n^{(\boldsymbol{h})}$ the following equality holds:*

$$Z_n^{(\boldsymbol{h})} = A_{\boldsymbol{h},n-1} Z_{n-1}^{(\boldsymbol{h})}, \tag{4.1}$$

where $A_{\boldsymbol{h},n} = \prod_{x\in W_n} a_{\boldsymbol{h}}(x)$, $\prod_{y\in S(x)} \sum_{j=1}^{q} exp\{J\delta_{i,j}\} h_{j,y} = a_{\boldsymbol{h}}(x) h_{i,x}$, $a_{\boldsymbol{h}}(x) \in \mathbb{Q}_p$, $i = 1, 2, ..., q$.

Using Lemma 4.1 we get the following statements.

Lemma 4.2. *Let $k = 2$. If $\boldsymbol{h}$ be a translation-invariant solution of* (2.8) *then for the corresponding partition function $Z_n^{(\boldsymbol{h})}$ the following equality holds:*

$$Z_n^{(\boldsymbol{h})} = (h + \theta + q - 2)^{2^{n+1}-2}(h + q - 1) \tag{4.2}$$

Proof. It is easy to check that $\mathbf{h} = (h, 1, ...1)$ is a translation-invariant solution of (2.8), where h is a fixed point of (3.3). Since $\theta = exp\{J\}$, using (2.6) we get $Z_1^{(h)} = (h + \theta + q - 2)^2(h + q - 1)$. Then by lemma 4.1 we obtain the following equalities:

$$a_h(x) = \frac{(\theta h_{1,y} + q - 1)^2}{h_{1,x}} = \frac{(\theta h + q - 1)^2}{h} = (h + \theta + q - 2)^2,$$

$$A_{h,n} = (h + \theta + q - 2)^{2^{n+1}},$$

$$Z_n^{(h)} = (h + \theta + q - 2)^{2^{n+1}-2}(h + q - 1),$$

where $h = h_0, h = h_1, h = h_2$. The lemma is proved.

Lemma 4.3. *Let $k = 2$. If $h_{3,4}$ be G_2-periodic (non translation-invariant) solutions of* (2.8) *then for the corresponding partition function $Z_n^{(h)}$ the following equalities hold:*
If n is odd, then

$$Z_n^{(\boldsymbol{h})} = (h_3 + \theta + q - 2)^{\frac{2^{n+2}-2}{3}}(h_4 + \theta + q - 2)^{\frac{2^{n+1}-4}{3}}(h_4 + q - 1); \tag{4.3}$$

If n is even, then

$$Z_n^{(\boldsymbol{h})} = (h_3 + \theta + q - 2)^{\frac{2^{n+1}-2}{3}}(h_4 + \theta + q - 2)^{\frac{2^{n+2}-4}{3}}(h_4 + q - 1). \tag{4.4}$$

Proof. Let $h_{\sigma(x),x} = \begin{cases} h_{1,x}, \ if \ \ \sigma(x) = 1; \\ 1, \ if \ \ \sigma(x) \neq 1. \end{cases}$, $h_{1,x} = \begin{cases} h_3, \ if \ \ |x| \ \ is \ odd; \\ h_4, \ if \ \ |x| \ \ is \ even. \end{cases}$
Then,

$$h_{1,x} = \begin{cases} h_3, \ if \ \ n \ \ is \ odd; \\ h_4, \ if \ \ n \ \ is \ even. \end{cases}$$

Due to (3.2), we have

$$\begin{cases} h_3 = \left(\dfrac{\theta h_4 + q - 1}{h_4 + \theta + q - 2}\right)^2; \\ h_4 = \left(\dfrac{\theta h_3 + q - 1}{h_3 + \theta + q - 2}\right)^2. \end{cases} \tag{4.5}$$

Using (2.6) and (4.5), we get $Z_1^{(h)} = (h_3+\theta+q-2)^2(h_4+q-1)$.
We consider following cases.
Case 1. Let n is odd. By Lemma 4.1 and (4.5) we get

$$a_h(x) = \frac{(\theta h_{1,y}+q-1)^2}{h_{1,x}} = \frac{(\theta h_4+q-1)^2}{h_3} = (h_4+\theta+q-2)^2,$$

$$A_{h,n-1} = (h_3+\theta+q-2)^{2^n}, A_{h,n} = (h_4+\theta+q-2)^{2^{n+1}},$$

$$Z_n^{(\mathbf{h})} = (h_3+\theta+q-2)^{\frac{2^{n+2}-2}{3}}(h_4+\theta+q-2)^{\frac{2^{n+1}-4}{3}}(h_4+q-1).$$

Case 2. Let n is even. By Lemma 4.1 and (4.5) we get

$$a_h(x) = \frac{(\theta h_{1,y}+q-1)^2}{h_{1,x}} = \frac{(\theta h_3+q-1)^2}{h_4} = (h_3+\theta+q-2)^2,$$

$$A_{h,n-1} = (h_4+\theta+q-2)^{2^n}, A_{h,n} = (h_3+\theta+q-2)^{2^{n+1}},$$

$$Z_n^{(\mathbf{h})} = (h_3+\theta+q-2)^{\frac{2^{n+1}-2}{3}}(h_4+\theta+q-2)^{\frac{2^{n+2}-4}{3}}(h_4+q-1)$$

The Lemma is proved.

4.1. $q=3$ Case

For the convenience of studying we consider the case $q=3$, i.e. spin values are 1,2,3. In this case $D_1(\theta) = -72-144(\theta-1)-99(\theta-1)^2-30(\theta-1)^3-3(\theta-1)^4$. Using (3.7) we can obtain following solutions:

$$\begin{aligned} h_{3,4} = &\frac{-36-72(\theta-1)-45(\theta-1)^2-12(\theta-1)^3-(\theta-1)^4}{2(3+3(\theta-1)+(\theta-1)^2)^2} \\ &\pm\frac{(\theta-1)(\theta+2)\sqrt{-72-144(\theta-1)-99(\theta-1)^2-30(\theta-1)^3-3(\theta-1)^4}}{2(3+3(\theta-1)+(\theta-1)^2)^2}. \end{aligned} \tag{4.6}$$

Lemma 4.4. *Let $|Per_2(f_\theta(h))|$ be number of 2-periodic points of $f_\theta(h)$. Then the following statements are true:*

1) if $p \equiv 1 (\mathrm{mod}\, 8)$ or $p\equiv 3(\mathrm{mod}\, 8)$, then $|Per_2(f_\theta(h))| = 5$;

2) if $p=2$ or $p\equiv 5(\mathrm{mod}\, 8)$ or $p\equiv 7(\mathrm{mod}\, 8)$, then $|Per_2(f_\theta(h))| = 1$.

Proof.

a) Let $p=2$. Due to $\theta\in\mathcal{E}_2$ we have $|(\theta-1)^2|_2 \le \frac{1}{16}$ (see (3.4)). Hence, using strong triangle inequality one gets

$$|-8+(\theta-1)^2|_2 = \frac{1}{8}.$$

This means that $|D(\theta)|_2 = \frac{1}{8}$. Then thanks to lemma 2.1 there does not exist $\sqrt{\Delta(\theta)}$ in $\mathbb{Q}_2$. Consequently, $h_1, h_2 \notin \mathbb{Q}_2$.

Now we will check the existence of solutions h_3 and h_4 in $\mathbb{Q}_2$. In order to check it we calculate the norm of $D_1(\theta)$. Again using $|\theta-1|_2 \le \frac{1}{4}$ we obtain the followings:

$$|-144(\theta-1)|_2 \le \frac{1}{2^6},\quad |-99(\theta-1)^2|_2 \le \frac{1}{2^4},$$
$$|-30(\theta-1)^3|_2 \le \frac{1}{2^7},\quad |-3(\theta-1)^4|_2 \le \frac{1}{2^8}.$$

Keeping in mind $|-72|_2 = \frac{1}{2^3}$, from the last inequalities due to strong triangle inequality one has $|D_1(\theta)|_2 = \frac{1}{8}$. Then according to Lemma 2.1 we conclude that $\sqrt{D_1(\theta)} \notin \mathbb{Q}_2$. This means that $h_3, h_4 \notin \mathbb{Q}_2$. Thus in this case we have shown that $|Per_2(f_\theta(h))| = 1$.

b) Let $p \geq 3$. Since $\theta \in \mathcal{E}_p$ we get $|\theta - 1|_p < 1$. Then thanks to Lemma 2.1 existence of $\sqrt{\Delta(\theta)}$ in $\mathbb{Q}_p$ equivalence to the existence of $\sqrt{-2}$. We notice that $\sqrt{-2} \in \mathbb{Q}_p$ if -2 is a quadratic residue by modulo p. -2 is a quadratic residue by modulo p if and only if $p \equiv 1(\mathrm{mod}\, 8)$ or $p \equiv 3(\mathrm{mod}\, 8)$ (see [40]). So, we infer that

$$h_1, h_2 \in \mathbb{Q}_p \quad \Longleftrightarrow \quad p \equiv 1(\mathrm{mod}\, 8) \text{ or } p \equiv 3(\mathrm{mod}\, 8).$$

Now we will check the existence of solutions h_3 and h_4 in $\mathbb{Q}_p$. Due to $\theta \in \mathcal{E}_p$ we have
$|(24q - 24q^2)(\theta - 1) + (36 - 36q - 3q^2)(\theta - 1)^2 - 10q(\theta - 1)^3 - 3(\theta - 1)^4|_p \leq p^{-1}$.
By Lemma 2.1, existence $\sqrt{D_1(\theta)}$ is equivalent to existence $\sqrt{-72} = 6\sqrt{-2}$ or existence $\sqrt{-2}$. We notice that $\sqrt{-2} \in \mathbb{Q}_p$ if -2 is a quadratic residue by modulo p. $\sqrt{-2}$ exists in $\mathbb{Q}_p$ if and only if $p \equiv 1(\mathrm{mod}\, 8)$ or $p \equiv 3(\mathrm{mod}\, 8)$. So, we infer that

$$h_3, h_4 \in \mathbb{Q}_p \quad \Longleftrightarrow \quad p \equiv 1(\mathrm{mod}\, 8) \text{ or } p \equiv 3(\mathrm{mod}\, 8).$$

In conclusion, we get

$$h_1, h_2, h_3, h_4 \in \mathbb{Q}_p \quad \Longleftrightarrow \quad p \equiv 1(\mathrm{mod}\, 8) \text{ or } p \equiv 3(\mathrm{mod}\, 8). \tag{4.7}$$

Thus in this case we have shown that if $p \equiv 1(\mathrm{mod}\, 8)$ or $p \equiv 3(\mathrm{mod}\, 8)$ then $|Per_2(f_\theta(h))| = 5$, if $p = 2$, $p \equiv 5(\mathrm{mod}\, 8)$ and $p \equiv 7(\mathrm{mod}\, 8)$, then $|Per_2(f_\theta(h))| = 1$.
Finally, we have finished to prove Lemma 4.4.

Remark 4.5. *Note that all prime numbers are in the forms of* $p = 2$, $p \equiv 1(\mathrm{mod}\, 8)$, $p \equiv 3(\mathrm{mod}\, 8)$, $p \equiv 5(\mathrm{mod}\, 8)$, $p \equiv 7(\mathrm{mod}\, 8)$. *If* $p \not\equiv 1(\mathrm{mod}\, 8)$, $p \not\equiv 3(\mathrm{mod}\, 8)$ *then* $p = 2$ *or* $p \equiv 5(\mathrm{mod}\, 8)$ *or* $p \equiv 7(\mathrm{mod}\, 8)$.

The following theorem is immediate from 4.4.

Theorem 4.6. *Let* $q = 3$. *Following statements are true for* p*-adic Potts model on a Cayley tree of order two*

1) *if* $p \equiv 1(\mathrm{mod}\, 8)$ *or* $p \equiv 3(\mathrm{mod}\, 8)$, *then there are three translation-invariant* p*-adic quasi Gibbs measures and there are two* G_2*-periodic, non translation-invariant* p*-adic quasi Gibbs measures;*

2) *if* $p = 2$ *or* $p \equiv 5(\mathrm{mod}\, 8)$ *or* $p \equiv 7(\mathrm{mod}\, 8)$, *then there is one translation-invariant* p*-adic quasi Gibbs measure.*

Lemma 4.7. *The norms of the solutions* h_i, $i = 0, 1, ..., 4$ *are equal to one.*

Proof. From $\theta \in \mathbb{Q}_p$ and properties of p-adic norm we can get easily the result of Lemma 4.7.

Theorem 4.8. *Let* $q = 3$. *Following statements are true for* p*-adic Potts model on a Cayley tree of order two*

1) *if* $p = 2$ *or* $p \equiv 5(\mathrm{mod}\, 8)$ *or* $p \equiv 7(\mathrm{mod}\, 8)$, *then* μ_{h_0} *measure is unbounded;*

2) *if* $p \equiv 1(\mathrm{mod}\, 8)$ *or* $p \equiv 3(\mathrm{mod}\, 8)$,$p \neq 3$ *then only* μ_{h_0} *measure is bounded,* μ_{h_1}, μ_{h_2}, μ_{h_3},μ_{h_4} *measures are unbounded;*

3) *if* $p = 3$, *then all* μ_{h_0}, μ_{h_1}, μ_{h_2}, μ_{h_3},μ_{h_4} *measures are unbounded.*

Proof. Case 1. If $p=2$ or $p\equiv 5(\mathrm{mod}\,8)$ or $p\equiv 7(\mathrm{mod}\,8)$, then exists only μ_{h_0} translation-invariant measure. Note that $|h_0+\theta+q-2|_p=|\theta+2|_p=1$. By Lemma 4.2 we get

$$\lim_{n\to\infty}|\mu_{h_0}^{(n)}|_p=\lim_{n\to\infty}\left|\frac{1}{3(\theta+2)^{2^{n+1}-2}}\exp\{H_n(\sigma)\}\prod_{x\in W_n}h_{\sigma(x),x}\right|_p=1.$$

Case 2. If $p\equiv 1(\mathrm{mod}\,8)$ or $p\equiv 3(\mathrm{mod}\,8)$, $p\neq 3$ then there exist μ_{h_i}, $i=0,...,5$ measures. For translation-invariant solutions we have $|h_0+\theta+1|_p$=1. Since

$$|\theta-1|_p<1,\quad |h_{1,2}+2|_p=\left|\frac{(\theta-1)^2\pm(\theta-1)\sqrt{(\theta-1)^2-4(q-1)}}{2}\right|_p<1,$$

then we get

$$|h_{1,2}+\theta+1|_p=|h_{1,2}+2+\theta-1|_p\leq max\{|h_{1,2}+2|_p,|\theta-1|_p\}<1.$$

By Lemma 4.2, we have

$$\lim_{n\to\infty}|\mu_{h_0}^{(n)}|_p=1,$$

$$\lim_{n\to\infty}|\mu_{h_{1,2}}^{(n)}|_p=\lim_{n\to\infty}\left|\frac{1}{(h_{1,2}+2)(h_{1,2}+\theta+1)^{2^{n+1}-2}}\exp\{H_n(\sigma)\}\prod_{x\in W_n}h_{\sigma(x),x}\right|_p=\infty.$$

For two periodic non translation-invariant solutions we have $|\theta-1|_p<1$,

$$|h_{3,4}+2|_p=\left|\frac{-72(\theta-1)-45(\theta-1)^2-12(\theta-1)^3-(\theta-1)^4\pm(\theta-1)(\theta+2)\sqrt{D_1(\theta)}}{2(3+3(\theta-1)+(\theta-1)^2)^2}\right|_p<1.$$

Consequently, we get $|h_{3,4}+\theta+1|_p=|h_{3,4}+2+\theta-1|_p\leq max\{|h_{3,4}+2|_p,|\theta-1|_p\}<1$. By Lemma 4.3 we have

$$\lim_{n\to\infty}|\mu_{h_{3,4}}^{(n)}|_p=\infty.$$

Case 3. If $p=3$, then there exist μ_{h_i}, $i=0,...,5$ measures. Note that $|h_0+\theta+q-2|_3=|\theta+2|_3<1$. By Lemma 4.2 we get

$$\lim_{n\to\infty}|\mu_{h_0}^{(n)}|_3=\lim_{n\to\infty}\left|\frac{1}{3(\theta+2)^{2^{n+1}-2}}\exp\{H_n(\sigma)\}\prod_{x\in W_n}h_{\sigma(x),x}\right|_3=\infty.$$

We can prove that μ_{h_1}, μ_{h_2}, μ_{h_3},μ_{h_4} measures are unbounded as in the case 2.

Theorem 4.9. *For p-adic 3-state Potts model on a Cayley tree of order two there exist the phase transition occurrence if and only if $p\neq 3$, $p\equiv 1(\mathrm{mod}\,8)$ or $p\equiv 3(\mathrm{mod}\,8)$.*

5. p-ADIC ART QUASI GIBBS MEASURES

It is always interesting to study non-periodic Gibbs measures. In [41] some non-periodic Gibbs measures that are called ART measures were investigated. In [38, 39] p-adic ART generalized Gibbs measures for the Ising model on the Cayley tree were studied. In this section we are going to study p-adic ART quasi Gibbs measures for 3-state Potts model using by translation-invariant and G_2-periodic solutions on the Cayley tree of order k $(k\geq 3)$.

Recall that each solution of (2.8) define a p-adic quasi Gibbs measure for Potts model. One can see that $h_x=1$ for all $x\in V$ is a solution of (2.8) for any $k\geq 2$. Now we construct new solutions of (2.8) for

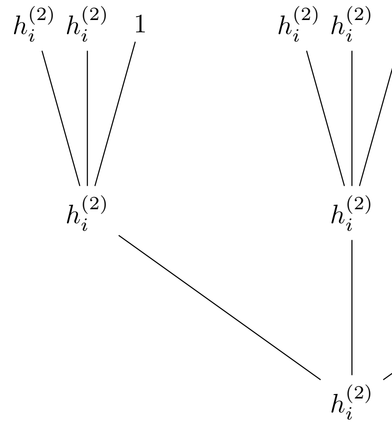

Fig. 1. The function $\widetilde{h}_x^{(i)}, i=\overline{0,2}$ on the Cayley tree of order three.

$k \geq 3$. In the case $k=2$ and $q=3$ all translation-invariant solutions of (2.8) had been found in [25]. For $k\geq 3$ we set up some solutions of (2.8) using (3.4) and (4.6).

(a_1). Let V^k be the set of all vertices of the Cayley tree Γ_+^k. Since $k>2$ one can consider V^2 as a subset of V^k. Define the following function

$$\widetilde{h}_x^{(i)} = \begin{cases} h_i^{(2)}, & \text{if } x\in V^2,\\ 1, & \text{if } x\in V^k\setminus V^2, \end{cases} \tag{5.1}$$

where $i=\overline{0,2}$. This function on the Cayley tree of order $k=3$ is shown in Fig.1.

Now we shall check that (5.1) satisfies (2.8) on Γ_+^k.

Let $x\in V^2\subset V^k$. For $i=\overline{0,2}$ we have

$$\begin{aligned}
\widetilde{h}_x^{(i)} &= \prod_{y\in S(x)} F(\widetilde{h}_y^{(i)},\theta) = \prod_{y\in S(x)}\left(\frac{\theta h_y^{(i)}+2}{h_y^{(i)}+\theta+1}\right)\\
&= \prod_{y\in S(x)\cap V^2}\left(\frac{\theta h_y^{(i)}+2}{h_y^{(i)}+\theta+1}\right)\times \prod_{y\in S(x)\cap(V^k\setminus V^2)}\left(\frac{\theta h_y^{(i)}+2}{h_y^{(i)}+\theta+1}\right)\\
&= \prod_{y\in S(x)\cap V^2}\left(\frac{\theta h_y^{(i)}+2}{h_y^{(i)}+\theta+1}\right) = \left(\frac{\theta h_i^{(2)}+2}{h_i^{(2)}+\theta+1}\right)^2 = h_i^{(2)}.
\end{aligned}$$

Here we used

$$\prod_{y\in S(x)\cap(V^k\setminus V^2)}\left(\frac{\theta h_y^{(i)}+2}{h_y^{(i)}+\theta+1}\right)^2 = 1.$$

Thus $\widetilde{h}_x^{(i)}, i=\overline{0,2}$ satisfies the functional equation (2.8). We denote by $\mu_{\widetilde{h}_x^{(i)}}, i=\overline{0,2}$ the Gibbs measures corresponding to $\widetilde{h}_x^{(i)}, i=\overline{0,2}$ and those measures we called p-adic ART quasi Gibbs measures.

(a_2). Let $k\geq 3$. We shall construct new p-adic (non periodic) Gibbs measures using by $h_0=1,\ h_3, h_4$. Define the following function

$$h_x^{(i)} = \begin{cases} h_i^{(1)}, & if\ x\in V^2\cap G_2,\\ F^2(h_i^{(1)},\theta), & if\ x\in V^2\cap(G^k\setminus G_2),\\ 1, & if\ x\in V^k\setminus V^2, \end{cases} \tag{5.2}$$

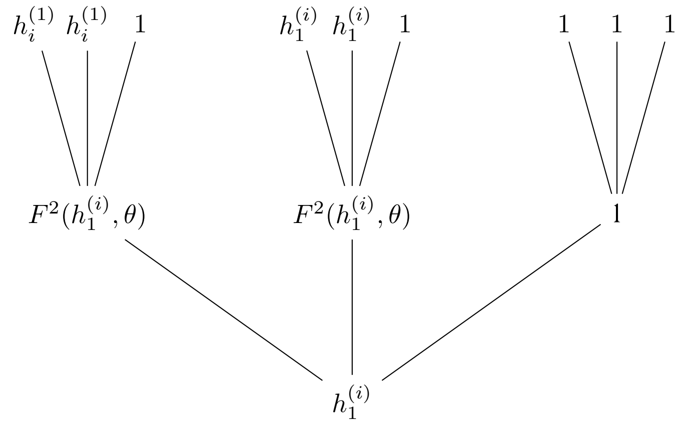

Fig. 2. The function $\widetilde{h}_x^{(i)}, i=0,3,4$ on the Cayley tree of order three.

where $i=0,3,4$. This function on the Cayley tree of order $k=3$ is shown in Fig. 2.

Now we shall check that (5.2) satisfies (2.8) on Γ_+^k.
Let $x\in V^2\cap G_2$. For $i=0,3,4$, we have

$$h_i^{(1)}=\prod_{y\in S(x)}F\left(h_y,\theta\right)=\prod_{y\in S(x)}F\left(F^2\left(h_y^{(1)},\theta\right)\right)=F^2\left(F^2\left(h_i^{(1)},\theta\right)\right).$$

Let $x\in V^2\cap\left(G^k\backslash G_2\right)$. $F^2\left(h_i^{(1)},\theta\right)=\prod\limits_{y\in S(x)}F\left(h_y,\theta\right)=F^2\left(h_1,\theta\right)$.

If $x\in V^k\backslash V^2$ then it is easy to see that $h_x^{(i)}$ satisfies (2.8).

Let $q=3$, $k>3$. Consequently, we have the following theorem.

Theorem 5.1. *If $p\equiv 1(\bmod 8)$ or $p\equiv 3(\bmod 8)$ then there exist at least four p-adic ART (non periodic) quasi Gibbs measures for the Potts model on a Cayley tree.*

Lemma 5.2. *Let $k\geq 3$. If $\mu_{\widetilde{h}}$ is p-adic ART quasi Gibbs measure constructed by rule (a_1) then for the corresponding partition function $Z_n^{(\boldsymbol{h})}$ the following equality holds:*

$$Z_n^{(\boldsymbol{h})}=(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}}(h+\theta+1)^{\frac{2(k^n-k)}{k-1}}[(\theta h+2)^k+2(h+\theta+1)^k]. \tag{5.3}$$

Proof. Since $\theta=exp\{J\}$, using (2.6) we get $Z_1^{(h)}=(\theta h+2)^k+2(h+\theta+1)^k$. Then by lemma 4.1 we obtain the following equalities:

$$a_h(x)=\frac{\prod\limits_{y\in S(x)}(\theta h_{1,y}+2)}{h_{1,x}}=\frac{\prod\limits_{y\in S(x)}(\theta h+2)}{h}=\frac{(\theta h+2)^2(\theta+2)^{k-2}}{h}=(h+\theta+1)^2(\theta+2)^{k-2},$$

$$A_{h,n}=(h+\theta+1)^{2k^n}(\theta+2)^{(k-2)k^n},$$

$$Z_n^{(\mathbf{h})}=(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}}(h+\theta+1)^{\frac{2(k^n-k)}{k-1}}[(\theta h+2)^k+2(h+\theta+1)^k],$$

where $h=h_0, h=h_1, h=h_2$. The lemma is proved.

Lemma 5.3. *Let $k\geq 3$. If $\mu_{\widetilde{h}}$ is p-adic ART quasi Gibbs measure constructed by rule (a_2) then for the corresponding partition function $Z_n^{(\boldsymbol{h})}$ the following equality holds:*
If n is odd, then

$$Z_n^{(\boldsymbol{h})}=(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}}(h_3+\theta+1)^{\frac{2(k^{n+1}-k^2)}{k^2-1}}(h_4+\theta+1)^{\frac{2(k^n-k)}{k^2-1}}Z_1^{(\boldsymbol{h})}; \tag{5.4}$$

If n is even, then

$$Z_n^{(\boldsymbol{h})}=(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}}(h_3+\theta+1)^{\frac{2(k^n-k^2)}{k^2-1}}(h_4+\theta+1)^{\frac{2(k^{n+1}-k)}{k^2-1}}Z_1^{(\boldsymbol{h})}; \tag{5.5}$$

where

$$Z_1^{(\boldsymbol{h})}=(\theta h_3+2)^k+2(h_3+\theta+1)^k.$$

Proof. Let

$$h_{1,x}=\begin{cases} h_3, \ if \quad n \ is\ odd; \\ h_4, \ if \quad n \ is\ even. \end{cases}$$

Due to (3.2), we have

$$\begin{cases} h_3=\left(\dfrac{\theta h_4+2}{h_4+\theta+1}\right)^2; \\ h_4=\left(\dfrac{\theta h_3+2}{h_3+\theta+1}\right)^2. \end{cases} \tag{5.6}$$

Using (2.6) and (5.6), we get $Z_1^{(h)}=(h_3+\theta+q-2)^2(h_4+q-1)$.
Consider following cases
Case 1. Let n is odd. By Lemma 4.1 and (5.6) we get the following equalities:

$$a_h(x)=\frac{\prod\limits_{y\in S(x)}(\theta h_{1,y}+2)}{h_{1,x}}=\frac{(\theta h_4+2)^2(\theta+2)^{k-2}}{h_3}=(h_4+\theta+1)^2(\theta+2)^{k-2},$$

$$A_{h,n-1}=(h_3+\theta+1)^{2k^{n-1}}(\theta+2)^{(k-2)k^{n-1}}, A_{h,n}=(h_4+\theta+1)^{2k^n}(\theta+2)^{(k-2)k^n},$$

$$Z_n^{(\mathrm{h})}=(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}}(h_3+\theta+1)^{\frac{2(k^{n+1}-k^2)}{k^2-1}}(h_4+\theta+1)^{\frac{2(k^n-k)}{k^2-1}}Z_1^{(\mathrm{h})}.$$

Case 2. Let n is even. By Lemma 4.1 and (5.6) we get the following equalities:

$$a_h(x)=\frac{\prod\limits_{y\in S(x)}(\theta h_{1,y}+2)}{h_{1,x}}=\frac{(\theta h_3+2)^2(\theta+2)^{k-2}}{h_4}=(h_3+\theta+1)^2(\theta+2)^{k-2},$$

$$A_{h,n-1}=(h_4+\theta+1)^{2k^{n-1}}(\theta+2)^{(k-2)k^{n-1}}, A_{h,n}=(h_3+\theta+1)^{2k^n}(\theta+2)^{(k-2)k^n},$$

$$Z_n^{(\mathrm{h})}=(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}}(h_3+\theta+1)^{\frac{2(k^n-k^2)}{k^2-1}}(h_4+\theta+1)^{\frac{2(k^{n+1}-k)}{k^2-1}}Z_1^{(\mathrm{h})}.$$

Finally, the Lemma 5.3 is proved.

Remark 5.4. *If $k=2$ in Lemma 5.2 and Lemma 5.3 then we get Lemma 4.2 and Lemma 4.3 respectively.*

Theorem 5.5. *Let $q=3$ and $k\geq 3$. Following statements are true for p-adic Potts model on a Cayley tree of order k*

1) *if $p=2$ or $p\equiv 5(\mathrm{mod}\,8)$ or $p\equiv 7(\mathrm{mod}\,8)$, then ART quasi Gibbs measure $\mu_{\widetilde{h_0}}$ is unbounded;*

2) *if $p\equiv 1(\mathrm{mod}\,8)$ or $p\equiv 3(\mathrm{mod}\,8)$, $p\neq 3$ then only ART quasi Gibbs measure $\mu_{\widetilde{h_0}}$ is bounded;*

3) *if $p=3$, then all ART quasi Gibbs measures constructed by rules (a_1) and (a_2) are unbounded.*

Proof. Case 1. If $p = 2$ or $p \equiv 5(\mathrm{mod}\, 8)$ or $p \equiv 7(\mathrm{mod}\, 8)$, then exists only ART quasi Gibbs measure $\mu_{\widetilde{h_0}}$. Note that $|h_0 + \theta + q - 2|_p = |\theta + 2|_p = 1$. By Lemma 5.2 we get

$$\lim_{n\to\infty} |\mu^{(n)}_{\widetilde{h_0}}|_p = \lim_{n\to\infty} \left| \frac{1}{3(\theta+2)^{\frac{k(k^n-1)}{k-1}}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x} \right|_p = 1.$$

Case 2. If $p \equiv 1(\mathrm{mod}\, 8)$ or $p \equiv 3(\mathrm{mod}\, 8)$, $p \neq 3$ then there exist ART quasi Gibbs measures constructed by rules (a_1) and (a_2). Since $\theta \in \mathcal{E}_p$, for h_0, h_1, h_2 we have

$$|\theta + 2|_p = 1, |h_0 + \theta + 1|_p = 1, |h_{1,2} + \theta + 1|_p < 1, 0 < |Z_1^{(h)}|_p < 1.$$

By Lemma 5.2, we have

$$\lim_{n\to\infty} |\mu^{(n)}_{\widetilde{h_0}}|_p = 1,$$

$$\lim_{n\to\infty} |\mu^{(n)}_{\widetilde{h_{1,2}}}|_p = \lim_{n\to\infty} \left| \frac{1}{(\theta+2)^{\frac{(k-2)(k^n-k)}{k-1}} (h+\theta+1)^{\frac{2(k^n-k)}{k-1}} Z_1^{(h)}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x} \right|_p = \infty.$$

For h_3, h_4 we have $|\theta + 2|_p = 1$, $|h_{3,4} + \theta + 1|_p < 1$, $0 < |Z_1^{(\mathbf{h})}|_p = |(\theta h_3 + 2)^k + 2(h_3 + \theta + 1)^k|_p < 1$. By Lemma 5.3 we have

$$\lim_{n\to\infty} |\mu^{(n)}_{\widetilde{h_{3,4}}}|_p = \infty.$$

Case 3. If $p = 3$, then there exist $\mu_{\widetilde{h_i}}$, $i = 0, ..., 5$ measures. Note that $|h_0 + \theta + 1|_3 = |\theta + 2|_3 < 1$. By Lemma 5.2 we obtain

$$\lim_{n\to\infty} |\mu^{(n)}_{\widetilde{h_0}}|_p = \lim_{n\to\infty} \left| \frac{1}{3(\theta+2)^{\frac{k(k^n-1)}{k-1}}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x} \right|_p = \infty.$$

We can prove that $\mu_{\widetilde{h_{1,2}}}, \mu_{\widetilde{h_{3,4}}}$ measures are unbounded as in the case 2.

The theorem is proved.

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