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M U N D A R I J A

22. Grinfunksiyasi. Grinfunksiyasi yordamida Dirixle va Neyman masalalarini yechish. To‘g‘ri to‘rtburchakli sohada Dirixle va Neyman masalalariga o‘xshash masalalarni Furye usulida yechish……………………………………………………………………………………..

Mavzu-1

Matematik fizika tenglamalar va ularning yechimlari to'g'risida tushunchalar.Xarakteristik forma.

Reja

1. Xususiy hosilali differensial tenglamalar.
2. Xususiy hosilalili differensial operator.
3. Xususiy hosilali ikkinchi tartibli differensial tenglama.
4. Xarakteristik forma. Kvadratik forma.
5. Ikkinchi tartibli xususiy hosilali differensial tenglamalarning klas-sifikatsiyasi va kanonik ko‘rinishi.

6. Kvazichiziqli tenglama. Tenglama tiplari.
7. Xususiy hosilali differensial tenglamalar sistemasi.
8. Kvazichiziqli sitema. Xarakteristik determinant.
9. Ikki o‘zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalar
10. Bir qiymatli almashtirishlar.
11. Asosiy tasdiqlar.Xarakteristik tenglama.
12. Ko‘p o‘zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglama.

Tayanch so‘z va iboralar: *xususiy hosilali differensial tenglama,xususiy hosilalili differensial operator, chiziqli differensial tenglama, kvazi chiziqli differensial tenglama, regulyar yechim, bir jinsl iva bir jinsli bo‘lmagan tenglama . Xarakteristik forma, elliptik tenglama giperbolik tenglama, ultragiperbolik tipdagi tenglama,parabolik tenglama, tekis elliptik tenglama, aralash tipdagi tenglama, xarakteristikalar tenglamasi, xarakteristik sirt.* x *nuqtada parabolik, elliptik, giperbolik tenglamalar, xususiy hosilali differensial tenglamalar sistemasi, kvazichiziqli sistema, Kanonik ko‘rinishga keltirish.Xarakteristik forma tushunchasi. Yuqori tartibli differentsial tenglamalarning va sistemalarning* *sinflari.Ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali differentsial tenglamalarni kanonik ko'rinishga keltirish.*

D orqali dekart ortogonal koordinatalari $x_1, x_2, ..., x_n,\ n \geq 2$ bo'lgan x nuqtalarning $n-$o'lchovli E^n Evklid fazosidagi sohani, ya'ni ochiq bog'langan (bo'sh bo'lmagan) to'plamni belgilaymiz. Tartiblangan manfiy bo'lmagan n ta butun sonning $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ ketma-ketligi n *tartibli multiindeks* deyiladi, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ son bu *multiindeksning uzunligi* deb ataladi.

$u(x) = u(x_1, x_2, ..., x_n)$ funksiyaning $x \in D$ nuqtadagi $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ tartibli hosilasini

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}},\ \ D^0 u = u(x)$$

ko'rinishda yozib olamiz. Xususiy holda $\alpha = \alpha_i$ bo'lganda

$$D^\alpha u = \frac{\partial^{\alpha_i} u}{\partial x_i^{\alpha_i}} = D_i^{\alpha_i},\ \ D_i u = \frac{\partial u}{\partial x_i} = u_{x_i},\ \ D_i^2 u = \frac{\partial^2 u}{\partial x_i^2} = u_{x_i x_i}\ .$$

$F = F(x, ..., p_\alpha, ...)$ funksiya D soha x nuqtalarning va $p_\alpha = p_{\alpha_1 \alpha_2 ... \alpha_n} = D^\alpha u$, $\alpha_i = 0,1,2,...$ haqiqiy o'zgaruvchining berilgan funksiyasi bo'lib, kamida bitta $\frac{\partial F}{\partial p_\alpha},\ |\alpha| = m > 0$ hosila noldan farqli bo'lsin.

Ushbu

$$F = F(x,\ ...,\ D^\alpha u,\ ...) = 0 \quad (1)$$

tenglik noma'lum $u(x) = u(x_1, x_2, ..., x_n)$ funksiyaga nisbatan m *tartibli xususiy hosilali differensial tenglama* deyiladi.

(1) tenglamaning chap tomoni esa *xususiy hosilalili differensial operator* deb yuritiladi.

Agar F barcha p_α $(|\alpha| = 0,1,2,..., m)$ o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa,

tenglama *chiziqli differensial tenglama* deyiladi.

Agar F, $|\alpha| = m$ bo'lganda barcha p_α o'zgaruvchilarga nisbatan chiziqli funksiya

bo‘lsa, tenglama *kvazi chiziqli differensial tenglama* deb yuritiladi.

D sohada aniqlangan $u(x)$ funksiya (1) tenglamada ishtirok etuvchi barcha hosilalari bilan uzluksiz bo‘lib, uni ayniyatga aylantirsa, $u(x)$ ni (1) tenglamaning *regulyar (klassik) yechimi* deyiladi.

Xususiy hosilali m- tartibli chiziqli differensial tenglamani ushbu

$$Lu \equiv \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = f(x) \quad (2)$$

ko‘rinishda yozib olish mumkin.

Barcha $x \in D$ lar uchun (2) tenglamaning o‘ng tomoni $f(x)$ nolga teng bo‘lsa, (2) tenglama *bir jinsli,* $f(x)$ funksiya nolga teng bo‘lmasa, *bir jinsli bo‘lmagan tenglama* deyiladi.

Agar $u(x)$ va $v(x)$ funksiyalar bir jinsli bo‘lmagan (2) tenglamaning yechinlari bo‘lsa, ravshanki $w(x) = u(x) - v(x)$ ayirma bir jinsli ($f = 0$) tenglamaning yechimi bo‘ladi.

Xususiy hosilali ikkinchi tartibli differensial tenglama

$$\sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x) \frac{\partial u}{\partial x_i} + C(x)u = f(x) \quad (3)$$

ko‘rinishda yoziladi, bu yerda $A_{ij}, B_i, C, f - D$ sohada berilgan haqiqiy funksiyalardir.

(3) tenglamaning barcha $A_{ij}, i, j = 1,...,n$ koeffitsiyentlari nolga teng bo‘lgan $x \in D$ nuqtalarda tenglama ikkinchi tartibli bo‘lmay qoladi, ya’ni bu nuqtalarda (3) tenglamaning tartibi buziladi. Bundan keyin biz (3) tenglama berilgan sohada uning tartibi ikkiga teng deb hisoblaymiz. (3) tenglamada $i \ne j$ bo‘lganda alohida-alohida $A_{ij}u_{x_i x_j}$, $A_{ij}u_{x_j x_i}$ qo‘shiluvchilar ishtirok etmay, balki ularning yig‘indisi $(A_{ij} + A_{ji})u_{x_i x_j}$ ishtirok etadi. Shu sababli ham umumiylikka zarar yetkazmay hamma vaqt $A_{ij} = A_{ji}$ deb hisoblaymiz.

Eslatib o‘tamiz, D sohada aniqlangan va k – tartibgacha xususiy hosilalari bilan uzluksiz bo‘lgan haqiaiy $u(x)$ funksiyalarning to‘plami $C^*(D)$ orqali belgilanadi.

Faraz qilaylik, 1-ma'ruzadagi (1) tenglamada ishtirok etayotgan $F = F(x,..., p_\alpha,...)$ funksiya, $p_\alpha = p_{\alpha_1\alpha_2...\alpha_n}$, $|\alpha| = m$o'zgaruvchilar bo'yicha uzluksiz birinchi tartibli hosilalariga ega bo'lsin. (1) tenglama nazariyasida $\lambda_1, \lambda_2, ..., \lambda_n$ haqiqiy o'zgaruvchilarga nisbatan ushbu

$$K(\lambda_1, \lambda_2,..., \lambda_n) = \sum_{\alpha=m} \frac{\partial F}{\partial p_\alpha} \lambda^\alpha, \ \lambda^\alpha = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} ... \lambda_n^{\alpha_n} \quad (4)$$

m-tartibli forma - m darajali bir jinsli ko'phad muhim rol o'ynaydi. Bu forma (1) tenglamaga mos bo'lgan *xarakteristik forma* deyiladi.

Ikkinchi tartibli kvazichiziqli

$$\sum_{i,j=1}^{n} A_{ij}(x) u_{x_i x_j} + \Phi(x, u, u_{x_1},..., u_{x_n}) = 0 \quad (5)$$

differensial tenglama uchun, bu yerda $A_{ij}(x) \in C(D)$, (4) forma

$$Q(\lambda_1, \lambda_2,..., \lambda_n) = \sum_{i,j=1}^{n} A_{ij}(x) \lambda_i \lambda_j \quad (6)$$

*kvadratik forma*dan iborat bo'ladi, shu jumladan (5) ko'rinishdagi ikkinchi tartibli tenglama tekshirilayotganda, iloji boricha erkli o'zgaruvchilarni almashtirib, tenglamalarni soddaroq ko'rinishga keltirishga harakat qilinadi va ayrim hollarda bunga erishiladi ham. Shu maqsadda, avvalo (5) tenglamani erkli o'zgaruvchilarni almashtirib uni soddaroq ko'rinishga keltirishga harakat qilamiz. $x = (x_1, x_2,..., x_n)$ o'rniga $y = y(x)$ ya'ni

$$y_k = y_k(x_1, x_2,..., x_n) \ k = \overline{1,n}$$

ya'ni

$$y_k = \beta_{k_1} x_1 + \beta_{k_2} x_2 + ... + \beta_{k_n} x_n \ k = \overline{1,n}$$

$y_k(x) \in C^2(D)$ va ushbu yakobian

$$\frac{D(y_1, y_2,..., y_n)}{D(x_1, x_2,..., x_n)} \neq 0$$

deb hisoblaymiz. U holda

$$\frac{\partial u}{\partial x_1}=\sum_{k=1}^{n}\frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_j \partial x_i}=\sum_{k,l=1}^{n}\frac{\partial^2 u}{\partial y_l \partial y_k}\cdot\frac{\partial y_l}{\partial x_j}\cdot\frac{\partial y_k}{\partial x_i}+\sum_{k=1}^{n}\frac{\partial u}{\partial y_k}\cdot\frac{\partial^2 y_k}{\partial x_j \partial x_i}.$$

Buni (5) tenglamaga qo‘yib ushbu tenglamaga kelamiz:

$$\sum_{i,j=1}^{n}A_{i,j}(x)\left\{\sum_{k,l=1}^{n}\frac{\partial^2 u}{\partial y_l \partial y_k}\cdot\frac{\partial y_l}{\partial x_j}\cdot\frac{\partial y_k}{\partial x_i}+\sum_{k=1}^{n}\frac{\partial u}{\partial y_k}\cdot\frac{\partial^2 y_k}{\partial x_j \partial x_i}\right\}+\overline{\Phi}_0(y,u,u_{y_1},\ldots,u_{y_n})=0\,.$$

Yoki

$$\sum_{k,l=1}^{n}\overline{A}_{k,l}(y)\frac{\partial^2 u}{\partial y_l \partial y_k}+\overline{\Phi}_1(y,u,u_{y_1},\ldots,u_{y_n})=0\,, \qquad (7)$$

bu yerda

$$\overline{A}_{k,l}(y)=\sum_{i,j=1}^{n}A_{i,j}(x)\cdot\frac{\partial y_l}{\partial x_j}\cdot\frac{\partial y_k}{\partial x_i}\ , \qquad (8)$$

$$\overline{\Phi}_1(y,u,u_{y_1},\ldots,u_{y_n})=\sum_{k=1}^{n}\frac{\partial u}{\partial y_k}\sum_{i,j=1}^{n}A_{i,j}(x)\cdot\frac{\partial^2 y_k}{\partial x_j \partial x_i}+\overline{\Phi}_0(y,u,u_{y_1},\ldots,u_{y_n}).$$

(5) tenglama tekshirilayotgan D sohada x_0 nuqtani olamiz va ushbu belgilashlarni kiritamiz.

$$y_0=y(x_0), \quad \beta_{ki}=\frac{\partial y_k(x_0)}{\partial x_i}.$$

U holda (8) forma x_0 nuqtada quyidagicha yoziladi

$$\overline{A}_{k,l}(y_0)=\sum_{i,j=1}^{n}A_{i,j}(x)\beta_{l,j}\beta_{k,i}\,. \qquad (9)$$

(6) kvadratik formani x_0 nuqtada yozib olamiz:

$$Q=\sum_{i,j=1}^{n}A_{i,j}(x_0)\lambda_i\cdot\lambda_j\,. \qquad (10)$$

Maxsus bo‘lmagan ushbu

$$\lambda_i = \sum_{k=1}^{n} \beta_{k,i} \xi_k , \ \det(\beta_{k,i}) \neq 0 \tag{11}$$

affin almashtirish yordamida (10) kvadratik forma

$$Q = \sum_{i,j=1}^{n} A_{i,j}(x_0) \left(\sum_{k=1}^{n} \beta_{k,i} \xi_k \right) \cdot \left(\sum_{l=1}^{n} \beta_{l,i} \xi_l \right) = \sum_{k,l=1}^{n} \xi_k \xi_l \left(\sum_{i,j=1}^{n} A_{i,j}(x_0) \cdot \beta_{ki} \cdot \beta_{lj} \right) =$$

$$= \sum_{k,l=1}^{n} \overline{A}_{kl}(y_0) \xi_k \xi_l \tag{12}$$

gakeladi. Bu kvadratik formaning koeffitsiyentlari ham (9) formula bilan aniqlanadi.

Shunday qilib, (5) tenglamani x_0 nuqtada x o‘zgaruvchilar o‘rniga yangi $y = y(x)$ o‘zgaruvchilar kiritib soddalashtirish uchun shu nuqtada (10) kvadratik formani maxsus bo‘lmagan (11) chiziqli almashtirish yordami bilan soddalashtirish yetarlidir.

Algebra kursida isbot qilinadiki, hamma vaqt shunday maxsus bo‘lmagan (11) almashtirish mavjud bo‘lib, uning yordami bilan (10) kvadratik forma quyidagi ko‘rinishga olib kelinadi:

$$Q = \sum_{k=1}^{n} \mu_k \xi_k^{\ 2} , \tag{13}$$

bu erda μ_k , $k = 1,\dots,n$ koeffitsiyentlar $1, -1, 0$ qiymatlarni qabul qiladi. Shu bilan birga masbat (manfiy) koeffitsiyentlar soni (inertsiya indeksi) va nolga teng bo‘lgan koeffitsiyentlar soni (forma defekti) affin almashtirishga nisbatan invuriant, ya’ni bu sonlar faqat (10) forma bilan aniqlanib, (11) almashtirishning tanlab olinishiga bog‘liq bo‘lmaydi.

Bu narsa (5) differensial tenglama $A_{i,j}(x)$ koeffitsiyentlarning x_0 nuqtada qabul qiladigan qiymatlariga qarab, klassifikatsiya qilish imkonini beradi.

Yuqorida aytilganlarga asosan (7) tenglama

$$\sum_{k=1}^{n} \mu_k u_{y_k y_k} + \overline{\Phi}(y, u, u_{y_1}, \dots, u_{y_n}) = 0 \tag{14}$$

ko‘rinishda yoziladi.

Ikkinchi tartibli differensial tenglamaning aralash hosilalar qatnashmagan bunday ko‘rinishi, odatda uning *kanonik korinishi* deyiladi.

(5) tenglamani bitta nuqtada emas, hech bo‘lmaganda $x_0 \in D$ nuqtaning biror kichik atrofida kanonik ko‘rinishga olib keluvchi o‘zgaruvchilarning almashtirilishini (affin bo‘lishi shart emas) topish mumkinmi degan savol tug‘uladi.

Bu savolga ijobiy javob faqat $n=2$ bo‘lgandagina ma’lum. Bu holni biz alohida ko‘ramiz. Agar barcha $\mu_k = 1$ yoki barcha $\mu_k = -1$, $k=1,\dots,n$ bolsa, ya’ni Q forma mos ravishda musbat yoki manfiy aniqlangan (gefinit) bo‘lsa, (5) tenglama $x \in D$ nuqtada *elliptik tipdagi* yoki *elliptik tenglama* deyiladi.

Agar μ_k koeffisiyentlardan bittasi manfiy, qolganlari musbat (yoki aksincha) bo‘lsa, (5) tenglama $x \in D$ nuqtada *giperbolik tenglama* deb ataladi.

μ_k koeffisiyentlardan l tasi, $1 < l < n-1$, musbat, qolgan $n-l$ tasi manfiy bo‘lsa, (5) tenglamaga *ultragiperbolik tipdagi tenglama* deyiladi.

Agar μ_k koeffisiyentlardan bittasi nolga teng bo‘lsa, qolganlari noldan farqli va bir xil ishorali bo‘lsa, (5) tenglama $x \in D$ nuqtada *parabolik tenglama* deb ataladi.

Agar koeffitsiyentlardan kamida bittasi nolga teng bo‘lsa, (5) tenglama keng manoda $x \in D$ nuqtada *parabolik tenglama* deb ataladi.

Agar (5) tenglama D sohaning har bir nuqtasida elliptik, giperbolik yoki parabolik bo‘lsa, u holda D sohada mos ravishda *elliptik*, *giperbolik yoki parabolik tipdagi tenglama* deb ataladi.

Agar noldan farqli bo‘lgan, bir xil ishorali k_0, k_1 haqiqiy sonlar mavjud bo‘lib, barcha $x \in D$ nuqtalar uchun ushbu

$$k_0 \sum_{i=1}^{n} \lambda_i^2 \le Q(\lambda_1, \lambda_2, ..., \lambda_n) \le k_1 \sum_{i=1}^{n} \lambda_i^2$$

tengsizlik bajarilsa, D sohada elliptik bo‘lgan tenglama *tekis elliptik tenglama* deyiladi.

Masalan, Trikomi nomi bilan yutitiladigan

$$x_2 u_{x_1 x_1} + u_{x_2 x_2} = 0$$

tenglama $x_2 > 0$ yarim tekislikning har nuqtasida elliptik bo‘lsa ham, bu yerda tekis elliptik emasdir.

D sohaning turli qismida (5) tenglama har xil tipga tegishli bo‘lsa, uni *aralash tipdagi tenglama* deyiladi.

Yuqorida keltirilgan Trikomi tenglamasi $x_2 = 0$ o‘qning ixtiyoriy qismini o‘zichiga olgan ixtiyoriy D sohada aralash tipdagi tenglamaga misol bo‘ladi.

Yuqorida bayon qilingan (5) tenglamaning klassifikasiyasini ekvivalent tarzida $A = \|A_{ij}\|$ matrisaning xarakteristik sonlariga asoslanib ham berish mumkin. Buning uchun algebradan ma’lum bo‘gan (10) kvadratik formaning (13) kanonik ko‘rinishidagi μ_k, $k = 1,2,...,n$ sonlar A matrisaning xarakteristik sonlaridan iborat ekanligini eslash kifoyadir. Ma’lumki simmetrik ($A_{ij} = A_{ji}$) matrisaning barcha xarakteristik sonlari haqiqiy sonlardan iboratdir.

Eslatib o‘tamiz, A matritsaning xarakteristik sonlar ushbu

$$\det(A - \lambda E) = 0$$

algebraik tenglamaning ildizlaridan iborat, bu erda E - birllik matritsa.

Demak, (5) tenglama berilgan D sohaning ixtiyoriy x nuqtasida A matritsa xarakteristik sonlarning ishorasini aniqlab, (5) tenglamaning qaysi tipga tegishli ekanligini darhol bilib olish mumkin.

Bu yerda yana bir tushuncha, xarakteristik sirtlar tushunchasini kiritib o‘tamiz.

Ushbu

$$\sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial \omega}{\partial x_i} \cdot \frac{\partial \omega}{\partial x_j} = 0$$

tenglama (5) differentsialtanglama*xarakteristikalariningtenglamasi*deyiladi.

Agar $\omega(x_1, x_2, ..., x_n)$ funktsiya xarakteristikalar tenglamasini qanoatlantirsa,

$$\omega(x_1, x_2, ..., x_n) = c\,, \; c = const$$

tenglik bilan aniqlangan sirt berilgan (5) differentsial tenglamani *xarakteristik sirti* yoki *xarakteristikasi* deyiladi.

O‘zgaruvchlar soni ikkita bo‘lganda xarakteristik egri chiziq haqida so‘z boradi.

Xarakteristikalar tenglamasi rasman bunday tuziladi: (5) differentsial tenglamaga mos bo‘lgan (6) kvadratik formani tuzib, unda $\lambda_i = \frac{\partial \omega}{\partial x_i}, \ \lambda_j = \frac{\partial \omega}{\partial x_j}$ deb, hosil bo‘lgan ifodani nolga tenglashtiramiz.

Faraz qilaylik, $\omega \in C^2$ bo‘lsin. (5) tenglamani soddalashtirish maqsadida x_i o‘zgaruvchilar o‘rniga kiritilgan y_i o‘zgaruvchilardan bittasini, masalan y_1 ni $y_1 = \omega(x_1, x_2, ..., x_n)$ desak, u holda xarakteristikalar tenglamasiga asosan $A_{11} = 0$ bo‘ladi. Shuning uchun ham differentsial tenglamaning bitta yoki bir nechta xarakteristikalar oilasini bilish, bu tenglamani soddaroq ko‘rinishga keltirish imkonini beradi.

1.Xususiy hosilali m-tartibli kvazichiziqli tenglama

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u + \Phi(x, u, ..., D^\beta u, ...) = 0 \tag{1.15}$$

ko‘rinishda yoziladi, bu yerda Φ ifoda noma’lum $u = u(x)$ funksiyaning $m-1$ dan yuqori bo‘lgan hosilalarini o‘z ichiga olmaydi.

(1.15) tenglamaga mos bo‘lgan xarakteristik forma ma’ruza-2 dagi (4) formulaga asosan

$$K(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{|\alpha|=m} a_\alpha \lambda^\alpha \tag{1.16}$$

ko‘rinishda yoziladi.

Agar D sohaning tayin x nuqtasida $\lambda_1, \lambda_2, ..., \lambda_n$ o‘zgaruvchilarning shunday $\lambda_i = \lambda_i(\mu_1, ..., \mu_n), \ i = 1, ..., n$ affin almashtirishini (affin almashtirish maxsus bo‘lmagan $\lambda = A\mu + B$ $(\det A \neq 0, \ B \in E^n)$ almashtirilishdan iboratdir) topish mumkin bo‘lsaki, natijada (1.16) formadan hosil bo‘lgan forma μ_i o‘zgaruvchilarning faqat l tasini, $0 < l < n$, o‘z ichiga olsa, (1) tenglama x *nuqtada parabolik* yoki *parabolik buziladigan* deb aytiladi.

Parabolik buzilish bo‘lmaganda, faqatgina $\lambda_1 = 0, ..., \lambda_n = 0$ bo‘lgandagina $K(\lambda_1, \lambda_2, ..., \lambda_n) = 0$ bo‘lsa, (1) tenglama x *nuqtada elliptik* deyiladi.

Agarda $\lambda_1,\lambda_2,...,\lambda_n$ o‘zgaruvchilar orasidan bittasini, masalan $\lambda_n=\lambda$ ni ajratib olish mumkin bo‘lsaki (zarur bo‘lgan holda bu o‘zgaruvchilarni affin almashtirishdan so‘ng), barcha $\lambda'=(\lambda_1,...,\lambda_{n-1})\in E^{n-1}$ nuqtalar uchun λ ga nisbatan xarakteristik

$$K(\lambda_1,...,\lambda_{n-1},\lambda)=0$$

tenglamaning barcha ildizlari haqiqiy bo‘lsa, (1) tenglama x *nuqtada giperbolik* deyiladi.

Agarda λ ildizlarining bir qismi haqiqiy, qolganlari kompleks bo‘lsa, (1.15) tenglama $x\in D$ *nuqtada qo‘shma tipdagi tenglama* deyiladi.

Bu ta’rifga asosan $\alpha\geq 3$ bo‘lgandagina (1.15) tenglama *qo‘shma* tip bo‘lishi mumkin.

Qo‘shma turdagi tenglamaga

$$\frac{\partial}{\partial x_1}(u_{x_1x_1}+u_{x_2x_2})=0$$

tenglama misol bo‘ladi.

Xuddi shunga o‘xshash, chiziqli bo‘lmagan ma’ruza-1 dagi (1) tenglama (4) (ma’ruza-2 dagi) forma xususiyatiga asosan tiplarga ajratiladi. (4) (ma’ruza-2 dagi) forma koeffitsiyentlari x nuqta bilan birga izlanayotgan $u(x)$ yechim va uning hosilalariga bog‘liq bo‘lgani sababli, tiplarga ajratish tekshirilayotgan holda faqat shu yechim uchungina ma’noga ega bo‘ladi.

Masalan,

$$u(x)u_{x_1x_1}+\sum_{i=2}^{n}u_{x_ix_i}=0$$

tenglama $u(x)>0$ bo‘lganda $x\in D$ nuqtalarda elliptik, $u(x)<0$ bo‘lganda giperbolik va $u(x)=0$ bo‘lganda $x\in D$ nuqtalarda parabolik buziladi.

Endi xususiy hosilali differensial tenglamalar sistemasining klassifikasiyasiga qisqacha to‘xtalib o‘tamiz.

(1) (ma’ruza-1 dagi) tenglamada qatnashyotgan F funksiya N o‘lchovli $F=(F_1,...,F_n)$ vektordan iborat bo‘lsin. Bu vektorning $F_1,...,F_n$ komponentalari D

soha x nuqtalarning hamda $p_0^j = u_j,\ p_\alpha^j = D^\alpha u_j,\ j = 1,...,M$ haqiqiy o‘zgaruvchilarning berilgan haqiqiy funksiyalari bo‘lsin.

Ushbu

$$F_i(x,\ ...,\ D^\alpha u_j,\ ...) = 0,\ i = 1,...,N,\ j = 1,...,M \quad (1.17)$$

ko‘rinishdagi tenglik, noma’lum $u_1,...,u_M$ funksiyalarga nisbatan *xususiy hosilali differensial tenglamalar sistemasi* deyiladi. (1.17) tenglamalar sistemasiga kirgan noma’lum funksiyalar hosilalarining eng yuqori tartibi shu *sistemaning tartibi* deyiladi.

(1.17) sistemaning tartibi m ga teng bo‘lsin. Agar hamma F_i funksiyalar barcha p_α^j o‘zgaruvchilarga nisbatan chiziqli bo‘lsa, (1.17) sistema chiziqli, agar F_i lar**,** $i = 1,...,N$**,** barcha $p_\alpha^j,\ |\alpha| = m,$ larga nisbatan chiziqli bo‘lsa, (1.17) *sistema kvazichiziqli* deyiladi.

Agar $N = M,\ N > M,\ N < M$ bo‘lsa, u holda (1.17) sistema mos ravishda *aniq, ortig‘i bilan aniqlangan, yetarlicha aniqlanmagan* deyiladi. (1.17) sistema aniq bo‘lib, uning har bir tenglamasining tartibi m ga teng bo‘lsin.

Ushbu

$$a_\alpha = \left\| \frac{\partial F_i}{\partial p_\alpha^j} \right\|,\ i, j = 1,...,N,\ \sum_{k=1}^{n} \alpha_k = m$$

kvadratik matrisani tuzamiz.

$$K(\lambda_1, \lambda_2,..., \lambda_n) = \det \sum_{|\alpha|=m} a_\alpha \lambda^\alpha = \det \sum_{|\alpha|=m} a_{\alpha_1...\alpha_n} \lambda_1^{\alpha_1},..., \lambda_n^{\alpha_n} \quad (1.18)$$

Ifoda haqiqiy skalyar $\lambda_1, \lambda_2,..., \lambda_n$ parametrlarga nisbatan Nm tartibli formadan iboratdir. Bu forma (1.17) sistemaning *xarakteristik determinanti* deyiladi.

(1.18) formaning xarakteriga qarab, xuddi (1.15) tenglamaga o‘xshash, (1.17) sistema ham tiplarga ajratiladi. Juda ko‘p hollarda amaliyotda uchraydigan tenglamalar sistemasini bitta

$$\sum_{|\alpha| \le m} a_\alpha D^\alpha u = f \quad (1.19)$$

matrisa tenglama ko‘rinishida yozish mumkin.

Bu yerda D^α differensial operator, $u = (u_1(x),..., u_N(x))$ vector-funksiya yoki

$u = \|u_j\|,\ j = 1,...,N$ matrisa-ustunning har bir komponentiga ta'sir qiladi, a_n – koeffitsiyentlar N – tartibli matrisadan iborat bo'lib, bular hamda (1.19) sistemaning o'ng tomoni $f = (f_1,..., f_N)$ yoki $f = \|f_j\|$ noma'lum $u_j(x)$ funksiyalarga va ularning tartibi $m-1$ dan katta bo'lmagan hosilalariga bog'liq bo'lishi mumkin. $m=1$ bo'lganda (1.19) sistemadan birinchi tartibli xususiy hosilali differensial tenglamalar sistemasi

$$\sum_{j=1}^{n} a_j D_j u + bu = f \tag{1.20}$$

kelib chiqadi, bu yerda $b - N$ -tartibli kvadratik matrisa.

(1.20) tenglamalar sistemasiga mos bo'lgan N -tartibli xarakteristik forma ushbu

$$K(\lambda_1, \lambda_2,..., \lambda_n) = \det \sum_{j=1}^{n} a_j \lambda_j$$

formula bilan beriladi.

Misol uchun (1.20) da $N = 2,\ b = 0,\ f = 0$,

$$a_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix},\ a_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}$$

bo'lsin. Bu holda ushbu sistemaga ega bo'lamiz:

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \frac{\partial}{\partial x_1} \begin{Vmatrix} u_1 \\ u_2 \end{Vmatrix} + \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} \frac{\partial}{\partial x_2} \begin{Vmatrix} u_1 \\ u_2 \end{Vmatrix} =$$

$$= \begin{Vmatrix} D_1 u_1 + 0 \\ 0 + D_1 u_2 \end{Vmatrix} + \begin{Vmatrix} 0 - D_2 u_2 \\ D_2 u_1 + 0 \end{Vmatrix} = \begin{Vmatrix} D_1 u_1 - D_2 u_2 \\ D_2 u_1 + D_1 u_2 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix}$$

yoki

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2},\ \frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1}. \tag{1.21}$$

Shunday qilib, kompleks o'zgaruvchili funksiyalar nazariyasidan ma'lum bolgan Koshi-Riman sistemasi hosil bo'ldi. (1.21) sistemaga mos bo'lgan xarakteristik forma

$$K = \det(a_1\lambda_1 + a_2\lambda_2) = \det \begin{Vmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{Vmatrix} = \lambda_1^2 + \lambda_2^2$$

ko‘rinishga ega bo‘ladi.

Demak, Koshi-Riman sistemasi elliptik tipdagi sistema ekan. $z = x_1 + ix_2$ kompleks o‘zgaruvchini hamda

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \quad \bar{z} = x_1 - ix_2$$

Differensial operatorni kiritib, (1.21) sistemani bitta differensial tenglama

$$\frac{\partial W}{\partial z} = 0$$

ko‘rinishda yozib olish mumkin, bu yerda

$$W(z) = u_1(x_1, x_2) + iu_2(x_1, x_2).$$

Mustaqil bajarish uchun mashqlar

1. Quyidagi tenglama xususiy hosilsli differensial tenglama bo’lishi yoki bo’lmasligini aniqlang:

1) $u_y(u_y - 4u_{xx}) - (u_y - 2u_{xx})^2 = 0.$

2) $\sin^2(u_{xx} + u_{xy}) + \cos^2(u_{xx} + u_{xy}) - u = 1.$

3) $\frac{\partial}{\partial x}\operatorname{tg} u - u_x \sec^2 u - 3u + 2 = 0.$

4) $\sin 2(u_{yy} + u_x) - 2\sin u_{yy}\cos u_x + u_{xy} + u = 0$

5) $u_{xx}^2 + u_{yy}^2 - \left(u_{xx} + u_{yy}\right)^2 = 0.$

6) $\cos(u_x + u_y) - \cos u_x \cos u_y + \sin u_x \sin u_y = 0.$

7) $2^{3u_x + 5u_y} 36^{u_y} = 3^{3u_x + 7u_y} 2^{2u_y}.$

8) $\frac{\partial}{\partial y}\operatorname{ctg} u_x - u_{xy}\operatorname{cosec}^2 u_x - 4u + 11 = 0$

9) $\log u_x u_y - \log u_x - \log u_y + 5u_x - 9u = 0$

10) $\frac{\partial}{\partial x} tg u - u_x \sec^2 u - 3u_y + 7u = 9$

11) $\sin(u_{xy} + u_x) - \sin u_{xy} \cos u_x - \cos u_{xy} \sin u_x + 2u = 0.$

12) $\sin^2(u_{xx} + u_{xy}) + \cos^2(u_{xx} + u_{xy}) - u_{xx} + u_{yy} + u_x = 1$

13) $\cos(u_{xx} + u_{yy}) - \cos u_{xx} \cos u_{yy} + \sin u_{xx} \sin u_{yy} = 0$

14) $u_{xx}^2 + u_{yy}^2 - (u_{xx} - u_{yy})^2 = 0.$

15) $\sin(u_{xy} + u_x) - \sin u_{xy} \cos u_x - \cos u_{xy} \sin u_x + 2u = 0.$

16) $\log|u_x u_y| - \log|u_x| - \log|u_y| + 5u - 6 = 0.$

17) $\log|u_{xx} u_{yy}| - \log|u_{xx}| - \log|u_{yy}| + u_x + u_y = 0.$

18) $u_x u_{xy}^2 + (u_{xx}^2 - 2u_{xy}^2 + u_y)^2 - 2xy = 0.$

19) $\cos^2 u_{xy} + \sin^2 u_{xy} - 2u_x^2 - 3u_y + u = 0.$

20) $2(u_x - 2u) u_{xy} - \frac{\partial}{\partial y}(u_x - 2u)^2 - xy = 0.$

21) $\frac{\partial}{\partial x}(u_{yy}^2 - u_y) - 2u_{yy} \frac{\partial}{\partial y}(u_{xy} - u_x) - 2u_x + 2 = 0.$

22) $2u_{xx} u_{xxy} - \frac{\partial}{\partial y}(u_{xx} - u_y)^2 - 2u_y u_{xxy} + u_x = 0.$

23) $u_x u_{xy}^2 + 2xu u_{yy} - 3xy u_y - u = 0.$

24) $u_y u_{xx} - 3x^2 u u_{xy} + 2u_x - f(x, y) u = 0.$

25) $2 \sin(x + y) u_{xx} - x \cos y \, u_{xy} + xy \, u_x - 3u + 1 = 0.$

26) $x^2 y \, u_{xxu} + 2e^x y^2 u_{xu} - (x^2 y^2 + 1) u_{xx} - 2u = 0.$

27) $3u_{xy} - 6u_{xx} + 7u_y - u_x + 8x = 0.$

28) $u_{xy} u_{xx} - 3u_{yy} - 6x u_y + xy \, u = 0.$

29) $a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + h(x, y) = 0.$

30) $a(x, y, u_x, u_{xy})u_{xyy} + b(x, y, u_{yy})u_{yyy} + 2uu_{xy}^2 - f(x, y) = 0.$

31) $u_{xy} + u_y + u^2 - xy = 0.$

32) $u_{xy} + 2\frac{\partial}{\partial x}(u_x^2 + u) - 6x\sin y = 0.$

33) $2xu_{xy} - 6\frac{\partial}{\partial x}(u^2 - xy) + u_{yy} = 0.$

34) $\frac{\partial}{\partial y}(yu_y + u_x^2) - 2u_x u_{xy} + u_x - 6u = 0.$

35) $\sin(u_{xy} - u_y) - \sin u_{xy}\cos u_y + \sin u_y \cos u_{xy} = 0$.

36) $\cos(u_{xy} - u_y) - \cos u_{xy}\cos u_y + \sin u_y \cos u_x = 0$.

37) $\lg|u_{xx}u_{yy}| - \lg|u_{xx}| - \lg|u_{yy}| + 6u_y^3 + u_x = 0$.

38) $\left(u_{xx} - u_{yy}\right)\left(u_{xx} + u_{yy}\right) - \frac{\partial}{\partial x}(u_x - u_y) - \frac{\partial}{\partial y}(u_x - u_y) + 6u_x = 0.$

39) $2\left(u_x - 2u\right)u_{xy} - \frac{\partial}{\partial y}\left(u_x - 2u\right)^2 - xy = 0.$

40) $\frac{\partial}{\partial x}\left(u_{yy}^2 - u_y\right) - 2u_{yy}\frac{\partial}{\partial y}(u_{xx} - u_x) - 2u_x + 2 = 0$.

41) $2u_{xx}u_{xxy} - \frac{\partial}{\partial y}(u_{xx} - u_y)^2 - 2u_y u_{xxy} + u_x = 0$.

42) $2u_{xx}u_{xxy} - \frac{\partial}{\partial y}(u_{xx} - u_y)^2 - 2u_y u_{xxy} + u_x = 0$.

43) $u_{xxy} - \frac{\partial}{\partial y}(u_{xx} - u_y) - 2u_y u_{xx} + u_x = 0$.

44) $\frac{\partial}{\partial x}\left(u_{yy}^2 - u_y\right) - 2u_{yy}\frac{\partial}{\partial x}(u_{yy} - u_x) - 5u_x = 0$.

45) $\cos 2u_{xx} - \cos^2 u_{xx} + \sin^2 u_{xx} - \sin u_{xx} + 8u_x + u = 0$.

46) $2u_{xy} - 6\frac{\partial}{\partial x}(u^2 - xy) + u_{yy} = 0$.

47) $\left(tgu_{xx} + ctgu_{yy}\right)^2 - tg^2 u_{xx} - ctg^2 u_{yy} + 6u_x = 0$.

48) $\frac{\partial}{\partial y}\left(yu_y + u_x^2\right) - 2u_x u_{xy} + u_x - 6u = 0$.

49) $\frac{\partial}{\partial x}\left(u_{yy}^2 - u_y\right) - 2u_{yy}\frac{\partial}{\partial y}(u_{xx} - u_x) - 2u_x + 2 = 0$.

50) $2u_{xy} - 6\frac{\partial}{\partial x}(u^2 - xy) + u_{yy} = 0$.

51) $\cos(u_x + u_y) - \cos u_x \cos u_y + \sin u_x \sin u_y = 0.$

52) $u_{xx}^2 + u_{yy}^2 - (u_{xx} - u_{yy})^2 = 0.$

53) $\sin^2(u_{xx} + u_{xy}) + \cos^2(u_{xx} + u_{xy}) - u = 1.$

2. Quyidagi tenglamalarning qaysi biri chiziqli(bir jinsli yoki bir jinsli bo`lmagan), kvazichiziqli yoki chiziqli bo`lmagan tenglamalar ekanligini aniqlang.

2) $2\sin(x+y)U_{xx} - x\cos(yU_{xy}) + xyU_x - 3U + 1 = 0$

3) $\frac{\partial}{\partial y}\left(yU_y + (U_x)^2\right) - 2U_xU_{xy} + U_x - 6U = 0$

3. $u_x u_{xy}^2 + 2xuu_{yy} - 3xyu_y - u = 0.$

4. $u_y u_{xx} - 3x^2 uu_{xy} + 2u_x - f(x, y)u = 0.$

5. $2\sin(x+y)u_{xx} - x\cos yu_{xy} + xyu_x - 3u + 1 = 0.$

6. $x^2yu_{xxy} + 2e^xy^2u_{xy} - (x^2y^2+1)u_{xx} - 2u = 0.$

7. $3u_{xy} - 6u_{xx} + 7u_y - u_x + 8x = 0.$

8. $u_{xy}u_{xx} - 3u_{yy} - 6xu_y + xyu = 0.$

19. $a(x,\ y)u_{xx}+b(x,\ y)u_{xy}+c(x,\ y)u_{yy}+d(x,\ y)u_x+$
$+e(x,\ y)u_y+h(x,\ y)=0.$

20. $a(x, y, u_x, u_{xy})\,u_{xyy}+b(x, y, u_{yy})\,u_{yyy}+2uu_{xy}^2-f(x,y)=0.$

21. $u_{xy}+u_y+u^2-xy=0.$

22. $u_{xy}+2\frac{\partial}{\partial x}(u_x^2+u)-6x\sin y=0.$

23. $2xu_{xy}-6\frac{\partial}{\partial x}(u^2-xy)+u_{yy}=0.$

24. $\frac{\partial}{\partial y}(yu_y+u_x^2)-2u_xu_{xy}+u_x-6u=0.$

Тенгламаларнинг тартибини аниқланг.

9. $u_xu_{xy}^2+(u_{xx}^2-2u_{xy}^2+u_y^2)-2u=0$

10. $\cos^2 u_{xy}-2u_xu_{xx}+3u_y+\sin^2 u_{xy}+u_x=3$

11. $\frac{\partial}{\partial y}(u_x-2u)^2+2(u_x-2u)u_{xy}-xy^2=0$

12. $\log u_{xx}-\log u_{xx}u_{yy}+log u_{yy}+2u_x=0$

13. $\frac{\partial}{\partial x}(u_{yy}^2-u_y)-2u_{yy}\frac{\partial}{\partial y}(u_{xy}-u_x)+2u_x+7=0$

14. $2u_{xx}u_{xxy}-\frac{\partial}{\partial y}(u_{xx}-u_y)^2-2u_yu_{xxy}+u_x=0.$

Tayanch iboralar.

Differentsial tenglama, oddiy differentsial tenglama, xususiy hosilali differentsial tenglam, tenglama tartibi, yechim, chiziqli, chiziqli bir jinsli, chiziqli bir jinsli bo’lmagan, kvazichiziqli, nochiziqli differentsial tenglama, matematik fizika tenglamalari, matematik fizikaning asosiy tenglamalari.

Nazorat uchun savollar.

1. Differentsial tenglamaga ta’rif bering.
2. Qachon differentsial tenglama oddiy va xususiy hosilali differentsial tenglama deyiladi.
3. Differentsial tenglama tartibi va yechilishiga ta’rif bering.
4. Qachon differentsial tenglama chiziqli (bir jinsli yoki bir jinsli bo’lmagan), kvazichiziqli, nochiziqli (chiziqli bo’lmagan) tenglama deyiladi.
5. Qaysi tenglamalar matematik fizika tenglamalari deyiladi.
6. Matematik fizikaning asosiy tenglamalarini yozing.

2-Mavzu: Ikkinchi tartibli xususiy hosilali differensial tenglamalarning klassifikasiyasi va kanonik ko‘rinishi.

Ikkinchi tartibli ikkinchi o‘zgaruvchili xususiy hosilali differensial tenglama umumiy holda

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

ko‘rinishda yoziladi, bu yerda $u = u(x, y)$ -noma’lum funksiya va xususiy hosilalar uchun quyidagicha belgilashlar kiritilgan:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

Ushbu

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x, y, u, u_x, u_y) = 0 \qquad (2.1)$$

ko‘rinishdagi tenglama ikkinchi tartibli hosilalarga nisbatan *chiziqli xususiy hosilali differensial tenglama* deyiladi, bu yerda a_{11}, a_{12}, a_{22} - x va y o‘zgaruvchilarga bog‘liq funksiyalar.

Agar (2.1) tenglama

$$a_{11}(x, y, u, u_x, u_y)u_{xx} + 2a_{12}(x, y, u, u_x, u_y)u_{xy} + a_{22}(x, y, u, u_x, u_y)u_{yy} + F_1(x, y, u, u_x, u_y) = 0 \qquad (2.2)$$

ko‘rinishida bo‘lsa, bunday tenglama *kvazichiziqli* tenglama deyiladi. Agar (2.1) tenglama

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f = 0 \qquad (2.3)$$

ko‘rinishida bo‘lsa, bunday tenglama *chiziqli tenglama* deyiladi, bu yerda $a_{11}, a_{12}, a_{22}, b_1, b_2, c, f$ - faqat x va y o‘zgaruvchilarning funksiyalari.

Agar (2.3) tenglamaning koeffitsiyentlari x va y o‘zgaruvchilarga bog‘liq bo‘lmasa, u holda (2.3) tenglama *o‘zgarmas koeffitsiyentli chiziqli tenglama* deyiladi. Agar $f(x) = 0$ bo‘lsa, (2.3) tenglama*chiziqli birjinslitenglama*deyiladi. Agar $f(x) \neq 0$ bo‘lsa, (2.3) tenglama*chiziqli birjinslibo‘lmagantenglama*deyiladi.

Endi (2.1) tenglamadan ham soddaroq tenglamani olish maqsadida (2.1) tenglamada

$$\begin{cases}\xi = \varphi(x,y)\\ \eta = \psi(x,y)\end{cases} \qquad (2.4)$$

bir qiymatli almashtirish bajaramiz. (2.4) almashtirish bir qiymatli bo‘lishi uchun

$$\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix} \neq 0 \qquad (2.4')$$

shartning bajarilishi zarur. (2.1) tenglamada (2.4) almashtirishni bajarsak, (2.1) tenglamaga ekvivalent bo‘lgan yangi tenglama hosil bo‘ladi. Biz buni quyida ko‘ramiz. Lekin quyidagicha savol tugiladi: ξ,η ni qanday tanlasak, yangi tenglama (2.1) tenglamaga qaraganda sodda ko‘rinishga keladi. Keyingi mulohazalarda bu savolga javob beramiz. Yangi ξ,η o‘zgaruvchilarda hosilalar quyidagicha bo‘ladi:

$$\begin{cases}u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x\\ u_y = u_\xi \cdot \xi_y + u_\eta \cdot \eta_y\\ u_{xx} = (u_{\xi\xi}\cdot \xi_x + u_{\xi\eta}\eta_x)\cdot \xi_x + \xi_{xx}u_\xi + (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta \eta_{xx} =\\ \qquad = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\eta \xi_{xx} + u_\eta \eta_{xx},\\ u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\eta\xi_{xy} + u_\eta\eta_{xy},\\ u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy},\end{cases}.$$

Bularni (2.1) tenglamaga ko‘yamiz:

$$a_{11}(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} + u_\xi\xi_{xx} + u_\eta\eta_{xx}) + 2a_{12}(u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y +$$
$$+ \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}) + a_{22}(u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}) +$$
$$+ F(\xi,\eta,u,u_\xi,u_\eta) = 0.$$

Natijada

$$\bar{a}_{11}u_{\xi\xi} + 2\bar{a}_{12}u_{\xi\eta} + \bar{a}_{22}u_{\eta\eta} + \bar{F}(\xi,\eta,u,u_\xi,u_\eta) = 0 \qquad (2.5)$$

tenglamaga kelamiz. Bu yerda

$$\begin{cases}\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2\\ \bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y\\ \bar{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2\\ \bar{F} = F + \bar{a}_{11}(u_\xi\xi_{xx} + u_\eta\eta_{xx}) + 2a_{12}(u_\xi\xi_{xy} + u_\eta\eta_{xy}) + a_{22}(u_\xi\xi_{yy} + u_\eta\eta_{yy})\end{cases} \qquad (2.6)$$

(2.6) formula uchun quyidagi tenglik o‘rinli

$$\bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22} = (a_{12}^2 - a_{11}a_{22})(\xi_x\eta_y - \xi_x\eta_y)^2 . \qquad (2.7)$$

Bu yerdan oldingi (4’) farazga ko‘ra (2.7) chap tomonining ishorasi $\bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22}$ ifodaning ishorasi bilan bir xil ekan.

Quyidagi birinchi tartibli xususiy hosilali differensial tenglamani ko‘ramiz:

$$a_{11}z_x^2 + 2a_{12}z_x z_y + a_{22}z_y^2 = 0 \qquad (2.8)$$

Shu bilan birga quyidagi birinchi tartibli oddiy differensial tenglamani ko‘ramiz:

$$a_{11}(dy)^2 - 2a_{12}dxdy + a_{22}(dx)^2 = 0 \qquad (2.9)$$

Buni yana,agar $dx \neq 0$ bo‘lganda

$$a_{11}(\frac{dy}{dx})^2 - 2a_{12}\frac{dy}{dx} + a_{22} = 0 \qquad (2.9')$$

ko‘rinishda ham yozish mumkin.

(2.8) va (2.9) tenglamalar orasida quyidagicha ekvivalent munosabat o‘rinli.

Lemma 1. *Faraz qilaylik,* $z = \varphi(x, y)$ *funksiya (2.8) tenglamaning xususiy yechimi bo‘lsin, u holda* $\varphi(x, y) = C$, $C \equiv const$, *munosabat (2.9) tenglamaning umumiy yechimi bo‘ladi.*

Uning teskarisi ham o‘rinli.

Lemma 2. *Faraz qilaylik,* $\varphi(x, y) = C$ *munosabat (2.9) tenglamaning umumiy yechimi bo‘lsin, u holda* $z = \varphi(x, y)$ *funksiya (2.8) tenglamaning xususiy yechimi bo‘ladi.*

Lemma 1 ning isboti. $z = \varphi(x, y)$ funksiyani (2.8) tenglamaga qo‘yamiz. Ixtiyoriy x va y da

$$a_{11}(\frac{\varphi_x}{\varphi_y})^2 - 2a_{12}(-\frac{\varphi_x}{\varphi_y}) + a_{22} = 0 \qquad (*)$$

bo‘ladi.

Faraz qilaylik, $\varphi(x, y) = C$ qandaydir $y = f(x, C)$ oshkormas ko‘rinishda berilgan bo‘lsin, u holda bu funksiyani hosilalarini hisoblasak,

$$\varphi_x + \varphi_y y_x = 0 \Rightarrow y_x = -\left[\frac{\varphi_x}{\varphi_y}\right]_{y=f(x,C)}$$

Buni (2.9) ga qo‘yamiz, u holda (*) dan

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \left[a_{11}\left(\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22}\right]_{y=f(x,c)} = 0 .$$

Demak, (2.8) tenglamaning xususiy yechimi (2.9) tenglamaning umumiy yechimi bo‘lar ekan.

Lemma 2 ning isboti.

Faraz qilaylik, $\varphi(x,y) = C$ (2.9) tenglamaning umumiy yechimi bo‘lsin. Ixtiyoriy (x_0, y_0) nuqtadan o‘tuvchi bitta yechimini olamiz:

$$C_0 = \varphi(x_0, y_0)$$

$y = f(x, C_0)$ funksiya (2.9) tenglamani qanoatlantiradi, chunki $\varphi(x,y) = C$ funksiyalar oilasining bittasi. Bundan

$$0 = a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \left[\frac{dy}{dx} = -\left[\frac{\varphi_x}{\varphi_y}\right]_{y=f(x,C_0)}\right] =$$

$$= \left[a_{11}\left(\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(\frac{\varphi_x}{\varphi_y}\right) + a_{22}\right]_{y=f(x,C_0)} = 0$$

$x = x_0$ qo‘yamiz, u holda $y = y_0$ bo‘ladi. Shundan so‘ng,

$$a_{11}\varphi_x^2(x_0, y_0) + 2a_{12}\varphi_x(x_0, y_0)\varphi_{y(}(x_0, y_0) + a_{22}\varphi_y^2(x_0, y_0) = 0$$

kelib chiqadi. (x_0, y_0) nuqtaning ixtiyoriyligidan $z = \varphi(x,y)$ (2.8) tenglamaning xususiy yechimi ekan.

Xulosa. (2.8) tenglamaning yechimini topish uchun (2.9) tenglamaning umumiy
yechimini topish yetarli ekan.

(2.9) yoki

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = 0 \qquad (2.9')$$

tenglama (2.1) tenglamaning *xarakteristik tenglamasi* deyiladi. (2.9) tenglama dy/dx ga nisbatan kvadrat tenglama bo‘lganligi uchun quyidagilarni topamiz:

$$\frac{dy}{dx} = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \qquad (2.10)$$

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \qquad (2.11)$$

(2.10) va (2.11) tenglamalarning yechimlari *xarakteriskalar* deyiladi.

Endi $\Delta = a_{12}^2 - a_{11}a_{22}$ belgilash kiritamiz. Δ ning ishorasiga qarab (2.1) tenglama quyidagi tiplarga bo‘linadi:

1–hol. Agar M nuqtada $\Delta = a_{12}^2 - a_{11}a_{22} > 0$ bo‘lsa, u holda (2.1) tenglama M nuqtada *giperbolik tipdagi tenglama* deyiladi.

2–hol. Agar M nuqtada $\Delta = 0$ bo‘lsa, u holda (2.1) tenglama M nuqtada *parabolik tipdagi tenglama* deyiladi.

3–hol. Agar M nuqtada $\Delta < 0$ bo‘lsa, u holda (2.1) tenglama M nuqtada *elliptik tipdagi tenglama* deyiladi.

(2.7) formuladan ko‘rinadiki, (2.1) tenglamada almashtirish bajargandan keyin ham tenglama tipi uzgarmas ekan.

Endi har bir holni alohida qarab chiqamiz.

1–hol. $\Delta > 0$ bo‘lsin. U holda (2.10) va (2.11) tenglamalar 2 ta har xil xarakteristikalarga ega, ya’ni

$$\begin{cases} \varphi(x, y) = C_1 \\ \psi(x, y) = C_2 \end{cases}.$$

U holda $\begin{cases} \xi = \varphi(x, y) \\ \eta = \psi(x, y) \end{cases}$ (2.8) ning xususiy yechimi bo‘ladi. Bularni (2.6) ifodaga qo‘yamiz. (2.8) dan

$$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = 0$$

$$\bar{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2 = 0$$

kelib chiqadi. (2.7) dan

$$\overline{a}_{12}^2 - \overline{a}_{11}\overline{a}_{22} = (a_{12}^2 - a_{11}a_{22})\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}^2 \neq 0 \Rightarrow$$

$$\Rightarrow \overline{a}_{12} \neq 0 \qquad \overline{a}_{11} = 0 \qquad \overline{a}_{22} = 0.$$

U holda (2.5) da

$$\overline{a}_{11}u_{\xi\xi} + 2\,\overline{a}_{12}u_{\xi\eta} + \overline{a}_{22}\,u_{\eta\eta} + \overline{F} = 0$$

tenglama

$$2\overline{a}_{12}u_{\xi\eta} + \overline{F} = 0 \quad => \quad u_{\xi\eta} = -\frac{\overline{F}}{2\,\overline{a}_{12}} \quad => \quad u_{\xi\eta} = \Phi_1$$

ko‘rinishga keladi, bu yerda

$$\Phi_1 = -\frac{\overline{F}}{2\overline{a}_{12}}.$$

Shunday qilib, giperbolik tipdagi tenglama $u_{\xi\eta} = \Phi_1$ ko‘rinishda bo‘lar ekan. Giperbolik tipdagi tenglamaning yana boshqacha kanonik ko‘rinishi mavjud bo‘lib, uni topish uchun quyidagicha almashtirish bajaramiz:

$$((\xi,\eta) \to (\alpha,\beta)) \quad \begin{cases} \xi = \alpha + \beta & \alpha = \dfrac{\xi + \eta}{2} \\ \eta = \alpha - \beta & \beta = \dfrac{\xi - \eta}{2} \end{cases}$$

$$u_\xi = \frac{1}{2}(u_\alpha + u_\beta)\,, \quad u_\eta = \frac{1}{2}(u_\alpha - u_\beta)\,, \; u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} - u_{\beta\beta})\,, \; u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} - u_{\beta\beta}) = \Phi_2$$

$$u_{\alpha\alpha} - u_{\beta\beta} = \Phi_2, \quad \Phi_2 = 4\Phi_1.$$

Bu giperbolik tipdagi tenglamaning ikkinchi kanonik ko‘rinishidir.

2–hol. Agar M nuqtada $\Delta = a_{12}^2 - a_{11}a_{22} = 0$ (parabolik tip) bo‘lsa, (2.9) tenglama ikkita ustma – ust tushadigan $\varphi(x,y) = C$ xarakteristikaga ega bo‘lamiz.

$\xi = \varphi(x,y)$ funksiya (2.8) ning yechimi bo‘ladi.

$\eta = \eta(x,y)$ funksiyani shunday tanlaymizki, $\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$ shart bajarilsin.

$\xi = \varphi(x,y)$ funksiya (2.8) ning yechimi bo‘lganligi uchun $\overline{a}_{11} = 0$ bo‘ladi.

$$\overline{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = a_{11}\xi_x^2 + 2\sqrt{a_{11}}\sqrt{a_{22}}\xi_x\xi_y + a_{22}\xi_y^2 =$$

$$= (\sqrt{a_{11}}\xi_x + \sqrt{a_{22}}\xi_y)^2 = 0$$

$$\bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y = a_{11}\xi_x\eta_x + \sqrt{a_{11}}\sqrt{a_{22}}\xi_x\eta_y +$$

$$+\sqrt{a_{11}}\sqrt{a_{22}}\xi_y\eta_x + a_{22}\xi_y\eta_y = \sqrt{a_{11}}\xi_x(\sqrt{a_{11}}\eta_x + \sqrt{a_{12}}\eta_y) +$$

$$+\sqrt{a_{12}}\xi_y(\sqrt{a_{11}}\eta_x + \sqrt{a_{22}}\eta_y) = (\sqrt{a_{11}}\eta_x + \sqrt{a_{12}}\eta_y)(\sqrt{a_{11}}\eta_x + \sqrt{a_{12}}\xi_y) = 0 .$$

Demak, agarMnuqtada $\Delta = 0$ bo‘lsa, u holda

$$\bar{a}_{11} = 0, \ \bar{a}_{12} = 0, \ \bar{a}_{22} \neq 0$$

bo‘lar ekan. Bulardan (2.5) tenglama

$$\bar{a}_{22}u_{\eta\eta} + \bar{F} = 0$$

ko‘rinishga keladi. Xulosashuki,

$$u_{\eta\eta} = \Phi_2$$

parabolik tenglamaning kanonik ko‘rinishidir, bu yerda $\Phi_2 = -\dfrac{\bar{F}}{a_{22}}$.

3–hol. Agar M nuqtada $\Delta = a_{12}^2 - a_{11}a_{22} < 0$ (elliptik tip) bo‘lsa, (2.9) tenglama ikkita har xil xarakteristikalarga ega, lekin bu xarakteristikalar uzaro qo‘shma, ya’ni

$$\begin{cases} \varphi(x, y) = C_1 \\ \varphi^*(x, y) = C_2 \end{cases}.$$

$\xi = \varphi(x, y), \quad \eta = \varphi^*(x, y)$ deb olsak, xuddi giperbolik tipga o‘xshab $\bar{a}_{11} = 0 \ \bar{a}_{22} = 0$ bo‘ladi, ya’ni

$$u_{\xi\eta} + \Phi = 0$$

kanonik ko‘rinishga keladi.

Lekin bu yerda kompleks funksiyalar ham ishtirok etiyapti. Biroq biz faqat haqiqiy funksiyalar bilan ish ko‘ramiz. Shuning uchun kompleks funksiyalardan qutilish uchun quyidagicha almashtirish bajaramiz:

$$\alpha = \frac{\varphi + \varphi^*}{2} = \frac{\xi - \eta}{2} \quad \text{(haqiqiy qism)},$$

$$\beta = \frac{\varphi - \varphi^*}{2i} = \frac{\xi - \eta}{2i} \quad \text{(mavhum qism)}.$$

Yuqoridagi belgilashlardan

$$\xi = \alpha + i\beta , \ \eta = \alpha - i\beta \ ; \ \xi_x = \alpha_x + i\beta_x , \ \xi_y = \alpha_y - i\beta_y \Rightarrow$$

$$0=\bar{a}_{11}=a_{11}\xi_x^2+2a_{12}\xi_x\xi_y+a_{22}\xi_y^2=a_{11}(\alpha_x+i\,\beta_x)^2+2a_{12}(\alpha_x+i\,\beta_x)(\alpha_y+i\,\beta_y)+$$

$$+a_{22}(\alpha_y+i\,\beta_y)^2=a_{11}(\alpha_x^2+2i\,\alpha_x\beta_x+\beta_x^2)+2a_{12}(\alpha_x\alpha_y+i\,(\alpha_x\beta_y+\alpha_y\beta_x)-$$

$$-\beta_x\beta_y)+a_{22}(\alpha_y^2+2i\,\alpha_y\beta_y+\beta_y^2)$$

Bundan haqiqiy va mavhum qismini ajratib olamiz.

$$a_{11}(\alpha_x^2-\beta_x^2)+2a_{12}(\alpha_x\alpha_y+\beta_x\beta_y)+a_{22}(\alpha_y^2-\beta_y^2)=0\rightarrow\bar{\bar{a}}_{11}-\bar{\bar{a}}_{22}=0\,,$$

$$\bar{\bar{a}}_{12}=a_{11}\alpha_x\beta_x+a_{12}\left(\alpha_x\beta_y+\alpha_y\beta_x\right)+a_{22}\alpha_y\beta_y=0,\;\;\bar{\bar{a}}_{12}=0,\;\;\bar{\bar{a}}_{11}=\bar{\bar{a}}_{22}\,.$$

Bundan elliptik tipdagi tenglamaning kanonik ko‘rinishi:

$$u_{\alpha\alpha}+u_{\beta\beta}=\Phi_3\,.$$

Shunday qilib,

- giperbolik tenglamaning kanonik ko‘rinishi:

$$\begin{matrix} u_{xx}-u_{yy}=\Phi_1 \\ u_{xy}=\Phi_2 \end{matrix}\;;$$

- parabolik tenglamaning kanonik ko‘rinishi:

$$u_{\underset{(yy)}{xx}}=\Phi_4\,;$$

-elliptik tenglamaning kanonik ko‘rinishi:

$$u_{xx}+u_{yy}=\Phi_3\,;$$

Asosiy adabiyotlar

1.Wolter A. Stranss. Partial differential equation; An introduction. Birkhhauzer. Germaniy, 2005

2.Davia D. Bleecker, George Csordes. Basic of partial Differential Equations. Birkhhauzer. Germaniy, 2009.

3.СалохиддиновМ.С. Математикфизикатенгламалари. Т.,«Ўзбекистон», 2002, 448 бет.

4. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М. 2004.

5. Бицадзе А.В., Калиниченко Д.Ф. Сборник задач по уравнениям математической физики. М. 1977.

Mustaqil ta'lim mavzulari

1.Xususiy hosilali differensial tenglamalar va ularning yechimlari to'g'ri-sida tushunchalar.

2. Koshi- Gursaning birinchi masalasi.

Glossariy

Ikkinchi tartibli ikkinchi o'zgaruvchili xususiy xosilali differensial tenglama - umumiy xolda

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

ko'rinishda yoziladi, bu yerda $u = u(x, y)$ -noma'lum funksiya va xususiy xosilalar uchun quyidagicha belgilashlar kiritilgan:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

Ikkinchi tartibli xosilalarga nisbatan chiziqli xususiy xosilali differensial tenglama - ushbu

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x, y, u, u_x, u_y) = 0 \qquad (1)$$

ko'rinishdagi tenglama ikkinchi tartibli xosilalarga nisbatan chiziqli xususiy xosilali differensial tenglama deyiladi, bu yerda a_{11}, a_{12}, a_{22} -x va y uzgaruvchilarga boglik funksiyalar.

Kvazichiziqli tenglama - agar (1) tenglama

$$a_{11}(x, y, u, u_x, u_y)u_{xx} + 2a_{12}(x, y, u, u_x, u_y)u_{xy} + a_{22}(x, y, u, u_x, u_y)u_{yy} + F_1(x, y, u, u_x, u_y) = 0 \qquad (2)$$

ko'rinishida bo'lsa, bunday tenglama *kvazichiziqli* tenglama deyiladi. Agar (1) tenglama

Chiziqli tenglama - agar (1) tenglama

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f = 0 \quad (3)$$

ko'rinishida bo'lsa, bunday tenglama chiziqli tenglama deyiladi, bu yerda $a_{11}, a_{12}, a_{22}, b_1, b_2, c, f$ - fakat x va y uzgaruvchilarning funksiyalari.

O'zgarmas koeffitsientli chiziqli tenglama - agar (3) tenglamaning koeffitsientlari x va y uzgaruvchilarga boglik bulmasa, u xolda (3) tenglama uzgarmas koeffitsientli chiziqli tenglama deyiladi.

Keyslar banki

Keys: Masala o`rtaga tashlanadi: Kanonik ko'rinishga keltiring

$36u_{xx} + 12u_{xy} + u_{yy} + 18u_x + 3u_y = 0.$

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni

  keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Nazorat uchun savollar

1. Xususiyhosilalidifferensiyaltenglamagata'rifbering, ularningyechimidebqandayfunksiyasigaaytamiz.
2. Kvazi chiziqli tenglama deb qanday tenglamaga aytiladi.
3. Ikkinchi tartibli chiziqli xususiy hosilali differensial tenglamani klassifikatsiyalang.
4. $\det(A - \lambda E) = 0$ xarakteristik sonlarining ishorasi nimani aniqlaydi.

Mustaqil bajarish uchun mashqlar

1. Quyidagi tenglamalarni tipini aniqlang.

$$3U_{xx} - U_{yy} + 4U_x - U_y + 1 = 0;$$

$5U_{xx} + 16U_{xy} + 16U_{yy} + 64U = 0$;

$U_{xx} - 6U_{xy} + 9U_{yy} + 2U_y = 0$;

$y^{2m+1}U_{xx} + U_{yy} - U = 0$, m – butun manfiy boʻlmagan son;

$xU_{xx} + yU_{yy} - U = 0$;

$xyU_{xx} + U_{yy} = 0$.

$yu_{xx} + u_{yy} = 0$ $u_{xx} + xyu_{yy} = 0$.

$yu_{xx} + xu_{yy} = 0$.

$xu_{xx} + yu_{yy} + 2u_x + 2u_y = 0$.

$x^2u_{xx} + y^2u_{yy} + 2u_x + 2u_y = 0$.

$y^2u_{xx} + 2yu_{xy} + u_{yy} = 0$.

$u_{xx} - 2xu_{xy} = 0$.

$xu_{xx} + 2xu_{xy} + (x-1)u_{yy} = 0$.

$(3x+4y)u_{xx} + 2(4x+y)u_{xy} - 4(3y-x)u_{yy} +$

$+(x+y)u_x + yu_y + 2x + 5y = 0$.

$xU_{xx} - U_{yy} - U = 0$. $(1+x^2)u_{xx} + (1+y^2)u_{yy} + yu_y = 0$.

$U_{xx}+4U_{xy}+U_{yy}+U_x+U_y+2U-x^2y=0;$

$U_{xx}+2U_{xy}+U_{yy}+U_x+U_y+3U-xy^2=0;$

$2U_{xx}+2U_{xy}+U_{yy}+2U_x+2U_y-U=0;$

Quyidagi tenglamalarning turini aniqlang:

1. $(y+1)\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}=0,\ 1<x<3,\qquad 0<y<1.$

2. $y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x^2}+2(x+y)\frac{\partial^2 u}{\partial x\partial y}=0,\ x^2+(y-6)^2<1.$

3. $2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}+y^2\frac{\partial^2 u}{\partial x^2}-x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial x}=0,\ |x|<1,\qquad |y|<1.$

4. $(x+y)\frac{\partial^2 u}{\partial x^2}+(x-y)\frac{\partial^2 u}{\partial y^2}+xu=0,\ (x+5)^2+y^2<1.$

5. $(y+1)\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}=0,\ 1<x<3,\qquad 0<y<1.$

6. $4\frac{\partial^2 u}{\partial x^2}-2(x-y)\frac{\partial^2 u}{\partial x\partial y}+(1-xy)\frac{\partial^2 u}{\partial y^2}=0,\ 2<x+y<5.$

7. $x^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0,\ 1<x^2+y^2<7.$

8. $x\frac{\partial^2 u}{\partial x^2}+6\frac{\partial u}{\partial x}+(x+y)\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}=0,\ 0<x<2,\qquad 0<y<2.$

9. $6\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=0,\ 1<x<2,\qquad 2<y<3.$

10. $2x\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x^2}-(x^2-2)\frac{\partial^2 u}{\partial y^2}-2y\frac{\partial^2 u}{\partial x\partial y}=0,\ x^2+y^2<1.$

11. $5x\frac{\partial^2 u}{\partial x^2}-4x\frac{\partial^2 u}{\partial x\partial y}+2y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-u=0,\ 1<x<3,\qquad 4<y<8.$

Quyida berilgan xususiy hosilali differensial tenglamalarni kanonik shaklga keltiring:

3.1. $4u_{xx}+2u_{xy}-12u_{yy}+u_x+u_y+5x-3y=0$.

3.2. $2u_{xx}+6u_{xy}+5u_{yy}+u_x+u_y+xy=0$.

3.3. $u_{xx}+4u_{xy}+4u_{yy}+3u_x+2u_y+2x-3y=0$.

3.4. $16u_{xx}+8u_{xy}-48u_{yy}+4u_x+4u_y+2y-20=0$.

3.5. $u_{xx}+2u_{xy}+5u_{yy}+2u_x-3u_y+x+2y=0$.

3.6. $9u_{xx}+6u_{xy}+u_{yy}+4u_x-3y=0$.

3.7. $28u_{xx}+14u_{xy}-81u_{yy}+4u_x+u+x+y=0$.

3.8. $u_{xx}+2u_{xy}+10u_{yy}+5u_x+4x+3y=0$.

3.9. $9u_{xx}+12u_{xy}+4u_{yy}+4u_y-3u_x+2xy=0$.

3.10. $3u_{xx}+8u_{xy}+4u_{yy}+u_x+2y+4=0$.

3.11. $4u_{xx}-12u_{xy}+13u_{yy}+2u_x+5u_y+3x-y=0$.

3.12. $4u_{xx} - 20u_{xy} + 25u_{yy} + 3u_x + 2x - 5y = 0$.

3.13. $15u_{xx} + 14u_{xy} - 32u_{yy} + 4u_x + 3u_y + 2x - 17 = 0$.

3.14. $u_{xx} + 4u_{xy} + 13u_{yy} + 3u_x - 2u_y + 3x = 0$.

3.15. $4u_{xx} - 12u_{xy} + 9u_{yy} + 5u_x + 5x + 4y = 0$.

3.16. $27u_{xx} + 20u_{xy} - 68u_{yy} + 3u_y + 3x - 2y = 0$.

3.17. $u_{xx} - 10u_{xy} + 26u_{yy} + 3u_x - 2u_y + 3y = 0$.

3.18. $4u_{xx} + 4u_{xy} + u_{yy} + 5u_y + 7x + y = 0$.

3.19. $39u_{xx} + 26u_{xy} - 104u_{yy} + 2u_x - 3u_y - 2y = 0$.

3.20. $4u_{xx} - 4u_{xy} + 2u_{yy} - 2u_y + 3xy = 0$.

3.21. $9u_{xx} - 30u_{xy} + 25u_{yy} + 5u_x - u_y + x - 2y = 0$.

3.22. $14u_{xx} + 20u_{xy} - 16u_{yy} + 4u_x - 5u_y + x = 0$.

3.23. $4u_{xx} - 20u_{xy} + 29u_{yy} - u_x - 4u_y + 3x = 0$.

3.24. $u_{xx} + 10u_{xy} + 25u_{yy} + 5u_x - 3u_y + 2x - 2y = 0$.

3.25. $26u_{xx} + 26u_{xy} - 52u_{yy} + 4u_x - 5u_y + x + y = 0$.

TEST

Matematik fizika tenglamalari uchun nechta tipdagi masalalar qo`yiladi?	*Uchta tipdagi masalalar;	Ikkita tipdagi masalalar;	Bittata tipdagi masalalar;	Juda ko`p tipdagi masalalar.
Matematik fizika tenglamalari uchun qanday tipdagi masalalar qo`yiladi?	*Koshi, chegaraviy va aralash masalalarni	Faqat Koshi va aralash masalalarni	Faqat Koshi masalalarni	Faqat chegaraviy masalalarni
Koshi masalasi qanday tipdagitenglamalar uchun qo`yiladi?	*Giperbolik va Parabolik.	Giperbolik va Elliptik;	Giperbolik, Para-bolik va Elliptik	Parabolik va Elliptik
Aralash masala qanday tipdagitenglamalar uchun qo`yiladi?	*Giperbolik va Parabolik	Giperbolik va Elliptik;	Parabolik va Elliptik;	Giperbolik, Elliptik va Parabolik
Aralash masala qanday tipdagitenglamalar uchun qo`yilmaydi?	* Elliptik	Giperbolik	Parabolik	Giperbolik va Parabolik
Qaysi masalalarda chegaraviy va boshlang'ich shartlar qatnashadi?	*Faqat aralash masalalarda;	Koshi, chegaraviy va aralash masala-larda;	Faqat Koshi masalasida;	Faqat chegaraviy masalalarda
Qanday masalalar chegaraviy masalalar tipiga kiradi?	*Dipixle, Neyman va Puankare	Gursa, Koshi va Puankare	Neyman va aralash;	Koshi va Neyman.
Chegaraviy masala qanday tipdagitenglamalar uchun qo`yiladi?	*Elliptik;	Giperbolik va Elliptik;	Giperbolik;	Giperbolik va Parabolik.

3-Mavzu: Matematik fizikaning asosiy tenglamalarini keltirib chiqarish: tor tebranish tenglamasi. Issiqlik tarqalish tenglamasi; statsionar tenglamalar; moddiy nuqtaning og‘irlik kuchi ta'siridagi harakati.

Reja

1.To‘lqin tenglamalari.

2. Issiqlik tarqalish tenglamalari.

3. Statsionar maydon tenglamalari.

4. Ikki o’zgaruvchili ikkinchi tartibli xususiy differensial tenglamalar

5.Bir qiymatli almashtirishlar.

6.Asosiy tasdiqlar.

Tayanch so‘z va iboralar:*torning erkin tebranish tenglamalari, torning majburiy tebranish tenglamalari, issiqlik tarqalish tenglamalari, statsionar maydon tenglamalari. Matematik fizikaning asosiy tenglamalari va ularni keltirib chiqarish, ikkinchi tartibli xususiy hosilali kvazichiziqli differentsial tenglamalarning sinflari, o’zgarmas koeffitsientli chiziqli tenglama.*

Fizik jarayonlarni o‘rganishda matematik modellashtirish samarali natijalar beradi. *Matematik model* tashqi dunyoda uchraydigan biror bir hodisalar sinfining matematik simvollar yordamida mohiyatini chuqur ochishga katta yordam beradi.

Matematik fizika masalalarining qo‘yilishi ma’lum sinfdagi fizik hodisalarning asosiy qonuniyatlarini hisobga olgan holda ularning matematik modellarini yozishdan boshlanadi. Bunday hollarda xususiy hosilali differensial tenglamalardan iborat bo‘ladi, va ular *matematik fizika tenglamalari* deyiladi. Matematik fizika tenglamalari qatoriga integral, integro- differensial va boshqa tenglamalarni ham qo‘yish mumkin.

a) Torning kichik tebranish tenglamalari.

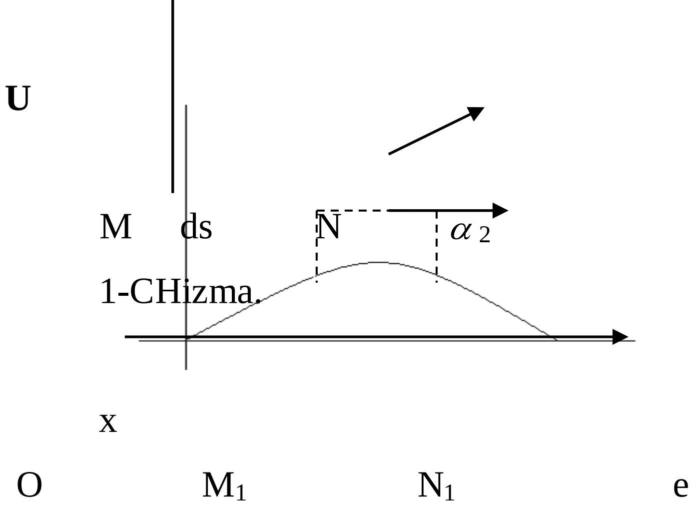
Ingichka ellastik (cho‘zilmaydigan, egilishga qarshilik ko‘rsatmaydigan) ipni *tor* deb ataymiz.

Faraz qilaylik, l ga teng bo‘lgan tor Ox o‘qning O va l nuqtalariga tortilib biriktirilgan bulib, $\vec{T}$ taranglik kuchi ta'siri ostida muvozanat holatda turgan bo'lsin.

Agar tashqi kuch ta'sirida tor muvozanatdan chetlashtirilsa, umuman olganda uning uzunligi va taranglik kuchi o'zgaradi. Lekin biz torning kichik tebranishini o'rganamiz. Bu holda torning uzunligi va taranglik kuchi $\vec{T}$ o'zgarmaydi. Bu fikrimizni oydinlashtiraylik. Agar tor 1- chizmada ko'rsatilgandek, muvozanat holatdan (x,u) tekistik bo'ylab chetlashtirilib, so'ngra qo'yib yuborilgan bo'lsa, u ko'ndalang tebranma harakat qiladi. Torning x absissali nuqtasiga mos kelgan siljishini u bilan belgilasak, u x koordinata va t vaqtning funksiyasi bo'ladi, ya'ni $u = u(x,t)$. Bu funksiyaning grafigi t ning har bir qiymati uchun torning formasini tasvirlaydi. Kichik tebranishlarni qarayotganimiz uchun u va $\frac{\partial u}{\partial x}$ lar juda kichik bo'lib, $\left(\frac{\partial u}{\partial x}\right)^2$ ni hisobga olmasligimiz mumkin. Bu holda muvozanat holatdan siljigan torning uzunligini l_1 desak,

$$l_1 = \int_0^l \sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\, dx \approx \int_0^l dx = l$$

bo'ladi, ya'ni deformatsiya natijasida torning uzunligi o'zgarmaydi, deb qaraymiz. Bundan Guk qonuniga asosan vaqt o'tishi bilan torning barcha nuqtalardagi taranglik kuchi $\vec{T}$ o'zgarmaydi, degan xulosaga kelamiz.



1-CHizma.

Faraz qilaylik, torning boshlangich holatdagi $dx = M_1N_1$ elementi deformatsiyadan keyin $ds = MN$ ga teng bo‘lsin. Tor juda ingichka deb qaralsa, uning har qanday elementining og’irligini hisobga olmasak ham bo‘ladi.

Demak, qaralayotgan holatda tor elementining ikkala tomoniga ta’sir etuvchi $\vec{T}$ taranglik kuchi e’tiborga olinadi. Bu kuch torning elastikligiga ko‘ra M nuqtaga urinma bo‘ylab chap tomonga, N nuqtaga urinma bo‘ylab esa o‘ng tomonga yo‘nalgan bo‘ladi (1- chizma).

Agar M nuqtaga qo‘yilgan kuchni $\vec{T}_1$, N nuqtaga qo‘yilgan kuchni $\vec{T}_2$ deb belgilasak, bu kuchlarning moduli T ga teng, yo‘nalishlari esa turlicha bo‘lib, nuqtadan nuqtaga o‘tishda o‘zgarib turadi. Nyuton ikkinchi qonuniga asoslanib torning muvozanat tenglamasini tuzamiz. Ko‘ndalang tebranishlarni qarayotganimiz uchun elementga ta’sir etayotgan kuchlarning faqat vertikal o‘qqa proektsiyalarining yigindisini qarasak bo‘ladi (kuchlarning gorizontal o‘qqa proektsiyalarining yigindisi 0 ga teng).

Agar M nuqtaga qo‘yilgan taranglik kuchini Ox o‘q bilan hosil qilgan burchagini α va N nuqtaga qo‘yilgan taranglik kuchining Ox o‘q bilan hosil qilgan burchagini α_2 bilan belgilasak, bu kuchlarning vertikal o‘qqa proektsiyalari mos ravishda $y_1 = -T\sin\alpha_1$ va $y_2 = T\sin\alpha_2$ lardan, gorizontal o‘qqa proektsiyalari esa mos ravishda $x_1 = -T\cos\alpha_1$ va $x_2 = T\cos\alpha_2$ lardan iborat bo‘ladi.

Demak, qaralayotgan kuchlarning o‘qlarga proektsiyalarining yigindisi

$$\Phi = T(\cos\alpha_2 - \cos\alpha_1) + T(\sin\alpha_2 - \sin\alpha_1)$$

ga teng.

Biz kichik tebranishlarni qarayotganimiz uchun α_1 va α_2 juda kichik bo‘lib,

$$\cos\alpha = \frac{1}{\sqrt{1+tg^2\alpha}} = \frac{1}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}} \approx 1,\ \sin\alpha = \frac{tg\alpha}{\sqrt{1+tg^2\alpha}} = \frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}} \approx \frac{\partial u}{\partial x},$$

ya’ni

$$\cos\alpha_1 \approx 1 \text{ va } \cos\alpha_2 \approx 1$$

hamda

$$\sin\alpha_1 \approx \operatorname{tg}\alpha_1 = \left(\frac{\partial u}{\partial x}\right)_x \quad \text{va} \quad \sin\alpha_2 \approx tg\alpha_2 = \left(\frac{\partial u}{\partial x}\right)_{x+\partial x}$$

taqribiy tengliklarni yozishimiz munkin.

Bularga ko‘ra (1) quyidagicha yoziladi:

$$\Phi = T(1-1) + T\left(\left(\frac{\partial u}{\partial x}\right)_{x+dx} - \left(\frac{\partial u}{\partial x}\right)_x\right),$$

yoki,

$$\Phi = T\left(\left(\frac{\partial u}{\partial x}\right)_{x+dx} - \left(\frac{\partial u}{\partial x}\right)_x\right).$$

Endi

$$\left(\frac{\partial u}{\partial x}\right)_{x+dx}$$

ifodani Teylor qatoriga yoyamiz:

$$\left(\frac{\partial u}{\partial x}\right)_{x+dx} = \left(\frac{\partial u}{\partial x}\right)_x + \left(\frac{\partial^2 u}{\partial x^2}\right)_x dx + \dots .$$

Bundan

$$\left(\frac{\partial u}{\partial x}\right)_{x+dx} \approx \left(\frac{\partial u}{\partial x}\right)_x + \left(\frac{\partial^2 u}{\partial x^2}\right)_x dx$$

taqribiy tenglikni hosil qilamiz.

Demak,

$$\Phi \approx T\frac{\partial^2 u}{\partial x^2}dx. \tag{2}$$

Nyutonning ikkinchi qonuniga ko‘ra, $dm = \rho(x)dx$ ($\rho(x)$ – torning zichligi) massaning

$\frac{\partial^2 u}{\partial t^2}$ – tezlanishga ko‘paytmasi qaralayotgan elementga qo‘yilgan kuchlarning o‘qlarga proeksiyalarining yigindisiga teng, ya’ni

$$\rho(x)dx\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}dx.$$

Bundan dx – ga qisqartirish natijasida

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{3}$$

tenglamaga ega bo‘lamiz.

Agar torni bir jinsli deb qarasak, u holda $\rho(x) = const$ bo‘ladi va

$$\frac{T}{\rho} = a^2 \tag{4}$$

deb belgilab olsak, tenglama

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \tag{5}$$

ko‘rinishga keladi. (5) tenglama torning *erkin tebranish tenglamasi* deyiladi. (5) tenglama qaralayotgan fizik jarayonning matematik modelidir.

Endi dx element gorizontal holatdan chiqarilgandan so‘ng unga taranglik kuchlaridan tashqari $F(t,x)dx$ tashqi kuch ham ta’sir etyapti deb faraz kilaylik. Bu holda (1) tenglama quyidagi ko‘rinishda bo‘ladi:

$$\rho(x)dx\frac{\partial^2 u}{\partial t^2} = \Phi + F(t,x)dx. \tag{6}$$

Bundan yuqoridagi kabi

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{7}$$

tenglamaga ega bo‘lamiz, bu yerda $f(x,t) = F(x,t)/\rho$. Bu tenglama bir jinsli torning *majburiy tebranish tenglamasi* deyiladi.

b) Membrananing kichik kundalang tebranish tenglamasi.

Yupqa ellastik (chuzilmaydigan, egilishga va siljishga qarshilik ko‘rsatmaydigan) plastinka *membrana* deyiladi. Membrananing qalinligi qolgan o‘lchamlariga nisbatan juda kichik bo‘lib, u doira, to‘g’ri to‘rtburchak, parallelogramm va x.k ko‘rinishda bo‘lishi mumkin.

Xuddi yuqoridagi o‘xshash, membrananing

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{8}$$

tenglamasini ham keltirib chiqarish mumkin. Bu tenglama membrananing *erkin* *tebranish tenglamasi* deyiladi.

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g(x, y, t) \tag{9}$$

tenglama membrananing *majburiy tebranish tenglamasi* deyiladi.

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^2 u}{\partial u^2} + \frac{\partial^2 u}{\partial g^2} \tag{10}$$

tenglama *akustika (gaz harakati) tenglamas*i yoki *uch ulchovli to'lqin tenglamasi* deyiladi.

Yuqoridagi (6), (8), (10) tenglamalar mos ravishda bir, ikki va uch o'lchovli to'lqin tenglamalari deyiladi. To'lqin tenglamalari *giperbolik tipdagi tenglamalar* sinfiga kiradi.

3. Issiqlik tarqalish tenglamalari.

Issiqlik tarqalish tenglamalari ham xuddi yuqoridagiga o'xshash keltirib chiqariladi.

Ushbu

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{11}$$

tenglama *sterjenda (simda) issiqlik tarqalish tenglamasi* deyiladi. Bunda $a = \sqrt{\frac{k}{c\rho}}$ issiqlik o'tkazuvchanlik koeffitsiyenti deyiladi.

Ushbu tenglama

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{12}$$

membranada issiqlik tarqalish tenglamasi deyiladi.

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{13}$$

tenglama esa *fazoviy jismlarda issiqlik tarqalish tenglamasi* deyiladi. Agar sterjenga, membranaga yoki fazoviy jismlarga qo'shimcha issiqlik manbai ulangan bo‘lsa, u holda issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t} = a^2 \ \Delta u + g(x, y, z, t), \tag{14}$$

bu yerda $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ bo‘ladi. Bu bir jinsli bo‘lmagan tenglamadir .

Yuqoridagi (11), (12), (13) tenglamalar mos ravishda *bir, ikki va uch ulchovli issiqlik tarqalish tenglamalari* deyiladi. Issiqlik tarqalish tenglamalari parabolik tipdagi tenglamalar sinfiga kiradi.

4. Statsionar maydonlar tenglamalari.

Yuqoridagi issiqlik maydonlari (sterjen, membrana va fazoviy jism) qaralib, bu maydonlarda issiqlik tarqalish masalalari ko‘rilgan edi. U maydonlar statsionar bo‘lagan maydonlar bo‘lib, issiqlik tarqalish jarayoni vaqtga bog’liq edi.

Endi issiqlik tarqalish jarayoni statsionar deb qaraymiz, yani vaqt o‘tishi bilan maydondagi temperatura o‘zgarmaydi. Bunday maydonlar *statsionar temperaturali maydonlar* deyiladi.

a)Bir jinsli sterjenda issiqlik tarqalish jarayonini statsionar deb qaraylik, u holda issiqlik tarqalish tenglamasida $u_t = 0$ bo‘lib, tenglama

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{15}$$

ko‘rinishga keladi. Agar sterjenga doimiy issiqlik manbalari ta’sir etsa, u holda tenglama

$$\frac{\partial^2 u}{\partial x^2} = -g(x) \tag{16}$$

ko‘rinishni oladi.

b) Agar bir jinsli membranada issiqlik tarqalish jarayoni statsionar bo‘lsa, issiqlik tarqalish tenglamasi

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{17}$$

ko‘rinishda yoziladi. Agar membranaga doimiy issiqlik manbalari ta’sir etsa, tenglama

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-g(x,y) \qquad (18)$$

ko‘rinishda bo‘ladi .

c) Bir jinsli qattiq jism uch o‘lchovli fazoda qaralayotgan bo‘lib, issiqlik tarqalish jarayoni statsionar bo‘lsa, bu holda issiqlik tarqalish tenglamasi

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad (19)$$

bo‘lib, agar unga issiqlik manbalari ta’sir etsa, uning ko‘rinishi

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-g(x,y,z) \qquad (20)$$

bo‘ladi.

Yuqoridagi keltirib chiqarilgan (15), (17) va (19) tenglamalar mos ravishda bir, ikki va uch o‘lchovli *Laplas tenglamalari* deyiladi. (16), (18) va (20) tenglamalar esa bir, ikki va uch o‘lchovli *Puasson tenglamalari* deyiladi.

Statsionar maydon tenglamalari *elliptik tipdagi tenglamalar* sinfiga kiradi. Yuqoridagi barcha tenglamalar *matematik fizika tenglamalari* deyiladi.

Asosiy adabiyotlar

1. Wolter A. Stranss. Partial differential equation; An introduction.
Birkhhauzer. Germaniy, 2005

2. Davia D. Bleecker, George Csordes. Basic of partial Differential Equations.
Birkhhauzer. Germaniy, 2009.

3. Салохиддинов М.С. Математик физика тенгламалари. Т., «Ўзбекистон»,
2002, 448 б.

4. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М. 2004.

5. Бицадзе А.В., Калиниченко Д.Ф. Сборник задач по уравнениям математической
физики. М. 1977.

Mustaqilta'limmavzulari

1. Matematik fizika tenglamalari uchun asosiy masalalarning qo‘yilishi: Koshi masalasi.
2. Ikkinchi tartibli ikki o‘zgaruvchili umumiy tenglama uchun Koshi masalasini ketma-ket yaqinlashish usuli bilan yechish.

Glossariy

Tor - Ingichka ellastik (chuzilmaydigan, egilishga karshilik kursatmaydigan) ipni tor deb ataymiz.

Torning erkin tebranish tenglamasitenglama

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

ko’rinishga keladi. (5) tenglama torning erkin tebranish tenglamasi deyiladi.

Bir jinsli torning majburiy tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

tenglamaga ega bo’lamiz. Bu yerda $f(x,t) = F(x,t)/\rho$. Bu tenglama bir jinsli torning majburiy tebranish tenglamasi deyiladi.

Membrana - Yupka elastik (chuzilmaydigan, egilishga va siljishga karshilik kursatmaydigan) plastinka membrana deyiladi.

Membranani erkin tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial u^2}\right) \qquad (8)$$

tenglamasini xam keltirib chikarish mumkin. Bu tenglama membranani erkin tebranish tenglamasi deyiladi.

Membrananing majburiy tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial u^2}\right) + g(x,u,t,)$$

tenglama membrananing majburiy tebranish tenglamasi deyiladi.

Akustika (gaz xarakati) tenglamasi yoki uch ulchovli tulkin tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^2 u}{\partial u^2} + \frac{\partial^2 u}{\partial g^2}$$

tenglama akustika (gaz xarakati) tenglamasi yoki uch ulchovli tulkin tenglamasi deyiladi.

Sterjen (sim) da issiklik tarkalish tenglamasi -

$$\frac{\partial u}{\partial t} = a^2\,\frac{\partial^2 u}{\partial x^2}$$

tenglama sterjen (sim) da issiklik tarkalish tenglamasi deyiladi.

Membranada issiklik tarkalish tenglamasi Ushbu tenglama

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

membranada issiklik tarkalish tenglamasi deyiladi.

Fazoviy jismlarda issiklik tarkalish tenglamasi -

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

tenglama esa fazoviy jismlarda issiklik tarkalish tenglamasi deyiladi.

Keyslar banki

Keys: Masala o`rtaga tashlanadi: Kanonik ko’rinishga keltiring $3u_{xx} + 32u_{xy} + 64u_{yy} = 0.$

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni

  keltiring (individual va kichik guruhlarda);

- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Nazorat uchun savollar

1. Tor tebranish tenglamasi qanday tenglama?

2. Dalamber yechimi va formulasi. Dalamber formulasi bilan aniqlangan yechimning fizik ma'nosini tushintirib bera olasizmi?

Mustaqil yechish uchun mashqlar

Quyidagi masalalarni yeching.

1. $a_1U_x + a_2U_y = 0$ differensial tenglamaning umumiy yechimini toping. (Bu yerda a_1 va a_2 koeffitsientlar kamida bittasi noldan farqli haqiqiy sonlardir).

2. $a_1U_x + a_2U_y = g(x, y)$ differensial tenglamaning umumiy yechimini toping. (Bu yerda a_1 va a_2 koeffitsientlar kamida bittasi noldan farqli haqiqiy sonlardir).

3. $a_1U_x + a_2U_y + g(x, y)U = 0$differensial tenglamaning umumiy yechimini toping. (Bu yerda a_1 va a_2 koeffitsientlar kamida bittasi noldan farqli haqiqiy sonlardir).

4. $a_1U_x + a_2U_y + g(x, y)U = \omega(x, y)$ differensial tenglamaning umumiy yechimini toping. (Bu yerda a_1 va a_2 koeffitsientlar kamida bittasi noldan farqli haqiqiy sonlardir).

5. $U_x + f(x, y)U = 0$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

6. $U_x + f(x, y)U = g(x, y)$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

7. $U_y + f(x, y)U = g(x, y)$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

8. $U_y + f(x, y)U = 0$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

9. $f(x)U_x + g(y)U_y = 0$ birinchi tartibli xususiy hosilali differensial tenglamani yeching.

10. $f(x)U_x + g(y)U_y = g(x,y)$ birinchi tartibli xususiy hosilali differensial tenglamani yeching.

11. $f(x)U_x + g(y)U_y + g(x,y)U = 0$ birinchi tartibli xususiy hosilali differensial tenglamani yeching.

12. $f(x)U_x + g(y)U_y + \varphi(x,y)U = \omega(x,y)$ birinchi tartibli xususiy hosilali differensial tenglamani yeching.

13. $U_{xy} + f(x,y)U_x = g(x,y)$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

14. $U_{xy} + f(x,y)U_x = 0$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

15. $U_{xy} + f(x,y)U_y = g(x,y)$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

16. $U_{xy} + f(x,y)U_y = 0$ differensial tenglamaning umumiy yechimini toping. Misollar keltiring.

17. $U = U(x,y)$ ikki o`zgaruvchili funksiyadan $\xi = \xi(x,y)$ va $\eta = \eta(x,y)$ teskari almashtirishlar orqaliquyidagi xususiy hosilalarni hosil qiling:
$U_x, U_y, U_{xx}, U_{xy}, U_{yy}$. 18. $U_x + U = y$. 19. $U_x + (x+y)U = 0$

20. $U_y + U = x - y$. 21. $U_y + 2xyU = 0$. 22. $U_{xy} + U_x = 1$

23. $U_{xy} + \frac{y}{x}U_x = 0$. 24. $U_{xy} + 2U_y = 1$ 25. $U_{xy} + (x-y)U_y = 0$

26. $2U_x - 3U_y = 4$. 27. $4U_x - 7U_y = 0$. 28. $3U_x - 2U_y + xU = 0$

29. $U_x + yU_y = x$. 30. $xU_x + U_y = 0$ 31. $U_x + yU_y + U = 0$

32. $\frac{1}{x}U_x - \frac{2}{y}U_y + U = 1$

Tenglamalarni yeching

$$x^2U_x - xyU_y + y^2 = 0;\quad xU_{xy} - U_y = 0$$

$$yU_{xy} = (1 + y\ln x)U_x;\quad (x - y)U_{xy} = U_y.$$

TEST

Ikki o‘lchovli Puasson tenglamasini ko‘rsating	* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f$	$\frac{d^2 u}{dx^2} = 0$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -f$	$\frac{d^2 u}{dx^2} = -g$
Uch o‘lchovli Puasson tenglamasini ko‘rsating	* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -f$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f$	$\frac{d^2 u}{dx^2} = 0$	$\frac{d^2 u}{dx^2} = -g$
Issiqlik o'tkazuvchanlik tenglamasini aniqlang?	* $u_t - a^2(u_{xx} + u_{yy}) = 0$	$u_{tt} = a^2 u_{xx}$	$u_{xx} + u_{yy} = f(x, y)$	$u_{xt} = g(x)$
Bir o‘lchovli to’lqin tenglamasini ko‘rsating	* $u_{tt} - a^2 u_{xx} = 0$	$u_{tt} - a^2(u_{xx} + u_{yy}) = 0$	$u_{tt} - a^2(u_{xx} + u_{yy} + u_{zz}) = 0$	$u_{xx} + u_{yy} + u_{zz} = 0$
Ikki o‘lchovli to’lqin tenglamasini ko‘rsating	* $u_{tt} - a^2(u_{xx} + u_{yy}) = 0$	$u_{tt} - a^2 u_{xx} = 0$	$u_{xx} + u_{yy} + u_{zz} = 0$	$u_{tt} - a^2(u_{xx} + u_{yy} + u_{zz}$
Biro‘lchovliissiqliktarqalishtenglamasiniko‘rsating	* $u_t - a^2 u_{xx} = 0$	$u_t - a^2(u_{xx} + u_{yy}) = 0$	$u_{xx} + u_{yy} + u_{zz} = 0$	$u_t - a^2(u_{xx} + u_{yy} + u_{zz}$
Ikkio‘lchovliissiqliktarqalishtenglamasiniko‘rsating	* $u_t - a^2(u_{xx} + u_{yy}) = 0$	$u_t - a^2 u_{xx} = 0$	$u_{xx} + u_{yy} + u_{zz} = 0$	$u_t - a^2(u_{xx} + u_{yy} + u_{zz}) = 0$
Quyidagitenglamalardanqaysibiriparaboliktipgategish	*	$u_{xx} + 2u_{xy} + 5u_{yy} -$ $-32u = 0$	$u_{xx} + u_{xy} - 2u_{yy} - 3u_x$ $-15u_y + 27x = 0$	$2u_{xx} + 3u_{xy} + u_{yy}$ $+7u_x + 4u_y - 2u$

li?	$u_{xx}-2u_{xy}+u_{yy}+9u_x+$ $+9u_y-u=0$			

4-Mavzu:Matematik fizika tenglamalari uchun asosiy masalalarning qo'yilishi. Koshi masalasi; chegaraviy masala va boshlang'ich-chegaraviy masalalar.

Reja

1. Matematik fizika masalasi.
2. To'lqin tenglamalari uchun boshlangiyach shartli masalalar.
3. Issiqlik tarqalish tenglamalari uchun boshlang'ich shartli masalalar.
4. Matematik fizika tenglamalariga qo'yilgan chegaraviy masalalar.
5. Matematik fizika masalalarining korrektligi. Adamar misoli.

Tayanch so'z va iboralar: *Matematik fizika masalasi. to'lqin tenglamalari uchun boshlangiyach shartli masalalar, issiqlik tarqalish tenglamalari uchun boshlang'ich shartli masalalar, Matematik fizika tenglamalariga qo'yilgan chegaraviy masalalar, matematik fizika masalalarining korrektligi, Adamar misoli.*

Matematik fizikaning masalalar doirasi turli xil fizik jarayonlarni urganish bilan chambarchas boglangan. Bularga tebranish nazariyasi, gazodinamika, gidrodinamika, elektrodinamika, akustika, issiqlik utkazuvchanlik va xakazolarda urganiladigan xodisalar kiradi.

Biz oldingi mavzuda ko'pgina fizik xodisalarning matematik modellarini ko`rib chiqdik. Ular ikkinchi tartibli o`zgarmas koeffitsientli chiziqli xususiy hosilali differensial tenglamalardan iborat edi. Bu tenglamalar uch xil (giperbolik, parabolik va elliptik) tipga ajralishi bilan tanishdik. Matematik fizika tenglamalari umuman olganda cheksiz kup yechimlarga ega. Bu yechimlar ichidan qo'yilgan fizik masalaga javob beradiganini ajratib olish kerak. Buning uchun tenglamalarga shu fizik masalaning moxiyatidan kelib chikib qo'shimcha shartlar qo'yiladi. Qo'shimcha shartlar, odatda, boshlang'ich, aralash va chegaraviy shartlardan iborat buladi. Biror matematik fizika tenglamasining berilgan qo'shimcha shart(lar)ni qanoatlantiruvchi yechimini topish matematik fizika masalasi deyiladi.

Quyida matematik fizika asosiy masalalarining qo'yilishi bilan tanishamiz va ularni klassifikatsiyalaymiz, sungra bu masalaning korrekt (tugri) qo'yilishi tushunchasini kiritamiz.

Matematik fizikaning bir jinsli va bir jinsli bulmagan tenglamalariga qo'yiladigan masalalar bir xil bulganligi uchun, biz fakatgina bir jinsli tenglamalarga boshlang'ich shartli, aralash va chegaraviy masalalarning qo'yilishi bilan cheklanamiz.

Avvalo matematik fizika tenglamalariga qo'yiladigan boshlang'ich shartli masalalarni ko`rib chikamiz. Bunday masalalar Koshi masalasi deyiladi.

Torning erkin tebranishini xarakterlovchi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

to`lqin tenglamasini karaylik. Bu tenglamaning cheksiz kup yechimlari ichidan torning xarakatini vaqtning ixtiyoriy momentida ifodalovchi yechimini ajratib olish uchun yechimga qo'shimcha shartlar berilishi kerak. Bu yerda torni cheksiz uzun deb karaymiz. Karalayotgan torning urtasida sodir buladigan tebranish uning chetlariga ta'sir kilmaydi, chunki u cheksiz uzun bulganligi uchun urtadan ikki tomonga karab yuguruvchi to`lqinlar ma'lum masofaga borganda asta-sekin suna boshlaydi. Demak, tor nuqtalarining xolatini vaqtning istalgan paytida aniqlash uchun vaqtning t=0 boshlang'ich momentida uning xar bir nuqtasining abstsissalar ukidan uzoqligi uni va boshlang'ich tezligini bilishimiz zarur.

Bir o'lchovli to`lqin tenglamasi uchun boshlang'ich shartli masalani ta'riflaymiz:

Ikkinchi tartibli xususiy hosilali (1) differensial tenglamaning $(-\infty < x < \infty, t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = F(x), \quad (-\infty < x < \infty) \quad (2)$$

shartlarni qanoatlantiruvchi yechimini toping. Bunda f(x) va F(x) berilgan funksiyalardir. Izlanayotgan u(x,y) funksiya t=0 da qanoatlantirishi kerak bulgan (2) shartlar boshlang'ich shartlar deyiladi.

Endi chegaralanmagan membrananing erkin kundalang tebranish jarayonini karaymiz. Bu jarayon (3) ko'rinishdagi to`lqin tenglamasi yordamida tavsiflanadi. Bu tenglama membrana xarakatini to`liq aniqlash uchun albatta uning t=0 momentdagi xolati, ya'ni nuqtalarning vaziyati U va ularning t=0 momentdagi $\partial u/\partial t$ tezligi ma'lum bulishi kerak. Membrana uchun uning boshlang'ich xolatini aniqlovchi shart $u|_{t=0} = f(x)$, boshlang'ich tezligini aniqlovchi shart esa $\left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x)$ funksiyalar orkali beriladi, bu yerda f(x) va F(x) berilgan funksiyalardir.

Ikki o'lchovli to`lqin tenglamasi uchun boshlang'ich shartli masala quyidagidan iborat:

Ikkinchi tartibli xususiy hosilali

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3}$$

differensial tenglamaning $(-\infty < x,\ y < \infty,\ t > 0)$ soxada aniqlangan va

$$u|_{t=0} = f(x,y), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x,y),\ (-\infty < x,\ y < \infty) \tag{4}$$

boshlang'ich shartlarni qanoatlantiruvchi u(x,y, t) yechimini toping.

Umumiy holda uch o'lchovli fazoda tutash muxit zarrachalarining kundalang tebranma xarakati karalayotgan bo'lsa, bunday xarakatlar uch o'lchovli to`lqin tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{5}$$

yordamida tavsiflanadi. Bu tenglama gaz tebranma xarakati karalayotgan bo'lsa, bunday xarakatlar akustik tebranishlar, vakuumda elektromagnit to`lqinlarning tarqalishi va xokazo xodisalarning matematik modelidir.

Tutash muxit zarrachalarining tebranishi xakidagi masala ham xuddi yuqorida kurilgan masalalarga uxshaydi, ya'ni uning xarakatini to`liq tasavvur qilish uchun t=0 boshlang'ich vaqt momentida u va $\partial u / \partial t$ larni bilishimiz kerak.

Uch o'lchovli to`lqin tenglamasi uchun boshlang'ich shartli masala quyidagicha buladi:

Ikkinchi tartibli xususiy hosilali (5) differensial tenglamaning $(-\infty < x,\ y,\ z < \infty,\ t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x, y, z), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y, z) \tag{6}$$

boshlang'ich shartlarni qanoatlantiruvchi u(x, y, z) yechimi topilsin.

CHegaralanmagan, ingichka va yon sirti tashki muxit bilan issiqlik almashmaydigan sterjenda issiqlik manbalarining ta'siri bulmaganda unda issiqlik tarqalish jarayoni

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{7}$$

tenglama bilan tavsiflangan edi. Sterjen juda uzun bulganligi uchun uning urta kismidagi issiqlik jarayonlariga temperaturaning boshlang'ich taqsimlanishi katta ta'sir etadi va bu temperatura vaqt utishi bilan sterjen uchlariga yetib bormasdan soviydi.

Ko'rinib turibdiki, sterjenning istalgan nuqtasidagi u(x, y) temperaturani vaqtning xar kanday momentida bilish uchun t=0momentda sterjen nuqtalaridagi temperaturani bilishimiz kerak. Bu temperatura u(x, 0)=f(x) boshlang'ich shart yordamida beriladi, bunda f(x) $(-\infty < x < \infty)$ aniqlangan uzluksiz funksiyadir. Issiqlik tarqalish tenglamasi uchun boshlang'ich shart to`lqin tenglamalariga qo'yiladigan boshlpngich shartdan fark kilib, bunda u izlanayotgan funksiyaning t=0momentda berilishidan iboratdir. Shunday kilib, bir o'lchovli issiqlik tarqalish tenglamasi uchun boshlang'ich shartli masala quyidagicha qo'yiladi:

Ikkinchi tartibli xususiy hosilali (7) differensial tenglamaning $(-\infty < x < \infty,\ t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x)\ (-\infty < x < \infty) \tag{8}$$

boshlang'ich shartlarni qanoatlantiruvchi u(x, t) yechimi topilsin.

CHegaralanmagan membranada issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (9)$$

ko'rinishda tavsiflanadi. Bu yerda ham issiqlik tarqalish masalasi yuqoridagiga uxshash buladi:

Ikkinchi tartibli xususiy hosilali (9) differensial tenglamaning $(-\infty < x, y < \infty, t > 0)$ soxada aniqlangan va

$$u|_{t=0} = f(x, y), \quad (-\infty < x,\ y < \infty) \qquad (10)$$

boshlang'ich shartlarni qanoatlantiruvchi u(x, u,t) yechimi topilsin.

Uch o'lchovli fazoda notekis isitilgan jismni karaymiz. Agar bu jism tashki muxit bilan issiqlik almashmasa va unda issiqlik manbalari bulmasa, u holda issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (11)$$

ko'rinishda bulishini aytib o'tgan edik. Bu tenglama uchun boshlang'ich shart vaqtning t=0 boshlang'ich momentida uning barcha nuqtalarining temperaturasi berilishidan iborat.

Boshlang'ich temperatura $u|_{t=0} = f(x, y, z)$, funksiya yordamida beriladi, bunda $f(x, y, z)$ funksiya jismning barcha nuqtalarida aniqlangan uzluksiz funksiya.

Bu uch o'lchovli isiklik tarqalish tenglamasi uchun boshlang'ich shartli masala quyidagicha qo'yiladi:

Ikkinchi tartibli xususiy hosilali (11) differensial tenglamaning $(-\infty < x,\ y,\ z < \infty, t > 0)$ soxada aniqlangan va

$$u|_{t=0} = f(x, y, z), \qquad (12)$$

boshlang'ich shartni qanoatlantiruvchi u(x, y, z, t) yechimi topilsin.

Biz keyinchalik ko'ramizki, matematik fizika tenglamalariga qo'yilgan yuqoridagi masalalarning xar biri yagona yechimga ega buladi.

Biz bu yerda statsionar issiqlik maydonida temperaturaning taqsimlanishini tavsiflovchi Laplas tenglamasi uchun chegaraviy masalalarning qo'yilishi bilan tanishamiz. Bunday masalalar Dirixle masalasi deyiladi.

Laplas tenglamasi

$$\Delta u = 0 \qquad (1)$$

berilgan bulsin.Bu yerda Δu laplasian bulib, u bir, ikki va uch o'lchovli Dekart koordinatalar sistemasida quyidagi ko'rinishlarga ega:

$$\Delta \mathrm{u} = \frac{\partial^2 u}{\partial x^2}, \qquad \Delta \mathrm{u} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad \Delta \mathrm{u} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Matematik analiz kursidan ma'lumki, laplasian bir, ikki va uch o'lchovli tsilindrik koordinatalar sistemasida quyidagicha ko'rinishda yoziladi: $x = r\cos\varphi,\ y = r\sin\varphi,\ z = z;$

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right); \quad \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2};$$

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2};$$

Xuddi shuningdek bir, ikki va uch o'lchovli sferik koordinatalar sistemasida $x = \rho\sin\theta\cos\varphi,\ y = \rho\sin\theta\sin\varphi,\ z = \rho\cos\varphi;$

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right); \quad \Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right);$$

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{\rho\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2};$$

buladi. Avval bir o'lchovli Laplas tenglamasi uchun Dirixlening chegaraviy masalasini keltiramiz.

U quyidagidan iborat.

Ikkinchi tartibli bir o'zgaruvchili

$$\frac{d^2 u}{dx^2} = 0 \qquad (2)$$

tenglamaning $0 \le x \le l$ kesmada aniqlangan va

$$u\big|_{x=0} = u_0, \ u\big|_{x=l} = u_l \qquad (3)$$

chegaraviy shartlarni qanoatlantiruvchi $u(x)$ echimi topilsin.

Bu masalaning fizik ma'nosi ingichka sterjeni uchlarida berilgan tepraturaga kura uning boshka nuqtalaridagi temperaturani aniqlashdan iboratdir. Sterjenning yon sirtida issiqlik almashinmaydi deb faraz kilaylik. Yuqoridagi masala bir o'lchovli tsilindrik koordinatalar sistemasida quyidagicha ta'riflanadi:

$$\frac{1}{r}\frac{d}{\partial r}\left(r\frac{du}{dr}\right)=0 \tag{4}$$

tenglamaning $a \le r \le b$ soxada aniqlangan va

$$u\big|_{r=a}=u_a, \quad u\big|_{r=b}=u_b \tag{5}$$

chegaraviy shartlarni qanoatlantiruvchi $u(r)$ echimini toping.

Bu masala fizik nuqtai nazardan umumiy ukka ega bulgan $r=a$ va $r=b$ radiusli $(\alpha<b)$ tsilindrlar sirtida uzgarmas tempratura saklanganda temperaturaning statsionar taqsimlanishini aniqlashdan iborat.

Nixoyat, bir o'lchovli Laplas tenglamasi uchun Dirixlening chegaraviy masalasini bir o'lchovli sferik koordinatalar sistemasida karaymiz.

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\right)=0 \tag{6}$$

tenglamaning $\alpha \le \rho \le b$ soxada aniqlangan.

$$u\Big|_{\rho=a}=u_a, u\Big|_{\rho=b}=u_b \tag{7}$$

chegaraviy shartlarni qanoatlantiruvchi $u(\rho)$ echimi topilsin.

Bu masalaning moxiyati umumiy markazga ega bulgan $\rho=a$ va $\rho=b$ radiusli $(a<b)$ sferalar orasida joylashgan fazoda sferalar ustida uzgarmas temperatura saklanganda temperaturaning statsionar taqsimlanishini aniqlashdan iborat.

Dirixlening chegaraviy masalasi ikki o'lchovli Laplas tenglamasi uchun quyidagicha qo'yiladi

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{8}$$

tenglamaning $(x,y)\in S$ soxada aniqlangan va

$$u\big|_r = f(x,y) \tag{9}$$

chegaraviy shartni qanoatlantiruvchi $u(x,y)$ yechimi topilsin. Bu yerda G tekislik kismi S ning (membraning) chegarasi. Bu masalaning fizik ma’nosi G chiziq bilan chegaralangan S tekislik kismidagi temperaturaning statsionar taqsimlanishini G chegarada berilgan temperaturaga kura aniqlashdan iborat.

Xuddi shuningdek uch o’lchovli Laplas tenglamasi uchun Dirixle masalasi quyidagidan iborat buladi.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad (10)$$

tenglamaning $(x,y,z)\in D$ soxada aniqlangan

$$u\big|_s = f(x,y,z) \qquad (11)$$

chegaraviy shartni qanoatlantiruvchi $u(x,\ y,\ z)$ yechimi topilsin, bu yerda S fazo kismi bulgan D ni (jismni) chegaralovchi sirt.

Bu masalaning fizik ma’nosi S sirt bilan chegaralangan D jismdagi temperaturaning statsionar taqsimlanishini S sirtda berilgan temperaturaga kura aniqlashdan iborat. Ikki va uch o’lchovli Laplas tenglamalari uchun Dirixle masalasini tsilindrik va sferik koordinatalar sistemasida ham karash mumkin.

Ulardan birini, ya’ni doira uchun Dirixle masalasini keltiramiz

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}=0 \qquad (12)$$

tenglamaning $\left(0\le r\le a, 0\le\varphi\le 2\pi\right)$ soxada aniqlangan va

$$u\big|_{r=a} = f\left(\varphi\right) \qquad (13)$$

chegaraviy shartni qanoatlantiruvchi $u\left(r,\varphi\right)$ yechimini toping. Fizik nuqtai nazardan bu masala doira chegarasida berilgan temperaturaga kura uning ichki nuqtalarida temperaturaning statsionar taqsimlanishi aniqlashdan iborat.

Yuqorida bir, ikki va uch o’lchovli Laplas tenglamasi uchun chegaraviy masalalarni ko`rib chikdik. Bu chegaraviy masalalarning moxiyati shundan iboratki, bunda karalayotgan sterjen membrana va uch o’lchovli fazodagi kattik jismlarning chegaralarida ma’lum bulgan temperaturaga kura ularning ichki nuqtalaridagi temperaturani aniqlash talab etiladi.

Bunday qo’yilgan masalalar Dirixlening ichki masalalari deyiladi. Sterjen, membrana va uch o'lchovli fazodagi kattik jismlarning chegaralarida ma'lum bulgan temperaturaga kura bir, ikki va uch o'lchovchi Laplas tenglamalarining jismning tashki nuqtalaridagi yechimini izlash mumkin, ya'ni jismlarning tashkaridagi nuqtalarning temperaturasining ham aniqlash mumkin. Bunday masalalar Dirixlening tashki masalalari deyiladi. Bunday masalalarda (2,8,10) tenglamalarning tashki nuqtalaridagi yechimlari izlanib sterjen uchlarida membrana chegasida va jism sirti ustida mos ravishda (3,9,11) chegaraviy shartlarning bajarilishi talab kilinadi. Tashki masala uchun yuqoridagi shartlardan tashkari xar bir xol uchun qo'shimcha ravishda cheksizlikda yechimning nolga intilishi ham talab kilinadi. Biz kelajakda fakat ichki masalalarning karaymiz va ularni kiska kilib chegaraviy masalalar yoki Dirixle masalalari deb ataymiz.

Biz yuqorida matematik fizika masalalarini ko`rib utdik. Bu yerda matematik fizika tenglamalariga eng sodda qo'shimcha shartlar kuyish bilan chegaralandik.

Matematik fizika tenglamasiga biror masala qo'yilgan bo'lsa, bu masalaning yechimi albatta boshlang'ich va chegaraviy shartlardagi funksiyalarga boglik buladi. Bu funksiyalar odatda tajriba yo`li bilan aniqlanadi va shuning uchun ular juda aniq topilishi mumkin emas, chunki fizik kattaliklarni o`lchashda muayyan o`lchash xatoligi mavjuddir.

Boshlang'ich va chegaraviy shartlarni hosil qilishda yo`l qo'yilgan xatolik yechimga kanchalik ta'sir qilishini aniqlash ham muxim ahamiyatga egadir. Boshlang'ich chegaraviy shartlarning ozgina o`zgarishiga yechimning juda katta o`zgarishi mos kelishi ham mumkin. Bu xollarda bunday yechimdan foydalanish amalda yaxshi natijalar bermasligi mumkin.

Ta'rif: Agar masalada boshlang'ich chegaraviy shartlarni va tenglama ozod xadining ozgina o`zgarishiga yechimning ham ozgina ugarishi mos kelsa, bunday masala yechimi turgun deyiladi. Agar matematik fizika masalasining yechimi mavjud, yagona va turgun bo'lsa, u holda bunday masala korrekt (tugri) qo'yilgan deyiladi.

Agar matematik fizika masalasining yechimi uchun bu shartlarning istalgan biri bajarilmasa bunday masala korrekt qo'yilmagan masala deyiladi.

Matematik fizika tenglamalari yuqorida qo'yilgan masalalarning hammasi ham korrekt qo'yilgandir. Bularning kupchiligiga kelgusida ishonch hosil kilamiz. Korrekt qo'yilmagan masalalar ham juda kup uchraydi. Ulardarn biri Adamar misolini ko`rib utamiz:

Misol (Adamar)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (14)$$

tenglamaning $(-\infty<x<\infty, y>0)$ soxada aniqlangan va

$$u\big|_{y=0}=0,\ \frac{\partial u}{\partial y}\Big|_{y=0}=0 \qquad (15)$$

shartlarni qanoatlantiruvchi $u(x,y)$ echimi topilsin. Bu masalaning yechimi $u(x,y)=0$ bulib, u yagonadir. Ya'ni korrektlik shartlaridan ikkitasi bajariladi. Uchinchi shartni tekshiramiz. Buning uchun boshlang'ich shartlardan birini ozgina uzgartirib yechimning kanday o`zgarishini aniqlashimiz kerak. Berilgan (14) tenglamaning

$$u\Big|_{y=0}=\frac{\cos nx}{n},\frac{\partial u}{\partial y}\Big|_{y=0}=0 \qquad (16)$$

shartlarni qanoatlantiruvchi yechimini topish kerak bulsin. Ko'rinib turibdiki, (15) dagi birinchi shart (16) dagi birinchi shartdan yetarlicha katta p lar uchun juda oz mikdorda fark kiladi.

Xakikatdan ham, $|u|_{y=0}=\left|\cos nx\le\frac{1}{n_{n\to\infty}}\to 0\right|$

tekshirib kurish mumkinki, (14), (16) masalaning yechimi

$$u(x,y)=\frac{\cos nxchny}{n}$$

funksiyadan iborat buladi. Bu esa $nx\ne\frac{\pi}{2}(2x-1), k=1,2,...$ holda,

chegaralanmagan funksiyadir.

Bundan ko'rinadiki, keyingi masalaning yechimi oldingi masalaning yechimidan absolyut kiymat jixatdan juda katta fark kiladi, ya'ni boshlang'ich shartni ozgina uzgartirishimiz bilan yechim juda katta uzgarib ketadi. Demak, yechim turgun emas, bu esa (14),(15) masalaning korrekt qo'yilmaganligini bildiradi.

Asosiy adabiyotlar

1. Wolter A. Stranss. Partial differential equation; An introduction. Birkhhauzer.
Germaniy, 2005
2. Davia D. Bleecker, George Csordes. Basic of partial Differential Equations.
Birkhhauzer. Germaniy, 2009.
3. Салохиддинов М.С. Математик физика тенгламалари. Т., «Ўзбекистон»,
2002, 448 б.
4. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М. 2004.
5. Бицадзе А.В., Калиниченко Д.Ф. Сборник задач по уравнениям математической
физики. М. 1977.

Mustaqilta'limmavzulari

1. Chegaraviy masala va boshlang'ich-chegaraviy masalalar.
2. Ikkinchitartibliikkio'zgaruvchiliumumiytenglamauchunKoshimasalasiniketma-ketyaqinlashishusulibilanyechish.

Glossariy

Bir o'lchovli to`lqin tenglamasi uchun boshlang'ich shartli masala - Ikkinchi tartibli xususiy hosilali (1) differensial tenglamaning $(-\infty < x < \infty,\ t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x), \quad (-\infty < x < \infty)$$

shartlarni qanoatlantiruvchi yechimini toping. Bunda f(x) va F(x) berilgan funksiyalardir.

Ikki o'lchovli to`lqin tenglamasi uchun boshlang'ich shartli masala -

Ikkinchi tartibli xususiy hosilali

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

differensial tenglamaning $(-\infty < x,\ y < \infty,\ t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x,y), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x,y),\ (-\infty < x,\ y < \infty)$$

boshlang'ich shartlarni qanoatlantiruvchi u(x,y, t) yechimini toping.

Uch o'lchovli to`lqin tenglamasi uchun boshlang'ich shartli masala - Ikkinchi tartibli xususiy hosilali (5) differensial tenglamaning $(-\infty < x,\ y,\ z < \infty,\ t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x,y,z), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x,y,z)$$

boshlang'ich shartlarni qanoatlantiruvchi u(x, y, z) yechimi topilsin.

Ikki o'lchovli to`lqin tenglamasi uchun qo'yiladigan aralash masala - Ikkinchi tartibli xususiy hosilali

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{5}$$

differensial tenglamaning $(x, y \leftarrow S, t > 0)$ (bunda S L kontur bilan chegaralangan soxa) soxada aniqlangan va

$$\left.\begin{aligned} u\big|_{t=0} &= f(x,y) \\ \frac{\partial u}{\partial t}\Big|_{t=0} &= F(x,y) \end{aligned}\right\}(x,y)\in S \tag{6}$$

boshlang'ich shartlarni hamda

$$u/_L = 0 \tag{7}$$

chegaraviy shartni qanoatlantiruvchi $u(x,y,t)$ yechimi topilsin.

Bir o'lchovli issiqlik tarqalish tenglamasi uchun eng sodda aralash masala - Ikkinchi tartibli xususiy hosilali

$$\frac{\partial^2 u}{\partial t^2}=a^2\,\frac{\partial^2 u}{\partial x^2} \tag{9}$$

differensial tenglamani $(0 \le x \le l,\ t > 0)$ soxada aniqlangan va

$$u\big|_{t=0} = f(x) \qquad (0 \le x \le l) \tag{10}$$

boshlang’ich shart hamda (8) chegaraviy shartlarni qanoatlantiruvchi u(x,t) yechimini toping.

Keyslar banki

Keys: Masala o`rtaga tashlanadi: Kanonik ko’rinishga keltiring

$$u_{xx} - 10u_{xy} + 25u_{yy} + 5u_x - 25u_y = 0.$$

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni

  keltiring (individual va kichik guruhlarda);

- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Nazorat uchun savollar

1. CHegaralangan tor.
2. To’lqin tenglamasi uchun Koshi masalasi yechimining yagonaligi.
3. Koshi masalasi yechimini beradigan formulalar va ularni tekshirishni bilasizmi?

Mustaqil bajarish uchun mashqlar

Quyidagi Koshi masalalarini yeching:

1. $u_{xy} = 0;$

$$u\big|_{y=x^2} = 0, \qquad u_y\big|_{y=x^2} = \sqrt{|x|}, \qquad |x| < 1.$$

2. $u_{xy} + u_x = 0;$

$$u\big|_{y=x} = \sin x, \qquad u_x\big|_{y=x} = 1 \qquad |x| < \infty.$$

3. $u_{xx} - u_{yy} + 2u_x + 2u_y = 0;$

$$u\big|_{y=0} = x, \qquad u_y\big|_{y=0} = 0, \qquad |x| < \infty.$$

4. $u_{xx} - u_{yy} - 2u_x - 2u_y = 4;$

$$u\big|_{x=0} = -y, \qquad u_x\big|_{x=0} = y - 1, \qquad |y| < \infty.$$

5. $u_{xx} + 2u_{xy} - u_{yy} = 2;$

$$u\big|_{y=0} = 0, \qquad u_y\big|_{y=0} = x + \cos x, \qquad |x| < \infty.$$

6. $u_{xy} + yu_x + xu_y + xyu = 0;$

$$u\big|_{y=3x} = 0, \quad u_y\big|_{y=3x} = e^{-5x^2}, \qquad x < 1.$$

7. $xu_{xx} + (x + y)u_{xy} + yu_{yy} = 0;$

$$u\Big|_{y=\frac{1}{x}} = x^3, \qquad u_x\Big|_{y=\frac{1}{x}} = 2x^2, \qquad x > 0.$$

8. $u_{xx} + 2(1 + 2x)u_{xy} + 4x(1 + x)u_{yy} + 2u_y = 0;$

$$u\big|_{x=0} = y, \qquad u_x\big|_{x=0} = 2, \qquad |y| < \infty$$

9. $x^2u_{xx} - y^2u_{yy} - 2yu_y = 0;$

$$u\big|_{x=1} = y, \quad u_x\big|_{x=1} = y, \qquad y < 0.$$

10. $x^2u_{xx} - 2xyu_{xy} - 3y^2u_{yy} = 0;$

$u|_{y=1} = 0, \quad u_y|_{y=1} = \sqrt[4]{x^7}, \quad x > 0.$

11. $yu_{xx} + x(2y-1)u_{xy} - 2x^2u_{yy} - \frac{y}{x}u_x = 0;$

$u|_{y=0} = x^2, \quad u_y|_{y=0} = 1, \quad x > 0.$

12. $yu_{xx} - (x+y)u_{xy} + xu_{yy} = 0;$

$u|_{y=0} = x^2, \quad u_y|_{y=0} = x, \quad x > 0$

13. $u_{xy} + 2u_x + u_y + 2u = 1, \quad x > 0, \; y < 1;$

9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)
10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$.
11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

1. $4y^2U_{xx} + 2(1-y^2)U_{xy} - U_{yy} - \frac{2y}{1+y^2}(2U_x - U_y) = 0;$

$U(x,0) = \varphi(x), \quad U_y(x,0) = \psi(x).$

2. $U_{xx} - 2U_{xy} + 4e^y = 0;$

$U(0,y) = \varphi(y), \quad U_x(0,y) = \psi(y).$

3. $U_{xx} + 2\cos x U_{xy} - \sin^2 x U_{yy} - \sin x U_y = 0;$

$U(x, \sin x) = x + \cos x, \quad U_y(x, \sin x) = \sin x.$

4. $3U_{xx} - 4U_{xy} + U_{yy} - 3U_x + U_y = 0;$

$U(x,0) = \varphi(x), \quad U_y(x,0) = \psi(x).$

5. $e^y U_{xy} - U_{yy} + U_y = 0;$

$U(x,0) = -\frac{x^2}{2}, \quad U_y(x,0) = -\sin x.$

6. $U_{xx} - 2\sin x U_{xy} - (3 + \cos^2 x) U_{yy} - \cos x U_y = 0;$

$U(x, \cos x) = \sin x, \quad U_y(x, \cos x) = \frac{e^x}{2}.$

7. $U_{xx} - 2\sin x U_{xy} - (3 + \cos^2 x) U_{yy} + U_x + (2 - \sin x - \cos x) U_y = 0;$

$U(x, \cos x) = 0, \quad U_y(x, \cos x) = e^{-\frac{x}{2}} \cos x.$

8. $U_{xx} + 2\sin x U_{xy} - \cos^2 x U_{yy} + U_x + (\sin x + \cos x + 1) U_y = 0;$

$U(x, -\cos x) = 1 + 2\sin x, \quad U_y(x, -\cos x) = \sin x.$

Quyidagi Koshi masalalarini yeching:

1. $xU_{xx} - U_{yy} + \frac{1}{2}U_x = 0;\ U|_{y=0} = x, U_y|_{y=0} = 0,\ x > 0.$
2. $U_{xy} = 0;\ U|_{y=x^2} = 0, U_y|_{y=x^2} = \sqrt{|x|},\ |x| < 1.$
3. $U_{xy} + U_x = 0;\ U|_{y=x} = \sin x, U_x|_{y=x} = 1,\ |x| < \infty.$
4. $U_{xx} - U_{yy} + 2U_x + 2U_y = 0;\ U|_{y=0} = x, U_y|_{y=0} = 0,\ |x| < \infty.$
5. $U_{xx} - U_{yy} - 2U_x - 2U_y = 4;\ U|_{x=0} = -y, U_x|_{x=0} = y - 1,$
   $|y| < \infty.$

TEST

Quyidagi tenglamalardan qaysi biri elliptik tipga tegishli?	* $u_{xx}+2u_{xy}+5u_{yy}-32u=0$	$u_{xx}-2u_{xy}+u_{yy}+9u_x$ $+9u_y-u=0$	$2u_{xx}+3u_{xy}+u_{yy}$ $+7u_x+4u_y-2u=0$	$u_{xx}+u_{xy}-2u_{yy}-3u$… $-15u_y+27x=0$
Quyidagitenglamalardanqaysibirigiperboliktipgategishli	* $2u_{xx}+3u_{xy}+u_{yy}+$ $+7u_x+4u_y-2u=0$	$u_{xx}-2u_{xy}+u_{yy}+$ $+9u_x+9u_y-u=0$	$u_{xx}+2u_{xy}+5u_{yy}-$ $-32u=0$	$u_{xx}+u_{xy}+2u_{yy}-3u$… $-15u_y+27x=0$
$u_{xx}-u_{yy}+\frac{2}{x}u_x-\frac{2}{y}u_y=0$ tengalama nechanchi tartibli	*2	1	3	4
$u_{xx}-u_{yy}+\frac{2}{x}u_x-\frac{2}{y}u_y=0$ tenglama uchun quyidagi fikrlarning qaysi to’g’ri ?	*Bir jinsli	kvazichiziqli	Chiziqli emas.	Bir jinsli emas
Quyidagi tenglama tipini aniqlang: $u_{xx}+xyu_{yy}=0$	*Aralash	Elliptik	Parabolik	Giperbolik
Tenglama tartibini aniqlang: $u_x u_{xy}^2+(u_{xx}^2-2u_{xy}^2+u_y)^2-$ $-2xy=0$	*2	1	3	4
Tenglama tartibini aniqlang: $\cos^2 u_{xy}+\sin^2 u_{xy}-$ $-2u_x^2-3u_y+u=0$	*1	2	3	4

5-Mavzu:Koshi masalasi va uning qo‘yilishida xarakteristikalarning roli. Korrekt qo‘yilgan masala tushunchasi.

Reja

Tayanch so’z va iboralar: *Chiziqli differensial operator,* q*o’shma differensial operatorlar, o’zi-o’ziga qo’shma operator, Riman funksiyasi, Gursa masalasi,Koshi masalasi*.

Koshi masalasi. D sohada yotuvchi uzluksiz egrillikka ega bo’lgan ochiq Jordan chi-zig’ini δ orqali belgilaymiz. Bu chiziq shunday xossaga ega bo’lsinki, o’zining hech bir nuqtasida (2) tenglamaning xarakteristikalari bilan urinishga ega bo’lmasin $l-\delta$ da berilgan vektor bo’lib, δ ning urinmasi bilan hech qanday nuqtada ustma-ust tushmasin.

Koshi masalasi bunday qo’yiladi. (2) *tenglamaning ushbu*

$$u\big|_{\delta}=\varphi, \qquad \frac{\partial u}{\partial l}\bigg|_{\delta}=\psi \tag{15}$$

boshlang’ich shartlarni qanoatlantiruvchi yechimi topilsin, bu yerda berilgan $\varphi(x)$ va $\psi(x)$ mos ravishda ikki marta va bir marta uzluksiz differensiallanuvchi funksiyalardir.

$P(x,y)$ nuqtadan chiquvchi $x_1=x,\ y_1=y$ xarakteristikalar δ egri chiziq bilan Q' va Q

nuqtalarda kesishadi deb faraz qilamiz.

PQ, PQ' to’g’ri chiziqlar va δ egri chiziqning QQ' qismi bilan chegaralangan sohani G orqali belgilab olamiz (20-chizma).

G sohadagi ixtiyoriy ikki marta uzluksiz differensiallanuvchi $u(x,y_1)$ va $\upsilon(x_1,y_1)$ funk-siyalar uchun quyidagi ayniyat o’rinli bo’ladi:

$$\begin{aligned}2(\upsilon Lu-uM\upsilon)=&\frac{\partial}{\partial y_1}\left(\frac{\partial u}{\partial x_1}\upsilon-\frac{\partial \upsilon}{\partial x_1}u+2bu\upsilon\right)+\\&+\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial y_1}\upsilon-\frac{\partial \upsilon}{\partial y_1}u+2au\upsilon\right)\end{aligned} \tag{16}$$

Bu ayniyatni G soha bo’yicha integrallab, Gauss – Ostrogradskiy formulasini qo’llash natijasida

$$2\int_G (\upsilon Lu - uM\upsilon)dx_1dy_1 = \int_S \left(\frac{\partial u}{\partial y_1}\upsilon - \frac{\partial \upsilon}{\partial y_1}u + 2au\upsilon\right)dy_1 -$$

$$-\left(\frac{\partial u}{\partial x_1}\upsilon - \frac{\partial \upsilon}{\partial x_1}u + 2bu\upsilon\right)dx_1.$$

tenglikni hosil qilamiz, bunda $S-G$ sohaning chegarasi, ya'ni

$$PQ + QQ' + Q'P.$$

PQ da $dy_1 = 0,$ PQ' da $dx_1 = 0$ bo'lgani uchun avvalgi tenglik quyidagi ko'rinishda yoziladi:

$$2\int_G (\upsilon Lu - uM\upsilon)dx_1dy_1 = \int_{QQ'} \left(\frac{\partial u}{\partial y_1}\upsilon - \frac{\partial \upsilon}{\partial y_1}u + 2au\upsilon\right)dy_1 -$$

$$-\left(\frac{\partial u}{\partial x_1}\upsilon - \frac{\partial \upsilon}{\partial x_1}u + 2bu\upsilon\right)dx_1 + \int_{Q'}^{P}\left(\frac{\partial u}{\partial y_1}\upsilon - \frac{\partial \upsilon}{\partial y_1}u + 2au\upsilon\right)dy_1 - \tag{17}$$

$$-\int_P^Q\left(\frac{\partial u}{\partial y_1}\upsilon - \frac{\partial \upsilon}{\partial y_1}u + 2au\upsilon\right)dx_1.$$

Bu ifodaning o'ng tomonidagi ikkinchi va uchinchi integrallarda $u(x_1, y_1)$ funksiyaning hosila-lari qatnashgan hadlarni bo'laklab integrallab, ushbu

$$\begin{aligned}
&\int_{Q'}^{P}\left(\upsilon\frac{\partial u}{\partial y_1} - u\frac{\partial \upsilon}{\partial y_1} + 2au\upsilon\right)dy_1 = (u\upsilon)_{Q'}^{P} - 2\int_Q^P u\left(\frac{\partial \upsilon}{\partial y_1} - a\upsilon\right)dy_1,\\
&\int_P^Q\left(\upsilon\frac{\partial u}{\partial x_1} - u\frac{\partial \upsilon}{\partial x_1} + 2bu\upsilon\right)dx_1 = (u\upsilon)_P^Q - 2\int_P^Q u\left(\frac{\partial \upsilon}{\partial x_1} - b\upsilon\right)dx_1
\end{aligned} \tag{18}$$

tengliklarga ega bo'lamiz.

(17) formulada $u(x_1, y_1)$ funksiya (2), (15) Koshi masalasining yechimi, $\upsilon(x_1, y_1)$ esa Riman funksiyasi, ya'ni $\upsilon(x_1, y_1) = \upsilon(P') = R(x_1, y_1; x, y) = R(P', P)$ bo'lsin deb hisoblaymiz.

U holda δ egri chiziqqa P' nuqtadan o'tkazilgan normalni n orqali belgilab, $dy_1 = \frac{dx_1}{dn}ds,\ dx_1 = -\frac{dy_1}{dn}ds$ formulalarni e'tiborga olsak, (17) dan (18) ga va Riman funksiyasining (8) xossalariga asosan

$$
\begin{aligned}
u(P) &= \frac{1}{2}u(Q)R(Q,P) + \frac{1}{2}u(Q')R(Q',P) - \\
&- \frac{1}{2}\int_{QQ'}\left[\frac{\partial u(P')}{\partial N}R(P',P) - u(P')\frac{\partial R(P',P)}{\partial N}\right]ds - \\
&- \frac{1}{2}\int_{QQ'}\left[a(P')\frac{dx_1}{dn} + b(P')\frac{dY_1}{dn}\right]R(P',P)u(P')dS + \\
&\int_G f(P')R(P')dx_1dy_1
\end{aligned}
\qquad (19)
$$

formulani hosil qilamiz, bunda

$$
\frac{\partial}{\partial N} = \frac{\partial x_1}{\partial n}\frac{\partial}{\partial y_1} + \frac{dy_1}{dn}\frac{\partial}{\partial x_1}.
$$

(15) boshlang'ich shartlarga asosan (19) formuladagi $\frac{\partial u}{\partial N}$ ni hamma vaqt bir qiymatli aniqlab olishimiz mumkin. (19) formula bilan aniqlangan $u(x,y)$ funksiyaning (2) tenglamani qanoat-lantirishini tekshirib ko'rish qiyin emas.

Shunday qilib, (19) *formula* (2), (15) *Koshi masalasining yechimidan iboratdir*. (19) formulani hosil qilish jarayonidan, bu masalaning yagonaligi va turg'unligi ham kelib chiqadi.

Asosiy adabiyotlar

4. Wolter A. Stranss. Partial differential equation; An introduction. Birkhhauzer.

Germaniy, 2005

5. Davia D. Bleecker, George Csordes. Basic of partial Differential Equations.

Birkhhauzer. Germaniy, 2009.

6. Салохиддинов М.С. Математик физика тенгламалари. Т., «Ўзбекистон»,

2002, 448 б.

4. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М. 2004.

5. Бицадзе А.В., Калиниченко Д.Ф. Сборник задач по уравнениям математической
физики. М. 1977.

Mustaqil ta'lim mavzulari

1. Qo'shma differensial operatorlar.
2. Riman usuli.

Glossariy

Chiziqli differensial operator-

$$Lu = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i} + A_0 u$$

chiziqli differensial operator deyiladi

Qo'shma differensial operatorlar-

$$M\upsilon = \sum_{i,j=1}^{n} \frac{\partial^2 (A_{ij}\upsilon)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (A_i \upsilon)}{\partial x_i} + A_0 \upsilon \qquad (1)$$

ifoda Lu ga *qo'shma differensial operator* deyiladi.

O*'zi-o'ziga qo'shma operator* **–** Agar

$$P_i = \sum_{j=1}^{n} \left[\upsilon A_{ij} \frac{\partial u}{\partial x_j} - u \frac{\partial (A_{ij}\upsilon)}{\partial x_j} \right] + A_i u \upsilon$$

belgilashni kiritsak, avvalgi tenglik quyidagi ko'rinishda yoziladi:

$$\upsilon Lu - uM\upsilon = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}$$

yoki

$$\upsilon Lu - uM\upsilon = \overline{P}, \qquad \overline{P} = (P_1, \ldots, P_n).$$

Agar $L \equiv M$ bo'lsa, L operator *o'zi-o'ziga qo'shma operator* deyiladi.

Riman funksiyasi-

$$Lu \equiv \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y) \qquad (2)$$

L operatorning *Riman funksiyasi* deb, quyidagi shartlarni qanoatlantiruvchi $\upsilon(x,y)$ funksiyaga aytiladi.

$$\begin{array}{l} 1. M\upsilon = 0, \\ 2.\ x = x_1,\quad y = y_1 \end{array} \text{ xarakteristikalarda} \qquad (3)$$

$$\upsilon(x_1, y) = e^{\int_{y_1}^{y} a(x_1,\tau)d\tau}, \quad \upsilon(x, y_1) = e^{\int_{x_1}^{x} b(t,y_1)dt}, \qquad (4)$$

bu yerda (x_1, y_1) nuqta (46) tenglama berilgan D sohaning tayin nuqtasidir.

Keyslar banki

Keys: Masala o`rtaga tashlanadi: Shturm – Liuvill masalasini yeching

$$\begin{cases} y'' + \lambda y = 0,\ 1/4 \le x \le 1/2, \\ \quad y'(1/4) = y'(1/2) = 0 \end{cases}$$

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni

  keltiring (individual va kichik guruhlarda);

- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Nazorat uchun savollar

1. Gyuygensprintsipi nima?
2. Gursa masalasiqanday masala?
3. Qo'shma differentsial operatorlar va Riman usulini bilasizmi?

Mustaqil ishlash uchun mashqlar

195. (73) тенгламанинг

$$u(x,0)=\varphi(x),\ 0\le x<\infty,$$
$$u(0,y)=\psi(y),\ 0\le y<\infty,$$
$$\varphi(0)=\psi(0);\ \varphi''(0)=\psi''(0)$$

шартларни қаноатлантирувчи ечимини топиш масаласи коррект қўйилганми?

196. (73) тенгламанинг

$$u(x,x)=\tau(x),\quad (u_x-u_y)|_{t=x}=\nu(x)$$

шартларни қаноатлантирувчи ечимини топиш масаласи коррект қўйилганми?

197. Ушбу

$$y^2u_{xx}+yu_{yy}+\frac{1}{2}u_y=0$$

тенглама учун

$$u(x,0)=\tau(x),\quad u_y(x,0)=\nu(x),\quad 0<x<1,$$

Коши масаласини $y<0$ да нокоррект эканлигини исботланг.

198. Ушбу

$$u_{xx}-u_{tt}=0,$$
$$u(x,x)=\tau(x),\quad u_y(x,x)=\nu(x),\quad -\infty<x<+\infty$$

масаланинг бир қийматли ечимини таъминловчи $\tau(x)$ ва $\nu(x)$ функ ялар орасидаги боғланишни топинг.

199. Ушбу

$$u_{xx}-u_{yy}=6(x+y),\quad -\infty<x,y<+\infty$$
$$u(x,x)=0,\quad u_x(x,x)=\tau(x)$$

масаланинг ечими ягона эмаслигини исботланг.

200. $D=\{(x,y):-\infty<x,y<+\infty\}$ соҳада қуйидаги

$$u_{xy}=0,\quad u(x,0)=\tau(x),\quad u_y(x,0)=\nu(x)$$

Коши масаласини нокоррект эканлигини исботланг.

TEST

Quyidagi tenglamaning tipi va tartibini aniqlang	*Giperbolik tipga tegishli,	Parabolik tipga tegishli,	Elliptik tipga tegishli, tartibi ikkiga	Giperbolik tipga tegishli,

$u_{xx} - 6u_{xy} + 3u_{yy} +$ $+2u_y - 2xyu_x = 0$	tartibi ikkiga teng	tartibi ikkiga teng;	teng	tartibi uchga teng
Quyidagi tenglamaning tipi va tartibini aniqlang $u_{xy} - u_y + u_x = 0$	*Giperbolik tipga tegishli, tartibi ikkiga teng	Giperbolik tipga tegishli, tartibi birga teng	Elliptik tipga tegishli, tartibi ikkiga teng	Parabolik tipga tegishli, tartibi ikkiga teng;
Quyidagi tenglamaning tipi va tartibini aniqlang $5u_{xx} + 4u_{yx} + 3u_{yy} +$ $+2u_y - u_x = 0$	*Elliptik tipga tegishli, tartibi ikkiga teng	Giperbolik tipga tegishli, tartibi ikkiga teng	Parabolik tipga tegishli, tartibi ikkiga teng;	Parabolik tipga tegishli, tartibi birga teng
Quyidagi $u_{xx} + u_{xy} - 2u_{yy} -$ $-u_y + u_x - u = 0$ tenglamaning tipi va tartibini aniqlang	*Giperbolik tipga tegishli, tartibi ikkiga teng	Elliptik tipga tegishli, tartibi birga teng	Elliptik tipga tegishli, tartibi ikkiga teng	Parabolik tipga tegishli, tartibi ikkiga teng;
Quyidagi $u_{xx} + 4u_{yy} + 5u_{xy} -$ $-6u_y + e^x u = 0$ tenglamaning tipi va tartibini aniqlang	*Elliptik tipga tegishli, tartibi ikkiga teng	Giperbolik tipga tegishli, tartibi ikkiga teng	Parabolik tipga tegishli, tartibi ikkiga teng;	Giperbolik tipga tegishli, tartibi birga teng
Tenglamaning kanonik tenglamasini aniqlang: $u_{xx} + 2u_{xy} +$ $+5u_{yy} - 32u = 0$	* $v_{\xi\xi} + v_{\eta\eta} - 8v = 0$	$v_{\xi\xi} + v_{\eta\eta} = 8v$	$v_{\xi\xi} - 8v = 0$	$v_{\eta\eta} - 32v = 0$
$u_{xx} - 9u_{yy} = 0$	* $y + 3x = const$	$2y + x = const$,	$y + 2x = const$,	$3y + x = const$,

tenglama xarakteristikalarini aniqlang?	, $y-3x=const$	$2y-x=const$	$y-2x=const$	$3y-x=const$
$4u_{xx}-u_{yy}=0$ tenglama xarakteristikalarini aniqlang?	* $2y+x=const$, $2y-x=const$	$y+2x=const$, $y-2x=const$	$y+3x=const$, $y-3x=const$	$3y+x=const$, $3y-x=const$

6-mavzu: Giperbolik tipdagi tenglamalar. Tor tebranish tenglamasi. Dalamber formulasi. Dalamber formulasi bilan aniqlangan yechimning fizik ma'nosi.

1. Ushbu

$$Lu = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i} + A_0 u$$

chiziqli differensial operatorni tekshiramiz.

D – E^n fazoda bo'laklari silliq S sirt bilan chegaralangan soha bo'lsin. Faraz qilamiz, A_{ij} koeffitsientlari ikkinchi ikkinchi tartibli, A_i - birinchi tartibli uzluksiz hosilalarga ega, A_{ij} koeffitsient esa uzluksiz bo'lsin.

$$M\upsilon = \sum_{i,j=1}^{n} \frac{\partial^2 (A_{ij}\upsilon)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (A_i \upsilon)}{\partial x_i} + A_0 \upsilon \qquad (1)$$

ifoda Lu ga *qo'shma differensial operator* deyiladi.

Bevosita hisoblash bilan

$$\upsilon Lu - uM\upsilon = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \sum_{j=1}^{n} \left[\upsilon A_{ij} \frac{\partial u}{\partial x_j} - u \frac{\partial (A_{ij}\upsilon)}{\partial x_j} \right] \right\} + A_i u \upsilon$$

tenglikning to'g'riligiga ishonch hosil qilamiz.

Agar

$$P_i = \sum_{j=1}^{n} \left[\upsilon A_{ij} \frac{\partial u}{\partial x_j} - u \frac{\partial (A_{ij}\upsilon)}{\partial x_j} \right] + A_i u \upsilon$$

belgilashni kiritsak, avvalgi tenglik quyidagi ko'rinishda yoziladi:

$$\upsilon Lu - uM\upsilon = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}$$

yoki

$$\upsilon Lu - uM\upsilon = \overline{P}, \qquad \overline{P} = (P_1, \ldots, P_n).$$

Agar $L \equiv M$ bo'lsa, L operator *o'zi-o'ziga qo'shma operator* deyiladi.

L operatorni quyidagi ko'rinishda yozib olamiz:

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial u}{\partial x_j} \right) - \sum_{i,j=1}^{n} \frac{\partial A_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i} + A_0 u$$

Agar

$$B_i = A_i - \sum_{j=1}^{n} \frac{\partial A_{ij}}{\partial x_j}, \qquad A_0 = C$$

belgilashlarni kiritsak, L bunday ko'rinishga keladi:

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu \qquad (0)$$

Bu holda M operator ushbu

$$M\upsilon = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial A_{ij}}{\partial x_j}\upsilon + A_{ij} \frac{\partial \upsilon}{\partial x_j} \right) - \sum_{i=1}^{n} \frac{\partial (A_i \upsilon)}{\partial x_i} + A_0 \upsilon$$

$$= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial \upsilon}{\partial x_j} \right) - \sum_{i=1}^{n} \frac{\partial (B_i \upsilon)}{\partial x_i} + C\upsilon \qquad (1)$$

`

ko'rinishga ega bo'ladi.

Agar M operator berilgan bo'lsa, unga qo'shma operator L bo'lishini tekshirib ko'rish qiyin emas. (0) va (1) formulalardan ko'rinyaptiki, qo'shma operatorlar faqat o'rta hadlari bilan bir-biridan farq qiladi.

Ravshanki, $B_i \equiv 0, \quad i = 1,2,...,n$ da faqat shu holdagina $M \equiv L$ bo'ladi. Bundan darhol o'zi-o'ziga qo'shma operatorni

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial u}{\partial x_j} \right) + Cu, \qquad A_{ij} = A_{ji}$$

ko'rinishga keltirish mumkin ekanligi kelib chiqadi.

Bizga ma’lum bo’lgan Laplas va to’lqin operatori o’zi-o’ziga qo’shma operatorlardir, lekin issiqlik tarqalish operatori esa o’zi-o’ziga qo’shma operator bo’lmaydi.

2. Riman funksiyasi. Birinchi bobdan bizga ma’lumki, ikkinchi tartibli ikki o’zgaruvchi xususiy hosilali differensial tenglamaning koeffitsientlari yetarli umumiy shartlarni qanoatlantir-ganda, uni

$$Lu \equiv \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y) \qquad (2)$$

ko’rinishda yoziladi.

L operatorning *Riman funksiyasi* deb, quyidagi shartlarni qanoatlantiruvchi $\upsilon(x,y)$ funksiyaga aytiladi.

1. $M\upsilon = 0,$
2. $x = x_1, \quad y = y_1$ xarakteristikalarda (3)

$$\upsilon(x_1, y) = e^{\int_{y_1}^{y} a(x_1,\tau)d\tau}, \quad \upsilon(x, y_1) = e^{\int_{x_1}^{x} b(t,y_1)dt}, \qquad (4)$$

bu yerda (x_1, y_1) nuqta (2) tenglama berilgan D sohaning tayin nuqtasidir.

Agar qo’shimcha $\frac{\partial a}{\partial x}, \frac{\partial b}{\partial y}$ va $c(x,y)$ funksiyalarning uzluksizligi talab qilinsa, u holda Riman funksiyasi mavjud bo’ladi.

Haqiqatan ham, (3) tenglamani x_1 dan x gacha va y_1 dan y gacha ikki marta integral-lash natijasida quyidagi tenglikni hosil qilamiz:

$$\begin{gathered}\upsilon(x,y) - \upsilon(x,y_1) - \upsilon(x_1,y) + \upsilon(x_1,y_1) - \int_{y_1}^{y} a(x,\tau)\upsilon(x,\tau)d\tau + \\ + \int_{y_1}^{y} a(x_1,\tau)\upsilon(x_1,\tau)d\tau - \int_{x_1}^{x} b(t,y)\upsilon(t,y)dt + \int_{x_1}^{x} b(t,y_1)\upsilon(t,y_1)dt + \\ + \int_{x_1}^{x} dt \int_{y_1}^{y} c(t,\tau)\upsilon(t,\tau)d\tau = 0.\end{gathered} \qquad (5)$$

(4) shartlarga asosan

$$\frac{\partial \upsilon(x_1, y)}{\partial y} = a(x_1, y)\upsilon(x_1, y), \quad \frac{\partial \upsilon(x, y_1)}{\partial x} = b(x, y_1)\upsilon(x, y_1)$$

yoki

$$\upsilon(x, y) - \int_{x_1}^{x} b(t, y_1)\upsilon(t, y_1)dt = 1$$

$$\upsilon(x_1, y) - \int_{y_1}^{y} a(x_1, \tau)\upsilon(x_1, \tau)d\tau = 1$$

$$\upsilon(x_1, y_1) = 1.$$

Bularni e'tiborga olsak, (5) tenglik $\upsilon(x, y)$ ga nisbatan Volterraning ikkinchi turdagi chiziqli integral tenglamasi ko'rinishida yoziladi:

$$\begin{gathered} \upsilon(x, y) - \int_{x_1}^{x} b(t, y)\upsilon(t, y)dt - \int_{x_1}^{x} a(x, \tau)\upsilon(x, \tau)d\tau + \\ + \int_{x_1}^{x} dt \int_{y_1}^{y} c(t, \tau)\upsilon(t, \tau)d\tau = 1. \end{gathered} \tag{6}$$

(6) tenglama izlanayotgan funksiyani bir qiymatli

$$\begin{gathered} \upsilon(x, y) = w(x, y) + \int_{x_1}^{x} w(t, y)b(t, y)\exp\left(\int_{t}^{x} b(t_1, y)dt_1\right)dt + \\ + \int_{y_1}^{y} w(x, \tau)a(x, \tau)\exp\left(\int_{t}^{y} a(x, \tau_1)d\tau_1\right)d\tau \end{gathered}$$

almashtirish natijasida quyidagi integral tenglamaga keladi:

$$w(x, y) + \int_{x_1}^{x} dt \int_{y_1}^{y} K_0(x, y; t, \tau)w(t, \tau)d\tau = 1, \tag{7}$$

bu yerda

$$K_0(x,y;t,\tau)=c(t,\tau)-b(t,y)a(t,\tau)\exp\left(\int_\tau^y a(t,\tau_1)d\tau_1\right)-$$
$$-a(x,\tau)b(t,\tau)\exp\left(\int_t^x b(t_1,\tau)dt_1\right)+$$
$$+b(t,\tau)\int_t^x c(t_1,\tau)\exp\left(\int_{t_1}^t b(t_2,\tau)dt_2\right)dt_1+$$
$$+a(t,\tau)\int_\tau^y c(t,\tau_1)\exp\left(\int_{\tau_1}^\tau a(t,\tau_2)d\tau_2\right)d\tau_1.$$

II bobdan ma'lumki, (7) tenglama yagona yechimga egadir.

Riman funksiyasi faqat x,y o'zgaruvchilarga bog'liq bo'lmay, x_1,y_1 o'zgaruvchilarga ham bog'liq bo'lgani uchun, uni

$$\upsilon=R(x,y;\ x_1,y_1)$$

ko'rinishda belgilab olish tabiiydir.

(4) ga asosan, ushbu

$$\begin{gathered}\frac{\partial R(x,y;x_1,y_1)}{\partial y}-a(x_1,y)R(x_1,y;x_1,y_1)=0,\\ \frac{\partial R(x,y_1;x_1,y_1)}{\partial x}-b(x,y_1)R(x,y_1;x_1,y_1)=0,\\ R(x_1,y_1;x_1,y_1)=1\end{gathered}\tag{8}$$

va

$$R(x,y;x,y_1)=e^{\int_{y_1}^{y}a(x,\tau)d\tau},\quad R(x,y;x_1,y)=e^{\int_{x_1}^{x}b(t,y)d\tau}$$

shartlarni e'tiborga olib,

$$\begin{gathered}\frac{\partial R(x,y;x,y_1)}{\partial y_1}-a(x,y_1)R(x,y;x,y_1)=0,\\ \frac{\partial R(x,y;x_1,y)}{\partial x_1}-b(x_1,y)R(x,y;x_1,y)=0,\\ R(x,y;x,y)=1\end{gathered}\tag{9}$$

tengliklarni hosil qilamiz.

Riman funksiyasi (3) tenglamaning yechimi bo’lgani uchun, ya’ni

$$MR(x_1, y_1; x, y) = 0$$

yoki

$$\frac{\partial^2 R}{\partial x_1 \partial y_1} - \frac{\partial}{\partial x_1}(aR) - \frac{\partial}{\partial y_1}(bR) = -cR(x_1, y_1; x, y)$$

tenglik o’rinlidir. Bunga asosan, D sohadagi yetarli silliq $u(x_1, y_1)$ funksiya uchun ushbu

$$\begin{gathered} \frac{\partial^2}{\partial x_1 \partial y_1}[u(x_1, y_1)R(x_1, y_1; x, y)] - R(x_1, y_1; x, y)Lu(x_1, y_1) = \\ \frac{\partial}{\partial x_1}\left[u\left(\frac{\partial R}{\partial y_1} - aR\right)\right] + \frac{\partial}{\partial y_1}\left[u\left(\frac{\partial R}{\partial x_1} - bR\right)\right] \end{gathered} \tag{10}$$

ayniyatning to’g’riligiga ishonch hosil qilish qiyin emas.

(10) ayniyatni x_1 va y_1 o’zgaruvchilar bo’yicha $x_0 \le x_1 \le x,\ y_0 \le y_1 \le y$ oraliqlarda integrallab, bu yerda (x_0, y_0) D sohaning ixtiyoriy nuqtasi, (8) ga asosan quyidagi teng-likni hosil qilamiz:

$$\begin{gathered} u(x, y) = u(x_0, y)R(x_0, y; x, y) + u(x, y_0)R(x, y_0; x, y) - \\ - u(x_0, y_0)R(x_0, y_0; x, y) + \\ + \int_{y_0}^{y}\left[a(x_0, y_1)R(x_0, y_1; x, y) - \frac{\partial R(x_0, y_1; x, y)}{\partial y_1}\right] \times \\ \times u(x_0, y_1)dy_1 + \int_{x_0}^{x}\left[b(x_1, y_0)R(x_1, y_0; x, y) - \right. \\ \left. - \frac{\partial R(x_1, y_0; x, y)}{\partial x_1}\right]u(x_1, y_0)dx_1 - \\ - \int_{x_0}^{x_1} dx_1 \int_{y_0}^{y} R(x_1, y_1; x, y)Lu(x_1, y_1)dy_1. \end{gathered} \tag{11}$$

Bu yerdagi Riman funksiyasining hosilalari qatnashgan integrallarni bo’laklab integral-laymiz:

$$\int_{y_0}^{y} u(x_0, y_1)\frac{\partial R(x_0, y_1; x, y)}{\partial y_1}dy_1 = u(x_0, y)R(x_0, y; x, y) -$$
$$- u(x_0, y_0)R(x_0, y_0; x, y) - \int_{y_0}^{y} R(x_0, y_1; x, y)\frac{\partial u(x_0, y_1)}{\partial y_1}dy_1,$$

$$\int_{x_0}^{x} u(x_1, y_0)\frac{\partial R(x_1, y_0; x, y)}{\partial x_1}dx_1 = u(x, y_0)R(x, y_0; x, y) -$$
$$- u(x_0, y_0)R(x_0, y_0; x, y) - \int_{x_0}^{x} R(x_1, y_0; x, y)\frac{\partial u(x_1, y_0)}{\partial x_1}dx_1.$$

Bularga asosan, (11) tenglik

$$\begin{gathered} u(x, y) = u(x_0, y_0)R(x_0, y_0; x, y) + \\ + \int_{x_0}^{x} R(x_1, y_0; x, y)\left[\frac{\partial u(x_1, y_0)}{\partial x_1} + b(x_1, y_0)u(x_1, y_0)\right]dx_1 + \\ + \int_{y_0}^{y} R(x_0, y_1; x, y)\left[\frac{\partial u(x_0, y_1)}{\partial y_1} + a(x_0, y_1)u(x_0, y_1)\right]dy_1 + \\ - \int_{x_0}^{x_1} dx_1 \int_{y_0}^{y} R(x_1, y_1; x, y)Lu(x_1, y_1)dy_1 \end{gathered} \quad (12)$$

ko’rinishda yoziladi.

Agar $u(x, y) = R(x_0, y_0; x, y)$ bo’lsa, (12) dan (9) ga asosan

$$\int_{x_0}^{x_1} dx_1 \int_{y_0}^{y} R(x_1, y_1; x, y)LR(x_0, y_0; x_1, y_1)dy_1 = 0 \quad (57)$$

ayniyat hosil bo’ladi.

(13) ayniyatdan $R(x, y; x_1, y_1)$ Riman funksiyasi oxirgi juft x_1, y_1 o’zgaruvchilarga nisba-tan bir jinsli

$$LR(x, y; x_1, y_1) = 0 \quad (14)$$

tenglamaning yechimi ekanligi kelib chiqadi. (2) tenglamaning o’ng tomonidagi $f(x, y)$ funk-siya uzluksiz bo’lganda, ushbu

$$u_0(x,y)=\int_{x_0}^{x_1} dx_1\int_{y_0}^{y} R(x_1,y_1;x,y)f(x_1,y_1)dy_1$$

funksiya uning xususiy yechimlaridan biri bo’ladi. Bunga (9) va (14) ga asosan, bevosita hisoblash bilan ishonch hosil qilish qiyin emas.

3. Gursa masalasi. (11) ayniyatdagi $u(x,y)$ funksiyani (2) tenglamaning yechimi deb hisoblaymiz. U holda, (11) formulaning o’ng tomonidagi $u(x,y_0)$ va $u(x_0,y)$ larni ixtiyoriy differensiallanuvchi funksiyalar bilan, $u(x_0,y_0)$ ni ixtiyoriy o’zgarmas, $Lu(x_1,y_1)$ ni $f(x_1,y_1)$ funksiya bilan almashtirsak, Riman funksiyasining xossalariga asosan (11) formula (2) tengla-maning regulyar yechimini beradi.

Demak, (2) tenglama uchun

$$u(x,y_0)=\varphi(x),\qquad u(x_0,y)=\psi(y).$$

Gursa masalas, bunda $\varphi(x)$ va $\psi(x)$- berilgan uzluksiz differensiallanuvchi, $\varphi(x_0)=\psi(x_0)$ shartni qanoatlantiruvchi funksiyalar, yagona turg’un yechimga ega bo’ladi va bu yechim ushbu

$$\begin{aligned}u(x,y)=&R(x,y_0;x,y)\varphi(x)+R(x_0,y;x,y)\psi(y)-\\&-R(x_0,y_0;x,y)\varphi(x_0)+\\&+\int_{x_0}^{x}\left[b(x_1,y_0)R(x_1,y_0;x,y)-\frac{\partial}{\partial x_1}R(x_1,y_0;x,y)\right]\varphi(x_1)dx_1+\\&+\int_{y_0}^{y}\left[a(x_0,y_1)R(x_0,y_1;x,y)-\frac{\partial}{\partial y_1}R(x_0,y_1;x,y)\right]\psi(y_1)dy_1+\\&+\int_{x_0}^{x}dx_1\int_{y_0}^{y}R(x_1,y_1;x,y)f(x_1,y_1)dy_1\end{aligned}$$

formula bilan aniqlanadi.

Nazorat uchun savollar.

1. Ikki o’zgaruvchili ikkinchi tartibli xususiy hosilali giperbolik tipdagi differentsial tenglama uchun Koshi masalasini qo’ying.

2. Ikki o’zgaruvchili ikkinchi tartibli xususiy hosilali giperbolik tipdagi differentsial tenglama uchun Gursa masalasini qo’ying
3. Koshi masalasini Dalamber ususli bilan yechishning bosqichlarini tushuntiring.
4. Koshi masalasining yechimi, berilgan tenglamaning qanday yechimi bo’ladi.
5. Gursa masalasini yechish usullarini tushuntiring.

Xarakteristikada berilgan masalalarni yeching:

88. $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$; $y+x=0$ da $u(x,y)=\varphi(x)$, $y-x=0$ da $u(x,y)=\psi(x)$, $\varphi(0)=\psi(0)$.

89. $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}=0$; $y-x=0$ da $u(x,y)=\varphi(x)$, $5x-y=0$ da $u(x,y)=\psi(x)$, $\varphi(0)=\psi(0)$.

90. $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}=0$; $y=5x+3$ da $u(x,y)=\varphi(x)$, $y=x-1$ da $u(x,y)=\psi(x)$, $\varphi(-1)=\psi(-1)$.

91. $\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+8\frac{\partial^2 u}{\partial y^2}=0$; $y+4x=0$ da $u(x,y)=\varphi(x)$, $y+2x+2=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

92. $3\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}=0$; $x-y-1=0$ da $u(x,y)=\varphi(x)$, $x+3y+1=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{2})=\psi(\frac{1}{2})$.

93. $4\frac{\partial^2 u}{\partial x^2}-8\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}=0$; $x+2y+1=0$ da $u(x,y)=\varphi(x)$, $3x+2y+2=0$ da $u(x,y)=\psi(x)$, $\varphi(-\frac{1}{2})=\psi(-\frac{1}{2})$.

94. $3\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}=0$; $x+3y+2=0$ da $u(x,y)=\varphi(x)$,

7-mavzu: Tor tebranish tenglamasi uchun Koshi masalasi yechimining yagonaligi va turg'unligi.

Reja:

1. Masalaning qo`yilishi.
2. Masala yechimining mavjudligi. Dalamber usuli.
3. Masala yechimining yagonaligi.
4. Masala yechimining turg'unligi.
5. Mavzuga doir masalalar yechish.

Koshi masalasi. $\Omega=\{(x,t): -\infty<x<+\infty,\ 0<t<+\infty\}$ sohadabir jinsli tor tebranish

$$\frac{\partial^2 U}{\partial t^2}-a^2\frac{\partial^2 U}{\partial x^2}=0 \qquad (1)$$

tenglamasining

$$U(x,0)=f_1(x), \qquad \frac{\partial U(x,0)}{\partial t}=f_2(x) \qquad (2)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Tor deganda ingichka elastik, ya'ni uzunligi o'zgarmaydigan, shaklining o'zgarishiga qarshilik ko'rsatmaydigan ip yoki sim tushuniladi. (1)- (2) Koshi masalasiga kuch ta'sirisiz erkin ravishda ko'ndalang tebranayotgan cheksiz uzun torning berilgan boshlang'ich xolati $f_1(x)$ va boshlang'ich tezligi $f_2(x)$ larga asosan torning keyingi vaqt va x nuqtalardagi xolati $U(x,t)$ ni aniqlash haqidagi fizik masala mos keladi.

Tarixiy ma'lumotlar.

Oddiy differentsial tenglamalar nazariyasiga oid dastlabki masalalar XVI asrning oxiri va XVII asrning boshlarida paydo bo'lgan. Shotland matematigi Jon Neper (1550-1617) haqiqiy sonlar uchun logarifm ta'rifini berishga asos qilib ikkita o'zaro bog'liq uzluksiz to'g'ri chiziqli harakatni olib, shu orqali differentsial tenglama bilan aniqlanuvchi logarifmik funktsiyani topdi. Keyinchalik differentsial tenglamalarga keluvchi masalalar fizika, optika va boshqa sahalarda ham paydo bo'la boshlagan.

Xususiy hosilali differentsial tenglamalar nazariyasi dastlab torning tebranishi haqidagi masalani yechishdan boshlab rivojlana boshlagan. 1715 yilda ingliz matematigi Bruk Teylor (1685-1731) tor tebranish tenglamasini ba'zi chegaraviy shartlar asosida ikkita ikkinchi tartibli o'zgarmas koeffitsientli chiziqli bir jinsli oddiy differentsial tenglamalarga keltirib, yechimlarni sinuslar yig'indisi shaklida topdi.

1749 yilda frantsuz matematigi Dalamber Jan Leron (1717-1783) tor tebranish tenglamasining umumiy yechimini ikkita ixtiyoriy funktsiyalarning yig'indisi ko'rinishida topdi.

Masala yechimining mavjudligi.

(1)–(2) masalani Dalamber (xarakteristikalar) usuli bilan yechamiz. (1) tenglamaning xarakteristik tenglamasi

$$dx^2-a^2dt^2=0$$

bo'lib, bu tenglama ikkita har xil

$$x-at=C_1, \quad x+at=C_2$$

echimlarga ega bo'ladi. (1) tenglamadagi x va t o'zgaruvchilarni

$$\xi = x - at, \quad \eta = x + at, \quad U(x,t)=V(\xi,\eta)$$

tengliklarga asosan almashtiramiz. U holda

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial \xi^2} - 2a^2 \frac{\partial^2 V}{\partial \xi \partial \eta} + a^2 \frac{\partial^2 V}{\partial \eta^2},$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial \xi^2} + 2\frac{\partial^2 V}{\partial \xi \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}$$

bo'lib, (1) tenglama ushbu

$$\frac{\partial^2 V}{\partial \xi \partial \eta} = 0 \qquad (3)$$

kanonik ko'rinishga keladi. (3) tenglamani

$$\frac{\partial}{\partial \xi}\left(\frac{\partial V}{\partial \eta}\right) = 0$$

ko'rinishda yozib, ξ bo'yicha integrallaymiz. Natijada birinchi tartibli

$$\frac{\partial V}{\partial \eta} = P(\eta)$$

('(η) – ixtiyoriy funksiya) tenglama hosil bo'ladi. Bu tenglamani η bo'yicha integrallab,

$$V(\xi,\eta) = \int P(\eta)d\eta + \varphi(\xi)$$

ifodaga ega bo'lamiz. Agar

$$\psi(\eta) = \int P(\eta)d\eta$$

deb belgilasak, u holda qaralayotgan kanonik tenglamaning umumiy yechimi

$$V(\xi,\eta) = \varphi(\xi) + \psi(\eta) \qquad (4)$$

ko'rinishida yoziladi. Bu yerda $\varphi(\xi)$, $\psi(\eta)$ ixtiyoriy funksiyalar. (4) ifodada ξ va η o'zgaruvchilardan eski x va t o'zgaruvchilarga qaytib, berilgan (1) tenglamaning umumiy yechimini hosil qilamiz:

$$U(x,t) = \varphi(x-at) + \psi(x+at). \qquad (5)$$

Bunda φ va ψ funksiyalarni ixtiyoriy, ikkinchi tartibligacha uzluksiz hosilalarga ega deb qaraymiz.

Umumiy yechimning (5) ifodasidan va (2) boshlang'ich shartlardan foydalanib, φ va ψ funksiyalarni topish uchun quyidagi sistemaga ega bo'lamiz:

$$\psi(x)+\varphi(x)=f_1(x), \tag{6}$$

$$\psi'(x)-\varphi'(x)=\frac{1}{a}f_2(x). \tag{7}$$

(7) tenglikni *x* bo‘yicha integrallab, sistemani

$$\psi(x)+\varphi(x)=f_1(x),$$
$$\psi(x)-\varphi(x)=\frac{1}{a}\int_0^x f_2(z)dz+C$$

ko‘rinishda yozamiz. Bunda *C*– ixtiyoriy o‘zgarmas son. Oxirgi sistemani yechib, φ va ψ funksiyalarni topamiz:

$$\varphi(x)=\frac{1}{2}f_1(x)-\frac{1}{2a}\int_0^x f_2(z)dz-\frac{C}{2}, \tag{8}$$

$$\psi(x)=\frac{1}{2}f_1(x)+\frac{1}{2a}\int_0^x f_2(z)dz+\frac{C}{2}. \tag{9}$$

(8) formulada *x* ni *x–at* bilan, (9) dagi *x* ni *x+at* bilan almashtiramiz va (5) ifodaga qo‘yib, quyidagini hosil qilamiz:

$$U(x,t)=\frac{f_1(x-at)+f_1(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at} f_2(z)dz. \tag{10}$$

Bu esa bir jinsli tor tebranish tenglamasi uchun Koshi masalasi yechimini ifodalovchi Dalamber formulasidir. Bu yerda $f_1(x)$ ikkinchi tartibli uzluksiz hosilaga, $f_2(x)$ birinchi tartibli uzluksiz hosilaga ega deb faraz qilinadi.

Masala yechimining yagonaligi.

Teorema. Agar (1) – (2) masalaning echimi mavjud bo’lsa, u xolda bu yechim yagona bo’ladi.

Isboti. Teskarisini faraz qilamiz. (1) – (2) masala ikkita $\overline{U}(x,t)\neq\overline{\overline{U}}(x,t)$ yechimlarga ega bo’lsin. U xolda $\upsilon(x,t)=\overline{U}(x,t)-\overline{\overline{U}}(x,t)$ funktsiya $\frac{\partial^2\upsilon(x,t)}{\partial t^2}-a^2\frac{\partial^2\upsilon(x,t)}{\partial x^2}=0$ (1’) tenglamaning $\upsilon(x,0)=0$, $\frac{\partial\upsilon(x,0)}{\partial t}=0$ (2’) boshlang’ich shartlarni qanoatlantiruvchi yechimi bo’ladi.

(1’) – (2’) masaga Dalamber (xarakteristikalar) usuli bilan yechish jarayonini takrorlab, $\upsilon(x,t)=0$ yechimni hosil qilamiz. Oxirgi tenglikdan $\overline{U}(x,t)=\overline{\overline{U}}(x,t)$ ekanligi kelib chiqadi. Teorema isbot bo’ldi.

Masala yechimining turg’unligi. Boshlang’ich shartlarda berilgan $f_1(x)$ va $f_2(x)$ funktsiyalarning kichik o’zgarishiga (1) – (2) masala echimining kichik o’zgarishi mos kelishini ko’rsatamiz. Faraz qilaylik ixtiyoriy kichik $\varepsilon>0$ son uchun $\left|f_1(x)\right|\le\varepsilon$, $\left|f_2(x)\right|\le\varepsilon$ tengsizliklar o’rinli bo’lsin. U xolda ixtiyoriy $t\le T$ uchun (1) – (2) masalaning yechimini ifodalovchi tenglikdan

$$\left|U(x,t)\right| \le \frac{\left|f_1(x-at)\right| + \left|f_1(x+at)\right|}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\left|f_2(z)\right|dz \le (1+T)\varepsilon$$ tengsizlikka ega bo'lamiz. Demak, (1) – (2) Koshi masalasi korrekt qo'yilgan masala ekan.

Bir jinsli bo‘lmagan tor tebranish tenglamasi

$$\frac{\partial^2 U}{\partial t^2} - a^2\frac{\partial^2 U}{\partial x^2} = f(x,t) \qquad (11)$$

uchun Koshi masalasi yechimini ifodalovchi Dalamber formulasi quyidagi ko‘rinishga ega bo‘ladi:

$$U(x,t) = \frac{f_1(x-at) + f_1(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} f_2(z)dz + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau \qquad (12)$$

Bunda $f(x,t)$ birinchi tartibli uzluksiz hosilalarga ega deb faraz qilindi.

<u>1–masala</u>. Bir jinsli tor tebranish tenglamasi $U_{tt}=U_{xx}$ uchun $U(x,0)=x^2$, $U_t(x,0)=0$ boshlang‘ich shartlarni qanoatlantiruvchi yechimi topilsin.

<u>Yechilishi</u>. Berilgan masalada $a=1$, $f_1(x)=x^2$, $f_2(x)=0$; $f(x,t)=0$ bo‘lganligi uchun (11) formulaga asosan izlangan yechim

$$U(x,t) = \frac{(x-t)^2 + (x+t)^2}{2} \quad \text{yoki} \quad U(x,t) = x^2 + t^2 \text{ bo‘ladi.}$$

<u>2–masala</u>. Bir jinsli bo‘lmagan tor tebranish tenglamasi $U_{tt}-4U_{xx}=t$ uchun $U(x,0)=0$, $U_t(x,0)=x$ boshlang‘ich shartlarni qanoatlantiruvchi yechimi topilsin.

<u>Yechilishi</u>. Bu masalada $a=2$, $f_1(x)=0$, $f_2(x)=x$, $f(x,t)=t$ bo‘lganligi uchun izlanayotgan yechimni (12) formuladan topamiz:

$$U(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} zdz + \frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \tau d\xi d\tau = \frac{1}{8}z^2\Big|_{x-2t}^{x+2t} + \frac{1}{4}\int_0^t \left(\xi\Big|_{x-2(t-\tau)}^{x+2(t-\tau)}\right)\tau d\tau =$$

$$= \frac{1}{8}\left[(x+2t)^2 - (x-2t)^2\right] + \int_0^t \left(t\tau - \tau^2\right)d\tau = xt + \left(t\frac{\tau^2}{2} - \frac{\tau^3}{3}\right)\Bigg|_0^t =$$

$$= xt + \frac{t^3}{2} - \frac{t^3}{3} = xt + \frac{t^3}{6}.$$

Demak, izlangan yechim $U(x,t) = xt + \dfrac{t^3}{6}$ bo‘ladi.

Tayanch iboralar.

Tor, chegaralanmagan tor, bir jinsli tor tebranish tenglamasi, boshlang'ich masala, Dalamber formulasi, bir jinsli bo'lmagan tor tebranish tenglamasi.

Nazorat uchun savollar.

1. Bir jinsli tor tebranish tenglamasi uchun Koshi masalasini qo'ying.
2. Bir jinsli tor tebranish tenglamasi uchun Koshi masalasiga qanday fizik masala mos keladi.

3. Koshi masalasini yechimini ifodalovchi Dalamber formulasini yozing.
4. Bir jinsli bo'lmagan tor tebranish tenglamasi uchun Koshi masalasini qo'ying.
5. Bir jinsli bo'lmagan tor tebranish tenglamasi uchun Koshi masalasi yechimini ifodalovchi Dalamber formulasini yozing.

Xususiy hosilali differensial tenglamalar almashtirish yordamida kanonik ko'rinishga keltirilgan, dastlabki tenglamaning berilgan boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini toping:

65. $\frac{\partial^2 u}{\partial\xi\partial\eta}-2\frac{\partial u}{\partial\xi}=0;\quad \xi=2x+3y,\ \eta=4x-5y,\ u|_{x=0}=1,\ \frac{\partial u}{\partial x}\Big|_{x=0}=2.$

66. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0;\quad \xi=3x+8y,\ \eta=4x-5y,\ u|_{x=0}=5,\ \frac{\partial u}{\partial x}\Big|_{x=0}=7.$

67. $\frac{\partial^2 u}{\partial\xi\partial\eta}-4\frac{\partial u}{\partial\xi}=0;\quad \xi=3x+7y,\ \eta=4x-5y,\ u|_{x=0}=1,\ \frac{\partial u}{\partial x}\Big|_{x=0}=2.$

68. $\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial u}{\partial\xi}=0;\quad \xi=3x-4y,\ \eta=5x+6y,\ u|_{x=0}=2,\ \frac{\partial u}{\partial x}\Big|_{x=0}=3.$

69. $\frac{\partial^2 u}{\partial\xi\partial\eta}-3\frac{\partial u}{\partial\xi}=0;\quad \xi=2x+3y,\ \eta=5x-4y,\ u|_{x=0}=1,\ \frac{\partial u}{\partial x}\Big|_{x=0}=1.$

70. $\frac{\partial^2 u}{\partial\xi\partial\eta}-2\frac{\partial u}{\partial\eta}=0;\quad \xi=5x-6y,\ \eta=x+2y,\ u|_{x=0}=4,\ \frac{\partial u}{\partial x}\Big|_{x=0}=1.$

71. $\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{4}\frac{\partial u}{\partial\eta}=0;\quad \xi=2x-3y,\ \eta=3x+4y,\ u|_{x=0}=2,\ \frac{\partial u}{\partial x}\Big|_{x=0}=1.$

72. $\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial u}{\partial\eta}=0;\quad \xi=4x-3y,\ \eta=5x+2y,\ u|_{x=0}=3,\ \frac{\partial u}{\partial x}\Big|_{x=0}=5.$

73. $\frac{\partial^2 u}{\partial\xi\partial\eta}-3\frac{\partial u}{\partial\eta}=0;\quad \xi=3x-4y,\ \eta=3x+5y,\ u|_{x=0}=y,\ \frac{\partial u}{\partial x}\Big|_{x=0}=1.$

74. $\frac{\partial^2 u}{\partial\xi\partial\eta}+2\frac{\partial u}{\partial\eta}=0;\quad \xi=2x+3y,\ \eta=3x+5y,\ u|_{y=0}=2x,\ \frac{\partial u}{\partial y}\Big|_{y=0}=3.$

75. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}-4\dfrac{\partial u}{\partial\xi}=0$; $\xi=3x+y,\ \eta=2y-5x,\ u\big|_{y=0}=3x+5,\ \dfrac{\partial u}{\partial y}\bigg|_{y=0}=4.$

76. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}=0$; $\xi=x^2y^3,\ \eta=y,\ u\big|_{x=1}=3y^3+5,\ \dfrac{\partial u}{\partial x}\bigg|_{x=1}=3y+1.$

77. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{1}{3\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^2y^3,\ \eta=x,\ u\big|_{y=1}=2x,\ \dfrac{\partial u}{\partial x}\bigg|_{y=1}=3x^2+1.$

78. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{2}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=y^2x^3,\ \eta=x,\ u\big|_{y=1}=2x^2,\ \dfrac{\partial u}{\partial y}\bigg|_{y=1}=3x+1.$

79. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=xy^3,\ \eta=y,\ u\big|_{x=1}=3y,\ \dfrac{\partial u}{\partial x}\bigg|_{x=1}=2+3y.$

80. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{1}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^3y^2,\ \eta=x,\ u\big|_{y=1}=2x^3,\ \dfrac{\partial u}{\partial y}\bigg|_{y=1}=3x\cdot$

81. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=xy^3,\ \eta=y,\ u\big|_{x=1}=1+2y,\ \dfrac{\partial u}{\partial x}\bigg|_{x=1}=3y^2.$

82. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{5}{3\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^3y^4,\ \eta=y,\ u\big|_{x=1}=3y^5,\ \dfrac{\partial u}{\partial x}\bigg|_{x=1}=3y^4.$

83. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{3\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^2y^4,\ \eta=y,\ u\big|_{x=1}=y,\ \dfrac{\partial u}{\partial x}\bigg|_{x=1}=3y+2.$

84. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{3}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^3y^2,\ \eta=y,\ u\big|_{x=1}=3y^2,\ \dfrac{\partial u}{\partial x}\bigg|_{x=1}=3y+2.$

85. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{9}{2\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^3y^2,\ \eta=x,\ u\big|_{y=1}=x^3,\ \dfrac{\partial u}{\partial x}\bigg|_{y=1}=x^2-2.$

86. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}-\dfrac{2}{\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^2y^3,\ \eta=x,\ u\big|_{y=1}=x^2+1,\ \dfrac{\partial u}{\partial x}\bigg|_{y=1}=x.$

87. $\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{4}{3\eta}\dfrac{\partial u}{\partial\xi}=0$; $\xi=x^3y^4,\ \eta=x,\ u\big|_{y=1}=4x^2,\ \dfrac{\partial u}{\partial x}\bigg|_{y=1}=6x.$

Xarakteristikada berilgan masalalarni yeching:

88. $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$; $y+x=0$ da $u(x,y)=\varphi(x)$, $y-x=0$ da $u(x,y)=\psi(x)$,

$\varphi(0)=\psi(0)$.

89. $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}=0$; $y-x=0$ da $u(x,y)=\varphi(x)$,

$5x-y=0$ da $u(x,y)=\psi(x)$, $\varphi(0)=\psi(0)$.

90. $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}=0$; $y=5x+3$ da $u(x,y)=\varphi(x)$,

$y=x-1$ da $u(x,y)=\psi(x)$, $\varphi(-1)=\psi(-1)$.

91. $\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+8\frac{\partial^2 u}{\partial y^2}=0$; $y+4x=0$ da $u(x,y)=\varphi(x)$,

$y+2x+2=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

92. $3\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}=0$; $x-y-1=0$ da $u(x,y)=\varphi(x)$,

$x+3y+1=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{2})=\psi(\frac{1}{2})$.

93. $4\frac{\partial^2 u}{\partial x^2}-8\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}=0$; $x+2y+1=0$ da $u(x,y)=\varphi(x)$,

$3x+2y+2=0$ da $u(x,y)=\psi(x)$, $\varphi(-\frac{1}{2})=\psi(-\frac{1}{2})$.

94. $3\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}=0$; $x+3y+2=0$ da $u(x,y)=\varphi(x)$,

$2x-y-1=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{7})=\psi(\frac{1}{7})$.

95. $$25\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}=0;$$ $2x-5y-4=0$ da $u(x,y)=\varphi(x)$,

$x+5y+3=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{3})=\psi(\frac{1}{3})$.

96. $$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-8\frac{\partial^2 u}{\partial y^2}=0;$$ $4x-y+3=0$ da $u(x,y)=\varphi(x)$,

$2x+y-4=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{6})=\psi(\frac{1}{6})$.

97. $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}-6\frac{\partial^2 u}{\partial y^2}=0;$$ $2x+y+1=0$ da $u(x,y)=\varphi(x)$,

$3x-y-2=0$ da $u(x,y)=\psi(x)$, $\varphi(\frac{1}{5})=\psi(\frac{1}{5})$.

98. $$2\frac{\partial^2 u}{\partial x^2}-7\frac{\partial^2 u}{\partial x\partial y}-4\frac{\partial^2 u}{\partial y^2}=0;$$ $4x+y+1=0$ da $u(x,y)=\varphi(x)$,

$x-2y+4=0$ da $u(x,y)=\psi(x)$, $\varphi(-\frac{2}{3})=\psi(-\frac{2}{3})$.

99. $$\frac{\partial^2 u}{\partial x^2}+2x\frac{\partial^2 u}{\partial x\partial y}-\frac{1}{x}\frac{\partial^2 u}{\partial y^2}=0, \quad (x>0);$$ $y-1=0$ da $u(x,y)=\varphi(x)$,

$x^2-y=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

100. $$\frac{\partial^2 u}{\partial x^2}+2x\frac{\partial^2 u}{\partial x\partial y}=0, \quad (y>0);$$ $y-x^2=0$ da $u(x,y)=\varphi(x)$,

$x-2=0$ da $u(x,y)=\psi(x)$, $\varphi(2)=\psi(4)$.

101. $$2y\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}=0, \quad (x>0);$$ $y-\sqrt{x}=0$ da $u(x,y)=\varphi(x)$,

$y-2=0$ da $u(x,y)=\psi(x)$, $\varphi(4)=\psi(4)$.

102. $$\frac{\partial^2 u}{\partial x^2}-4x^2\frac{\partial^2 u}{\partial y^2}-\frac{1}{x}\frac{\partial u}{\partial x}=0, \quad (x>0),$$ $y-x^2=0$ da $u(x,y)=\varphi(x)$,

$y+x^2+2=0$ da $u(x,y)=\psi(x)$, $\varphi(1)=\psi(1)$.

103. $$\frac{\partial^2 u}{\partial x^2}+2shx\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{1}{chx}\frac{\partial u}{\partial y}-thx\frac{\partial u}{\partial x}=0;$$ $y-e^x=0$ da $u(x,y)=\varphi(x)$,

$y-e^{-x}=0$ da $u(x,y)=\psi(x)$, $\varphi(0)=\psi(0)$.

8-mavzu:Tor tebranish tenglamasi uchun Gursa va Darbu masalasi masalasini yechish

Koshi masalasi. D_1 sohada (1) tenglamaning

$$U(x,y)|_L = f_1(x,y), \qquad \left.\frac{dU(x,y)}{dn}\right|_L = f_2(x,y) \tag{2}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi to'ilsin, bu yerda $f_1(x,y)$, $f_2(x,y)$ – berilgan funksiyalar.

(1)-(2) Koshi masalasini Dalamber usuli bilan yechish mumkin.

Gursa masalasi. $D_2 = \{(x,y): 0 < x < +\infty,\ 0 < y < +\infty\}$ sohada

$$u_{xy} = f(x,y) \tag{3}$$

tenglamaning

$$u(x,0) = \varphi_1(x),\ u(0,y) = \varphi_2(y) \tag{4}$$

shartlarni qanaotlantiruvchi $u(x,y)$ echimi topilsin. Bu yerda $x=0$ va $y=0$ chiziqlar (3) tenglamaning xarakteristkalari. $\varphi_1(x)$, $\varphi_2(y)$ funktsiyalar uzluksiz differentsiallanuvchi funktsiyalar bo'lib, $\varphi_1(0)=\varphi_2(0)$ shartni qanoatlantiruvchi berilgan funktsiyalar. $f(x,y)$ berilgan uzluksiz funktsiya.

(3) tenglamani x va y bo'yicha ketma-ket integrallab, quyidagi tengliklarni hosil qilamiz:

$$u_y(x,y) = u_y(0,y) + \int_0^x f(\xi,y)d\xi$$

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + \int_0^y d\eta \int_0^x f(\xi,\eta)d\xi$$

Oxirgi tenglikdan (4) shartlarga asosan (3)-(4) Gursa masalasining yechimiga ega bo'lamiz:

$$u(x,y) = \varphi_1(x) + \varphi_2(y) - \varphi_1(0) + \int_0^y \int_0^x f(\xi,\eta)d\xi d\eta$$

(3) tenglamadan umumiyroq bo'lgan tenglama uchun Gursa masalasini qaraymiz.

Gursa masalasi. D_2 sohada

$$u_{xy}=a(x,y)u_x+b(x,y)u_y+c(x,y)u+f(x,y) \qquad (3')$$

$$u(x,0)=\varphi_1(x), \qquad u(0,y)=\varphi_2(y) \qquad (4)$$

shartlarni qanoatlantiruvchi $u(x,y)$ funktsiya topilsin, bu yerda a,b,c,f koeffitsientlar x va y o'zgaruvchilarning berilgan uzluksiz funktsiyalari.

(3') tenglamani x va y bo'yicha ketma-ket integrallab, (4) shartlarga asosan quyidagi integro-differentsial tenglamaga ega bo'lamiz:

$$u(x,y)=\varphi_1(x)+\varphi_2(y)-\varphi_1(0)+\int_0^y\int_0^x f(\xi,\eta)d\xi d\eta+\int_0^y\int_0^x\left[a(\xi,\eta)u_\xi+b(\xi,\eta)u_\eta+c(\xi,\eta)u\right]d\xi d\eta$$

Bu integro-differentsial tenglamani ketma-ket yaqinlashish usuli bilan yechish mumkin.

Agar (3') tenglamada a,b,c koeffitsientlar o'zgarmas sonlar bo'lsa, u holda bu tenglamani $u(x,y)=ve^{\lambda x+\mu y}$ almashtirish yordamida

$$v_{xy}+c_1v=f_1(x,y) \qquad (3'')$$

ko'rinishga keltirish mumkin.

(3'') tenglamada $c_1=0$ bo'lsa, u holda (3)-(4) Gursa masalasi hosil bo'ladi.

<u>1–masala</u>.

$$x^2U_{xx}-y^2U_{yy}=0 \qquad (x>0,\ y>0) \qquad (3)$$

tenglamaning

$$U(x,1)=f_1(x),\ U_y(x,1)=f_2(x) \qquad (4)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini Dalamber usuli bilan to'ing.

<u>Yechilishi</u>. Berilgan tenglamani kanonik ko'rinishga keltirib integrallaymiz. Natijada kanonik tenglamaning umumiy yechimi hosil bo'ladi. Hosil bo'lgan yechimda eski x va y o'zgaruvchilarga qaytib, berilgan tenglamaning umumiy yechimiga ega bo'lamiz (1.3–§ dagi 3–misolga qarang):

$$U(x,y)=\varphi(xy)+\sqrt{xy}\,\psi\left(\frac{y}{x}\right). \qquad (5)$$

Umumiy yechimning (5) ifodasidan va (4) boshlang‘ich shartlardan foydalanib, ixtiyoriy φ va ψ funksiyalarni topish uchun quyidagi sistemani hosil qilamiz

$$\varphi(x)+\sqrt{x}\psi\left(\frac{1}{x}\right)=f_1(x), \tag{6}$$

$$x\varphi'(x)+\frac{\sqrt{x}}{2}\psi\left(\frac{1}{x}\right)+\frac{1}{\sqrt{x}}\psi'\left(\frac{1}{x}\right)=f_2(x). \tag{7}$$

(6) tenglamaning ikkala tomonini x bo‘yicha differensillaymiz, (7) tenglamaning ikkala tomonini esa x ga bo‘lamiz. Natijada

$$\varphi'(x)+\frac{1}{2\sqrt{x}}\psi\left(\frac{1}{x}\right)-\frac{1}{x\sqrt{x}}\psi'\left(\frac{1}{x}\right)=f_1'(x), \tag{8}$$

$$\varphi'(x)+\frac{1}{2\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{x\sqrt{x}}\psi'\left(\frac{1}{x}\right)=\frac{1}{x}f_2(x) \tag{9}$$

sistemaga ega bo‘lamiz. (8)–(9) sistemadan $\psi'\left(\frac{1}{x}\right)$ funksiyani topamiz

$$\psi'\left(\frac{1}{x}\right)=-\frac{x^{\frac{3}{2}}}{2}f_1'(x)+\frac{\sqrt{x}}{2}f_2(x)$$

va uni $[x_0, x]$ $(x\neq 0)$ oraliqda integrallab, $\psi\left(\frac{1}{x}\right)$ ni topamiz:

$$\psi\left(\frac{1}{x}\right)=\frac{1}{2}\int_{x_0}^{x}\frac{f_1'(z)}{\sqrt{z}}dz-\frac{1}{2}\int_{x_0}^{x}\frac{f_2(z)}{\sqrt{z^3}}dz+C, \tag{10}$$

bu yerda C– ixtiyoriy o‘zgarmas son.

(10) ni e’tiborga olib (6) dan $\varphi(x)$ ni topamiz:

$$\varphi(x)=f_1(x)-\frac{\sqrt{x}}{2}\int_{x_0}^{x}\frac{f_1'(z)}{\sqrt{z}}dz+\frac{\sqrt{x}}{2}\int_{x_0}^{x}\frac{f_2(x)}{\sqrt{z^3}}dz-C\sqrt{x}. \tag{11}$$

To’ilgan φ va ψ funksiyalarning (10) va (11) ifodalarini (5) tenglikka qo‘yib,

$$U(x,y)=f_1(xy)-\frac{\sqrt{xy}}{2}\int_{xy}^{\frac{x}{y}}\frac{f_1'(z)}{\sqrt{z}}dz+\frac{\sqrt{xy}}{2}\int_{xy}^{\frac{x}{y}}\frac{f_2(x)}{\sqrt{z^3}}dz \tag{12}$$

yechimni hosil qilamiz. (12) ifodadagi birinchi integralni bo‘laklab integrallab, berilgan (3) tenglamaning (4) boshlang‘ich shartlarni qanoatlantiruvchi yechimini quyidagi ko‘rinishda yozamiz:

$$U(x,y)=\frac{1}{2}f_1(xy)+\frac{y}{2}f_1\left(\frac{x}{y}\right)+\frac{\sqrt{xy}}{4}\int_{xy}^{\frac{x}{y}}\frac{f_1(z)-2f_2(z)}{\sqrt{z^3}}dz\,.$$

I. $D=\{(x,t):0<x<l,\ 0<t<+\infty\}$ sohada bir jinsli $U_{tt}=a^2U_{xx}$ tor tebranish tenglamasi uchun quyidagi aralash masalalar yechilsin:

1. $U(0,t)=U(l,t)=0,\ U(x,0)=0, U_t(x,0)=\sin(\frac{2\pi x}{l})$

2. $U(0,t)=U(l,t)=0,\ U(x,0)=5\sin(\frac{3\pi x}{l})-\frac{1}{2}\sin\left(\frac{8\pi x}{l}\right), U_t(x,0)=0$

3. $U(0,t)=U(l,t)=0,\ U(x,0)=0, U_t(x,0)=6\sin(\frac{\pi x}{l})-\sin\left(\frac{3\pi x}{l}\right)+\sin\left(\frac{7\pi x}{l}\right)$

4. $U(0,t)=U(l,t)=0,\ U(x,0)=\frac{1}{3}\sin(\frac{2\pi x}{l})+4\sin\left(\frac{5\pi x}{l}\right)-\frac{1}{4}\sin\left(\frac{8\pi x}{l}\right), U_t(x,0)=A\sin(\frac{s\pi x}{l})+B\sin\left(\frac{p\pi x}{l}\right), A,B=const, s,p\in N$

5. $U(0,t)=U(l,t)=0,\ U(x,0)=Ax, U_t(x,0)=0$

6. $U(0,t)=U(l,t)=0,\ U(x,0)=\frac{16h}{5}\left(\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)\right),\ h>0, U_t(x,0)=0$

7. $U(0,t)=U(l,t)=0,\ U(x,0)=\frac{4hx(l-x)}{l^2},\ h>0, U_t(x,0)=0$

Tayanch iboralar.

Aralash masala, chegaralangan sterjen, o’zgaruvchilarni ajratish, xos sonlar, xos funktsiyalar, Furg’e qatori, Furg’e koeffitsientlari, funktsional qator, tekis yaqinlashish.

Nazorat uchun savollar.

1. Bir jinsli tor tebranish tenglamasi uchun aralash masalani qo’ying.
2. O’zgaruvchilarni ajratish usulining moxiyatini tushuntiring.
3. Qaysi masala xos sonva xos funktsiyalar xaqidagi masala deyiladi.

4. Bir jinsli tor tebranish tenglamasining bir jinsli chegaraviy shartlarni qanoatlantiruvchi aralash masala yechimining ko'rinishini yozing.
5. Bir jinsli bo'lmagan tor tebranish tenglamasining bir jinsli chegaraviy shartlarni qanoatlantiruvchi aralash masala yechimining ko'rinishini yozing.

9-mavzu: Tor tebranish tenglamasi uchun birinchi tur chegaraviy masala yechimining yagonaligi.

Reja:

1. Aralash masalaning qo'yilishi.
2. Yagonalik teoremasi.
3. Energiya integrali usuli.

Birdan birlik teoremasi. Energiya integrali. Tenglamani va unga qo'shimcha shartlarni shunday tanlaylikki, isbot qilinadigan teoremamamiz keyinchalik issiqlik tarqalish masalasiga ham tegishli bo'lsin . teoremani bir o'lchovli fazoda ko'ramiz . quyida tenglamani ko'raylik.

$$\alpha\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left\{\gamma\frac{\partial u}{\partial x}\right\}+\delta u=f(x,t) \qquad (1)$$

Bunda $\alpha,\beta,\delta,\gamma$ lar umuman x ning uzliksiz funksiyalari $0\le x\le l$, $\alpha\ge 0,\ \alpha=0$ bo'lganda $\beta>0,$ umuman $\beta\ge 0,\ \gamma<0,\ \delta\ge 0.$ qo'shimcha shartlar:

$$u(x,0)=f_1(x),\quad u_t(x,0)=f_2(x) \qquad (2)$$

$$u(0,t)=\mu_1(t),\quad u(l,t)=\mu_2(t) \qquad (3)$$

$\mu_1(t)\ge 0,\ \mu_2(t)\ge 0,$ ikki uchi biriktirilgan tor masalasi uchun $\mu_1(t)\equiv 0,\quad \mu_2(t)\equiv 0,$ (2) ning ikkinchisi $\alpha\ne 0$ holi uchun.

(1) tenglamaning (2) va (3) shartlarni qanoatlantiruvchi yechimi birdan biri bo'ladi. Birdan-birlik teoremas isboti. Faraz qilaylik tenglamani qo'yilgan shartlarni qanoatlantiruvchi yechimi ikkita bo'lsin: $u_1(x,t)$ va $u_2(x,t)$. U holda bu yechimlar ayirmasi $v(x,t)=u_1(x,t)-u_2(x,t),$

$$\alpha\frac{\partial^2 v}{\partial t^2}+\beta\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}\left\{\gamma\frac{\partial v}{\partial x}\right\}+\delta v=0 \qquad (4)$$

Tenglamani qanoatlantiradi va

$$v(x,0)=0,\quad v_t(x,0)=0 \qquad (5)$$

$$v(0,t)=0,\quad v(l,t)=0 \qquad (6)$$

Bir jinsli shartlarga bo'ysinadi (5) ning ikkinchisi $\alpha\ne 0$ holi uchun. Endi bizning vazifamiz bunday $v(x,t)$ funksiyaning aynan nolga teng ekanligini ko'rsatishdir. (4) ning ikki tomonini $\frac{\partial v}{\partial t}$ ga ko'paytirib, x bo'yicha $(0,l)$ oraliqda integrallaymiz:

$$\int_0^l \alpha \frac{\partial^2 v}{\partial t^2}\frac{\partial v}{\partial t}dx + \int_0^l \beta\left(\frac{\partial v}{\partial t}\right)^2 dx + \int_0^l \frac{\partial v}{\partial t}\frac{\partial}{\partial x}\left(\gamma\frac{\partial v}{\partial x}\right)dx + \int_0^l \delta\frac{\partial v}{\partial t}v dx = 0 \qquad (7)$$

Bundagi integrallardan birinchisi

$$\int_0^l \alpha \frac{\partial^2 v}{\partial t^2}\frac{\partial v}{\partial t}dx = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\alpha\left(\frac{\partial v}{\partial t}\right)^2\right]dx$$

Uchinchi integrallarni bo'laklb integrallasak,

$$\int_0^l \frac{\partial v}{\partial t}\frac{\partial}{\partial x}\left(\gamma\frac{\partial v}{\partial x}\right)dx = \gamma\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}\Big|_0^l - \int_0^l \gamma\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial t}dx \qquad (8)$$

(6) shartlarga asosan o'ng tomandagi integraldan boshqa hadlar nolga teng. Integral ostidagi ifodani

$$\gamma\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left[\gamma\left(\frac{\partial v}{\partial x}\right)^2\right]$$

desak bo'ladi va to'rtinchi integral

$$\int_0^l \delta\frac{\partial v}{\partial t}v dx = \frac{1}{2}\int_0^l \delta\frac{\partial}{\partial t}\left(v^2\right)dx$$

U holda (7) quyidagicha yoziladi:

$$\frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\alpha\left(\frac{\partial v}{\partial t}\right)^2 + \delta v^2 - \gamma\left(\frac{\partial v}{\partial x}\right)^2\right]dx = -\int_0^l \beta\left(\frac{\partial v}{\partial t}\right)^2 dx,$$

$-\gamma = k^2$ desak va

$$\frac{1}{2}\int_0^l \left[\alpha\left(\frac{\partial v}{\partial t}\right)^2 + \delta v^2 + k^2\left(\frac{\partial v}{\partial x}\right)^2\right]dx = E(t), \qquad (9)$$

$$\int_0^l \beta\left(\frac{\partial v}{\partial t}\right)^2 dx = A \geq 0$$

Belgilarni qabul qilsak, (9) dan:

$$\frac{dE(t)}{dt} = -A \leq 0.$$

Bunda E(t) funksiya t o'sishi bilan o'suvchi emas degan hulosaga kelamiz. t=0 da E(t)=0, demak t>0 bo'lganda E(t)=0 bo'lishi kerak. Lekin E(t) ni ifodalovchi integral ostidagi funksiyalar manfiy bo'lmagani uchun E(t)>0 bo'ladi. Demak , $E(t)\equiv 0$ ekan, shuningdek integral ostidagi qo'shiluvchilarning har biri ham nol bo'lishi kerak, yani

$$\alpha\left(\frac{\partial v}{\partial t}\right)^2=0,\quad \delta v^2=0,\quad k^2\left(\frac{\partial v}{\partial x}\right)^2=0.$$

Agar $\delta\neq 0$ bo'lsa, u holda $v\equiv 0$ ekanligiga birdan erishamiz; $\delta=0$ bo'lsa, $\frac{\partial v}{\partial t}=0$ va $\frac{\partial v}{\partial x}=0$

ligidan $v=const$ desak, v uchun yozilgan boshlang'ich asosan $v=0$ deymiz. Demak, $u_1=u_2$ ekan.

Endi $E(t)$ integralning qanday ma'noni anglatishini ko'raylik. Buni tor tebranishining

masalasida tekshirish qulay, bunda $\delta=0$, $\alpha=\rho,\quad k^2=T,\quad \beta=0$ edi, u holda

$$E(t)=\frac{1}{2}\int_0^l\left[\rho\left(\frac{\partial v}{\partial t}\right)^2+T\left(\frac{\partial v}{\partial x}\right)^2\right]dx$$

bo'lib, energiya integrali deyiladi. O'ng tomondagiintegralni ikkita integral yig'indisi shaklida yozsak : birinchisi $\frac{1}{2}\int_0^l T\left(\frac{\partial v}{\partial x}\right)dx$ torning potentsial energiyasini ifodalaydi. Haqiqatan, membrana tebranishining tenglamasini chiqarganda potensial energiya haqida gapirilgan edi. Tor uchun faqat mebranadagi v_y ni deyish mumkin. Demak, E(t) torning tebranishida hosil bo'lgan kinetik va patentsial energiyalarning yig'indisini ko'rsatar ekan.

Bundan tashqari, $A\equiv 0$ bo'lishi $\frac{dE}{dt}\equiv 0$ ni, ya'ni E=const ekanligini tasvirlab, torning tebranish protsessida energiyaning saqlanish qonuni ifodalaydi.

<u>1–masala</u>. $D=\{(x,t)\colon 0<x<l,\ 0<t<+\infty\}$ sohada $U_{tt}=a^2U_{xx}$ tenglamaning

$$U(x,0)=\frac{4h}{l^2}x(l-x)\quad (h>0),\ U_t(x,0)=0,$$

$U(0,t)=0$, $U(l,t)=0$ shartlarni qanoatlantiruvchi yechimi to'ilsin.

Yechilishi. Berilgan masalada $f_1(x)=\frac{4h}{l^2}x(l-x)$, $f_2(x)=0$. Masala yechimini (8) qator ko‘rinishida izlaymiz. Bu qatorning koeffitsientlarini (11) va (12) formulalar yordamida topamiz:

$$a_k=\frac{2}{l}\int_0^l f_1(x)\sin\frac{k\pi x}{l}dx=\frac{8h}{l^3}\int_0^l\left(lx-x^2\right)\sin\frac{k\pi x}{l}dx,\quad b_k=0.$$

a_k koeffitsientni topish uchun o‘ng tomondagi integralni ikki marta bo‘laklab integrallaymiz:

$$U_1=lx-x^2,\ dV_1=\sin\frac{k\pi x}{l}dx,\quad dU_1=(l-2x)dx,$$

$$V_1=-\frac{l}{k\pi}\cos\frac{k\pi x}{l};\ a_k=-\frac{8h}{l^3}\left(lx-x^2\right)\frac{l}{k\pi}\cos\frac{k\pi x}{l}\bigg|_0^l+\frac{8h}{k\pi l^2}\int_0^l(l-2x)\cos\frac{k\pi x}{l}dx$$

yoki $$a_k=\frac{8h}{k\pi l^2}\int_0^l(l-2x)\cos\frac{k\pi x}{l}dx;$$

$$U_2=l-2x,\ dV_2=\cos\frac{k\pi x}{l}dx,\ dU_2=-2dx,\ V_2=\frac{l}{k\pi}\sin\frac{k\pi x}{l};$$

$$a_k=\frac{8h}{k^2\pi^2l^2}(l-2x)\sin\frac{k\pi x}{l}\bigg|_0^l+\frac{16h}{k^2\pi^2l}\int_0^l\sin\frac{k\pi x}{l}dx=-\frac{16h}{k^3\pi^3}\cos\frac{k\pi x}{l}\bigg|_0^l=$$
$$=-\frac{16h}{k^3\pi^3}(\cos k\pi-1)=\frac{16h}{k^3\pi^3}[1-(-1)^k].$$

To’ilgan a_k va b_k koeffitsientlarning qiymatlarini (8) tenglikka qo‘yib, masala yechimini hosil qilamiz:

$$U(x,t)=\sum_{k=1}^{\infty}\frac{16h}{k^3\pi^3}[1-(-1)^k]\cos\frac{k\pi at}{l}\sin\frac{k\pi x}{l}.$$

Agar $k=2n$ bo‘lsa, $1-(-1)^k=0$, agar $k=2n+1$ bo‘lsa, $1-(-1)^k=2$ bo‘lganligi uchun yechimni quyidagi ko‘rinishda yozish mumkin:

$$U(x,t)=\frac{32h}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\frac{(2n+1)\pi at}{l}\sin\frac{(2n+1)\pi x}{l}.$$

2–masala. $D=\{(x,t):\ 0<x<l,\ 0<t<+\infty\}$ sohada

$$U_{tt}=a^2U_{xx},\quad U(x,0)=0,\ U_t(x,0)=\sin\frac{5\pi x}{l},\ U(0,t)=0;\ U(l,t)=0$$

aralash masalaning yechimi to’ilsin.

Yechilishi. (8) funksional qatorning koeffitsientlarini topamiz. $f_1(x)=0$, $f_2(x)=\sin\frac{5\pi x}{l}$ ekanligidan

$$a_k=0; \quad b_k=\frac{2}{k\pi a}\int_0^l \sin\frac{5\pi x}{l}\sin\frac{k\pi x}{l}dx$$

bo‘ladi. $X_k(x)=\sqrt{\frac{2}{l}}\sin\frac{k\pi x}{l}$ $(k=1,2,...)$– xos funksiyalar $(0,l)$ oraliqda normallashgan ortogonal funksiyalar sistemasini tashkil qilganligi uchun

$$\frac{2}{l}\int_0^l \sin\frac{k\pi x}{l}\sin\frac{n\pi x}{l}dxdx=\begin{cases}0, & k\neq n\\ 1, & k=n\end{cases}$$

bo‘ladi. Bundan $k\neq5$ bo‘lganda $b_k=0$, $k=5$ bo‘lganda $b_5=\frac{l}{5\pi a}$ ekanligi kelib chiqadi.

Demak, masalaning izlangan yechimi

$$U(x,t)=\frac{l}{5\pi a}\sin\frac{5\pi at}{l}\sin\frac{5\pi x}{l}$$

bo‘ladi.

3–masala. $D=\{(x,t):\ 0<x<l,\ \ 0<t<+\infty\}$ sohada

$$U_{tt}=a^2U_{xx}+\sin\frac{\pi x}{l}, \quad U(x,0)=0,\ U_t(x,0)=0, \quad U(0,t)=U(l,t)=0$$

aralash masalaning yechimi to’ilsin.

Yechilishi. Berilgan masalada $f(x,t)=\sin\frac{\pi x}{l}$. Masala yechimini ifodalovchi (15) funksional

$$U(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin\frac{k\pi x}{l}$$

qatorning koeffitsienti (20) formulaga asosan

$$T_k(t)=\frac{2}{k\pi a}\int_0^t\left[\int_0^l \sin\frac{\pi\xi}{l}\sin\frac{k\pi a}{l}(t-\tau)\sin\frac{k\pi\xi}{l}d\xi\right]d\tau=$$
$$=\frac{l}{k\pi a}\int_0^t \sin\frac{k\pi a}{l}(t-\tau)\left[\frac{2}{l}\int_0^l \sin\frac{\pi\xi}{l}\sin\frac{k\pi\xi}{l}d\xi\right]$$

bo‘ladi. $X_k(\xi)=\sqrt{\frac{2}{l}}\sin\frac{k\pi\xi}{l}$ funksiyalarning (0,l) oraliqda ortonormallik shartidan $k\neq 1$ da $T_k(t)=0$,

$$T_1(t)=\frac{l}{\pi a}\int_0^t \sin\frac{\pi a}{l}(t-\tau)d\tau=\frac{l}{\pi a}\cos\frac{\pi a}{l}(t-\tau)\Big|_0^t=\frac{l}{\pi a}\left(1-\cos\frac{\pi a}{l}t\right)$$

ekanligi kelib chiqadi.

Shunday qilib, berilgan masala yechimi $U(x,t)=\frac{l}{\pi a}\left(1-\cos\frac{\pi a}{l}t\right)\sin\frac{\pi x}{l}$ ko‘rinishda yoziladi.

Tekislikdagi D_1 sohada ikki o‘zgaruvchili ikkinchi tartibli chiziqli gi’erbolik tipdagi $\left(D=a_{12}^2-a_{11}a_{22}>0\right)$

$$a_{11}(x,y)U_{xx}+2a_{12}(x,y)U_{xy}+a_{22}(x,y)U_{yy}+a_{13}(x,y)U_x+$$

$$a_{23}(x,y)U_y+a_{33}(x,y)U=f(x,y) \qquad (1)$$

tenglamani qaraymiz. D_1 sohada L chiziq berilgan bo‘lib, bu chiziq (1) tenglamaning xarakteristik chiziqlari bilan ustma–ust tushmasin. L chiziq D_1 soha chegarasining qismi bo‘lishi ham mumkin, n orqali L chiziqning normalini belgilaymiz.

II. $D=\{(x,t): 0<x<l,\ 0<t<+\infty\}$ sohada bir jinsli bo‘lmagan $U_{tt}=a^2U_{xx}+f(x,t)$ tor tebranish tenglamasining bir jinsli $U(x,0)=0$, $U_t(x,0)=0$ boshlang‘ich shartlarni va quyidagi chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin:

1. $U(0,t)=U(l,t)=0$, $f(x,t)=Ae^{-t}\sin\frac{\pi x}{l}$;
2. $U(0,t)=U(l,t)=0$, $f(x,t)=Axe^{-t}$;
3. $U(0,t)=U(l,t)=0, f(x,t)=sin\frac{\pi x}{l}$
5. $U(0,t)=U(l,t)=0, f(x,t)=A\sin t$
6. $U(x,0)=\frac{\pi}{2}, U_t(x,0)=0, U(0,t)=U(l,t)=0, f(x,t)=\frac{\pi^2 x}{2l^2}$
7. $u_{tt}=u_{xx}+t\sin x$, $0<x<\pi$, $t>0$, $U(x,0)=0$, $Ut(x,0)=0, U(0,t)=U(\pi,t)=0$
8. $U(0,t)=U(l,t)=0$, $f(x,t)=b\ sh\ x$

Tayanch iboralar.

Aralash masala, yechimning yagonaligi, yagonalik teoremasi, energiya integrali, kinetik energiya, potentsial energiya, energiyaning saqlanish qonuni.

Nazorat uchun savollar.

1. Aralash masalani qo'ying.
2. Yagonalik teoremasini ayting.
3. Energiya integralini yozing.
4. Kinetik energiya integralini yozing.
5. Potentsial energiya integralini yozing.
6. Energiyaning saqlanish qonunini qaysi tenglik ifodalaydi.

10-mavzu:Tor tebranish tenglamasi uchun birinchi chegaraviy masalani Furye usuli bilan yechish. Xos sonlar va xos funksiyalar.

Reja:

1. Masalaning qo`yilishi.
2. O‘zgaruvchilarni ajratish usuli.
3. Mavzuga doir masalalar yechish.

Tekislikdagi $D=\{(x,t):\, 0<x<l,\; 0<t<\infty\}$ sohada bir jinsli

$$U_{tt}=a^2U_{xx} \tag{1}$$

tor tebranish tenglamasining

$$U(x,0)=f_1(x), \qquad U_t(x,0)=f_2(x) \tag{2}$$

boshlang‘ich shartlarni va

$$U(0,t)=0, \qquad U(l,t)=0 \tag{3}$$

bir jinsli chegaraviy shartlarni qanoatlantiruvchi yechimi to’ilsin.

Bu masalani o‘zgaruvchilarni ajratish (yoki Fure) usuli bilan yechamiz. (1) tenglama yechimini

$$U(x,t)=X(x)\cdot T(t) \tag{4}$$

ko‘rinishda izlaymiz. Bu yerda $X(x)$ va $T(t)$ noma’lum funksiyalar. (4) ifodani (1) tenglamaga qo‘yib, $X(x)$ va $T(t)$ noma’lum funksiyalarni topish uchun

$$T''(t)+a^2\lambda T(t)=0\,, \tag{5}$$

$$X''(x)+\lambda X(x)=0 \tag{6}$$

tenglamalarga ega bo‘lamiz. Bunda $\lambda=const$. (4) ifodadan va (3) chegaraviy shartlardan

$$X(0)=0, \qquad X(l)=0 \tag{7}$$

chegaraviy shartlar kelib chiqadi.

(6)–(7) masala xos son va xos funksiyalarni topish haqidagi Shturm–Liuvill masalasidir. (6)–(7) masalaning xos sonlari

$$\lambda_k=\left(\frac{\pi k}{l}\right)^2 \quad (k=1,2,...),$$

bu xos sonlarga mos trivial bo‘lmagan (aynan nolga teng bo‘lmagan) normallashgan xos funksiyalari

$$X_k(x)=\sqrt{\frac{2}{l}}\sin\frac{\pi kx}{l}$$

bo‘ladi. $\lambda=\lambda_k$ bo‘lganda (5) tenglamaning umumiy yechimi

$$T_k(t)=a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}$$

ko‘rinishga ega bo‘lib,

$$U(x,t)=X_k(x)T_k(t)=\left(a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}$$

funksiya (a_k, b_k– ixtiyoriy o‘zgarmas sonlar) (1) tenglamani va (3) chegaraviy shartlarni qanoatlantiradi.

(1) tenglamaning (2)–(3) shartlarni qanoatlantiruvchi yechimini

$$U_k(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l} \tag{8}$$

qator ko‘rinishda izlaymiz. Agar (8) funktsional qator va uning ikkinchi tartibli hosilalari tekis yaqinlashuvchi bo‘lsa, u holda bu qator yig‘indisi (1) tenglamani hamda (3) chegaraviy shartlarni qanoatlantiradi.

a_k va b_k o‘zgarmas sonlarni (8) qatorning yig‘indisi (2) boshlang‘ich shartlarni qanoatlantiradigan qilib tanlaymiz. U holda (2) shartlardan

$$f_1(x)=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi x}{l}, \tag{9}$$

$$f_2(x)=\sum_{k=1}^{\infty}\frac{k\pi a}{l}b_k\sin\frac{k\pi x}{l} \tag{10}$$

tengliklarga ega bo‘lamiz. (9) va (10) tengliklar mos ravishda $f_1(x)$ va $f_2(x)$ funksiyalarning $(0,l)$ oraliqdagi sinuslar bo‘yicha Fure qatoriga yoyilmalaridir. (9) va (10) Fure qatorlarining koeffitsientlari

$$a_k=\frac{2}{l}\int_0^l f_1(x)\sin\frac{k\pi x}{l}dx, \tag{11}$$

$$b_k=\frac{2}{k\pi a}\int_0^l f_2(x)\sin\frac{k\pi x}{l}dx \tag{12}$$

formulalar bo‘yicha to’iladi.

Tekislikdagi D sohada bir jinsli bo‘lmagan

$$U_{tt}=a^2U_{xx}+f(x,t) \tag{13}$$

tor tebranish tenglamasining (2) boshlang‘ich shartlarni va (3) chegaraviy shartlarni qanoatlantiruvchi yechimi to’ilsin.

(13), (2), (3) masala yechimini

$$U(x,t)=V(x,t)+W(x,t)$$

ko‘rinishda yozish mumkin. Bu yerda $V(x,t)$ bir jinsli bo‘lmagan (13) tenglamaning bir jinsli

$$V(x,0)=0, \quad V_t(x,0)=0 \tag{14}$$

boshlang‘ich shartlarni va (3) chegaraviy shartlarni qanoatlantiruvchi yechimi, $W(x,t)$ esa bir jinsli (1) tenglamaning (2) boshlang‘ich shartlarni va (3) chegaraviy shartlarni qanoatlantiruvchi yechimi.

$V(x,t)$ funksiyani

$$V(x,t)=\sum_{k=1}^{\infty} T_k(t)\sin\frac{k\pi x}{l} \tag{15}$$

qator ko‘rinishda izlaymiz. Bunda $T_k(t)$ noma’lum funksiyalar.

(15) ifodani (13) tenglamaga qo‘yib,

$$\sum_{k=1}^{\infty}\left[T_k''(t)+\left(\frac{k\pi a}{l}\right)^2 T_k(t)\right]\sin\frac{k\pi x}{l}=f(x,t) \tag{16}$$

tenglikka ega bo‘lamiz. $f(x,t)$ funksiyani $(0,l)$ oraliqda sinuslar bo‘yicha Fure qatoriga yoyamiz:

$$f(x,t)=\sum_{k=1}^{\infty} f_k(t)\sin\frac{k\pi x}{l} \tag{17}$$

va (16) bilan (17) ni taqqoslab, noma’lum $T_k(t)$ funksiyalarga nisbatan

$$T_k''(t)+\left(\frac{k\pi a}{l}\right)^2 T_k(t)=f_k(t) \tag{18}$$

differensial tenglamalarni hosil qilamiz.

Bu yerda

$$f_k(t)=\frac{2}{l}\int_0^l f(\xi,t)\sin\frac{k\pi\xi}{l}d\xi \quad (k=1,2,...).$$

(14) boshlang‘ich shartlardan, (15) ifodaga asosan

$$T_k(0)=0, \quad T_k'(0)=0 \ (k=1,2,...) \tag{19}$$

boshlang‘ich shartlar kelib chiqadi.

(18) tenglamaning (19) bir jinsli boshlang‘ich shartlarni qanoatlantiruvchi yechimi

$$T_k(t)=\frac{2}{k\pi a}\int_0^t\left[\int_0^l f(\xi,\tau)\sin\frac{k\pi a}{l}(t-\tau)\sin\frac{k\pi\xi}{l}d\xi\right]d\tau \tag{20}$$

ko‘rinishga ega bo‘ladi.

Shunday qilib, (13), (2), (3) masalaning yechimi quyidagi ko‘rinishda yoziladi:

$$U(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin\frac{k\pi x}{l}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}. \tag{21}$$

Bu yerda $T_k(t)$ (20) formuladan, a_k va b_k koefitsientlar esa mos ravishda (11) va (12) formulalar yordamida aniqlanadi.

Tekislikdagi D sohada bir jinsli bo‘lmagan (13) tenglamaning (2) boshlang‘ich shartlarni va bir jinsli bo‘lmagan

$$U(0,t)=\mu_1(t),\quad U(l,t)=\mu_2(t) \tag{22}$$

chegaraviy shartlarni qanoatlantiruvchi yechimi to’ilsin.

Bu masala yechimini

$$U(x,t)=\overline{V}(x,t)+\overline{W}(x,t)$$

ko‘rinishda yozish mumkin. Bu yerda $\overline{W}(x,t)$ yordamchi funksiya bo‘lib, uni

$$\overline{W}(x,t)=(\alpha_1x+\beta_1)\mu_1(t)+(\alpha_2x+\beta_2)\mu_2(t) \tag{23}$$

ko‘rinishda izlab, (22) chegaraviy shartlarni qanoatlantiradigan qilib tanlaymiz. U holda $\overline{W}(x,t)$ quyidagi ko‘rinishga ega bo‘ladi:

$$\overline{W}(x,t)=\mu_1(t)+\frac{x}{l}(\mu_2(t)-\mu_1(t)). \tag{24}$$

$\overline{V}(x,t)$ funksiya esa bir jinsli bo‘lmagan

$$\overline{V}_{tt}=a^2\overline{V}_{xx}+g(x,t) \tag{25}$$

tor tebranish tenglamasining bir jinsli bo‘lmagan

$$\overline{V}(x,0)=f_1(x)-\overline{W}(x,0),\quad \overline{V}_t(x,0)=f_2(x)-\overline{W}(x,0) \tag{26}$$

boshlang‘ich shartlarni va bir jinsli

$$\bar{V}(0,t)=0,\ \ \bar{V}(l,t)=0 \tag{27}$$

chegaraviy shartlarni qanoatlantiruvchi yechimi. Bu yerda

$$g(x,t)=f(x,t)-\left(\overline{W}_{tt}-\alpha^2\overline{W}_{xx}\right).$$

(25), (26), (27) masala oldin yechilgan (13), (2), (3) masalaga o‘xshashdir.

<u>1–masala</u>. $D=\{(x,t):\ 0<x<l,\ \ 0<t<+\infty\}$ sohada $U_{tt}=a^2U_{xx}$ tenglamaning

$$U(x,0)=\frac{4h}{l^2}x(l-x)\quad (h>0),\ U_t(x,0)=0,$$

$U(0,t)=0$, $U(l,t)=0$ shartlarni qanoatlantiruvchi yechimi to’ilsin.

<u>Yechilishi</u>. Berilgan masalada $f_1(x)=\dfrac{4h}{l^2}x(l-x)$, $f_2(x)=0$. Masala yechimini (8) qator ko‘rinishida izlaymiz. Bu qatorning koeffitsientlarini (11) va (12) formulalar yordamida topamiz:

$$a_k=\frac{2}{l}\int_0^l f_1(x)\sin\frac{k\pi x}{l}dx=\frac{8h}{l^3}\int_0^l\left(lx-x^2\right)\sin\frac{k\pi x}{l}dx,\quad b_k=0.$$

a_k koeffitsientni topish uchun o‘ng tomondagi integralni ikki marta bo‘laklab integrallaymiz:

$$U_1=lx-x^2,\ dV_1=\sin\frac{k\pi x}{l}dx,\quad dU_1=(l-2x)dx,$$

$$V_1=-\frac{l}{k\pi}\cos\frac{k\pi x}{l};\ a_k=-\frac{8h}{l^3}\left(lx-x^2\right)\frac{l}{k\pi}\cos\frac{k\pi x}{l}\Big|_0^l+\frac{8h}{k\pi l^2}\int_0^l(l-2x)\cos\frac{k\pi x}{l}dx$$

yoki $$a_k=\frac{8h}{k\pi l^2}\int_0^l(l-2x)\cos\frac{k\pi x}{l}dx;$$

$$U_2=l-2x,\ dV_2=\cos\frac{k\pi x}{l}dx,\ dU_2=-2dx,\ V_2=\frac{l}{k\pi}\sin\frac{k\pi x}{l};$$

$$a_k=\frac{8h}{k^2\pi^2l^2}(l-2x)\sin\frac{k\pi x}{l}\Big|_0^l+\frac{16h}{k^2\pi^2 l}\int_0^l\sin\frac{k\pi x}{l}dx=-\frac{16h}{k^3\pi^3}\cos\frac{k\pi x}{l}\Big|_0^l=$$
$$=-\frac{16h}{k^3\pi^3}(\cos k\pi-1)=\frac{16h}{k^3\pi^3}[1-(-1)^k].$$

To’ilgan a_k va b_k koeffitsientlarning qiymatlarini (8) tenglikka qo‘yib, masala yechimini hosil qilamiz:

$$U(x,t)=\sum_{k=1}^{\infty}\frac{16h}{k^3\pi^3}[1-(-1)^k]\cos\frac{k\pi at}{l}\sin\frac{k\pi x}{l}.$$

Agar k=2n bo‘lsa, 1–(–1)^k=0, agar k=2n+1 bo‘lsa, 1–(–1)^k=2 bo‘lganligi uchun yechimni quyidagi ko‘rinishda yozish mumkin:

$$U(x,t)=\frac{32h}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\frac{(2n+1)\pi at}{l}\sin\frac{(2n+1)\pi x}{l}.$$

<u>2–masala</u>. $D=\{(x,t):\ 0<x<l,\ 0<t<+\infty\}$ sohada

$$U_{tt}=a^2U_{xx},\quad U(x,0)=0,\ U_t(x,0)=\sin\frac{5\pi x}{l},\ U(0,t)=0;\ U(l,t)=0$$

aralash masalaning yechimi to’ilsin.

<u>Yechilishi</u>. (8) funksional qatorning koeffitsientlarini topamiz. $f_1(x)=0$, $f_2(x)=\sin\frac{5\pi x}{l}$ ekanligidan

$$a_k=0;\quad b_k=\frac{2}{k\pi a}\int_0^l\sin\frac{5\pi x}{l}\sin\frac{k\pi x}{l}dx$$

bo‘ladi. $X_k(x)=\sqrt{\frac{2}{l}}\sin\frac{k\pi x}{l}$ $(k=1,2,...)$– xos funksiyalar (0,l) oraliqda normallashgan ortogonal funksiyalar sistemasini tashkil qilganligi uchun

$$\frac{2}{l}\int_0^l\sin\frac{k\pi x}{l}\sin\frac{n\pi x}{l}dxdx=\begin{cases}0, & k\neq n\\ 1, & k=n\end{cases}$$

bo‘ladi. Bundan $k\neq5$ bo‘lganda b_k=0, k=5 bo‘lganda $b_5=\frac{l}{5\pi a}$ ekanligi kelib chiqadi.

Demak, masalaning izlangan yechimi

$$U(x,t)=\frac{l}{5\pi a}\sin\frac{5\pi at}{l}\sin\frac{5\pi x}{l}$$

bo‘ladi.

<u>3–masala</u>. $D=\{(x,t):\ 0<x<l,\ 0<t<+\infty\}$ sohada

$$U_{tt}=a^2U_{xx}+\sin\frac{\pi x}{l},\quad U(x,0)=0,\ U_t(x,0)=0,\quad U(0,t)=U(l,t)=0$$

aralash masalaning yechimi to’ilsin.

Yechilishi. Berilgan masalada $f(x,t)=\sin\frac{\pi x}{l}$. Masala yechimini ifodalovchi (15) funksional

$$U(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin\frac{k\pi x}{l}$$

qatorning koeffitsienti (20) formulaga asosan

$$T_k(t)=\frac{2}{k\pi a}\int_0^t\left[\int_0^l\sin\frac{\pi\xi}{l}\sin\frac{k\pi a}{l}(t-\tau)\sin\frac{k\pi\xi}{l}d\xi\right]d\tau=$$
$$=\frac{l}{k\pi a}\int_0^t\sin\frac{k\pi a}{l}(t-\tau)\left[\frac{2}{l}\int_0^l\sin\frac{\pi\xi}{l}\sin\frac{k\pi\xi}{l}d\xi\right]$$

bo‘ladi. $X_k(\xi)=\sqrt{\frac{2}{l}}\sin\frac{k\pi\xi}{l}$ funksiyalarning $(0,l)$ oraliqda ortonormallik shartidan $k\neq1$ da $T_k(t)=0$,

$$T_1(t)=\frac{l}{\pi a}\int_0^t\sin\frac{\pi a}{l}(t-\tau)d\tau=\frac{l}{\pi a}\cos\frac{\pi a}{l}(t-\tau)\Big|_0^t=\frac{l}{\pi a}\left(1-\cos\frac{\pi a}{l}t\right)$$

ekanligi kelib chiqadi.

Shunday qilib, berilgan masala yechimi $U(x,t)=\frac{l}{\pi a}\left(1-\cos\frac{\pi a}{l}t\right)\sin\frac{\pi x}{l}$ ko‘rinishda yoziladi.

Tekislikdagi D_1 sohada ikki o‘zgaruvchili ikkinchi tartibli chiziqli gi’erbolik tipdagi $\left(D=a_{12}^2-a_{11}a_{22}>0\right)$

$$a_{11}(x,y)U_{xx}+2a_{12}(x,y)U_{xy}+a_{22}(x,y)U_{yy}+a_{13}(x,y)U_x+$$
$$a_{23}(x,y)U_y+a_{33}(x,y)U=f(x,y) \qquad (1)$$

tenglamani qaraymiz. D_1 sohada L chiziq berilgan bo‘lib, bu chiziq (1) tenglamaning xarakteristik chiziqlari bilan ustma–ust tushmasin. L chiziq D_1 soha chegarasining qismi bo‘lishi ham mumkin, n orqali L chiziqning normalini belgilaymiz.

II. $D=\{(x,t):\,0<x<l,\ 0<t<+\infty\}$ sohada bir jinsli bo‘lmagan $U_{tt}=a^2U_{xx}+f(x,t)$ tor tebranish tenglamasining bir jinsli $U(x,0)=0$, $U_t(x,0)=0$ boshlang‘ich shartlarni va quyidagi chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin:

1. $U(0,t)=U(l,t)=0$, $f(x,t)=Ae^{-t}\sin\frac{\pi x}{l}$;

2. $U(0,t)=U(l,t)=0, \quad f(x,t) = Axe^{-t}$;

3. $U(0,t) = U(l,t) = 0, f(x,t) = sin\frac{\pi x}{l}$

5. $U(0,t) = U(l,t) = 0, f(x,t) = A\,\mathrm{sin}t$

6. $U(x,0)=\frac{\pi}{2}, U_t(x,0) = 0, U(0,t) = U(l,t) = 0, f(x,t) = \frac{\pi^2 x}{2l^2}$

7. $u_{tt} = u_{xx} + t\sin x\,,\ 0 < x < \pi,\ t > 0,\ U(x,0) = 0,\ Ut(x,0) = 0, U(0,t) = U(\pi,t) = 0$

8. $U(0,t)=U(l,t)=0, \ f(x,t) = b\ sh\ x$

Tayanch iboralar

To'lqin tenglamasi, Koshi masalasi, Kechuvchi ‘otentsial, Kirxgof formulasi, Puasson formulasi, chegaralanmagan membrana, tushish usuli, ‘orsevol formulasi.

Nazorat uchun savollar.

1. Fazoda to'lqin tenglamasi uchun Koshi masalasi qanday qo'yiladi?
2. Kechuvchi ‘otentsialni ifodalovchi ifodalovchi funktsiyani yozing?
3. Kirgaf formulasini yozing?
4. Puasson formulasini yozing?
5. Tekislikda to'lqin tenglamasi uchun Koshi masalasiga qanday fizik masala mos keladi?
6. Tushish metodini tushintiring .
7. ‘arsevol formulasini yozing?

11-Mavzu. Tor tebranish tenglamasi uchun ikkinchi tur chegaraviy masalani Furye usuli bilan yechish.

I. Asosiy tushuncha

1–MASALA. To'g'ri to'rtburchakli D sohada

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, l), \quad t > 0, \tag{54}$$

tenglamaning quyidagi

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le l, \tag{55}$$

boshlang'ich va

$$u_x(0,t) = 0, \quad u_x(l,t) = 0, \quad 0 \le t \le T, \tag{56}$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

YECHISH. Berilgan bir jinsli tor tebranish tenglamasining $u_x(0,t) = u_x(l,t) = 0$ chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini $u(x,t) = X(x)T(t)$ ko'rinishda izlaymiz. Bundan $X(x)$ funksiya uchun ubshu

$$X'(0) = 0, \quad X'(l) = 0, \tag{57}$$

chegaraviy shartlarni olamiz.

Endi $u(x,t)$ ko'paytmani (54) tenglamaga qo'ysak,

$$X(x)T''(t) = a^2 X''(x)T(t)$$

tenglikka ega bo'lamiz.

Oxirgi tenglikni $a^2 X(x)T(t) \neq 0$ ifodaga bo'lib,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

ifodani olamiz. Bundan esa $X(x)$ funksiyaga nisbatan

$$X''(x) + \lambda X(x) = 0, \tag{58a}$$

$$X'(0) = 0, \; X'(l) = 0, \tag{58b}$$

Shturm–Liuvill masalaga, $T(t)$ funksiyaga nisbatan esa

$$T''(t) + a^2\lambda T(t) = 0, \quad t > 0, \tag{59}$$

tenglamaga ega bo'lamiz.

(58a) tenglamaning umumiy yechimi quyidagi

$$\begin{aligned} X(x) =& c_1 e^{\sqrt{-\lambda}}x + c_2 e^{-\sqrt{-\lambda}}x, && \text{agar} \quad \lambda < 0; \\ X(x) =& c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, && \text{agar} \quad \lambda > 0; \\ X(x) =& c_1 x + c_2, && \text{agar} \quad \lambda = 0, \end{aligned}$$

ko'rinishda bo'ladi.

Agar $\lambda < 0$ bo'lsa, u holda $X(x) \equiv 0$ bo'lishini ko'rsatish qiyin emas.

Agar $\lambda > 0$ bo'lsa, u holda yuqoridagi umumiy yechimdan

$$X'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

bo'ladi. (58b) chegaraviy shartlarga asosan $c_2 = 0$ yoki $X(x) = c_1\cos\sqrt{\lambda}x = 0$ kelib chiqadi. Bundan $X'(x) = -c_1\lambda\cos\sqrt{\lambda}x = 0$ bo'ladi va $X'(l) = 0$ chegaraviy shartga ko'ra $\sqrt{\lambda}l = k\pi$ yoki Shturm–Liuvill masalasi cheksiz ko'p

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \quad k = 0, 1, 2, ..., \tag{60}$$

xos qiymatlarga ega ekanligi kelib chiqadi. Bularga mos xos funksiyalar

$$X_k(x) = \cos\frac{k\pi}{l}x, \quad k = 0, 1, 2, ..., \tag{61}$$

bo'ladi.

Agar $\lambda = 0$ bo'lsa, u holda (58a) tenglamaning umumiy yechimidan yuqoridagi kabi $c_1 = 0$ va $X(x) = c_2$ ekanligi kelib chiqadi, bundan esa $X'(l) = 0$ chegaraviy shart aynan bajariladi. Demak, (58) Shturm–Liuvill masalasi uchun $\lambda = 0$ xos qiymat va unga mos xos funksiya $X_0(x) = 1$ bo'ladi, ya'ni

$$\lambda_0 = 0, \qquad X_0(x) = 1.$$

(58) masalaning λ_k xos sonlarini (60) va xos funksiyalarini esa $k = 0$ bo'lganda (61) ko'rininshda

$$\lambda_0 = \left(\frac{\pi 0}{l}\right)^2 = 0, \qquad X_0(x) = \cos\frac{\pi 0}{l}x = 1$$

yozish mumkin.

Demak, (58) Shturm–Liuvill masalasi uchun

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \qquad X_k(x) = \cos\frac{k\pi}{l}x, \quad k = 0, 1, 2, \ldots$$

xos qiymat va xos funksiyalarga ega bo'ldik.

Endi (59) tenglamani qaraylik. Bu tenglama $\lambda = \lambda_k$ bo'lganda ham ma'noga ega va

$$T_k''(t) + a^2\lambda_k T_k(t) = 0, \quad t > 0, \tag{62}$$

tenglamani qaraymiz. Agar $k = 0$ bo'lsa, oxirgi tenglamaning umumiy yechimi

$$T_0(t) = A_0 + B_0 t$$

bo'ladi, bu yerda A_0, B_0 – ixtiyoriy o'zgarmaslar.

Agar $k > 0$ bo'lsa, (62) tenglamaning umumiy yechimi

$$T_k(t) = A_k \cos\left(\frac{ka\pi}{l}\right)t + B_k \sin\left(\frac{ka\pi}{l}\right)t, \quad t > 0 \tag{63}$$

ko'rinishda bo'ladi, bunda A_k va B_k – ixtiyoriy o'zgarmaslar.

Endi qaralayotgan (54)–(56) aralash masalaning yechimini

$$u(x,t) = \sum_{k=0}^{\infty} X_k(x)T_k(t)$$

ko'rinishda izlaymiz, ya'ni

$$u(x,t) = A_0 + B_0 t + \sum_{k=1}^{\infty}\left[A_k \cos\left(\frac{ka\pi}{l}\right)t + B_k \sin\left(\frac{ka\pi}{l}\right)t\right]\cos\left(\frac{k\pi}{l}\right)x, \tag{64}$$

Qaralayotgan masalaning boshlang'ich shartlariga ko'ra

$$\varphi(x) = \sum_{k=0}^{\infty} X_k(x)T_k(0) = A_0 + \sum_{k=1}^{\infty} A_k X_k(x), \qquad (65)$$

$$\psi(x) = \sum_{k=0}^{\infty} X_k(x)T_k'(0) = B_0 + \sum_{k=1}^{\infty} \frac{ka\pi}{l} B_k X_k(x), \qquad (66)$$

bo'ladi.

Faraz qilaylik, $\varphi(x)$ va $\psi(x)$ funksiyalar kosinuslar bo'yicha Fur'e qatoriga yoyilsin, ya'ni

$$\varphi(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos\left(\frac{k\pi x}{l}\right), \quad \psi(x) = \frac{\beta_0}{2} + \sum_{k=1}^{\infty} \beta_k \cos\left(\frac{k\pi x}{l}\right).$$

Bu yerda α_k va β_k koeffitsiyentlar.

$$\alpha_k = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{k\pi x}{l}\right)dx, \quad \beta_k = \frac{2}{k\pi a}\int_0^l \psi(x)\cos\left(\frac{k\pi x}{l}\right)dx,$$

ko'rinishda aniqlanadi.

Shunday qilib, Fur'e qatorlari uchun standart formulalardan foydalanib, (64) formuladagi A_k va B_k koeffitsiyentlar uchun quyidagi

$$A_k = \alpha_k = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{k\pi x}{l}\right)dx, \quad k > 0,$$

$$B_k = \frac{l}{k\pi a}\beta_k = \frac{2}{k\pi a}\int_0^l \psi(x)\cos\left(\frac{k\pi x}{l}\right)dx, \quad k > 0,$$

$$A_0 = \frac{\alpha_0}{2} = \frac{1}{l}\int_0^l \varphi(x)dx, \quad B_0 = \frac{\beta_0}{2} = \frac{1}{k\pi a}\int_0^l \psi(x)dx,$$

formulalarni olamiz.

Endi topilgan A_k va B_k koeffitsiyentlarni (64) formulaga qo'yib, (54)–(56) aralash masalaning $u(x,t)$ yechimini hosil qilamiz.

Mustaqil yechish uchun misollar

1. Ω={(x,t): 0< x < l, 0 < t < + ∞} sohada bir jinsli ut = a2uxx issiqlik tarqalish tenglamasi uchun quyidagi aralash masalalar yechilsin

$$1. u(0,t) = u(l,t) = 0, \quad u(x,0) = Ax;$$

$$2. u(0,t) = u_x(l,t) = 0, \quad u(x,0) = \sin\frac{\pi}{2l}x;$$

$$3. u_x(0,t) = u(l,t) = 0, \quad u(x,0) = \cos\frac{\pi x}{2l};$$

$$4. u_x(0,t) = u_x(l,t) = 0, \quad u(x,0) = C, \quad C = const;$$

$$5. u(0,t) = u_x(l,t) = 0, \quad u(x,0) = \begin{cases} 0, & 0 < x < l/2 \\ u_0, & l/2 < x < l \end{cases}$$

2. $\Omega=\{(x,t): 0<x<l,\ 0<t<+\infty\}$ sohadabirjinslibo‘lmagan $u_t= a^2u_{xx}+f(x,t)$ issiqliktarqalishtenglamasiningbirjinsli $u(x,0)=0$ boshlang‘ichvaquyidagichegaraviyshartlarniqanoatalantiruvchiyechimitopilsin:

$$6. u(0,t) = u(l,t) = 0, \quad f(x,t) = \sin\frac{\pi x}{l};$$

$$7. u(0,t) = u(l,t) = 0, \quad f(x,t) = xe^{-t};$$

$$8. u(0,t) = u_x(l,t) = 0, \quad f(x,t) = \sin\frac{\pi x}{l} + \sin\frac{2\pi}{l}x;$$

$$9. u_x(0,t) = u(l,t) = 0, \quad f(x,t) = xt;$$

$$10. u_x(0,t) = u_x(l,t) = 0, \quad f(x,t) = f_0(x)$$

3. Quyidagi aralash masalalar yechilsin:

$$11. u_t = u_{xx} + u + 2\sin 2x \sin x, \quad 0 < x < \frac{\pi}{2}, \quad t > 0;$$

$$u_x\big|_{x=0} = u\Big|_{x=\frac{\pi}{2}} = u\big|_{t=0} = 0;$$

$$12. u_t = u_{xx} + u + 2\sin 2x \cos x, \quad 0 < x < \frac{\pi}{2}, \quad t > 0;$$

$$u\big|_{x=0} = u_x\Big|_{x=\frac{\pi}{2}} = 1, \quad u\big|_{t=0} = x;$$

13. $u_t = u_{xx} + 4u + x^2 - 2t - 4x^2 t + 2\cos^2 x,$

$0 < x < \pi, \quad 0 < t < +\infty.$

$u_x\big|_{x=0} = 0, \quad u_x\big|_{x=\pi} = 2\pi t; \quad u\big|_{t=0} = 0$

14. $u_t - u_{xx} - u = xt(2-t) + 2\cos t, 0 < x < \pi, \quad t > 0$

$u_x\big|_{x=0} = t^2, \quad u_x\big|_{x=\pi} = t^2; \quad u\big|_{t=0} = \cos 2x$

15. $u_t - u_{xx} - 9u = 4\sin^2 t \cos 3x - 9x^2 - 2, \quad 0 < x < \pi, \quad 0 < t < +\infty$

$u_x\big|_{x=0} = 0, \quad u_x\big|_{x=\pi} = 2\pi; \quad u\big|_{t=0} = x^2 + 2x$

Tayanch iboralar.

Giperbolik tipdagi tenglama, Koshi masalasi, Riman usuli, Grin formulasi, Riman funktsiyasi, Riman formulasi.

Nazorat uchun savollar.

1. Giperbolik tipdagi tenglama uchun Koshi masalasini qo’ying.
2. Riman usulining moxiyatini tushuntiring.
3. Grin formulasini yozing.
4. Qaysi masalaning yechimiga Riman funktsiyasi deyiladi.
5. Riman formulasini yozing.

12-Mavzu. Issiqlik tarqalish tenglamasi. Ekstremum prinsipi. Birinchi tur chegaraviy masala yechimining yagonaligi.

Asosiy tushuncha

MASALANING QO'YILISHI. YECHIMNING YAGONALIGI. Berilgan chekli $Q = \{(x,t) : 0 < x < l,\ 0 < t < T\}$ sohada

$$Lu \equiv \frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad (1)$$

tenglamaning

$$u|_{t=0} = \varphi(x), \qquad 0 \le x \le l; \qquad (2)$$

boshlang'ich va

$$u|_{x=0} = \mu_1(t), \quad u|_{x=l} = \mu_2(t), \qquad 0 \le t \le T, \qquad (3)$$

chegaraviy shartlarni qanoatlantiradigan yechimini topish masalasi *birinchi chegaraviy masala* deb yuritiladi.

Bu yerda l uchi koordinat boshida bo'lgan sterjenining uzunligini, T esa shu fizik jarayonni o'rganish qancha vaqt davom etishini bildiradi, $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ – berilgan funksiyalar.

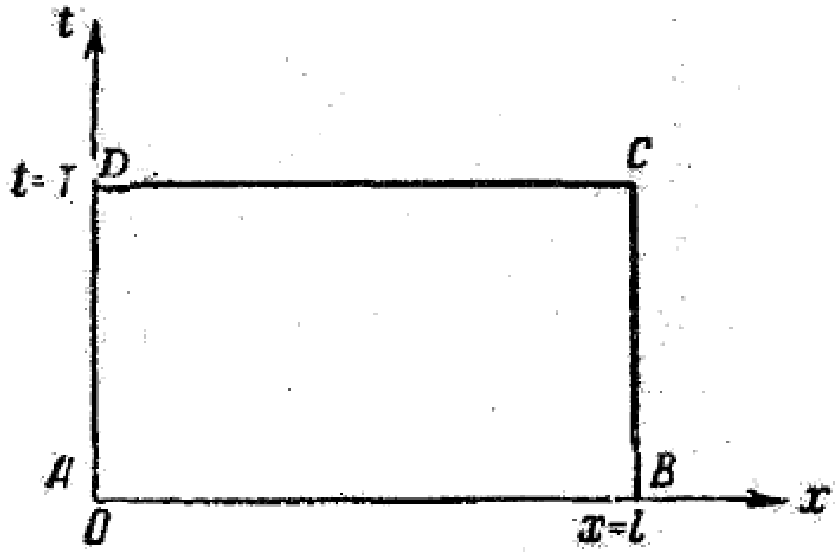
Biz izlanayotgan $u(x,t)$ yechimni $\overline{Q}$ yopiq sohada uzluksiz funksiya deb faraz qilamiz va shuning uchun berilgan $f(x,t)$, $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ funksiyalarni uzluksizligini va demak,

$$\varphi(0) = \mu_1(0), \quad \varphi(l) = \mu_2(0)$$

bo'lishini talab qilamiz.

I bobdan ma'lumki, (1)–(3) chegaraviy masala biror qattiq jismning boshlang'ich harorati $\varphi(x)$, uning chegarasidagi harorati ma'lum bo'lsa, bu jismning $\forall t \in [0,T]$ vaqtdagi $u(x,t)$ haroratni aniqlaydi.

Yuqorida qo'yilgan (1)–(3) masalani yechishda $t > 0$ bo'lishi juda muhim, chunki tor tebranish tenglamasidan farqli o'laroq t vaqtni $-t$ ga almashtirsak, (1) tenglama tubdan o'zgarib ketadi. Shuni ta'kidlash muhimki, agar (1) tenglamada t vaqt $-t$ ga almashtirilsa, qaralayotgan boshlang'ich–chegaraviy masala umuman yechimga ega bo'lmaydi. Bu fizikaviy jarayonlarga va tenglamani keltirib chiqarilishiga bevosita bog'liq.



17 — shakl.

Biz qo'yilgan (1)–(3) birinchi boshlang'ich–chegaraviy masalani to'liq o'rganish bilan cheklanamiz. Avval bu masala yechimining yagona ekanligini ekstremum prinsipi yordamida ko'rsatamiz va yechimning turg'unligini isbotlaymiz. (1)–(3) masala yechimining mavjudligi esa matematik fizikada keng ko'llaniladigan usullardan biri o'zgaruvchilarni ajratish, Fur'e usuli bilan ko'rsatamiz.

EKSTREMUM PRINSIPI.

1–TEOREMA. Yopiq $\overline{Q}$ sohada uzluksiz bo'lgan va Q soha ichida bir jinsli

$$u_t = a^2 u_{xx} \tag{6}$$

issiqlik o'tkazuvchanlik tenglamasini qanoatlantiruvchi $u(x,t)$ funksiya o'zining eng katta va eng kichik qiymatlariga Γ chiziqda erishadi.

Bu yerda Γ qaralayotgan Q to'rtburchakning $t = 0$, $x = 0$ va $x = l$ chiziqlar ustida yotgan chegaralarining yig'indisi.

ISBOT. $u(x,t)$ funksiyaning Q to'rtburchakdagi eng katta qiymati M, ya'ni $\max\limits_D |u(x,t)| = M$ va chiziq ustidagi eng katta qiymati esa m, ya'ni $\max\limits_\Gamma |u(x,t)| = m$ deb belgilaymiz.

Faraz qilaylik, Q to'rtburchakda shunday (x^*, t^*) ichki nuqta topilsinki, bu nuqtada $M > m$ bo'lsin, bu yerda $t^* > 0$, $0 < x^* < l$.

Quyidagi yordamchi

$$v(x,t) = u(x,t) + \frac{M-m}{4l^2}(x-x^*)^2$$

funksiyani qaraylik. Q to'rtburchakning asosi $t=0$ da, yon tomonlari $x=0$ va $x=l$ da

$$v(x,t) \le m + \frac{M-m}{4} = \frac{M}{4} + \frac{3m}{4} = \theta M < M, \quad 0 < \theta < 1$$

bo'lishini ko'rish qiyin emas.

Shu bilan birga $v(x^*,t^*) = u(x^*,t^*) = M$. Demak, yordamchi $v(x,t)$ funksiya ham $u(x,t)$ funksiya kabi o'zining eng katta qiymatiga Γ da erishmaydi.

Shunday ekan, $v(x,t)$ funksiya o'zining eng katta qiymatiga biror (x_1,t_1) $(0 < x_1 < l,\ 0 < t_1 < T)$ nuqtada erishsin.

U holda, matematik analiz kursidan ma'lumki, shu nuqtada $\frac{\partial^2 v}{\partial x_1^2} \le 0$ va $\frac{\partial v}{\partial t_1} \ge 0$ bo'ladi.

Agar $t_1 < T$ bo'lsa, u holda bu nuqtada $\frac{\partial v}{\partial t_1} = 0$ bo'ladi.

Agar $t_1 = T$ bo'lganda esa, $\frac{\partial v}{\partial t_1} \ge 0$ munosabat o'rinli bo'ladi. Demak, (x_1,t_1) nuqtada

$$v_t - a^2 v_{xx} \ge 0 \tag{7}$$

tengsizlik bajariladi.

Ikkinchi tomondan esa

$$v_t - a^2 v_{xx} = u_t - a^2 u_{xx} - \frac{M-m}{2l^2} = -\frac{M-m}{2l^2} < 0$$

bo'lishi kerak. Bu esa (7) tengsizlikka zid.

Demak, $M > m$ bo'ladigan nuqta topilsin, degan farazimiz noto'g'ri ekan. Ekstremum prinsipini eng katta qiymat uchun isbotlandi. Eng kichik qiymat uchun ham ekstremum prinsipi xuddi shunday isbotlanadi.

1–XULOSA. Agar $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lib, yopiq $\overline{Q}$ sohada eng katta (eng kichik) qiymatiga ega bo'lsa, u holda bu funksiya $\overline{Q}$ sohada o'zgarmasdir.

ISBOT. Faraz qilaylik, $\forall (x,t) \in \overline{Q}$ da $u(x,t) \neq 0$ bo'lsin. U holda ekstremum prinsipiga asosan $u(x,t)$ funksiya $\forall (x,t) \in \overline{Q}$ sohada o'zining eng katta (eng kichik) qiymatiga Γ chegarada erishadi. Bu esa shartga zid.

2–XULOSA. Agar $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lsa, u holda $\forall (x,t) \in \overline{Q}$ uchun quyidagi tengsizliklar o'rinli:

1) $\min\limits_{\Gamma} u(x,t) \leq u(x,t) \leq \max\limits_{\Gamma} u(x,t)$;

2) $|u(x,t)| \leq \max\limits_{\Gamma} |u(x,t)|$.

3–XULOSA. Faraz qilaylik $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lsin. Agar $\forall (x,t) \in \Gamma$ uchun $u(x,t) \geq 0 (\leq 0)$ bo'lsa, u holda $u(x,t) \geq 0 (\leq 0)$, $\forall (x,t) \in \overline{Q}$ bo'ladi.

Ekstremum prinsipidan foydalanib, (1)–(3) birinchi chegaraviy masala yechimining yagonaligi va turg'unligini isbot qilaylik.

2–TEOREMA. Agar issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning yechimi mavjud bo'lsa, u holda bu yechim yagona bo'ladi.

ISBOT. Haqiqatan ham, agar qaralayotgan masala bir xil boshlang'ich va chegaraviy shartlarni qanoatlantiruvchi ikkita $u_1(x,t)$ va $u_2(x,t)$ yechimlarga ega bo'lsa, ularning ayirmasi $u(x,t) = u_1(x,t) - u_2(x,t)$ bir jinsli (6) issiqlik o'tkazuvchanlik tenglamasini hamda bir jinsli boshlang'ich

$$u(x,0) = u_1(x,0) - u_2(x,0) = \varphi(x) - \varphi(x) = 0,$$

va

$$u(0,t) = u_1(0,t) - u_2(0,t) = \mu_1(t) - \mu_1(t) = 0,$$

$$u(l,t) = u_1(l,t) - u_2(l,t) = \mu_2(t) - \mu_2(t) = 0,$$

chegaraviy shartlarni qanoatlantiradi. U holda ekstremum prinsipiga ko'ra $\min\limits_{\overline{Q}} u(x,t)$ va $\max\limits_{\overline{Q}} u(x,t)$ Q sohaning Γ chegarasida erishadi.

Bir jinsli shartlarga asosan Q sohaning chegarasida $u(x,t)$ funksiya nolga teng. Demak $\forall (x,t) \in \overline{Q}$ uchun $u(x,t) \equiv 0$ bo'ladi. Bundan $u_1(x,t) \equiv u_2(x,t)$ ekanligi kelib chiqadi.

3–TEOREMA. Issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning yechimi $f(x,t)$, $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ funksiyalarga uzluksiz bog'liq bo'ladi.

ISBOT. Faraz qilaylik, $u_1(x,t)$ funksiya (1)–(3) birinchi chegaraviy masalaning $f(x,t)$, $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ funksiyalarga bog'liq bo'lgan yechimi, $u_2(x,t)$ funksiya esa $f^*(x,t)$, $\varphi^*(x)$, $\mu_1^*(t)$ va $\mu_2^*(t)$ funksiyalarga bog'liq bo'lgan yechimi bo'lsin. Berilgan funksiyalar uchun

$$|f(x,t) - f^*(x,t)| < \varepsilon; \quad \forall (x,t) \in Q;$$

$$|\varphi(x) - \varphi^*(x)| < \varepsilon, \quad 0 \le x \le l;$$

$$|\mu_i(t) - \mu_i^*(t)| < \varepsilon, \quad i = 1,2, \quad 0 \le t \le T;$$

tengsizliklar bajarilsin. U holda $\forall (x,t) \in Q$ uchun ekstremum prinsipidan kelib chiqqan 2–xulosaga ko'ra

$$|u_1(x,t) - u_2(x,t)| \le \max_{\Gamma} |u_1(x,t) - u_2(x,t)| =$$

$$= \max\{\max_{\overline{Q}} |f(x,t) - f^*(x,t)|,$$

$$\max_{x\in[0,l]} |\varphi(x) - \varphi^*(x)|, \max_{t\in[0,T]} |\mu_i(t) - \mu_i^*(t)|\}$$

bo'ladi. Bundan Q sohada $|u_1(x,t) - u_2(x,t)| < \varepsilon$ tengsizlikni olamiz. Bu tengsizlik (1)–(3) masala yechimining turg'un ekanligini bildiradi.

Mustaqil yechish uchun misollar

1. $\Omega=\{(x,t)$: $0< x < l$, $0 < t <+ \infty\}$ sohada bir jinsli bo'lmagan $u_t= a^2u_{xx}+f(x,t)$ issiqlik tarqalish tenglamasining bir jinsli $u(x,0)=0$ boshlang'ich va quyidagi chegaraviy shartlarni qanoatalantiruvchi yechimi topilsin:

6. $u(0,t) = u(l,t) = 0, \quad f(x,t) = \sin\frac{\pi x}{l};$

7. $u(0,t) = u(l,t) = 0, \quad f(x,t) = xe^{-t};$

8. $u(0,t) = u_x(l,t) = 0, \quad f(x,t) = \sin\frac{\pi x}{l} + \sin\frac{2\pi}{l}x;$

9. $u_x(0,t) = u(l,t) = 0, \quad f(x,t) = xt;$

10. $u_x(0,t) = u_x(l,t) = 0, \quad f(x,t) = f_0(x)$

3. Quyidagi aralash masalalar yechilsin:

$$11.\ u_t = u_{xx} + u + 2\sin 2x \sin x, \quad 0 < x < \frac{\pi}{2}, \quad t > 0;$$

$$u_x\big|_{x=0} = u\Big|_{x=\frac{\pi}{2}} = u\big|_{t=0} = 0;$$

$$12.\ u_t = u_{xx} + u + 2\sin 2x \cos x, \quad 0 < x < \frac{\pi}{2}, \quad t > 0;$$

$$u\big|_{x=0} = u_x\Big|_{x=\frac{\pi}{2}} = 1, \quad u\big|_{t=0} = x;$$

$$13.\ u_t = u_{xx} + 4u + x^2 - 2t - 4x^2 t + 2\cos^2 x,$$

$$0 < x < \pi, \quad 0 < t < +\infty.$$

$$u_x\big|_{x=0} = 0, \quad u_x\big|_{x=\pi} = 2\pi t; \quad u\big|_{t=0} = 0$$

$$14.\ u_t - u_{xx} - u = xt(2-t) + 2\cos t,\ 0 < x < \pi, \quad t > 0$$

$$u_x\big|_{x=0} = t^2, \quad u_x\big|_{x=\pi} = t^2; \quad u\big|_{t=0} = \cos 2x$$

$$15.\ u_t - u_{xx} - 9u = 4\sin^2 t \cos 3x - 9x^2 - 2, \quad 0 < x < \pi, \quad 0 < t < +\infty$$

$$u_x\big|_{x=0} = 0, \quad u_x\big|_{x=\pi} = 2\pi; \quad u\big|_{t=0} = x^2 + 2x$$

Tayanch iboralar.

Ekstremum printsipi, maksimum, minimum, chegaralangan sterjen, issiqlik tarqalishi, aralash masala, Fure usuli, regulyar yechim.

Nazorat uchun savollar.

1. Ekstremum printsipini ifodalovchi teoremani ayting.
2. Bir jinsli issiqlik tarqalish tenglamasi uchun aralash masalani qo’ying.
3. Regulyar yechimni ta’riflang.
4. Bir jinsli issiqlik tarqalish tenglamasi uchun aralash masala yechimini ifodalovchi formulani yozing.
5. 1- teoremani ayting.

13, 14-Mavzu. Issiqlik tarqalish tenglamasi uchun chegaraviy masalani Furye usuli bilan yechish. Bir jinsli bo'lmagan parabolik tenglama uchun chegaraviy masalani yechish. Issiqlik tarqalish tenglamasi uchun ikkinchi tur chegaraviy masalani Fuyre usuli yordamida yechish.

I. Asosiy tushunchalar

I.1 Aralash masala: Tekislikdagi $\Omega = \{(x,t): 0 < x < l, 0 < t < T\}$ sohada bir jinsli

$$u_t = a^2 u_{xx} \tag{1}$$

issiqlik tarqalish tenglamasining

$$u(x,0) = \varphi(x), \quad 0 \le x \le l \tag{2}$$

boshlang'ich va

$$u(0,t) = 0, \; u(l,t) = 0, \; 0 \le t < T \tag{3}$$

bir jinsli chegaraviy shartlarni qanoatlantiruvchi regulyar yechimi topilsin.

Ta'rif: (1) tenglamaning regulyar yoki klassik yechimi deb Ω sohada, tenglamada qatnashuvchi o'zining hosilalari bilan uzluksiz va tenglamani ayniyatga aylantiruvchi *u=u(x,y)* funksiyaga aytiladi.

Aralash masalani o'zgaruvchilarni ajratish (yoki Fure) usuli bilan yechamiz. Bu usulga asosan (1) tenglamaning yechimini

$$u(x,t) = X(x)T(t) \tag{4}$$

shaklda izlasak, quyidagi

$$X''(x) + \lambda X(x) = 0, \tag{5}$$

$$T'(x) + a^2 \lambda T(t) = 0 \tag{6}$$

ikkita oddiy differensial tenglama hosil bo'ladi, bunda λ = const. (4) ifoda va (3) chegaraviy shartlardan (5) tenglama uchun quyidagi

$$X(0) = X(l) = 0 \tag{7}$$

chegaraviy shartlar kelib chiqadi.

(5), (7) masala - xos son va xos funksiyalarni topish xaqidagi Shturm-Liuvill masalasi bo‘lib, u tor tebranish tenglamasi uchun aralash masalani yechishda ham kurilgan edi.

Bu masalaning xos sonlari $\lambda_n = \left(\frac{\pi n}{l}\right)^2, (n = 1, 2, ...),$ bu xos sonlarga mos trivial bo‘lmagan xos funksiyalari $X_n(x) = \sin\frac{\pi n}{l}x$ ko‘rinishda ekanligini aniqlagan edik. $\lambda = \lambda_n$ bo‘lganda (6) tenglamaning umumiy yechimi

$$T_n(t) = a_n e^{-(a\pi n/l)^2 t}$$

ko‘rinishga ega bo‘lib, (4) tenglikka asosan

$$U_n(x,t) = X_n(x)T_n(t) = a_n e^{-\left(\frac{\pi na}{l}\right)^2 t} \sin\frac{\pi n}{l}x$$

funksiyalar (a_n-ixtiyoriy, o‘zgarmas sonlar) (1) tenglamani va (3) chegaraviy shartni qanoatlantiradi. Tenglama bir jinsli bo‘lgani uchun bu yechimlar yig‘indisi yana yechim bo‘ladi. Shuning uchun (1) tenglamaning (2), (3) shartlarni qanoatlantiruvchi yechimini

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\pi na}{l}\right)^2 t} \sin\frac{\pi n}{l}x \qquad (8)$$

qator ko‘rinishida izlaymiz. Agar (8) funktsional qator va uning *t* bo‘yicha birinchi, *x* bo‘yicha ikkinchi tartibli hosilalari tekis yaqinlashuvchi bo‘lsa, u holda bu qator yig‘indisi (1) tenglamani va (3) chegaraviy shartlarni qanoatlantiradi. Boshlang‘ich shartni ham qanoatlantirishini talab qilsak,

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} a_n \sin\frac{\pi n}{l}x$$

tenglikka ega bo‘lamiz. Bu tenglikni $\varphi(x)$ funksiyaning $(0,l)$ oraliqdagi sinuslar bo‘yicha Fure qatoriga yoyilmasi desak, u holda a_n Fure koeffitsienti bo‘lib,

$$a_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{\pi n}{l}xdx \qquad (9)$$

formulabo‘yichatopiladi.

(9) tenglikkaasosan (1)-(3) masalaning (8) yechiminiquyidagiko‘rinishdayozishmumkin

$$u(x,t) = \int_0^l G(x,y,t)\varphi(y)dy, \qquad (10)$$

bu yerda

$$G(x,y,t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi na}{l}\right)^2 t} \sin\frac{\pi n}{l}x \sin\frac{\pi n}{l}y.$$

Bu funksiya oniy manbaning ta’sir funksiyasi deyiladi.

1- Teorema. *Agar [0,l] kesmada φ(x) funksiya*

1. *uzluksiz;*
2. *bo‘lakli-uzluksiz hosilaga ega va*
3. *φ(0)=φ(l)=0*

shartni qanoatlantirsa, u holda (8) qator aralash masalaning $\overline{\Omega}$ *da uzluksiz va cheksiz differensiallanuvchi yechimi bo‘ladi.*

***Izoh*.** Yuqoridagi teoremadan ko‘rinadiki, (10) funksiya aralash masalaning yechimi bo‘lishi uchun, (2) boshlang‘ich shartda berilgan *φ(x)* funksiya uzluksiz, bo‘lakli silliq va boshlang‘ich hamda chegaraviy shartlarning moslashganlik shartiga *(φ(0)=φ(l)=0)* bo‘ysunishi kerak. Lekin *φ(x)* funksiyaning uzluksizligi va moslashganlik shartini qanoatlantirishi amaliyot uchun og‘ir shartdir.

Masalan, $U_0 = const$ temperaturagacha isitilgan va chetlarida nol temperaturaga ega bo‘lgan, soviyotgan sterjenda issiqlik tarqalish masalasida, boshlang‘ich va chegaraviy shartlarning moslashganlik sharti bajarilmaydi, ya’ni $\varphi(0)= \varphi(l)=0=U_0 \neq 0$. Bu holda quyidagi teorema o‘rinlidir.

2- Teorema. *Agar [0,l] kesmada φ(x) funksiya bo‘lakli-uzluksiz (I - tur uzilishlarga ega) bo‘lsa, u xolda (10) funksiya:*

1) *Ω sohada (1) tenglamasining yechimi bo‘ladi;*
2) $\overline{\Omega} = \{(x,t): 0 \le x \le l, 0 \le t \le T\}$ *sohada chegaralangan;*
3) *(3) chegaraviy shartlarni qanoatlantiradi;*
4) *t=0 da φ(x) funksiyaning uzluksiz nuqtalarida uzluksiz va* $u(x,0)=\varphi(x)$ *bo‘ladi.*

I.2 Endi bir jinsli bo‘lmagan issiqlik tarqalish tenglamasini qaraymiz.

Tekislikdagi Ω sohada bir jinsli bo‘lmagan

$$u_t = a^2 u_{xx} + f(x,t) \tag{11}$$

issiqlik tarqalish tenglamasining (2) boshlang‘ich va (3) chegaraviy shartlarni qanoatlantiruvchi regulyar yechimi topilsin.

(11), (2), (3) macala yechimini *u(x,t)=v(x,t)+w(x,t)* ko‘rinishda izlaymiz, bundagi *v(x,t)* funksiyani bir jinsli bo‘lmagan (10) tenglamaning bir jinsli

$$v(x,0) = 0, \quad 0 \le x \le l \tag{12}$$

boshlang‘ich shartni va (3) chegaraviy shartlarni qanoatlantiruvchi yechimi, *w(x,t)* ni esa bir jinsli (1) tenglamaning (2) boshlang‘ich va (3) chegaraviy shartlarni qanoatlantiruvchi yechimi deb hisoblaymiz.

$v(x,t)$ funksiyani

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{xn}{l} x \tag{13}$$

qator ko‘rinishda izlaymiz. Bunda $T_n(t)$ noma’lum funksiyalar. (11) tenglamadan *f(x,t)* funksiyani ham $\sin\frac{\pi n}{l}x$ lar bo‘yicha Fure qatoriga yoyib yozsak,

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{\pi n}{l} x \tag{14}$$

bo‘lib, bunda

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{\pi n}{l} x dx$$

ko‘rinishda bo‘ladi.

Endi (13) va (14) ni (11) tenglamaga qo‘yib, noma’lum $T_n(t)$ funksiyalarga nisbatan

$$T_n'(t)+\left(\frac{\pi n}{l}a\right)^2 T_n(t)= f_n(t), \quad n=1,2,3,\dots \qquad (15)$$

differensial tenglamalarni hosil qilamiz.

(12) boshlang‘ich shartdan (13) ga asosan

$$T_n(0)=0, \quad n=1,2,3,\dots \qquad (16)$$

boshlang‘ich shartlar kelib chiqadi.

(15) tenglamaning (16) bir jinsli boshlang‘ich shartni qanoatlantiruvchi yechimi

$$T_n(t)=\frac{2}{l}\int_0^t\int_0^l f(x,t)\sin\frac{\pi n}{\ell}x e^{-\left(\frac{\pi n}{l}a\right)^2(t-\tau)} dx d\tau, \qquad (17)$$

ko‘rinishga ega bo‘ladi.

Shunday qilib, (11), (12), (3) masalaning yechimi

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{\pi n}{l}x+\sum_{n=1}^{\infty}a_n e^{-\left(\frac{\pi n}{l}a\right)^2 t}\sin\frac{\pi n}{l}x, \qquad (18)$$

ko‘rinishda bo‘ladi. Bu erda $T_n(t)$ funksiyalar (17) formuladan, a_n koeffitsient (9) formula orqali aniqlanadi.

I.3 Endi aralash masalaning umumiy ko‘rinishini qarab chiqaylik:

Ω sohada bir jinsli bo‘lmagan (11) tenglamaning (2) boshlang‘ich shartni va bir jinsli bo‘lmagan

$$u(0,t) = \mu_1(t), \quad u(l,t)=\mu_2(t), \qquad (19)$$

chegaraviy shartlarni qanoatlantiruvchi regulyar yechimi topilsin.

Bu masala yechimini $u(x,t)=z(x,t)+\omega(x,t)$ ko‘rinishda yozish mumkin. Bu erda ω*(x,t)* yordamchi funksiya bo‘lib, uni

ω(x,t)=A(t)x+B(t) ko‘rinishda izlaymiz, bunda *A(t)* va *V(t)* noma’lum funksiyalar. Bu noma’lum funksiyalarniω*(x,t)* funksiyani (19) chegaraviy shartlarni qanoatlantiradigan qilib tanlash natijasida topamiz.

Bunda *ω(x,t)* funksiya quyidagi ko‘rinishga ega bo‘ladi.

$$\omega(x,t)=\mu_1(t)+\frac{x}{l}\left[\mu_2(t)-\mu_1(t)\right].$$

z(x,t) funksiya esa bir jinsli bo‘lmagan

$$z_t=a^2z_{xx}+g(x,t) \tag{20}$$

issiqlik tarqalish tenglamasining bir jinsli bo‘lmagan

$$z(x,0)=\varphi(x)-\omega(x,0) \tag{21}$$

boshlang‘ich va bir jinsli

$$z(0,t)=z(l,t)=0 \tag{22}$$

chegaraviy shartlarni qanoatlantiruvchi yechimi. Bu erda

$$g(x,t)=f(x,t)-(\omega_t-a^2\omega_{xx})$$

(20), (21), (22) masala oldin echilgan (11), (2), (3) masalaga o‘xshashdir.

Izoh: Agar chegaraviy shartlarda nomahlum *u(x,t)* funksiyaning hosilasi ham qatnashsa, ba’zi hollarda *ω(x,t)* funksiyani

$$\omega(x,t)=A(t)x^2+B(t)x+C(t)$$

ko‘rinishda izlash mumkin, bunda *A(t), B(t), C(t)* noma’lum funksiyalar.

Ushbu

$$u_t-a^2u_{xx}-bu_x-cu=F(x,t)$$

tenglama uchun (2) va (3) shartlarni qanoatlantiruvchi aralash masala

$$u(x,t)=e^{\alpha x+\beta t}v(x,t),$$

almashtirish yordamida

$$v_t-a^2v_{xx}=e^{-\alpha x-\beta t}F(x,.t)$$

tenglamani va $v(x,0)=e^{-\alpha x}\varphi(x)$ boshlang‘ich hamda $v\left(0,t\right)=v(l,t)=0$ chegaraviy shartlarni qanoatlantiruvchi aralash masalaga keltirib yechiladi, bu yerda

$$\alpha=-\frac{b}{2a^2},\ \beta=c-\frac{b^2}{4a^2}.$$

II. Masalalarni yechish namunalari

1-masala. $\Omega=\{(x,t)\colon 0< x<l,\ 0<t<+\infty\}$ sohada $u_t = a^2 u_{xx}$ tenglamaning

$$u(x,0) = \begin{cases} x, & 0 < x \le l/2; \\ l-x, & \ell/2 \le x < l \end{cases}$$

boshlang'ichva$u(0,t)=u(l,t)=0$ chegaraviyshartlarniqanoatlantiruvchiyechimitopilsin.

Yechilishi: Masala yechimini (8) qator ko'rinishda izlaymiz. Bu qatorning koeffitsientini (9) formula yordamida topamiz:

$$a_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{\pi n}{l}xdx = \frac{2}{l}\left\{\int_0^{l/2} x\sin\frac{\pi n}{l}xdx + \int_{l/2}^{l}(l-x)\sin\frac{\pi n}{l}xdx\right\}$$

Ikkinchi integralda l-x=y almashtirish bajarib, ba'zi hisob-kitoblardan keyin, y ni yana x bilan almashtirib, ushbu

$$a_n = \frac{2}{l}\left[1-(-1)^n\right]\int_0^{l/2} x\sin\frac{\pi n}{l}xdx$$

tenglikka ega bo'lamiz. Bo'laklab itegrallash natijasida ushbuni topamiz:

$$a_n = 2\left[1-(-1)^n\right]\frac{l}{\pi n}\left\{-\frac{1}{2}\cos\frac{\pi n}{2} + \frac{1}{\pi n}\sin\frac{\pi n}{2}\right\}.$$

Topilgan a_n koeffitsientning qiymatini (8) qatorga qo'yib, masala yechimini hosil qilamiz:

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n]\left(-\frac{1}{2n}\cos\frac{\pi n}{2} + \frac{1}{\pi n^2}\sin\frac{\pi n}{2}\right)e^{-\left(\frac{\pi n}{l}a\right)^2 t}\sin\frac{\pi n}{l}x.$$

Agar $n=2k$ bo'lsa, $1-(-1)^n=0$, agar $n=2k+1$ bo'lsa, $1-(-1)^n=2$ va $\cos\frac{\pi n}{2} = \cos(\pi k + \frac{\pi}{2}) = 0$, $\sin\frac{\pi n}{2} = \sin(\pi k + \frac{\pi}{2}) = (-1)^k$ bo'lganligi uchun yechimni quyidagi ko'rinishda yozish mumkin:

$$u(x,t) = \frac{4l}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}e^{-\left(\frac{\pi(2k+1)}{l}a\right)^2 t}\sin\frac{\pi(2k+1)}{l}x.$$

2-masala. $\Omega=\{(x,t): 0 < x<1, 0 < t< +\infty\}$ sohada

$$u_t - u_{xx} = t(x+1) \tag{23}$$

tenglamaning

$$u(x,0)=0, \quad 0\leq x\leq 1 \tag{24}$$

boshlang‘ich va

$$u_x(0,t)=t^2, \quad u(1,t)=t^2 \tag{25}$$

chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin.

Yechilishi: Berilgan masalada $a=1$, $l=1$, $\varphi(x)=0$ bo‘lib, (25) chegaraviy shartlarda noma’lum $u(x,t)$ funksiyaning hosilasi qatnashganligi hamda bu shartning bir jinsli bo‘lmaganligi sababli bu masalani quyidagicha yechamiz.

Masala echimini $u(x,t)=\omega(x,t)+z(x,t)$ ko‘rinishda izlaymiz, bunda $\omega(x,t)$ funksiyani ushbu $\omega(x,t)=A(t)x+B(t)$ ko‘rinishda izlab, (25) chegaraviy shartlardan $A(t)=t^2$, $B(t)=0$ bo‘lishini, hamda $\omega(x,t)=xt^2$ ekanligini topamiz.

U holda

$$z(x,t)=u(x,t)- xt^2 \tag{26}$$

funksiya

$$z_t-z_{xx}=(1-x)t \tag{27}$$

tenglamani va

$$z(x,0)=0, \quad z_x(0,t)=z(1,t)=0 \tag{28}$$

shartlarni qanoatlantiruvchi aralash masalaning yechimi bo‘ladi.

(1)-(3) aralash masalani yechishdagi kabi $z_t-z_{xx}=0$ bir jinsli tenglamaning (28) dagi chegaraviy shartlarni qanoatlantiruvchi yechimini $z(x,t)=X(x)T(t)$ ko‘rinishda izlab,

$$X''(x)+\lambda^2 X(x)=0,$$
$$X'(0)=X(1)=0$$

Shturm-Liuvill masalasiga kelamiz. Bu masalaning xos sonlari $\lambda_n = \frac{\pi}{2} + \pi n, \quad n = 0,1,2,3,\ldots$ va bularga mos trivial bo‘lmagan xos funksiyalari $X_n(x)=c’s\lambda_n x$ ko‘rinishda ekanligini topamiz.

U holda (27), (28) masalaning yechimini

$$z(x,t) = \sum_{n=0}^{\infty} P_n(t) X_n(x) = \sum_{n=0}^{\infty} P_n(t) \cos \lambda_n x \tag{29}$$

ko‘rinishda izlaymiz va uni (27) tenglamaga qo‘yib,

$$\sum_{n=0}^{\infty} \left[T_n'(t) + \lambda_n^2 T_n(t) \right] \cos \lambda_n x = (1-x)t \tag{30}$$

tenglikni hosil qilamiz. 1-x funksiyani $X_n(x)=\cos\lambda_n x$ xos funksiyalar sistemasi bo‘yicha (0;1) intervalda Fure qatoriga yoyamiz.

$$\begin{aligned} 1-x &= \sum_{n=0}^{\infty} a_n \cos \lambda_n x \\ a_n &= 2\int_0^1 (1-x)\cos \lambda_n x dx = \frac{2}{\lambda_n^2} \end{aligned} \tag{31}$$

U holda (30) va (31) ni taqqoslab, noma’lum $T_n(t)$ funksiyalarga nisbatan

$$T_n'(t) + \lambda_n^2 T_n(t) = \frac{2t}{\lambda_n^2}, \quad n = 0,1,2,3,\ldots \tag{32}$$

differensial tenglamalarni hosil qilamiz.

(32) tenglamaning $T_n(0)=0$ boshlang‘ich shartni qanoatlantiruvchi yechimi

$$T_n(t) = 2\lambda_n^{-6} \left(e^{-\lambda_n^2 t} + \lambda_n^2 t - 1 \right) \tag{33}$$

ko‘rinishda bo‘ladi.

Shunday qilib, (26), (29) va (33) ga asosan (23)-(25) aralash masalaning yechimi

$$u(x,t) = xt^2 + 2\sum_{n=0}^{\infty} \lambda_n^{-6} \left(e^{-\lambda_n^2 t} + \lambda_n^2 t - 1 \right) \cos \lambda_n x$$

ko‘rinishda ekanligini topamiz, bu yerda $\lambda_n = \frac{\pi}{2} + \pi n$.

3-masala.$\Omega=\{(x,t):\ 0< x < 1,\ 0 < t < +\infty\}$ cohada $u_t=u_{xx}-2u_x+x+2t$, $u(x,0)=e^x \sin \pi x$, $u(0,t)=0$, $u(1,t)=t$, aralash masalaning yechimi topilsin.

***Yechilishi*:** Berilgan masalada

$a=1$, $l=1$, $b=-2$, $c=0$, $F(x,t)= x+2t$, $\varphi(x)=e^x\sin\pi x$, $\mu_1(t)=0$, $\mu_2(t)=t$ chegaraviy shart bir jinsli bo‘lmaganligi sababli masala yechimini

$$u(x,t) = z(x,t) + \omega(x,t) \tag{34}$$

ko‘rinishda izlaymiz. Bu yerda $\omega(x,t)$ yordamchi funksiya bo‘lib, uni faqat, chegaraviy shartlarni qanoatlantiradigan qilib tanlaymiz.

$\omega(x,t)=\mu_1(t)+\frac{x}{2}[\mu_2(t)-\mu_2(t)]$ ga asosan $\omega(x,t)=xt$ bo‘ladi.

U holda $z(x,t) = u(x,t)-xt$ funksiya uchun quyidagi $z_t=z_{xx}-2z_x$,

$$z(x,0)=e^x\sin\pi x,\quad z(0,t) = z(1,t)=0$$

aralash masalaga kelamiz. Bu masalada

$$z(x,t) = e^{x-t}\, v(x,t) \tag{35}$$

almashtirish bajarsak $v(x,t)$ noma’lum funksiyaga nisbatan ushbu

$$v_t = v_{xx}, \tag{36}$$

$$v(x,0)= e^{-x} z(x,0) = \sin \pi x,\quad v(0,t)= v(1,t)=0 \tag{37}$$

aralash masala hosil bo‘ladi. Bu masala yechimini (8) qator ko‘rinishida izlaymiz va uning koeffitsientini (9) formula yordamida topamiz:

$$a_n = 2\int_0^1 \sin\pi x \sin\pi n x\, dx = \begin{cases} 0, & агар\ \ n\neq 1, \\ 1, & агар\ \ n=1. \end{cases}$$

Demak, $a_n=0$, $n\neq 1$ bo‘lsa va $a_1=1$.

U holda (36)-(37) masalaning yechimi

$$v(x,t)=e^{-\pi^2 t}\sin\pi x \tag{38}$$

ko‘rinishda bo‘ladi.

Shunday qilib (34), (35) va (38) ga asosan berilgan masalani yechimi

$$u(x,t) = xt+\sin\pi x\cdot e^{x-t-\pi^2 t}$$

ko‘rinishda ekanligini topamiz.

Amaliy

1. Ω={(x,t): 0< x < l, 0 < t < + ∞} sohada bir jinsli ut = a2uxx issiqlik tarqalish tenglamasi uchun quyidagi aralash masalalar yechilsin

$$1. u(0,t) = u(l,t) = 0, \quad u(x,0) = Ax;$$

$$2. u(0,t) = u_x(l,t) = 0, \quad u(x,0) = \sin\frac{\pi}{2l}x;$$

$$3. u_x(0,t) = u(l,t) = 0, \quad u(x,0) = \cos\frac{\pi x}{2l};$$

$$4. u_x(0,t) = u_x(l,t) = 0, \quad u(x,0) = C, \quad C = const;$$

$$5. u(0,t) = u_x(l,t) = 0, \quad u(x,0) = \begin{cases} 0, & 0 < x < l/2 \\ u_0, & l/2 < x < l \end{cases}$$

2. Ω={(x,t): 0<x<l, 0 <t<+ ∞} sohadabirjinslibo‘lmaganu_t= $a^2u_{xx}+f(x,t)$ issiqliktarqalishtenglamasiningbirjinsli$u(x,0)=0$ boshlang‘ichvaquyidagichegaraviyshartlarniqanoatalantiruvchiyechimitopilsin:

$$6. u(0,t) = u(l,t) = 0, \quad f(x,t) = \sin\frac{\pi x}{l};$$

$$7. u(0,t) = u(l,t) = 0, \quad f(x,t) = xe^{-t};$$

$$8. u(0,t) = u_x(l,t) = 0, \quad f(x,t) = \sin\frac{\pi x}{l} + \sin\frac{2\pi}{l}x;$$

$$9. u_x(0,t) = u(l,t) = 0, \quad f(x,t) = xt;$$

$$10. u_x(0,t) = u_x(l,t) = 0, \quad f(x,t) = f_0(x)$$

3. Quyidagi aralash masalalar yechilsin:

$$11. u_t = u_{xx} + u + 2\sin 2x \sin x, \quad 0 < x < \frac{\pi}{2}, \quad t > 0;$$

$$u_x\big|_{x=0} = u\Big|_{x=\frac{\pi}{2}} = u\big|_{t=0} = 0;$$

$$12. u_t = u_{xx} + u + 2\sin 2x \cos x, \quad 0 < x < \frac{\pi}{2}, \quad t > 0;$$

$$u\big|_{x=0} = u_x\Big|_{x=\frac{\pi}{2}} = 1, \quad u\big|_{t=0} = x;$$

13. $u_t = u_{xx} + 4u + x^2 - 2t - 4x^2 t + 2\cos^2 x,$

$0 < x < \pi, \quad 0 < t < +\infty.$

$u_x\big|_{x=0} = 0, \quad u_x\big|_{x=\pi} = 2\pi t; \quad u\big|_{t=0} = 0$

14. $u_t - u_{xx} - u = xt(2-t) + 2\cos t, 0 < x < \pi, \quad t > 0$

$u_x\big|_{x=0} = t^2, \quad u_x\big|_{x=\pi} = t^2; \quad u\big|_{t=0} = \cos 2x$

15. $u_t - u_{xx} - 9u = 4\sin^2 t \cos 3x - 9x^2 - 2, \quad 0 < x < \pi, \quad 0 < t < +\infty$

$u_x\big|_{x=0} = 0, \quad u_x\big|_{x=\pi} = 2\pi; \quad u\big|_{t=0} = x^2 + 2x$

Tayanch iboralar.

CHegaralangan sterjen, issiqlik tarqalishi, aralash masala, Furg'e usuli, regulyar yechim.

Nazorat uchun savollar.

6. Bir jinsli issiqlik tarqalish tenglamasi uchun aralash masalani qo'ying.
7. Regulyar yechimni ta'riflang.
8. Bir jinsli issiqlik tarqalish tenglamasi uchun aralash masala yechimini ifodalovchi formulani yozing.
9. 1- teoremani ayting.
10. 2- teoremani ayting.
11. Bir jinsli bo'lmagan issiqlik tarqalish tenglamasining bir jinsli chegaraviy shartlarni qanoatlantiruvchi yechimining ko'rinishini yozing.

15-Mavzu: Koshi masalasi va uning yechimini yagonaligi, mavjudligi va -turg‘unligi. Fundamental yechim

I. Asosiy tushunchalar.

I.1 Koshi masalasi $\Omega=\{(x,t): 0< x < l , 0 < t < + \infty\}$ sohada

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

tenglamaning

$$u\big|_{t=0} = u(x,0) = f(x), \quad -\infty < x < +\infty \tag{2}$$

boshlang‘ich shartni qanoatlantiruvchi yechimi topilsin.

Bu masalani yechish uchun Furening o‘zgaruvchilarni ajratish usuli va xususiy yechimlar superpozitsiyasidan foydalanamiz.

(1) tenglamaning yechimini $u(x,t)=X(x)T(t)$ ko‘rinishda qidiramiz. Bu ko‘paytmadan hosilalar olib (1) tenglamaga qo‘ysak

$$\frac{T'}{a^2 T} = \frac{X''}{X} = \mu, \quad \mu = const$$

tenglikhosilbo‘ladi. Butengliklardanquyidagiikkitatenglamalarnihosilqilamiz:

$$T'(t)-\mu a^2\, T(t)=0$$

$$X''(x)-\mu\, X(x)=0$$

Bularni yechib,

$$T(t) = c\, e^{a^2 \mu t}, \quad X(x) = A\cos\sqrt{\mu}\, x + B\sin\sqrt{\mu}\, x$$

larni topamiz. $t\to\infty$ da issiqlik cheksizga intilishi mumkin emas. Shuning uchun $\mu = -\lambda^2$ deb olamiz. U holda (1) tenglamaning xususiy yechimlari quyidagiga teng bo‘ladi:

$$u_\lambda(x,t) = (\alpha\cos\lambda x + \beta\sin\lambda x)e^{-\lambda^2 a^2 t}$$

Bu yerda α,β,λ lar ixtiyoriy o‘zgarmas sonlar. λ ning har bir qiymatida turli α va β larni aniqlash mumkin, ya’ni α va β lar λ ning ixtiyoriy funksiyalari $\alpha = \alpha(\lambda)$, $\beta = \beta(\lambda)$ bo‘ladi. U holda xususiy yechimlar ushbu ko‘rinishni oladi.

$$u_\lambda(x,t)=\left[\alpha(\lambda)\cos\lambda x+\beta(\lambda)\sin\lambda x\right]e^{-a^2\lambda^2 t}$$

Bu yerda λ parametr - ∞ dan + ∞ gacha qiymatlarni oladi. Berilgan (1) tenglama chiziqli va bir jinsli. Uning cheksiz ko‘p xususiy yechimlari mavjud va bu yechimlar uzluksiz o‘zgaruvchi λ parametrga bog‘liq. Shuning uchun xususiy yechimlar superpozitsiyasiga asosan $u_\lambda(x,t)$ yechimlarning λ uzluksiz parametr bo‘yicha integrali ham yechim bo‘ladi:

$$u(x,t)=\int_{-\infty}^{+\infty}u_\lambda(x,t)d\lambda=\int_{-\infty}^{+\infty}\left[\alpha(\lambda)\cos\lambda x+\beta(\lambda)\sin\lambda x\right]e^{-a^2\lambda^2 t}d\lambda. \qquad (3)$$

Boshlang‘ich (2) shartdan foydalanib, noma’lum α(λ) va β(λ) larni aniqlaymiz:

$$u\big|_{t=0}=\int_{-\infty}^{+\infty}\left[\alpha(\lambda)\cos\lambda x+\beta(\lambda)\sin\lambda x\right]d\lambda=f(x). \qquad (4)$$

Bu yerda berilgan *f(x)* funksiya (-∞;+ ∞) intervalda berilgan va absolyut integrallanuvchi, ya’ni $\int_{-\infty}^{+\infty}|f(x)|dx$ integral yaqinlashuvchi bo‘lsin.

Matematik analiz kursidan ma’lumki, yuqoridagi shartlar bajarilganda *f*(*x*) uchun Furening integral formulasi

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\lambda\int_{-\infty}^{+\infty}f(\xi)e^{i\lambda(x-\xi)}d\xi \quad \text{yoki}$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\lambda\int_{-\infty}^{+\infty}f(\xi)\cos\lambda(\xi-x)d\xi=$$

$$=\int_{-\infty}^{\infty}\left\{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\cos\lambda\xi d\xi\right)\cos\lambda x+\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\sin\lambda\xi d\xi\right)\sin\lambda x\right\}d\lambda$$

o‘rinli bo‘ladi. Bu tenglikni (4) bilan taqqoslab, ushbuni hosil qilamiz:

$$\left.\begin{aligned}\alpha(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\cos\lambda\xi d\xi \\ \beta(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\sin\lambda\xi d\xi\end{aligned}\right\}$$

Bu topilgan $\alpha(\lambda)$ va $\beta(\lambda)$ larni (3) yechimga qo‘ysak, (1) tenglama va (2) boshlang‘ich shartni qanoatlantiruvchi funksiyani hosil qilamiz:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} f(\xi)\cos\lambda(x-\xi)e^{-a^2\lambda^2 t}d\xi .$$

Bunda integrallash tartibini o‘zgartirib quyidagini hosil qilamiz:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\left\{\int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t}\cos\lambda(x-\xi)d\lambda\right\}d\xi$$

Ushbu

$$\int_{-\infty}^{+\infty} e^{-\alpha\lambda^2}\cos\lambda\beta d\lambda = 2\int_{-\infty}^{+\infty} e^{-\alpha\lambda^2}\cos\beta\lambda d\lambda = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\beta^2}{4\alpha}};(\alpha > 0) \qquad (5)$$

formuladan foydalanib, katta qavs ichidagi integralni hisoblaymiz va o‘rniga qo‘yib, quyidagini hosil qilamiz:

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4a^2 t}}d\xi \qquad (6)$$

Puasson integrali deb ataluvchi bu integral (1), (2) Koshi masalasining yechimidir.

(6) formulada ishtirok etgan

$$G(x,t,\xi) = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-\xi)^2}{4a^2 t}} \qquad (7)$$

funksiya (1) tenglamaning fundamental yechimi deb nomlanadi.

Teorema. *Agar f(x) funksiya sonlar o’qida uzluksiz va chegralangan bo‘lsa, u holda* (6) *Puasson integrali* (1), (2) *masalaning uzluksiz va chegaralangan u*(*x,t*) *yagona yechimini aniqlaydi.*

II. Masalalarni yechish namunalari

1-masala. $4\cdot u_t = u_{xx}$ bir jinsli issiqlik tarqalish tenglamasining $u\big|_{t=0} = \sin x e^{-x^2}$ boshlang‘ich shartni qanoatlantiruvchi yechimi topilsin.

***Yechilishi*.** Berilgan masalada $a = 1/2$, $f(x) = \sin x\, e^{-x^2}$ bo‘lganligi uchun (6) formulaga asosan yechim

$$u(x,t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} \sin \xi\, e^{-\xi^2} \cdot e^{-\frac{(x-\xi)^2}{t}} d\xi$$

ko‘rinishda bo‘ladi. Bu integralda $\dfrac{x-(1+t)\xi}{\sqrt{t(1+t)}} = \mu$ almashtirish bajarsak, quyidagiga ega bo‘lamiz:

$$u(x,t) = \frac{1}{\sqrt{\pi}\sqrt{1+t}} \cdot e^{-\frac{x^2}{1+t}} \int_{-\infty}^{+\infty} e^{-\mu^2} \sin\left[\frac{x-\sqrt{t(1+t)}\mu}{1+t}\right] d\mu =$$

$$= \frac{1}{\sqrt{\pi}(1+t)} \cdot e^{-\frac{x^2}{1+t}} \left\{ \sin\frac{x}{1+t} \int_{-\infty}^{+\infty} e^{-\mu^2} \cos\sqrt{\frac{t}{1+t}}\mu d\mu - \cos\frac{x}{1+t} \int_{-\infty}^{+\infty} e^{-\mu^2} \sin\sqrt{\frac{t}{1+t}}\mu d\mu \right\}$$

Birinchi integralning (5) formulaga asosan

$$\int_{-\infty}^{+\infty} e^{-\mu^2} \cos\sqrt{\frac{t}{1+t}}\mu d\mu = \sqrt{\pi} e^{-\frac{t}{4(1+t)}}$$

ga tengligini hamda ikkinchi integralning integral ostidagi funksiyasi μ - ga nisbatan toq funksiya bo‘lganligi uchun nolga tengligini hisobga olsak, izlangan *u(x,t)*yechim quyidagi

$$u(x,t) = (1+t)^{-1/2} \sin\frac{x}{1+t} \quad e^{-\frac{4x^2+t}{4(1+t)}}$$

ko‘rinishda bo‘lishini topamiz.

Bu misoldan ko‘rinadiki, agar boshlang‘ich $f(x)$ funksiya sonlar o‘qida uzluksiz va chegaralangan bo‘lsa, u holda Koshi masalasining yechimi ham uzluksiz va chegaralangan ekanligini ko‘ramiz. Lekin ko‘pincha amaliy masalalarda *f(x)* funksiya chekli sondagi uzilish nuqtalariga ega bo‘ladi.

Agar *f(x)* funksiya uzilish nuqtalariga ega bo‘lsa, u holda Koshi masalasining yechimi qanday ko‘rinishda bo‘ladi? degan tabiiy savol tug‘iladi. Misol sifatida quyidagi masalani qaraylik.

2-masala. $u_t=a^2u_{xx}$ tenglamaning ushbu

$$u\big|_{t=0} = f(x) = \begin{cases} T_1, & x \geq 0; \\ T_2, & x < 0; \quad T_1, T_2 = const, T_1 \neq T_2 \end{cases}$$

boshlang‘ich shartni qanoatlantiruvchi yechimi topilsin.

Yechilishi. Berilgan masalada boshlang‘ich *f(x)* funksiya *x*=0 nuqtada uzilishga ega. Bu holda ham masalaning yechimini (6) formula ko‘rinishida izlaymiz:

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi = \frac{T_2}{2a\sqrt{\pi t}} \int_{-\infty}^{0} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi + \frac{T_1}{2a\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi =$$

$$= \frac{T_2}{\sqrt{\pi}} \int_{-\infty}^{-\frac{x}{2a\sqrt{t}}} e^{-\mu^2} d\mu + \frac{T_1}{\sqrt{\pi}} \int_{-\frac{x}{2a\sqrt{t}}}^{-\infty} e^{-\mu^2} d\mu = \frac{T_1+T_2}{2} + \frac{T_1-T_2}{2} \int_{0}^{-\frac{x}{2a\sqrt{t}}} e^{-\mu^2} d\mu$$

(11) formulaga asosan izlangan yechimini ushbu

$$u(x,t) = \frac{T_1+T_2}{2} + \frac{T_1-T_2}{2} \Phi\left(\frac{x}{2a\sqrt{t}}\right)$$

ko‘rinishda yoza olamiz.

Bu formulada ko‘rinadiki, *u(x,t)* yechimning ixtiyoriy *t* uchun *x*=0 nuqtadagi qiymati $u(0,t) = \frac{T_1+T_2}{2}$ ga teng ekan. Demak, agar *f(x)* funksiya chekli sondagi uzilish nuqtalariga ega bo‘lsa, u holda (6) Puasson integrali Koshi masalasining *f(x)* funksiya uzilish nuqtalaridan tashqari, barcha nuqtalarda uzluksiz va chegaralangan yechimini berar ekan.

III. Mustaqil yechish uchun masalalar.

Issiqlik tarqalish tenglamasi uchun Koshi masalasi yechimini Puasson inetegralidan foydalanib toping:

1. $u_t - u_{xx}, \quad u(x,0) = xe^{-x^2};$

2. $4u_t - u_{xx}, \quad u(x,0) = e^{2x-x^2};$

3. $u_t = a^2 u_{xx}, \quad u(x,0) = Te^{-\left(x/a\right)^2}, T = const;$

4. $u_t - u_{xx} + 3t^2, \quad u(x,0) = \sin x;$

5. $u_t - u_{xx} + e^{-t}, \cos x, \quad u(x,0) = \cos x; \backslash$

6. $u_t = u_{xx} + e^t \sin x, \quad u(x,0) = \sin x;$

7. $u_t = 4u_{xx} + t + e^t, \quad u(x,0) = \begin{cases} T, & x_1 < x < x_2 \\ 0, & x < x_1 \text{ ёки } x > x_2 \end{cases}$

8. $u_t = 2u_{xx}, \quad u(x,0) = \begin{cases} 1030, & |x| < l \\ 30, & |x| > l; \end{cases}$

9. $4u_t = u_{xx} + 8t, \quad u(x,0) = \begin{cases} 0, |x| \geq h; \\ -T, -h < x \leq 0; \\ T, \ 0 < x < h \end{cases}$

10. $u_t = a^2 u_{xx} + bu_x + u + 1,, \quad u(x,0) = 1;$

11. $u_t = 2u_{xx} + 2u + e^t, \quad u(x,0) = \cos x;$

12. $u_t = u_{xx} + u + t \sin x, \quad u(x,0) = 1;$

13. $u_t - a^2 u_{xx} - bu_x - cu = 0, \quad u(x,0) = e^{-x^2};$

14. $u_t - u_{xx} - u_x = 0, \quad u(x,0) = \sin x;$

15. $u_{xx} - u_t + \lambda^2 u = 0, \quad u(x,0) = \varphi(x).$

Tayanch iboralar

CHegaralanmagan sterjen, integral almashtirish, Furening integral almashtirish usuli, absolyut integrallanuvchanlik, Fure integrali, Puasson integrali, fundamental yechim.

Nazorat uchun savollar.

1. Bir jinsli issiqlik tarqalish tenglamasi uchun Koshi masalasini qo’ying.
2. Absolyut integrallanuvchanlikshartini yozing.
3. Fure integralini yozing.
4. Fure kosinus va sinus almashtirishlari formulalarini yozing.
5. Bir jinsli issiqlik tarqalish tenglamasi uchun Koshi masalasi yechimini ifodalovchi Puasson integralini yozing.
6. Fundamental yechim ko’rinishini yozing.

16-Mavzu. Bir jinsli bo‘lmagan parabolik tenglama uchun Koshi masalasi.

I. Asosiy tushunchalar.

Bir jinsli bo‘lmagan issiqilik tarqalish tenglamasi.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial u}{\partial x^2} + g(x,t) \tag{8}$$

uchun Koshi masalasining yechimi quyidagi ko‘rinishda bo‘ladi:

$$u(x,t) = \int_{-\infty}^{+\infty} f(\xi)G(x,t,\xi)d\xi + \int_0^t d\tau \int_{-\infty}^{+\infty} G(x,t,-\tau,\xi)g(\xi,\tau)d\xi \tag{9}$$

Bunda $g(x,t)$ va $g_x(x,t)$ uzluksiz va chegaralangan deb faraz qilinadi.

Quyidagi

$$u_t - a^2u_{xx} - bu_x - cu=F(x,t)$$

tenglama uchun (2) boshlang‘ich shartni qanoatlantiruvchi Koshi masalasi.

$$u\ (x,t) = e^{\,c\tau}v(y,\tau) \tag{10}$$

almashtirish yordamida

$$v_\tau = a^2v_{yy} + e^{-c\tau}F(y,-b\tau,\tau)$$

tenglama uchun $v(y,0)=f(y)$ boshlang‘ich shartni qanoatlantiruvchi Koshi masalasiga keltirib yechiladi, bu yerda $y=x+bt,\ \tau=t,\ b,c=const$.

<u>Izoh</u>. Koshi masalasini yechish davomida, (6) va (9) formulalardan foydalanganda, ba’zan shunday integrallar hosil bo‘ladiki, ularni elementar funksiyalar yordamida hisoblab bo‘lmay qoladi, lekin ko‘p hollarda ularni extimollar integrali deb ataluvchi.

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\mu^2}d\mu \tag{11}$$

integral orqali ifodalash mumkin bo‘ladi, Bunda *F(x)* funksiya sonlar o‘qida o’suvchi, $F(0)=0$, $\lim\limits_{x\to-\infty}\Phi(x)=-1,\ \lim\limits_{x\to+\infty}\Phi(x)=1$ bo‘lib, uning boshqa x nuqtalardagi qiymatlari ko‘pgina jadvallarda keltirilgan.

3-masala. $u_t=u_{xx}+\sin t$ tenglamaning $u\big|_{t=0}=e^{-x^2}$ boshlang'ich shartni qanoatlantiruvchi yechimi topilsin.

***Yechilishi*:** Bu masalada $a=1, f(x)=e^{-x^2}$, $g(x,t)=\sin t$ bo'lganligi uchun izlanayotgan yechimni (9) formuladan topamiz:

$$u(x,t)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\xi^2}\,e^{-\frac{(x-\xi)^2}{4t}}d\xi+\frac{1}{2\sqrt{\pi}}\int_0^t\frac{\sin\tau}{\sqrt{t-\tau}}d\tau\cdot\int_{-\infty}^{+\infty}e^{-\frac{(x-\xi)^2}{4\sqrt{t-\upsilon}}}d\xi=$$

$$=\frac{1}{\sqrt{\pi}(1+4t)}e^{-\frac{x^2}{1+4t}}\int_{-\infty}^{+\infty}e^{-\mu^2}d\mu+\frac{1}{\sqrt{\pi}}\int_0^t\sin\tau\,d\tau\int_{-\infty}^{+\infty}e^{-\mu^2}d\mu,$$

$\int_{-\infty}^{+\infty}e^{-\mu^2}d\mu=\sqrt{\pi}$ ekanligini hisobga olsak, izlanayotgan yechimni quyidagi ko'rinishda topamiz:

$$u(x,t)=(1+4t)^{-\frac{1}{2}}e^{-\frac{x^2}{1+4t}}-\cos t+1$$

4-masala $u_t-a^2u_{xx}-bu_x-u=e^t\sin x$ tenglamaning $u\big|_{t=0}=A$, $(A=const)$ boshlang'ich shartni qanoatlantiruvchi yechimi topilsin.

***Yechilishi*:** Bu masalada $c=1$, $F(x,t)=e^t\sin x, f(x)=A$, bo'lib uni (10) almashtirish yordamida $v_\tau=a^2v_{yy}+\sin(y-b\tau)$, $v(y,0)=A$ masalaga keltiramiz. Bu yangi masalada $f(y)=A$, $g(y,\tau)=\sin(y-b\tau)$ bo'lganligi uchun uning yechimini (9) formuladan topamiz.

$$v(y,\tau)=\frac{A}{2a\sqrt{\pi\tau}}\int_{-\infty}^{+\infty}e^{-\frac{(y-\xi)^2}{4a^2\tau}}d\xi+\frac{1}{2a\sqrt{\pi}}\int_0^\tau\frac{d\eta}{\sqrt{\tau-\eta}}\int_{-\infty}^{+\infty}\sin(\xi-b\eta)e^{-\frac{(y-\xi)^2}{4a^2(\tau-\eta)}}d\xi$$

(-∞;+∞) bo'yicha olingan inegralni oldingi misoldagi kabi almashtirish bajarib hisoblaganimizdan keyin, quyidagi

$$v=(y,\tau)=A+e^{-a^2\tau}\int_0^\tau e^{a^2\eta}\sin(y-b\eta)d\eta$$

formulaga ega bo‘lamiz. Bundagi integralni ikki marta bo’laklab integrallab, ushbu

$$v(y,\tau) = A + \frac{b}{a^4+b^2}\left[\cos(y-b\tau) + \frac{a^2}{b}\sin(y-b\tau)\right] - \frac{b}{a^4+b^2}e^{-a^2\tau}\left[\cos y + \frac{a^2}{b}\sin y\right]$$

yechimni hosil qilamiz. Endi eski o‘zgaruvchilarga qaytib, berilgan masalaning yechimini

$$u(x,t) = e^t\left\{A + \frac{b}{a^4+b^2}\left[\cos x + \frac{a^2}{b}\sin x\right] - \frac{b}{a^4+b^2}e^{-a^2\tau}\left[\cos(x+bt) + \frac{a^2}{b}\sin(x+bt)\right]\right.$$

ko‘rinishda ekanligini topamiz.

III. Mustaqil yechish uchun masalalar.

Issiqlik tarqalish tenglamasi uchun Koshi masalasi yechimini Puasson inetegralidan foydalanib toping:

1. $u_t - u_{xx},\quad u(x,0) = xe^{-x^2};$

2. $4u_t - u_{xx},\quad u(x,0) = e^{2x-x^2};$

3. $u_t = a^2u_{xx},\quad u(x,0) = Te^{-\left(x/a\right)^2},\ T = const;$

4. $u_t - u_{xx} + 3t^2,\quad u(x,0) = \sin x;$

5. $u_t - u_{xx} + e^{-t}, \cos x,\quad u(x,0) = \cos x;\backslash$

6. $u_t = u_{xx} + e^t\sin x,\quad u(x,0) = \sin x;$

7. $u_t = 4u_{xx} + t + e^t,\quad u(x,0) = \begin{cases} T, & x_1 < x < x_2 \\ 0, & x < x_1 \text{ ёки } x > x_2 \end{cases}$

8. $u_t = 2u_{xx},\quad u(x,0) = \begin{cases} 1030, & |x| < l \\ 30, & |x| > l; \end{cases}$

9. $4u_t = u_{xx} + 8t,\quad u(x,0) = \begin{cases} 0, |x| \ge h; \\ -T, -h < x \le 0; \\ T,\ 0 < x < h \end{cases}$

10. $u_t = a^2 u_{xx} + bu_x + u + 1,, \quad u(x,0) = 1;$

11. $u_t = 2u_{xx} + 2u + e^t, \quad u(x,0) = \cos x;$

12. $u_t = u_{xx} + u + t \sin x, \quad u(x,0) = 1;$

13. $u_t - a^2 u_{xx} - bu_x - cu = 0, \quad u(x,0) = e^{-x^2};$

14. $u_t - u_{xx} - u_x = 0, \quad u(x,0) = \sin x;$

15. $u_{xx} - u_t + \lambda^2 u = 0, \quad u(x,0) = \varphi(x).$

Tayanch iboralar

CHegaralanmagan sterjen, integral almashtirish, Furg'ening integral almashtirish usuli, absolyut integrallanuvchanlik, Furg'e integrali, Puasson integrali, fundamental yechim.

Nazorat uchun savollar.

7. Bir jinsli issiqlik tarqalish tenglamasi uchun Koshi masalasini qo'ying.
8. Absolyut integrallanuvchanlikshartini yozing.
9. Furg'e integralini yozing.
10. Furg'e kosinus va sinus almashtirishlari formulalarini yozing.
11. Bir jinsli issiqlik tarqalish tenglamasi uchun Koshi masalasi yechimini ifodalovchi Puasson integralini yozing.

Asosiy adabiyotlar

1. Saloxiddinov M.S. Matematik fizika tenglamalari. Toshkent. «O‘zbekiston», 2002.
2. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М. Изд-во МГУ. 2004.
3. Бицадзе А.В., Калиниченко Д.Ф. Сборник задач по уравнениям матема¬тичес¬кой физики. М. 1985.
4. Salohiddinov M., Islomov B. Matmatik fizika tenglamalari fanidan masalalar to’plami. –Toshkent, Universitet. 2017. 370 b.
5. Zikirov O. S. Matmatik fizika tenglamalari. Toshkent – “Fan va texnologiya”. 2017. 320 b.

Qo’shimcha adabiyotlar

6. Wolter A.Stranss. Partial Differential Equations; An introduction. Birkhhauzer. Germany, 2005.
7. Davia D.Bleecker, George Csordes. Basic of Partial Differential Equations. Birkhhauzer. Germany, 2009.
8. Будак Б.М., Самарский А.А., Тихонов А.Н. Сборник задач по математической физике. М. 1972.
9. Михлин С.Г. “Курс математической физики”.М.: Наука. 1968.
10. Владимиров В.С., Жаринов В.В. Уравнения математической физики. Учебник для ВУЗов. М.: ФИЗМАТЛИТ. 2004.
11. Владимиров В.С., и др. Сборник задач по уравнениям математической физики. М.: ФИЗМАТЛИТ. 2004. 286 с.
12. Сабитов К.Б. Уравнения математической физики. Учебник для ВУЗов. М.: ФИЗМАТЛИТ. 2013. 352 с.
13. Кошляков В.С., Глинер Э.Б., Смирнов М.М. Основные дифференциаль-ные уравнения математической физики. М. 1962.
14. Петровский И.Г. Лекции об уравнениях с частными производными. М., 1961.
15. Владимиров В.С. Обобщенные функции в математической физике. М. “Наука”.1979.
16. Ильин А.М. Уравнения математической физики. М.: ФИЗ-МАТЛИТ. 2009. 192 с.
17. Pinsky Mark A. Partial differential equations and boundary-value problems with applications.Amerkan Mathematical Society. 2011. 527 p.
18. Зикиров О.С. Хусусий ҳосилали дифференциал тенгламалар. Тошкент, “Университет”. 2012. 260 бет.

Axborot manbaalari

1. www.lib.homelinex.org/math
2. www.eknigu.com/lib/Mathematics/
3. www.eknigu.com/info/M_Mathematics/MC
4. http://www.raai.org/libraryhttp://www.raai.org/library
   http://www.intuit.ru